

STUDIES IN LOGIC

AND

THE FOUNDATIONS OF MATHEMATICS

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A TRANSFINITE TYPE THEORY WITH TYPE VARIABLES

by

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PREFACE

The purpose of this book is to present a new system of transfinite type theory. The system constitutes a formal language in which one can express certain concepts, and use certain modes of reasoning, which naturally occur in informal discussions of the interpretations of type theories (and, I might add, set theories). While the formation rules contain certain novel features, the reader will see that the deductive structure of the system is founded upon very familiar logical principles. It is my hope that the reader who takes the trouble to become genuinely familiar with the workings of the system will come to regard it as a rather simple and natural system.

It may then be asked how the system should be extended to higher transfinite types. It is clear that the basic ideas underlying this system can be used to formulate higher transfinite type theories, but at this time it is not clear to me that there is a single most appropriate way of extending the system to higher types. Therefore it has seemed best to postpone this question, and present only the basic system which one would expect to have as a part of any transfinite type theory constructed along the present lines.

The book should be comprehensible to students who have had a good course in mathematical logic, and a brief introduction to type theory with λ -conversion. It is to be hoped that the rather detailed treatment of a number of topics, such as the formalization of the semantics of type theory, will be of educational value to students.

I would like to express my deep appreciation to the North-Holland Publishing Company for its expert and sympathetic handling of the many difficulties presented by the manuscript.

September, 1965

Peter B. Andrews

INTRODUCTION

As the human quests for ever deeper and broader knowledge and understanding, and for increasingly abstract insights into the structure and unity of that which is discovered, carry the intellect into realms farther and farther removed from naive intuitive comprehension, the need for ever more precise expression of thought will undoubtedly grow continually. We may suppose that expanding research in metamathematics, increasing sophistication in the use of digital computers, and other factors will eventually render desirable the actual formalization of certain fields of knowledge which have attained the requisite precision. In particular one may anticipate eventual attempts to formalize certain scientific theories.

It seems appropriate to ask, therefore, what sort of logistic system might appropriately serve as an underlying logic to which the constants and postulates of the particular theory might be added to obtain the formalized theory. Presumably the theory will presuppose the existence of certain fundamental entities from which all other structures and concepts in the theory are constructed. If particular names are assigned to certain entities, structures, or concepts, these will be added to the logistic system as constants, and the fundamental assumptions and definitions of the theory will be added as postulates.

Clearly the underlying logistic system should impose no assumptions about the cardinality of the domain of fundamental entities, except that the domain be non-empty, since such an assumption might contradict or restrict the theory being formalized. Moreover the logistic system should impose no assumptions about the structure of the fundamental entities, since this would constitute a perhaps undesirable extension of the theory. The formalized theory should express simply and directly the content of the intuitive theory, without artificial conventions or encum-

branches except insofar as these are necessary to insure against fallacious reasoning.

Thus set theories which take the axiom of extensionality in the form $\forall x[x \in y \equiv x \in z] \supset [y = z]$ are not appropriate, since this axiom assures the existence of no more than one non-set (commonly called the empty set), and there is no reason to suppose that the fundamental entities will be sets. (Indeed, if they are sets, they are presumably composed of still more fundamental entities.) Much more seriously, the great freedom of expression permitted in set theory might lead to expressions in the formalized language which would be difficult to interpret in the intuitive theory, since they would correspond to nothing in the intuitive theory. We shall leave to others further consideration of the problem of using some form of set theory as the underlying logic, and henceforth turn our attention to functional calculi of appropriately high orders and simple type theory.

Since in the development of a scientific theory one feels free at any time to use any mathematical theory which is known or may be developed, it seems desirable that the underlying logic should be such that any branch of mathematics can be developed within it. This suggests that a functional calculus of finite order might not be adequate.

As a result of the research of Whitehead and Russell culminating in *Principia Mathematica* [19]¹⁾, it is clear that extensive mathematics can be developed within type theory if an Axiom of Infinity is assumed. However, the domain of individuals of the type theory is most naturally taken to be the domain of fundamental entities of the scientific theory, and this may not be infinite. Of course one might postulate the existence of another and infinite domain of individuals, but this would prompt questions concerning the nature of, and justification for, these individuals.

From an intuitive point of view the natural thing to do is to pass to transfinite type theory, for the domain with the first transfinite type consists of (or at least contains) the union of all domains of finite type, and is therefore infinite. Thus the natural numbers and other mathematical concepts can be defined at the transfinite level while the "real world" objects of the scientific theory exist at the level of finite types.

¹⁾ Numbers in brackets refer to entries in the bibliography.

It is not immediately clear, however, whether a satisfactory formulation of transfinite type theory which meets these requirements can be found, for each of the principal systems of transfinite type theory which has appeared in the literature makes some assumption about the cardinality of the domain of individuals. (Bustamante's system in [3], and Kemeny's system T_γ in [12], which is essentially a simplification of Bustamante's system, each have axioms asserting that the domain of lowest type is finite²), L'Abbé's system of transfinite type theory (see [13] or [14]) takes the domain of individuals to be the natural numbers, and thus requires that the domain of individuals be infinite.) Bustamante does succeed in showing that the natural numbers can be defined at the transfinite level, and the Peano postulates proved. However Bustamante's system has axioms asserting the existence of certain entities of transfinite type for which one would hope and expect to be able to prove existence in a type-theoretic context. The systems of Kemeny and L'Abbé also have such axioms³).

The present writer has been unable to examine Bruner's work [2], but McKinsey's rather critical review [15] suggests that it is not altogether satisfactory. According to McKinsey, Bruner does remark that the introduction of transfinite types allows one to demonstrate the axiom of infinity (for entities of transfinite type); McKinsey does not state, however, whether Bruner assumes any special axioms concerning entities of transfinite type.

Of course the idea of transfinite type theory has been referred to by

²) Indeed, Kemeny requires that the domain of lowest type be empty, and his axiom 9 is $\sim \exists a_0. a_0 = a_0$. Unfortunately this axiom renders his system inconsistent.

³) We refer to Bustamante's axiom A13:

$\vdash (x_\alpha) (\exists y_\alpha). y_\alpha = Px_\alpha$, where α is any ordinal of the second kind (one sees from Bustamante's definitions that Px_α can be interpreted as the power set of x_α);

to axiom 11 of Kemeny's system T_γ :

$(\forall a_\alpha) (\exists b_\alpha). [b_\alpha c_\alpha] \equiv c_\alpha [c_\alpha \subseteq a_\alpha]$, where α is of the second kind;

and to L'Abbé's axiom A16:

$(a_1) (CTS_{01}^1 a_1 \supset . (b_2) (\Phi_{021}(F_{11}^2 a_1) b_2 \supset . \Phi_{021}(F_{11}^1 a_1) M_2) \supset . (\exists c_2) (\Phi_{021} a_1 c_2 \ \& \ (b_2) (\Phi_{021}(F_{11}^2 a_1) b_2 \supset . c_2 b_2 = M_2)))$,

where M_2 is any wff of type 2 containing neither a_1 nor c_2 as free variables.

writers other than those just mentioned⁴), but none of these other writers makes any attempt to describe a system of transfinite type theory in any detail.

The apparent need for special axioms concerning the existence of certain entities of transfinite type in [3], [12], [13] and [14] suggests that something may be lacking in the fundamental logical structure of these theories. A clue as to what may be lacking is perhaps found in the fact that L'Abbé finds it useful to assign Gödel numbers to type symbols, and his axiom A16 is intuitively meaningful only in the light of this Gödel numbering. Evidently in a system of transfinite type theory it is useful to be able to discuss type symbols in some way.

The reader will recall that the usual motivation for an interest in transfinite type theory is a desire for a formal system constructed along type-theoretic lines in which one can formalize the semantics of finite type theories⁵). Since in discussing the semantics of type theory one often refers to type symbols, it seems only natural that within transfinite type theories there should be some means of referring to type symbols, and making arguments concerning them. But instead of doing this indirectly by referring to Gödel numbers, why not introduce variables ranging over type symbols, permit quantification with these variables, and introduce expressions for notions such as equality between type symbols?

A little reflection shows that certain difficulties attend such a program. To discuss these difficulties informally, let us suppose that we have a system such as L'Abbé's Σ_2 (in which 2 is the first transfinite type symbol) modified by the introduction of type variables ranging over finite type symbols and augmented by the logic of first order functional calculus (with equality) for type symbols. As one of the theorems about equality one might expect to have $(\alpha = \beta) \supset .(f_{\gamma\alpha}x_\alpha) = (f_{\gamma\beta}x_\alpha)$. However, if the formation rules permit such formulas as $(f_{\gamma\beta}x_\alpha)$ to be well-formed, the system may become inconsistent, and will certainly lose some of the appeal which type theory derives from the simplicity and naturalness of its interpretation. Even without permitting wffs of such a dubious nature,

⁴) See for example [11] p. 183 ff.; [7] p. 191 or [8] p. 62, footnote 48a; [16] p. 393 ff. or [17] p. 268 ff.; and [18] p. 110.

⁵) See, for example, [17], especially pp. 268–278.

however, contradictions might easily arise. For example, we would expect to have $(x_\alpha \stackrel{\alpha}{=} y_\alpha) \equiv (x_\alpha \stackrel{2}{=} y_\alpha)$ as a theorem, and so to be able to prove $x_\alpha \stackrel{2}{=} x_\alpha$. Also we would expect to have as a theorem $(\alpha = \beta) \supset .(x_\alpha \stackrel{2}{=} x_\alpha) \supset (x_\alpha \stackrel{2}{=} x_\beta)$. Hence by propositional calculus we would obtain $(\alpha = \beta) \supset .x_\alpha \stackrel{2}{=} x_\beta$, and by substitution we would obtain $(\alpha = \beta) \supset .x_\alpha \stackrel{2}{=} y_\beta$, and then $(\alpha = \alpha) \supset .x_\alpha \stackrel{2}{=} y_\alpha$, from which would follow $x_\alpha \stackrel{\alpha}{=} y_\alpha$. Then by substitution we would obtain first $x_0 \stackrel{0}{=} y_0$ and then $x_0 \stackrel{0}{=} \sim x_0$, from which follows $x_0 \equiv \sim x_0$, which is a contradiction.

Nevertheless, we shall construct a system of transfinite type theory, which we shall call Q, along the lines we have suggested, and we will show that it is consistent. (It will be clear that our consistency proof could be formalized in Zermelo-Fraenkel Set Theory with the Axiom of Replacement, and other systems of comparable power.) No assumptions will be made about the cardinality of the domain of individuals (except that it is not zero), but we shall prove a Theorem of Infinity⁶⁾ at the first transfinite type. Thus the system Q will satisfy the preliminary requirements mentioned above for an underlying logic in a formalized scientific theory. Of course it remains to be seen whether the system may be found unsatisfactory for this purpose for other reasons, or whether some yet superior system will be developed.

We remark that some philosophical interest may be attached to the fact that the Theorem of Infinity in Q follows from axioms which we feel the reader will agree can be regarded as purely logical. In this connection we quote the following comments by Professor Church:

"To avoid the paradoxes, Russell, and later Whitehead and Russell in *Principia Mathematica*, introduced the theory of types, and this in turn compelled the use of the axiom of infinity. The term "axiom of infinity" is due, not to Russell, but to C.J. Keyser. And in an early paper Russell had argued against Keyser that the axiom of infinity is not a special assumption which is required above and beyond the laws of logic, because the axiom (or supposed axiom) can be proved on logical grounds alone. Thus when Russell later reversed himself and adopted an axiom of infinity, it is almost his own admission, it might be said, that he thus went

⁶⁾ Theorem 1009 of Chapter III.

beyond pure logic. Indeed the axiom of infinity might be described as half-logical in character, since it can be stated in the same vocabulary that is used to state the laws of pure logic, but is not analytic according to any known theory of deductive reasoning that I believe can be accepted as adequately and naturally representative of existing standard practice in mathematical reasoning. And though it is known that elementary arithmetic can be obtained without the axiom of infinity if based on a set-theoretic rather than a type-theoretic approach, it does not appear that mathematics as a whole can dispense with such an axiom if otherwise based only on a standard and naturally acceptable formulation of pure logic.”⁷⁾

One of the traditional tests of the adequacy of a system of transfinite type theory is whether it can be used to formalize the semantics of finite type theory. We shall therefore show that this can indeed be done in Q.

Since Q, like L’Abbé’s systems, is a transfinite type theory with λ -conversion, it is natural to compare Q with L’Abbé’s systems, and we have at certain points chosen our notation so as to facilitate this comparison. It will become clear to the reader that many of the ideas which we use below originated in L’Abbé’s work, and we here express our appreciation of his pioneering contributions. L’Abbé defined⁸⁾ the set K^γ of type symbols to be the least class of symbols containing the symbols denoting the ordinals δ for which $\gamma \geq \delta$ and closed under the operation of forming the symbol $(\alpha\beta)$ from the symbols α and β . Thus K^1 is the set of finite type symbols. To formalize the semantics of finite type theory Σ_0 , L’Abbé found it necessary to use a system Σ_2 of transfinite type theory which takes the set K^3 for type symbols, and is thus two “transfinite jumps” above Σ_0 . However the system Q takes the smaller set K^2 , augmented appropriately by the addition of variables ranging over K^1 , for type symbols, and so should be compared with L’Abbé’s system Σ_1 , which takes K^2 for type symbols. The fact that the semantics of finite type theory can be formalized in Q, even in the absence of any axiom comparable to L’Abbé’s axiom A16, but apparently cannot⁹⁾ be formalized in Σ_1 , can be attri-

⁷⁾ [6], pp.183–184.

⁸⁾ [14], p.210.

⁹⁾ See [14], p.223.

buted directly to the greater freedom of expression which one obtains in Q with the aid of variables ranging over type symbols and quantification with respect to these variables.

The system Q_0 of finite type theory with λ -conversion which we have chosen to extend to transfinite type theory is not the original system due to Church (see [4] and [9]), but is in all essentials a recent modification of that system by Henkin [10], with further minor modifications by the present writer [1]. We have felt it best to develop the system Q from first principles, but it should be understood that the proofs of many of the early theorems of Q (especially those concerned with propositional calculus) are due to Henkin. We here express our appreciation to Professor Henkin for the opportunity to see his elegant work [10] before publication, and for his encouragement of our interest in it.

The present work was originally developed as a Ph.D. thesis at Princeton University. We wish to express our deep gratitude to Professor Church, both for the inspiration of his writings, and for his patience, encouragement, and guidance as teacher and advisor.

CHAPTER I

THE SYSTEM Q

I.1. The system Q_0

In order that the reader may more readily understand the system Q of transfinite type theory which we shall introduce, we first set forth the system Q_0 of finite type theory. In Chapter IV we shall show that the semantics of Q_0 can be formalized in Q.

The type symbols of Q_0 are the symbols in K^1 , where K^1 is the least class of symbols containing 0 and 1 and containing $(\alpha\beta)$ whenever it contains α and β . We shall use the small Greek letters α, β, γ and δ as syntactical variables ranging over type symbols.

The primitive symbols of Q_0 are as follows (where α is any type symbol in K^1):

Improper symbols: λ [].

Constants: $Q_{((0\alpha)\alpha)} \quad I_{(1(01))}$.

Variables: $x_\alpha^1 \ x_\alpha^2 \ x_\alpha^3 \ x_\alpha^4 \dots$ (infinitely many).

The formation rules are the following:

- (1) A constant or variable standing alone is a wff of the type indicated by its subscript.
- (2) If A_α is a wff of type α and $B_{(\beta\alpha)}$ is a wff of type $(\beta\alpha)$, then $[B_{(\beta\alpha)}A_\alpha]$ is a wff of type β .
- (3) If A_α is a wff of type α and x_β^i is a variable of type β then $[\lambda x_\beta^i A_\alpha]$ is a wff of type $(\alpha\beta)$.
- (4) Nothing is a wff of any type unless its being so follows from the above rules.

We shall use $A_\alpha, B_\alpha, C_\alpha, D_\alpha$, etc., as syntactical variables ranging over wffs of type α . We shall often abbreviate wffs and type symbols by omitting parentheses and brackets when this can be done without introducing ambiguity. In restoring omitted parentheses and brackets, formulas must be well-formed, and otherwise association is to the left. In addition a dot in an abbreviated wff (or type symbol) shall stand for a left bracket (or parenthesis), and its mate shall be understood to be as far to the right of the dot as is consistent with the entire expression being an abbreviation of a wff (or type symbol), with the pairing of left and right brackets (or parentheses) already present being unchanged. Thus our conventions are those of [4].

We also introduce the following definitions and abbreviations:

- $[A_\alpha = B_\alpha]$ stands for $[[Q_{\alpha\alpha}A_\alpha] B_\alpha]$.
 T_0 stands for $[\lambda x_0^1 x_0^1] = [\lambda x_0^1 x_0^1]$.
 F_0 stands for $[\lambda x_0^1 x_0^1] = [\lambda x_0^1 T_0]$.
 $[\forall x_\alpha^1 A_0]$ stands for $[\lambda x_\alpha^1 A_0] = [\lambda x_\alpha^1 T_0]$ (where $i = 1, 2, 3, \dots$).
 $\wedge_{000}$ stands for $[\lambda x_0^1 . \lambda x_0^2 . [\lambda x_{000}^1 . x_{000}^1 x_0^1 x_0^2] = [\lambda x_{000}^1 . x_{000}^1 T_0 T_0]]$.
 $[A_0 \wedge B_0]$ stands for $[[\wedge_{000} A_0] B_0]$.
 $\supset_{000}$ stands for $[\lambda x_0^1 . \lambda x_0^2 . [x_0^1 \wedge x_0^2] = x_0^1]$.
 $[A_0 \supset B_0]$ stands for $[[\supset_{000} A_0] B_0]$.

An occurrence of a variable x_α^1 in a wff A_β is said to be *free* if the occurrence is contained in no wf part of A_β of the form $[\lambda x_\alpha^1 C_\gamma]$. Otherwise the occurrence is said to be *bound*.

The system Q_0 has a single rule of inference, which is the following:
 RULE R: From C_0 and $[A_\alpha = B_\alpha]$ to infer D_0 , where D_0 is the result of replacing one occurrence of A_α in C_0 by an occurrence of B_α , and the occurrence of A_α in C_0 is not an occurrence of a variable immediately preceded by λ .

The axioms and axiom schemata for Q_0 are the following:

- 1 $[x_{00}^1 T_0 \wedge x_{00}^1 F_0] = \forall x_0^1 . x_{00}^1 x_0^1$.
- 2 $[x_\alpha^1 = x_\alpha^2] \supset . x_{0\alpha}^1 x_\alpha^1 = x_{0\alpha}^1 x_\alpha^2$.
- 3 $[x_{\alpha\beta}^1 = x_{\alpha\beta}^2] = \forall x_\beta^1 [x_{\alpha\beta}^1 x_\beta^1 = x_{\alpha\beta}^2 x_\beta^1]$.

4₁ $[[\lambda x_\alpha^1 B_\beta] A_\alpha] = B_\beta$, provided the variable x_α^1 has no free occurrence in B_β .

4₂ $[[\lambda x_\alpha^1 x_\alpha^1] A_\alpha] = A_\alpha$.

4₃ $[[\lambda x_\alpha^1 . B_{\beta\gamma} C_\gamma] A_\alpha] = .[[\lambda x_\alpha^1 B_{\beta\gamma}] A_\alpha] [[\lambda x_\alpha^1 C_\gamma] A_\alpha]$.

4₄ $[[\lambda x_\alpha^1 . \lambda x_\beta^1 B_\gamma] A_\alpha] = [\lambda x_\beta^1 . [\lambda x_\alpha^1 B_\gamma] A_\alpha]$, provided the variables x_α^1 and x_β^1 are distinct, and x_β^1 is not free in A_α .

5 $[\iota_{1(01)} . Q_{011} x_1^1] = x_1^1$.

In the above schemata, i and j stand for arbitrary positive integers.

In the intended interpretations of Q_0 , variables of type 0 range over the domain $\mathfrak{D}_0$ which contains the truth values "truth" and "falsehood", and nothing else, variables of type 1 range over some non-empty domain $\mathfrak{D}_1$ of individuals, and variables of type $(\alpha\beta)$ range over the domain $\mathfrak{D}_{\alpha\beta}$ consisting of all functions from $\mathfrak{D}_\beta$ to $\mathfrak{D}_\alpha$. λ serves as the abstraction operator, $Q_{0\alpha\alpha}$ denotes the equality relation between members of $\mathfrak{D}_\alpha$, and $\iota_{1(01)}$ denotes a selection operator. It may be seen that in these interpretations, T_0 denotes "truth", F_0 denotes "falsehood", $\wedge_{000}$ denotes conjunction, and $\supset_{000}$ denotes implication. For more details see [10].

We shall not develop the system Q_0 , but when we develop the system Q in Chapter II the reader will see that many of the theorems, meta-theorems, and derived rules of inference stated for Q may be modified and simplified so as to apply to Q_0 . Alternatively, the reader may refer to [10] and [1] and note that most of the results there apply to Q_0 with little or no change. In either case it will become apparent that all the ordinary laws of type-theoretic logic can be derived in Q_0 , and hence that the completeness of Q_0 in the sense of [9] can be established.

I.2. The primitive basis of Q

The primitive symbols of Q are the following:

Improper symbols: λ [] () $\forall$.

Constants: Q ι 0 1 2.

Primitive variables: $f g h i j k m n p q r s t u v w x y z$

$\begin{matrix} 1 & 1 & 1 \\ f & g & h \end{matrix} \quad \dots \quad \begin{matrix} 1 \\ z \end{matrix}$
 $\begin{matrix} 2 & 2 & 2 \\ f & g & h \end{matrix} \dots$ et cetera (infinitely many).

Type variables: $a b c d e \begin{matrix} 1 & 1 & 1 & 1 & 2 & 2 \\ a & b & c & d & e & a & b \end{matrix} \dots \begin{matrix} 3 & 3 \\ a & b \end{matrix} \dots$ et cetera
(infinitely many).

For typographical convenience we shall sometimes write $\bar{x}$ for $\overset{1}{x}$, $\bar{\bar{x}}$ for $\overset{2}{x}$, etc.

From the primitive symbols of Q we construct two sets of type symbols, called K_+^1 and K_+^2 , which are defined as follows: K_+^1 is the least class of symbols containing 0 and 1 and all type variables and containing $(\alpha\beta)$ whenever it contains α and β ; K_+^2 is the least class of symbols containing 2 and all members of K_+^1 and containing $(\alpha\beta)$ whenever it contains α and β .

As in the case of Q_0 , we shall use small Greek letters as syntactical variables ranging over type symbols. (But here of course they range over K_+^2 .) We shall use $f, g, \dots x, y, z$, etc., to denote arbitrary primitive variables, and a, b, c, d, e , etc., to denote arbitrary type variables.

We define the relation $>$ to be the smallest relation between type symbols (of K_+^2) satisfying the following conditions:

(1) If α is in K_+^1 , then $2 > \alpha$.

(2) If $\alpha > \beta$, then $(\alpha\gamma) > (\beta\gamma)$.

It would seem natural to require also that $>$ be transitive. However, as we shall show later (in Lemma 9 of I.5), there are no type symbols α, β and γ such that $\alpha > \beta$ and $\beta > \gamma$. Thus $>$ is transitive in a trivial sense.

It is clear that there is an effective procedure for determining of arbitrary type symbols α and β whether or not they stand in the relation $\alpha > \beta$. It should be noted that $2 > \alpha$ if and only if α is in K_+^1 .

We shall write $\alpha \geq \beta$ to indicate that $\alpha > \beta$ or $\alpha = \beta$. The formation rules of Q are the following:

- (1) If x is any primitive variable and α is any type symbol, x_α is a wff of type α and a wff-variable of type α . x is called the primitive variable of x_α .

- (2) $Q_{((0\alpha)\alpha)}$ is a wff of type $((0\alpha)\alpha)$.
- (3) $\iota_{(\alpha(0\alpha))}$ is a wff of type $(\alpha(0\alpha))$.
- (4) If α and β are type symbols in K_+^1 , then $[Q\alpha\beta]$ is a wff of type 0.
- (5) If A_γ is a wff of type γ and $B_{(\beta\alpha)}$ is a wff of type $(\beta\alpha)$ and $\alpha \geq \gamma$, then $[B_{(\beta\alpha)}A_\gamma]$ is a wff of type β .
- (6) If A_α is a wff of type α and x_β is a wff-variable of type β , then $[\lambda x_\beta A_\alpha]$ is a wff of type $(\alpha\beta)$, provided that for every occurrence of every wff-variable y_γ in A_α which is contained in no wf part of A_α of the form $[\lambda y_\gamma B_\delta]$, y_γ is either the same as x_β (so y is x and γ is β) or the primitive variable y of y_γ is distinct from the primitive variable x of x_β .
- (7) If A_0 is a wff of type 0 and a is any type variable, then $[\forall a A_0]$ is a wff of type 0, provided that every occurrence in A_0 of every wff-variable x_β , where a occurs in β , is in a wf part of A_0 of the form $[\lambda x_\beta B_\delta]$.

We shall use $A_\alpha, B_\alpha, C_\alpha, D_\alpha, E_\alpha, G_\alpha, H_\alpha, J_\alpha, L_\alpha, M_\alpha$ and P_α to denote arbitrary wffs of type α .

We shall refer to both type variables and wff-variables simply as variables. However primitive variables will always be referred to as primitive variables. Thus by "the variables occurring in a wff" we shall mean "the type variables and the wff-variables occurring in a wff". If x_α is any wff-variable with primitive variable x , then x_α shall be called an *x-variable*. Thus $z_{(01)}$ and z_c are *z-variables*, but $\overset{2}{z}_c$ is not a *z-variable*. It is, however, a $\overset{2}{z}$ -variable. For uniformity of terminology, we shall also call a type variable a an *a-variable*.

An occurrence of a wff or a variable in a wff A_α is said to be *free for* the type variable a in the wff A_α if that occurrence is contained in no wf part of A_α of the form $[\forall a B_0]$. An occurrence of a wff or a variable in a wff A_α is said to be *free for* the wff-variable x_β in the wff A_α if that occurrence is contained in no wf part of A_α of the form $[\lambda x_\beta B_\delta]$ and in no wf part of A_α of the form $[\forall b C_0]$ such that the type variable b occurs in β . An occurrence of a variable in a wff is *free* in that wff if it is free for itself; otherwise it is *bound*. By examining the formation rules one sees that a

wff-variable x_β is bound at an occurrence in a wff A_α if and only if that occurrence is in a wf part of A_α of the form $[\lambda x_\beta B_\beta]$.

Using this terminology we can restate formation rules (6) and (7) as follows:

- (6') If A_α is a wff of type α and x_β is a wff-variable of type β , then $[\lambda x_\beta A_\alpha]$ is a wff of type $(\alpha\beta)$, provided that no x -variable distinct from x_β occurs free in A_α .
- (7') If A_0 is a wff of type 0 and a is any type variable, then $[\forall a A_0]$ is a wff of type 0, provided that no wff-variable x_β , where a occurs in β , has a free occurrence in A_0 .

If A_α is any wff, X is any type variable (or wff-variable), and W is any type symbol (or wff), then the notation $[S_W^X A_\alpha]$ shall stand for the result of substituting W for X at all free occurrences of X in A_α (including free occurrences of X in subscripts, if X is a type variable). Similarly $[S_{W_1 \dots W_n}^{X_1 \dots X_n} A_\alpha]$, where the variables $X_1, \dots, X_n$ are distinct, shall stand for the result of simultaneously substituting W_1 for $X_1, \dots, W_n$ for X_n at all free occurrences of $X_1, \dots, X_n$ in A_α .

As in Q_0 , we shall abbreviate wffs and type symbols by omitting parentheses and brackets, and our conventions for restoring omitted parentheses and brackets are the same as in the case of Q_0 (see page 2). We also introduce the following definitions and abbreviations:

$[A_\alpha \stackrel{\gamma}{=} B_\beta]$	stands for $[[Q_{0\gamma\gamma} A_\alpha] B_\beta]$ provided $\gamma \geq \alpha$ and $\gamma \geq \beta$.
$[A_\alpha = B_\alpha]$	stands for $[A_\alpha \stackrel{\alpha}{=} B_\alpha]$.
$[\alpha = \beta]$	stands for $[Q\alpha\beta]$ provided α and β are in K_+^1 .
T_0	stands for $[\lambda x_0 x_0] = [\lambda x_0 x_0]$.
F_0	stands for $[\lambda x_0 x_0] = [\lambda x_0 T_0]$.
$\sim_{00}$	stands for $[\lambda x_0 . x_0 = F_0]$.
$[\forall x_\beta A_0]$	stands for $[\lambda x_\beta A_0] = [\lambda x_\beta T_0]$.
$[\exists x_\beta A_0]$	stands for $\sim_{00} \forall x_\beta \sim_{00} A_0$.
$[\exists a A_0]$	stands for $\sim_{00} \forall a \sim_{00} A_0$.
$\wedge_{000}$	stands for $[\lambda x_0 . \lambda y_0 . [\lambda g_{000} . g_{000} x_0 y_0] = [\lambda g_{000} . g_{000} T_0 T_0]]$.
$[A_0 \wedge B_0]$	stands for $[[\wedge_{000} A_0] B_0]$.
$\supset_{000}$	stands for $[\lambda x_0 . \lambda y_0 . [x_0 \wedge y_0] = x_0]$.

$[A_0 \supset B_0]$ stands for $[[\supset_{000} A_0] B_0]$.

$\vee_{000}$ stands for $[\lambda x_0 . \lambda y_0 . \sim_{00} . [\sim_{00} x_0] \wedge [\sim_{00} y_0]]$.

$[A_0 \vee B_0]$ stands for $[[\vee_{000} A_0] B_0]$.

$[x_\beta A_0]$ stands for $[x_{\beta(0\beta)} [\lambda x_\beta . A_0]]$.

We shall from time to time make additions to this list of definitions and abbreviations. We shall also feel free to leave the subscripts off abbreviations of particular wffs. Thus we may write T in place of T_0 , F in place of F_0 , $\sim A_0$ in place of $\sim_{00} A_0$, etc. We supplement our conventions for restoring brackets in abbreviations of wffs as follows: The brackets enclosing parts of the forms $[\sim A_0]$, $[\forall a A_0]$, $[\forall x_\beta A_0]$, $[\exists a A_0]$, $[\exists x_\beta A_0]$ and $[x_\beta A_0]$ are to be restored first, in such a way that they have the smallest possible scope (consistent with the entire formula being well-formed, of course). Then the brackets enclosing parts of the form $[A_0 \wedge B_0]$ are to be restored in such a way that they have the smallest possible scope, the law of association to the left governing the relationship of brackets between several such parts. Then brackets enclosing parts of the form $[A_0 \vee B_0]$ are to be similarly restored. Finally brackets enclosing parts of the forms $[A_0 \supset B_0]$ and $[A_0 \overset{0}{=} B_0]$ are to be restored, using the convention of association to the left. (Thus our conventions for restoring omitted brackets are essentially those found on pages 79 and 171 of [5].)

The system Q has a single rule of inference, which is the following:

RULE R: From C_0 and $A_\alpha \overset{\gamma}{=} B_\beta$ to infer D_0 , where $\gamma \geq \alpha$, $\gamma \geq \beta$, D_0 is a wff of type 0 which is the result of replacing one occurrence of A_α in C_0 by an occurrence of B_β , and the occurrence of A_α in C_0 is not an occurrence of a variable immediately preceded by λ .¹⁰⁾

The axioms and axiom schemata of Q are the following:

$$1 \quad [g_{00} T_0 \wedge g_{00} F_0] = \forall x_0 . g_{00} x_0.$$

$$2 \quad [x_\alpha \overset{\beta}{=} y_\alpha] \supset . h_{0\alpha} x_\alpha = h_{0\alpha} y_\alpha \text{ provided } \beta \geq \alpha.$$

$$3 \quad [f_{\alpha\beta} \overset{\alpha\beta}{=} g_{\alpha\beta}] = \forall x_\beta [f_{\alpha\beta} x_\beta \overset{\alpha}{=} g_{\alpha\beta} x_\beta].$$

¹⁰⁾ Of course if any of these conditions is not satisfied, Rule R may not be applied. Note in particular that the result of replacing A_α by B_β in C_0 may not be a wff.

- 4₁ $[[\lambda x_\gamma B_\beta]A_\alpha] \stackrel{\beta}{=} B_\beta$ provided $\gamma \geq \alpha$ and no x -variable occurs free in B_β .
- 4₂ $[[\lambda x_\gamma x_\gamma]A_\alpha] \stackrel{\gamma}{=} A_\alpha$ provided $\gamma \geq \alpha$.
- 4₃ $[[\lambda x_\gamma [B_{\beta\delta} C_\epsilon]]A_\alpha] \stackrel{\beta}{=} . [[\lambda x_\gamma B_{\beta\delta}]A_\alpha] [[\lambda x_\gamma C_\epsilon]A_\alpha]$ provided $\gamma \geq \alpha$ and $\delta \geq \epsilon$ and this is a wff.
- 4₄ $[[\lambda x_\gamma . \lambda y_\delta B_\beta]A_\alpha] \stackrel{\beta\delta}{=} [\lambda y_\delta . [\lambda x_\gamma B_\beta]A_\alpha]$ provided x is not y and y_δ is not free in A_α and this is a wff.
- 4₅ $[[\lambda x_\gamma . \forall c B_0]A_\alpha] \stackrel{0}{=} [\forall c . [\lambda x_\gamma B_0]A_\alpha]$ provided c does not occur free in A_α or γ and this is a wff.
- 5 $[\iota_{\alpha(0\alpha)} . Q_{0\alpha\alpha} y_\alpha] \stackrel{\alpha}{=} y_\alpha$.
- 6 $\forall a T_0$.
- 7 $[\forall a A_0] \supset [S_\gamma^a A_0]$ provided $2 > \gamma$ and all free occurrences of a in A_0 are free for all variables in γ and this is a wff.
- 8 $[\alpha = \beta] \supset . [S_\alpha^c A_0] \stackrel{0}{=} [S_\beta^c A_0]$ provided $2 > \alpha$ and $2 > \beta$ and all free occurrences of c in A_0 are free for all variables in α and β .
- 9 $[(ab) = (cd)] \stackrel{0}{=} [a = c \wedge b = d]$.
- 10 $[x_a \stackrel{2}{=} y_b] \supset [a = b]$.

I.3. The interpretation and consistency of Q

In this section we shall define a model $\mathfrak{M}$ for Q, interpret Q in this model, and then show that this interpretation is sound. As a consequence we will see that Q is consistent.

In the model $\mathfrak{M}$ the domain $\mathfrak{D}_1$ of individuals will contain exactly one individual. We have chosen to discuss this particular model in order that our consistency proof for Q may be as explicit as possible. But of course other models for Q may be obtained by starting from different non-empty domains of individuals, and otherwise proceeding as in the case of $\mathfrak{M}$. Our description of the interpretation of Q in $\mathfrak{M}$ and our proof of the soundness of this interpretation will also apply to these other models

with virtually no changes. We shall call all these interpretations of Q *principal* interpretations. It is clear that there are many sound non-principal interpretations of Q, and much work remains to be done investigating them.

Since classical mathematics (without the Axiom of Choice) can be formalized in Q, there may be some interest in carefully examining upon what assumptions our consistency proof for Q rests. To facilitate such an examination we shall carry out the consistency proof in rather tedious detail. Our meta-language will not be formalized, but it will be clear that our proof could be carried out in Zermelo-Fraenkel Set Theory with the Axiom of Replacement, and other systems of comparable power. Of course such systems are much stronger than Q.

The set K^2 is the set of all type symbols in K_+^2 in which no type variables occur. The model $\mathfrak{M}$ consists of K^2 and a domain $\mathfrak{D}_\alpha$ for each α in K^2 . These domains are defined as follows. We first choose three distinct objects t , f and r which are not functions. (One may appropriately take "truth" for t and "falsehood" for f .) $\mathfrak{D}_0$ contains t and f and nothing else. $\mathfrak{D}_1$ contains r and nothing else. If $(\alpha\beta)$ is a type symbol in K^1 , $\mathfrak{D}_{\alpha\beta}$ consists of all functions from $\mathfrak{D}_\beta$ to $\mathfrak{D}_\alpha$. $\mathfrak{D}_2$ consists of the union of the domains $\mathfrak{D}_\alpha$ for which α is in K^1 . If $(\alpha\beta)$ is in K^2 but not in K^1 , $\mathfrak{D}_{\alpha\beta}$ consists of all functions from $\mathfrak{D}_\beta$ to $\mathfrak{D}_\alpha$.

Since if $\mathfrak{D}_\alpha \supseteq \mathfrak{D}_\beta$ any function in $\mathfrak{D}_{\beta\gamma}$ must also be in $\mathfrak{D}_{\alpha\gamma}$, we easily see that if α and β are any type symbols in K^2 , $\alpha \geq \beta$ if and only if $\mathfrak{D}_\alpha \supseteq \mathfrak{D}_\beta$.

DEFINITION: The *length* of a wff or type symbol of Q is the total number of occurrences of primitive symbols in it.

We next interpret our language Q in the model $\mathfrak{M}$ which we have just defined.

DEFINITION: An *assignment* of values in $\mathfrak{M}$ to the primitive and type variables of Q is a function π such that:

- (1) If a is any type variable, then πa is in K^1 .
- (2) If x is any primitive variable, then πx is some function with domain K^2 and with values in $\mathfrak{M}$ such that for each type symbol α in K^2 , $(\pi x)\alpha$ is in $\mathfrak{D}_\alpha$.

For each assignment π we define a mapping W_π of wffs and type symbols of Q into $\mathfrak{M}$. The definition is by induction with respect to the lengths of the wffs and type symbols of Q . The reader will note that if α is any type symbol of Q , then $W_\pi\alpha$ is in K^2 , and if A_α is any wff of type α , then $W_\pi A_\alpha$ is in $\mathfrak{D}_{W_\pi\alpha}$. The definition is as follows:

- (1) $W_\pi 0 = 0$; $W_\pi 1 = 1$; $W_\pi 2 = 2$.
- (2) If a is any type variable, $W_\pi a = \pi a$.
- (3) If α and β are type symbols of Q , $W_\pi(\alpha\beta) = (W_\pi\alpha W_\pi\beta)$.
- (4) If x is a primitive variable and α is a type symbol, $W_\pi x_\alpha = \pi x(W_\pi\alpha)$.
- (5) $W_\pi Q_{0\alpha\alpha}$ is that function in $\mathfrak{D}_{((0W_\pi\alpha)W_\pi\alpha)}$ such that for each member $\mathfrak{x}$ of $\mathfrak{D}_{W_\pi\alpha}$, $(W_\pi Q_{0\alpha\alpha})\mathfrak{x}$ is that function in $\mathfrak{D}_{(0W_\pi\alpha)}$ such that for each member $\mathfrak{y}$ of $\mathfrak{D}_{W_\pi\alpha}$, $((W_\pi Q_{0\alpha\alpha})\mathfrak{x})\mathfrak{y}$ is $\mathfrak{t}$ if $\mathfrak{y} = \mathfrak{x}$, and is $\mathfrak{f}$ otherwise. (Briefly, $W_\pi Q_{0\alpha\alpha}$ is the identity relation on $\mathfrak{D}_{W_\pi\alpha}$.)
- (6) For each α in K^2 we choose an element r_α in $\mathfrak{D}_\alpha$ as follows: r_0 is $\mathfrak{f}$, r_1 is $\mathfrak{r}$, r_2 is $\mathfrak{r}$, $r_{\beta\gamma}$ is that function from $\mathfrak{D}_\gamma$ to $\mathfrak{D}_\beta$ whose value is always r_β . Then we define $W_{\pi^t\alpha(0\alpha)}$ to be that function in $\mathfrak{D}_{W_\pi\alpha(0W_\pi\alpha)}$ such that for each element $\mathfrak{x}$ of $\mathfrak{D}_{(0W_\pi\alpha)}$, $(W_{\pi^t\alpha(0\alpha)})\mathfrak{x} = r_{W_\pi\alpha}$ unless there is some member $\mathfrak{y}$ of $\mathfrak{D}_{W_\pi\alpha}$ such that $\mathfrak{x}\mathfrak{y} = \mathfrak{t}$, while $\mathfrak{x}\mathfrak{z} = \mathfrak{f}$ for each member $\mathfrak{z}$ of $\mathfrak{D}_{W_\pi\alpha}$ which is distinct from $\mathfrak{y}$; in the latter case such a $\mathfrak{y}$ is unique, and we define $(W_{\pi^t\alpha(0\alpha)})\mathfrak{x} = \mathfrak{y}$.
- (7) Let α and β be in K_+^1 . $W_\pi[Q\alpha\beta] = \mathfrak{t}$ if $W_\pi\alpha = W_\pi\beta$; otherwise $W_\pi[Q\alpha\beta] = \mathfrak{f}$.
- (8) If A_γ and $B_{\beta\alpha}$ are wffs and $\alpha \geq \gamma$, we define $W_\pi[B_{\beta\alpha}A_\gamma]$ to be the value $(W_\pi B_{\beta\alpha})(W_\pi A_\gamma)$ of the function $(W_\pi B_{\beta\alpha})$ for the argument $(W_\pi A_\gamma)$. It is easy to see that $W_\pi\alpha \geq W_\pi\gamma$, so $\mathfrak{D}_{W_\pi\alpha} \supseteq \mathfrak{D}_{W_\pi\gamma}$, and $W_\pi A_\gamma$ is in $\mathfrak{D}_{W_\pi\gamma}$ while $W_\pi B_{\beta\alpha}$ is in $\mathfrak{D}_{(W_\pi\beta W_\pi\alpha)}$, so this definition makes sense.
- (9) If $[\lambda x_\beta A_\alpha]$ is a wff, $W_\pi[\lambda x_\beta A_\alpha]$ is that function in $\mathfrak{D}_{(W_\pi\alpha W_\pi\beta)}$ whose value for each argument $\mathfrak{x}$ in $\mathfrak{D}_{W_\pi\beta}$ is $W_\varphi A_\alpha$, where φ is that assignment which has the same values as π for all type variables and for all

primitive variables except x , while φx is that function such that $\varphi x(W_\pi\beta) = \mathfrak{x}$ and $\varphi x\gamma = \pi x\gamma$ for any member γ of K^2 distinct from $W_\pi\beta$.

- (10) If $[\forall a A_0]$ is a wff, $W_\pi[\forall a A_0] = \mathfrak{t}$ if $W_\varphi A_0 = \mathfrak{t}$ for every assignment φ which has the same values as π for all primitive variables and for all type variables other than a ; otherwise $W_\pi[\forall a A_0] = \mathfrak{f}$.

This completes the definition of W_π for any assignment π .

LEMMA 1: If U is a wff or type symbol and X is a primitive variable or type variable such that no X -variable has free occurrences in U , and if π and φ are assignments which have the same values for all primitive and type variables except X , then $W_\pi U = W_\varphi U$.

Proof: This follows directly from the definition above and the restrictions in formation rules (6) and (7).

LEMMA 2: $W_\pi T_0 = \mathfrak{t}$ for any assignment π .

Proof: $W_\pi T_0 = W_\pi[Q_{0(00)(00)}[\lambda x_0 x_0] [\lambda x_0 x_0]]$
 $= ((W_\pi Q_{0(00)(00)})W_\pi[\lambda x_0 x_0])W_\pi[\lambda x_0 x_0],$

which is clearly $\mathfrak{t}$ by the definition of $W_\pi Q_{0(00)(00)}$.

LEMMA 3: $W_\pi F_0 = \mathfrak{f}$ for any assignment π .

Proof: Using Lemma 2 it is easy to see that $W_\pi[\lambda x_0 x_0] \neq W_\pi[\lambda x_0 T_0]$, by considering the values of these functions on $\mathfrak{f}$ as determined by clause (9) of the definition above. Then Lemma 3 follows easily from the definition of F_0 .

LEMMA 4: Let $\mathfrak{x}$ and $\mathfrak{y}$ be members of $\mathfrak{D}_0$. For any assignment π , $((W_\pi \wedge_{000})\mathfrak{x})\mathfrak{y} = \mathfrak{t}$ if $\mathfrak{x} = \mathfrak{t}$ and $\mathfrak{y} = \mathfrak{t}$, but otherwise $((W_\pi \wedge_{000})\mathfrak{x})\mathfrak{y} = \mathfrak{f}$.

Proof: For convenience in this proof let α denote (000).

$$\begin{aligned} & ((W_\pi \wedge_{000})\mathfrak{x})\mathfrak{y} \\ &= ((W_\pi[\lambda x_0 \cdot \lambda y_0 \cdot Q_{0(0\alpha)(0\alpha)}[\lambda g_\alpha \cdot g_\alpha x_0 y_0] [\lambda g_\alpha \cdot g_\alpha T_0 T_0]])\mathfrak{x})\mathfrak{y} \\ &= ((W_\varphi Q_{0(0\alpha)(0\alpha)})W_\varphi[\lambda g_\alpha \cdot g_\alpha x_0 y_0])W_\varphi[\lambda g_\alpha \cdot g_\alpha T_0 T_0], \end{aligned}$$

where $(\varphi x) 0 = \xi$ and $(\varphi y) 0 = \eta$ so $W_\varphi x_0 = \xi$ and $W_\varphi y_0 = \eta$. If ξ and η are both t we easily see that $W_\varphi[\lambda g_\alpha . g_\alpha x_0 y_0] = W_\varphi[\lambda g_\alpha . g_\alpha T_0 T_0]$, so $((W_\pi \wedge_{000})\xi)\eta = t$. If ξ and η are not both t , we see that $W_\varphi[\lambda g_\alpha . g_\alpha x_0 y_0] \neq W_\varphi[\lambda g_\alpha . g_\alpha T_0 T_0]$ (for example by computing the values of these functions for the arguments $W_\varphi[\lambda u_0 . \lambda v_0 v_0]$ and $W_\varphi[\lambda u_0 . \lambda v_0 u_0]$), so $((W_\pi \wedge_{000})\xi)\eta = f$. This completes the proof of Lemma 4.

LEMMA 5: Let ξ and η be members of $\mathfrak{D}_0$. For any assignment π , $((W_\pi \supset_{000})\xi)\eta = f$ if $\xi = t$ and $\eta = f$, but otherwise $((W_\pi \supset_{000})\xi)\eta = t$.

Proof: $((W_\pi \supset_{000})\xi)\eta = ((W_\pi[\lambda x_0 . \lambda y_0 . [x_0 \wedge y_0] = x_0])\xi)\eta$
 $= ((W_\varphi Q_{000}) W_\varphi[x_0 \wedge y_0]) W_\varphi x_0$, where $(\varphi x) 0 = \xi$ and $(\varphi y) 0 = \eta$, so $W_\varphi x_0 = \xi$ and $W_\varphi y_0 = \eta$. Thus $W_\varphi[x_0 \wedge y_0] = ((W_\varphi \wedge_{000}) W_\varphi x_0) W_\varphi y_0$
 $= ((W_\varphi \wedge_{000})\xi)\eta$. Using Lemma 4 we see that $((W_\varphi \wedge_{000})\xi)\eta = \xi$ if $\xi = f$ or $\eta = t$, but $((W_\varphi \wedge_{000})\xi)\eta \neq \xi$ if $\xi = t$ and $\eta = f$. Hence Lemma 5 follows by the definition of $W_\varphi Q_{000}$.

DEFINITION: If α and γ are type symbols and c is a type variable, $(S_\gamma^c \alpha)$ shall denote the result of substituting γ for c at all occurrences of c in α .

LEMMA 6: Let γ be any type symbol in K_+^1 , let a be any type variable, and let A_δ be any wff in which all free occurrences of a are free for all type variables in γ . Let π be any assignment, and let φ be that assignment which has the same values as π for all variables except a , while $\varphi a = W_\pi \gamma$. Then $W_\pi[S_\gamma^a A_\delta] = W_\varphi A_\delta$.

Proof: Note first that $W_\pi \gamma$ is in K^1 , so φ is properly defined. Also note that if α is any type symbol, $W_\pi(S_\gamma^a \alpha) = W_\varphi \alpha$; this is easily established by induction on the length of α . Then we proceed by induction on the length of A_δ , considering the following cases:

(1) A_δ is a wff variable x_δ . Then $W_\pi[S_\gamma^a x_\delta] = W_\pi x_{(s_\gamma^a \delta)} = \pi x_{(s_\gamma^a \delta)} = \pi x(W_\pi S_\gamma^a \delta) = \pi x(W_\pi \delta) = \varphi x(W_\pi \delta) = W_\varphi x_\delta$.

(2) A_δ is $Q_{0\alpha\alpha} \cdot W_\pi[S_\gamma^a Q_{0\alpha\alpha}]$ is $W_\pi Q_{0(s_\gamma^a \alpha)(s_\gamma^a \alpha)}$, which is the equality relation between entities of type $W_\pi(S_\gamma^a \alpha)$. $W_\varphi Q_{0\alpha\alpha}$ is the equality relation between entities of type $W_\varphi \alpha$. But $W_\pi(S_\gamma^a \alpha) = W_\varphi \alpha$, so $W_\pi[S_\gamma^a Q_{0\alpha\alpha}] = W_\varphi Q_{0\alpha\alpha}$.

(3) A_δ is $\iota_{\alpha(0\alpha)}$. The proof for this case is similar to the proof for case (2).

(4) A_δ is $[Q\alpha\beta]$. $W_\pi[S_\gamma^a[Q\alpha\beta]] = W_\pi[Q(S_\gamma^a\alpha)(S_\gamma^a\beta)]$, which is $\dagger$ if $W_\pi(S_\gamma^a\alpha) = W_\pi(S_\gamma^a\beta)$, and $\dagger$ otherwise. But $W_\pi(S_\gamma^a\alpha) = W_\varphi\alpha$ and $W_\pi(S_\gamma^a\beta) = W_\varphi\beta$, so in either case $W_\pi[S_\gamma^a[Q\alpha\beta]] = W_\varphi[Q\alpha\beta]$.

(5) A_δ is $[B_{\delta\alpha}C_\beta]$, where $\alpha \geq \beta$. By inductive hypothesis $W_\pi[S_\gamma^a B_{\delta\alpha}] = W_\varphi B_{\delta\alpha}$ and $W_\pi[S_\gamma^a C_\beta] = W_\varphi C_\beta$, so $W_\pi[S_\gamma^a[B_{\delta\alpha}C_\beta]] = W_\pi[[S_\gamma^a B_{\delta\alpha}][S_\gamma^a C_\beta]] = (W_\pi[S_\gamma^a B_{\delta\alpha}]) (W_\pi[S_\gamma^a C_\beta]) = (W_\varphi B_{\delta\alpha}) (W_\varphi C_\beta) = W_\varphi[B_{\delta\alpha}C_\beta]$.

(6) A_δ is $[\lambda x_\beta C_\alpha]$. Since $[S_\gamma^a[\lambda x_\beta C_\alpha]] = [\lambda x_{(S_\gamma^a\beta)}[S_\gamma^a C_\alpha]]$ and $W_\pi(S_\gamma^a\beta) = W_\varphi\beta$ and $W_\pi(S_\gamma^a\alpha) = W_\varphi\alpha$, $W_\pi[S_\gamma^a[\lambda x_\beta C_\alpha]]$ is that function in $\mathcal{D}_{(W_\varphi\alpha W_\varphi\beta)}$ whose value for each argument ε in $\mathcal{D}_{W_\varphi\beta}$ is $W_{\pi'}[S_\gamma^a C_\alpha]$, where π' is that assignment which agrees with π except on x , while $\pi'x(W_\varphi\beta) = \varepsilon$ and $\pi'x\varepsilon = \pi x\varepsilon$ for each ε distinct from $W_\varphi\beta$ in K^2 . Now $W_\varphi[\lambda x_\beta C_\alpha]$ is that function in $\mathcal{D}_{(W_\varphi\alpha W_\varphi\beta)}$ whose value for each argument ε in $\mathcal{D}_{W_\varphi\beta}$ is $W_{\varphi'}C_\alpha$, where φ' is that assignment which agrees with φ except on x , while $\varphi'x(W_\varphi\beta) = \varepsilon$ and $\varphi'x\varepsilon = \varphi x\varepsilon$ for each ε distinct from $W_\varphi\beta$ in K^2 . But then for each ε in $\mathcal{D}_{W_\varphi\beta}$, φ' agrees with π' except on a , while $\varphi'a = \varphi a = W_\pi\gamma = W_{\pi'}\gamma$, so by inductive hypothesis $W_{\pi'}[S_\gamma^a C_\alpha] = W_{\varphi'}C_\alpha$. Thus the functions described are the same, which is to say $W_\pi[S_\gamma^a[\lambda x_\beta C_\alpha]] = W_\varphi[\lambda x_\beta C_\alpha]$.

(7) A_δ is $[\forall c B_0]$. If c is a or a does not occur free in B_0 , then a has no free occurrences in A_δ , so $W_\pi[S_\gamma^a A_\delta] = W_\pi A_\delta = W_\varphi A_\delta$ by Lemma 1.

Suppose, on the other hand, that c is not a and that a does occur free in B_0 . Then by the conditions on A_δ , c does not occur in γ . Also $[S_\gamma^a[\forall c B_0]] = [\forall c[S_\gamma^a B_0]]$.

Suppose $W_\pi[\forall c[S_\gamma^a B_0]] = \dagger$; then there is an assignment π' which agrees with π except on c such that $W_{\pi'}[S_\gamma^a B_0] = \dagger$. Let φ' be that assignment which agrees with π' except on a , while $\varphi'a = W_{\pi'}\gamma$. By inductive hypothesis $W_{\varphi'}B_0 = W_{\pi'}[S_\gamma^a B_0] = \dagger$. Now c does not occur in γ , so $\varphi'a = W_{\pi'}\gamma = W_\pi\gamma = \varphi a$, so φ' agrees with φ except on c , so $W_\varphi[\forall c B_0] = \dagger$.

Conversely suppose $W_\varphi[\forall c B_0] = \dagger$; then there is an assignment φ' which agrees with φ except on c such that $W_{\varphi'}B_0 = \dagger$. Let π' be that assignment which agrees with φ' except on a , while $\pi'a = \pi a$. Then π'

agrees with π except on c , which does not occur in γ , so $W_\pi \gamma = W_\pi \gamma = \varphi a = \varphi' a$, so by inductive hypothesis $W_\pi[S_\gamma^a B_0] = W_{\varphi'} B_0 = \dagger$; hence $W_\pi[\forall c[S_\gamma^a B_0]] = \dagger$.

Thus $W_\pi[\forall c[S_\gamma^a B_0]] = \dagger$ if and only if $W_\varphi[\forall c B_0] = \dagger$, so $W_\pi[S_\gamma^a[\forall c B_0]] = W_\varphi[\forall c B_0]$.

This completes the proof of Lemma 6.

THEOREM 7: If P_0 is a theorem of Q, then $W_\pi P_0 = \dagger$ for every assignment π .

We prove this by showing that the property of the theorem is preserved by the rule of inference of Q and is possessed by all the axioms. Consider the following cases:

(0) P_0 is inferred from C_0 and from $Q_{0\gamma\gamma} A_\alpha B_\beta$ by Rule R.

We are given that for any assignment π , $W_\pi C_0 = \dagger$ and $W_\pi[Q_{0\gamma\gamma} A_\alpha B_\beta] = \dagger$, so by the definition of $W_\pi Q_{0\gamma\gamma}$ we see that $W_\pi A_\alpha = W_\pi B_\beta$ for each assignment π .

Let G be any wf part of C_0 containing the occurrence of A_α in question, and let G' be the corresponding part of P_0 with the occurrence of A_α replaced by an occurrence of B_β . Since P_0 is a wff evidently G' must be a wff also. We prove by induction on the length of G that $W_\pi G' = W_\pi G$ for each assignment π .

If G is A_α then G' is B_β and we have already seen that $W_\pi G' = W_\pi B_\beta = W_\pi A_\alpha = W_\pi G$. Hence we need only consider cases where G is a wff which properly contains A_α , and hence is well-formed according to formation rule (5), (6) or (7).

If G has the form $[D_{\delta\epsilon} E_\sigma]$, where $\epsilon \geq \sigma$ and the occurrence of A_α is in E_σ , then G' has the form $[D_{\delta\epsilon} E']$, and since G' is a wff, E' must be a wff of some type τ , where $\epsilon \geq \tau$. By inductive hypothesis $W_\pi E' = W_\pi E_\sigma$, so $W_\pi G' = (W_\pi D_{\delta\epsilon})(W_\pi E') = (W_\pi D_{\delta\epsilon})(W_\pi E_\sigma) = W_\pi G$.

If G has the form $[D_{\delta\epsilon} E_\sigma]$, where $\epsilon \geq \sigma$ and the occurrence of A_α is in $D_{\delta\epsilon}$, then G' has the form $[D' E_\sigma]$, and since G' is a wff, D' must be a wff of some type $(\rho\tau)$, where $\tau \geq \sigma$. By inductive hypothesis $W_\pi D' = W_\pi D_{\delta\epsilon}$, so $W_\pi G' = (W_\pi D')(W_\pi E_\sigma) = (W_\pi D_{\delta\epsilon})(W_\pi E_\sigma) = W_\pi G$.

If G has the form $[\lambda x_\gamma D_\delta]$ then G' has the form $[\lambda x_\gamma D'_\delta]$, and by inductive

hypothesis $W_\varphi D'_\varepsilon = W_\varphi D_\delta$ for each assignment φ . Hence clearly $W_\pi[\lambda x_\gamma D'_\varepsilon] = W_\pi[\lambda x_\gamma D_\delta]$.

If G has the form $[\forall a D_0]$ then G' has the form $[\forall a D'_0]$, and by inductive hypothesis $W_\varphi D'_0 = W_\varphi D_0$ for each assignment φ . Hence clearly $W_\pi[\forall a D'_0] = W_\pi[\forall a D_0]$.

Now C'_0 is P_0 , so we have $W_\pi P_0 = W_\pi C'_0 = W_\pi C_0 = t$ for each assignment π .

(1) P_0 is Axiom 1, which is

$$[g_{00}T_0 \wedge g_{00}F_0] = .[\lambda x_0 . g_{00}x_0] = [\lambda x_0 T_0].$$

If $(W_\pi g_{00})\varepsilon = t$ for each ε in $\mathfrak{D}_0$, then by Lemma 4 $W_\pi[g_{00}T_0 \wedge g_{00}F_0] = t$. Also for each ε in $\mathfrak{D}_0$ we have $(W_\pi[\lambda x_0 . g_{00}x_0])\varepsilon = (W_\pi g_{00})\varepsilon = t$ and $(W_\pi[\lambda x_0 T_0])\varepsilon = W_\pi T_0 = t$, so $W_\pi[\lambda x_0 . g_{00}x_0] = W_\pi[\lambda x_0 T_0]$, so we see that $W_\pi P_0 = t$.

On the other hand if there is some ε in $\mathfrak{D}_0$ such that $(W_\pi g_{00})\varepsilon = f$, then by Lemmas 2, 3 and 4 we see that $W_\pi[g_{00}T_0 \wedge g_{00}F_0] = f$. Also $(W_\pi[\lambda x_0 . g_{00}x_0])\varepsilon = (W_\pi g_{00})\varepsilon = f$, but $(W_\pi[\lambda x_0 T_0])\varepsilon = t$ so $W_\pi[[\lambda x_0 . g_{00}x_0] = [\lambda x_0 T_0]] = f$. Hence $W_\pi P_0 = t$ in this case also.

(2) P_0 is an instance of Axiom Schema 2, and so has the form $[x_\alpha \stackrel{\beta}{=} y_\alpha]$ $\supset . h_{0\alpha}x_\alpha = h_{0\alpha}y_\alpha$, where $\beta \geq \alpha$.

If $W_\pi[x_\alpha \stackrel{\beta}{=} y_\alpha] = f$ then $W_\pi P_0 = t$ by Lemma 5.

On the other hand if $W_\pi[x_\alpha \stackrel{\beta}{=} y_\alpha] = t$, then by the definition of $W_\pi Q_{0\beta\beta}$ we see that $W_\pi x_\alpha = W_\pi y_\alpha$, so

$$W_\pi[h_{0\alpha}x_\alpha] = (W_\pi h_{0\alpha}) (W_\pi x_\alpha) = (W_\pi h_{0\alpha}) (W_\pi y_\alpha) = W_\pi[h_{0\alpha}y_\alpha],$$

so $W_\pi[h_{0\alpha}x_\alpha = h_{0\alpha}y_\alpha] = t$ and by Lemma 5 $W_\pi P_0 = t$.

(3) P_0 is an instance of Axiom Schema 3, and so has the form

$$[f_{\alpha\beta} \stackrel{\alpha\beta}{=} g_{\alpha\beta}] = .[\lambda x_\beta . f_{\alpha\beta}x_\beta \stackrel{\alpha}{=} g_{\alpha\beta}x_\beta] = [\lambda x_\beta T_0].$$

If $W_\pi[f_{\alpha\beta} \stackrel{\alpha\beta}{=} g_{\alpha\beta}] = t$ then $W_\pi f_{\alpha\beta} = W_\pi g_{\alpha\beta}$, so for each ε in $\mathfrak{D}_{W_\pi\beta}$ we have $(W_\pi[\lambda x_\beta . f_{\alpha\beta}x_\beta \stackrel{\alpha}{=} g_{\alpha\beta}x_\beta])\varepsilon = ((W_\pi Q_{0\alpha\alpha}) ((W_\pi f_{\alpha\beta})\varepsilon)) ((W_\pi g_{\alpha\beta})\varepsilon) = t$. Also $(W_\pi[\lambda x_\beta T_0])\varepsilon = t$, so $W_\pi[[\lambda x_\beta . f_{\alpha\beta}x_\beta \stackrel{\alpha}{=} g_{\alpha\beta}x_\beta] = [\lambda x_\beta T_0]] = t$. Hence $W_\pi P_0 = t$.

If $W_\pi[f_{\alpha\beta} \stackrel{\alpha\beta}{=} g_{\alpha\beta}] = \mathfrak{f}$, then $W_\pi f_{\alpha\beta} \neq W_\pi g_{\alpha\beta}$, so there must be some element $\mathfrak{r}$ in $\mathfrak{D}_{W_\pi\beta}$ such that $(W_\pi f_{\alpha\beta})\mathfrak{r} \neq (W_\pi g_{\alpha\beta})\mathfrak{r}$. Hence $(W_\pi[\lambda x_\beta \cdot f_{\alpha\beta} x_\beta \stackrel{\alpha}{=} g_{\alpha\beta} x_\beta])\mathfrak{r} = \mathfrak{f}$ while $(W_\pi[\lambda x_\beta T_0])\mathfrak{r} = \mathfrak{t}$, so $W_\pi[\lambda x_\beta \cdot f_{\alpha\beta} x_\beta \stackrel{\alpha}{=} g_{\alpha\beta} x_\beta]$ and $W_\pi[\lambda x_\beta T_0]$ are distinct functions, so $W_\pi[[\lambda x_\beta \cdot f_{\alpha\beta} x_\beta \stackrel{\alpha}{=} g_{\alpha\beta} x_\beta] = [\lambda x_\beta T_0]] = \mathfrak{f}$. Thus $W_\pi P_0 = \mathfrak{t}$ in this case also.

(4) P_0 is an instance of one of the axiom schemata 4₁–4₅.

In each case below, we are given a wff A_α and a wff-variable x_γ occurring in P_0 and an assignment π . Define φ to be that assignment which agrees with π on all type variables and on all primitive variables except x , while $\varphi x(W_\pi\gamma) = W_\pi A_\alpha$ and $\varphi x\delta = \pi x\delta$ if $\delta \neq W_\pi\gamma$. (Since in each case $\gamma \geq \alpha$, we see that $W_\pi A_\alpha$ is in $\mathfrak{D}_{W_\pi\gamma}$, so this makes sense.) Then $W_\varphi x_\gamma = W_\pi A_\alpha$.

CASE 4₁: P_0 is an instance of Axiom Schema 4₁, and so has the form $[[\lambda x_\gamma B_\beta]A_\alpha] \stackrel{\beta}{=} B_\beta$, where $\gamma \geq \alpha$ and no x -variable occurs free in B_β .

$W_\pi[[\lambda x_\gamma B_\beta]A_\alpha] = (W_\pi[\lambda x_\gamma B_\beta])(W_\pi A_\alpha) = W_\varphi B_\beta = W_\pi B_\beta$ by Lemma 1, so $W_\pi P_0 = \mathfrak{t}$.

CASE 4₂: P_0 is an instance of Axiom Schema 4₂, and so has the form $[[\lambda x_\gamma x_\gamma]A_\alpha] \stackrel{\gamma}{=} A_\alpha$, where $\gamma \geq \alpha$.

$W_\pi[[\lambda x_\gamma x_\gamma]A_\alpha] = (W_\pi[\lambda x_\gamma x_\gamma])(W_\pi A_\alpha) = W_\varphi x_\gamma = W_\pi A_\alpha$, so $W_\pi P_0 = \mathfrak{t}$.

CASE 4₃: P_0 is an instance of Axiom Schema 4₃, and so has the form $[[\lambda x_\gamma \cdot B_{\beta\delta} C_\epsilon]A_\alpha] \stackrel{\beta}{=} .[[\lambda x_\gamma B_{\beta\delta}]A_\alpha][\lambda x_\gamma C_\epsilon]A_\alpha]$, where this is a wff.

$$\begin{aligned} W_\pi[[\lambda x_\gamma \cdot B_{\beta\delta} C_\epsilon]A_\alpha] &= (W_\pi[\lambda x_\gamma \cdot B_{\beta\delta} C_\epsilon])(W_\pi A_\alpha) = W_\varphi[B_{\beta\delta} C_\epsilon] \\ &= (W_\varphi B_{\beta\delta})(W_\varphi C_\epsilon) \\ &= ((W_\pi[\lambda x_\gamma B_{\beta\delta}])W_\pi A_\alpha)((W_\pi[\lambda x_\gamma C_\epsilon])W_\pi A_\alpha) \\ &= (W_\pi[[\lambda x_\gamma B_{\beta\delta}]A_\alpha])(W_\pi[[\lambda x_\gamma C_\epsilon]A_\alpha]) \\ &= W_\pi[[[\lambda x_\gamma B_{\beta\delta}]A_\alpha][[\lambda x_\gamma C_\epsilon]A_\alpha]] \end{aligned}$$

so $W_\pi P_0 = \mathfrak{t}$.

CASE 4₄: P_0 is an instance of Axiom Schema 4₄, and so is a wff of the form $[[\lambda x_\gamma \cdot \lambda y_\delta B_\beta]A_\alpha] \stackrel{\beta\delta}{=} [\lambda y_\delta \cdot [\lambda x_\gamma B_\beta]A_\alpha]$, where x and y are distinct prim-

itive variables and y_δ is not free in A_α . By virtue of formation rule (6) it is clear that no y -variable whatsoever is free in A_α .

Let η be an arbitrary member of $\mathfrak{D}_{W_\pi\delta}$. Let π' be that assignment which agrees with π on all type variables and on all primitive variables except y , while $\pi'y(W_\pi\delta) = \eta$ and $\pi'y\varepsilon = \pi y\varepsilon$ if $\varepsilon \neq W_\pi\delta$. Let φ' be that assignment which agrees with φ on all type variables and on all primitive variables except y , while $\varphi'y(W_\pi\delta) = \eta$ and $\varphi'y\varepsilon = \varphi y\varepsilon$ if $\varepsilon \neq W_\pi\delta$. (Note that $W_\varphi\delta = W_\pi\delta$.) Thus $W_\pi y_\delta = \eta$, $W_{\varphi'} y_\delta = \eta$, and $W_{\varphi'} x_\gamma = W_\varphi x_\gamma = W_\pi A_\alpha = W_{\pi'} A_\alpha$ by Lemma 1. (Here of course we use the fact that x and y are distinct.) Note that φ' agrees with π' on all type variables and on all primitive variables except x .

Now

$$\begin{aligned} (W_\pi[[\lambda x_\gamma . \lambda y_\delta B_\beta]A_\alpha])\eta &= (W_\varphi[\lambda y_\delta B_\beta])\eta = W_\varphi B_\beta = (W_{\pi'}[\lambda x_\gamma B_\beta])W_\pi A_\alpha \\ &= W_{\pi'}[[\lambda x_\gamma B_\beta]A_\alpha] = (W_\pi[\lambda y_\delta . [\lambda x_\gamma B_\beta]A_\alpha])\eta. \end{aligned}$$

Since this is true for every member η of $\mathfrak{D}_{W_\pi\delta}$, the functions $W_\pi[[\lambda x_\gamma . \lambda y_\delta B_\beta]A_\alpha]$ and $W_\pi[\lambda y_\delta . [\lambda x_\gamma B_\beta]A_\alpha]$ are the same. Hence $W_\pi P_0 = t$.

CASE 4₅: P_0 is an instance of Axiom Schema 4₅, and so is a wff of the form $[[\lambda x_\gamma . \forall c B_0]A_\alpha] \stackrel{0}{=} [\forall c . [\lambda x_\gamma B_0]A_\alpha]$, where c does not occur free in A_α or in γ .

$$W_\pi[[\lambda x_\gamma . \forall c B_0]A_\alpha] = W_\varphi[\forall c B_0].$$

If $W_\varphi[\forall c B_0] = t$, then $W_{\varphi'} B_0 = t$ for every assignment φ' which agrees with φ on all primitive variables and on all type variables other than c . In this case let π' be any assignment which agrees with π on all primitive variables and on all type variables except c , and let φ' be that assignment which agrees with π' on all type variables and on all primitive variables except x , while $\varphi'x(W_\pi\gamma) = W_{\pi'} A_\alpha$ and $\varphi'x\varepsilon = \pi'x\varepsilon$ if $\varepsilon \neq W_\pi\gamma$. Since c does not occur free in A_α by Lemma 1 we see that $W_{\pi'} A_\alpha = W_\pi A_\alpha$, and since c does not occur in γ we also see that $W_{\pi'}\gamma = W_\pi\gamma$. Hence it is easy to see that φ' agrees with φ on all primitive variables and on all type variables except c , so $W_{\varphi'} B_0 = t$. But then $W_{\pi'}[[\lambda x_\gamma B_0]A_\alpha]$

$= (W_{\pi}[\lambda x_{\gamma} B_0])W_{\pi'} A_{\alpha} = W_{\varphi'} B_0 = t$, so $W_{\pi}[\forall c. [\lambda x_{\gamma} B_0] A_{\alpha}] = t$. Thus $W_{\pi} P_0 = t$ in this case.

On the other hand suppose that $W_{\varphi}[\forall c B_0] = f$. Then there must be some assignment φ' which agrees with φ on all primitive variables and on all type variables except c such that $W_{\varphi'} B_0 = f$. Let π' be that assignment which agrees with π on all primitive variables and on all type variables except c , while $\pi' c = \varphi' c$. Again by Lemma 1 we see that $W_{\pi'} A_{\alpha} = W_{\pi} A_{\alpha}$ and $W_{\pi'} \gamma = W_{\pi} \gamma$. It is clear that φ' agrees with π' on all type variables and on all primitive variables except x , while $\varphi' x(W_{\pi'} \gamma) = \varphi' x(W_{\pi} \gamma) = \varphi x(W_{\pi} \gamma) = W_{\pi} A_{\alpha} = W_{\pi'} A_{\alpha}$ and $\varphi' x\varepsilon = \pi' x\varepsilon$ if $\varepsilon \neq W_{\pi} \gamma$. Hence

$$W_{\pi'}[[\lambda x_{\gamma} B_0] A_{\alpha}] = (W_{\pi'}[\lambda x_{\gamma} B_0])W_{\pi'} A_{\alpha} = W_{\varphi'} B_0 = f.$$

Therefore $W_{\pi}[\forall c. [\lambda x_{\gamma} B_0] A_{\alpha}] = f$, so $W_{\pi} P_0 = t$ in this case also.

(5) P_0 is an instance of Axiom Schema 5, and so has the form

$$[l_{\alpha(0\alpha)} \cdot Q_{0\alpha\alpha} y_{\alpha}] \stackrel{\alpha}{=} y_{\alpha}.$$

Let η be in $\mathfrak{D}_{W_{\pi\alpha}}$. $(W_{\pi}[Q_{0\alpha\alpha} y_{\alpha}])\eta = ((W_{\pi} Q_{0\alpha\alpha}) W_{\pi} y_{\alpha})\eta$, which is t if $\eta = W_{\pi} y_{\alpha}$, and f otherwise, by the definition of $W_{\pi} Q_{0\alpha\alpha}$. Hence by the definition of $W_{\pi} l_{\alpha(0\alpha)}$, $W_{\pi}[l_{\alpha(0\alpha)} \cdot Q_{0\alpha\alpha} y_{\alpha}] = (W_{\pi} l_{\alpha(0\alpha)}) W_{\pi}[Q_{0\alpha\alpha} y_{\alpha}] = W_{\pi} y_{\alpha}$. So $W_{\pi} P_0 = t$.

(6) P_0 is an instance of Axiom Schema 6, and so has the form $\forall a T_0$.

By Lemma 2 $W_{\varphi} T_0 = t$ for any assignment φ , so clearly $W_{\pi} P_0 = t$ for any assignment π .

(7) P_0 is an instance of Axiom Schema 7, and so is a wff of the form

$$[\forall a A_0] \supset [S_{\gamma}^a A_0], \text{ where } 2 > \gamma \text{ and all free occurrences of } a \text{ in } A_0 \text{ are free for all variables in } \gamma.$$

If $W_{\pi}[\forall a A_0] = f$, then $W_{\pi} P_0 = t$ by Lemma 5.

If $W_{\pi}[\forall a A_0] = t$, then $W_{\varphi} A_0 = t$, where φ is that assignment which agrees with π on all primitive variables and on all type variables except a , while $\varphi a = W_{\pi} \gamma$. Now γ is in K_+^1 , so by Lemma 6 $W_{\pi}[S_{\gamma}^a A_0] = W_{\varphi} A_0$. Hence $W_{\pi} P_0 = t$ by Lemma 5.

- (8) P_0 is an instance of Axiom Schema 8, and so has the form $[\alpha = \beta] \supset . [S_\alpha^c A_0] = [S_\beta^c A_0]$, where α and β are in K_+^1 and all free occurrences of c in A_0 are free for all variables in α and β .

If $W_\pi[\alpha = \beta] = \text{f}$, then $W_\pi P_0 = \text{t}$ by Lemma 5.

If $W_\pi[\alpha = \beta] = \text{t}$, then $W_\pi \alpha = W_\pi \beta$. Let φ be that assignment which agrees with π on all primitive variables and on all type variables except c , while $\varphi c = W_\pi \alpha$. By Lemma 6 $W_\pi[S_\alpha^c A_0] = W_\varphi A_0 = W_\pi[S_\beta^c A_0]$, so $W_\pi[[S_\alpha^c A_0] = [S_\beta^c A_0]] = \text{t}$. Hence $W_\pi P_0 = \text{t}$ by Lemma 5.

- (9) P_0 is Axiom 9, which is $[(ab) = (cd)] \overset{0}{=} [a = c \wedge b = d]$.

$W_\pi[(ab) = (cd)] = \text{t}$ if and only if $W_\pi(ab) = W_\pi(cd)$. But $W_\pi(ab) = (W_\pi a W_\pi b)$ and $W_\pi(cd) = (W_\pi c W_\pi d)$, and from the definition of K^1 it is clear that $(W_\pi a W_\pi b) = (W_\pi c W_\pi d)$ if and only if $W_\pi a = W_\pi c$ and $W_\pi b = W_\pi d$. Now by Lemma 4 $W_\pi[a = c \wedge b = d] = \text{t}$ if and only if $W_\pi[a = c] = \text{t}$ and $W_\pi[b = d] = \text{t}$, and this occurs if and only if $W_\pi a = W_\pi c$ and $W_\pi b = W_\pi d$. Hence we see that in all cases $W_\pi[(ab) = (cd)] = W_\pi[a = c \wedge b = d]$, so $W_\pi P_0 = \text{t}$.

- (10) P_0 is Axiom 10, which is $[x_a \overset{2}{=} y_b] \supset [a = b]$.

If $W_\pi[x_a \overset{2}{=} y_b] = \text{f}$, then $W_\pi P_0 = \text{t}$ by Lemma 5.

If $W_\pi[x_a \overset{2}{=} y_b] = \text{t}$, then from the definition of $W_\pi Q_{022}$ it is clear that $W_\pi x_a = W_\pi y_b$. Now $W_\pi x_a$ is in $\mathcal{D}_{W_\pi a}$, where $W_\pi a = \pi a$ is in K^1 . Similarly $W_\pi y_b$ is in $\mathcal{D}_{W_\pi b}$, where $W_\pi b = \pi b$ is in K^1 . The domains $\mathcal{D}_\gamma$, where γ is in K^1 , are disjoint, so evidently $W_\pi a = W_\pi b$. Hence $W_\pi[a = b] = \text{t}$, so $W_\pi P_0 = \text{t}$ by Lemma 5.

This completes the proof of Theorem 7.

THEOREM 8: The system Q is consistent.

Proof:

We first remark that assignments do indeed exist¹¹). Indeed, we can describe certain assignments explicitly. For example, we can describe an

¹¹) Of course by using the Axiom of Choice in our meta-language we can easily find assignments. The purpose of this paragraph is to make it clear that it is not necessary to use the Axiom of Choice to show that assignments exist. It will then be noted that the Axiom of Choice is not used at any point in our proof of the consistency of Q. (The elements τ_α referred to in this paragraph are of course those introduced on page 10.)

assignment θ by specifying that $\theta a = 1$ for every type variable a and that $\theta x\alpha = r_\alpha$ for each primitive variable x and for each type symbol α in K^2 .

Now if Q were not consistent, every wff of type 0 would be a theorem of Q. (As we will see in Chapter II, all the laws of propositional calculus can be derived in Q.) In particular F_0 would be a theorem. Then by Theorem 7, $W_\pi F_0$ would be t for every assignment π . However this contradicts Lemma 3.

I.4. Independence results

It is easy to see that Rule R is independent, since it is the sole rule of inference of Q. The wff T_0 (i.e. $[[Q_{0(00)(00)}[\lambda x_0 x_0]] [\lambda x_0 x_0]]$) is not an axiom of Q, but is a theorem, as we shall see in Chapter II (Theorem 100). Hence Rule R cannot be dispensed with.

We can also show the independence of certain axioms and axiom schemata of Q. We outline these proofs below, leaving most of the details for the reader to verify.

AXIOM 1

We construct a model $\mathcal{M}'$ for Q which is similar to the model $\mathcal{M}$ described in Section 3, except that $\mathcal{D}_0$ now contains three objects. For a complete description of the model $\mathcal{M}'$ and the interpretation of Q in $\mathcal{M}'$, simply replace the sentences "We first choose three distinct objects t, f and r which are not functions. $\mathcal{D}_0$ contains t and f and nothing else." in our treatment of $\mathcal{M}$ on page 9 by the sentences "We first choose four distinct objects t, f, u and r which are not functions. $\mathcal{D}_0$ contains t, f, u and nothing else." Reading through the rest of Section 3 with the new interpretation of Q in mind, we see that Lemmas 1-4 and 6 still apply without change, while Lemma 5 must be modified as follows: $((W_\pi \supset_{000})x)y = t$ if x is f or $\bar{x}$ and y are both t; otherwise $((W_\pi \supset_{000})x)y = f$. Also Theorem 7 applies to the system obtained when Axiom 1 is deleted from the list of axioms of Q. Hence Axiom 1 cannot be derived in this system, since $W_\pi[Axiom\ 1]$ is f for any assignment π such that $(W_\pi g_{00})t = (W_\pi g_{00})f = t$, and $(W_\pi g_{00})u = f$. Thus Axiom 1 is independent.

AXIOM SCHEMA 3

We construct a model $\mathfrak{M}^3$ for Q by another modification on the construction of $\mathfrak{M}$. Let $\mathfrak{D}_0$ and $\mathfrak{D}_1$ be defined as for $\mathfrak{M}$. Let $\mathfrak{D}_{11}$ contain *two* objects, b and g . (Note that in the case of $\mathfrak{M}$, the domain $\mathfrak{D}_{11}$ contained only one element, namely the identity function on $\mathfrak{D}_1$. b and g may be regarded respectively as the “blue” and the “green” identity functions on $\mathfrak{D}_1$.) If $(\alpha\beta)$ is neither (11) nor (21), $\mathfrak{D}_{\alpha\beta}$ consists of all functions from $\mathfrak{D}_\beta$ to $\mathfrak{D}_\alpha$, as before. (Of course there are now many more such functions if (11) or 2 is part of α or β ; for example $\mathfrak{D}_{0(11)}$ now contains four elements, whereas in the case of the model $\mathfrak{M}$ it contained only two.) $\mathfrak{D}_{21}$ consists of b , g , and all functions from $\mathfrak{D}_1$ to $\mathfrak{D}_2$ which do not have value r . (Thus if we regard b and g as functions, $\mathfrak{D}_{21}$ simply consists of all functions from $\mathfrak{D}_1$ to $\mathfrak{D}_2$.) $\mathfrak{D}_2$ is again the union of the domains $\mathfrak{D}_\alpha$ for which α is in K^1 .

The interpretation of Q in $\mathfrak{M}^3$ is made essentially as in the case of $\mathfrak{M}$. In clause (8) of the definition of W_π (page 10), if $W_\pi B_{\beta\alpha}$ is in $\mathfrak{D}_{11}$, then $W_\pi A_\gamma$ must be in $\mathfrak{D}_1$, and we define $W_\pi[B_{\beta\alpha}A_\gamma]$ to be r , the sole member of $\mathfrak{D}_1$. In clauses (6) and (9) when there is ambiguity as to which function is meant (as in the choice of r_{11} , r_{21} , or the value of $W_\pi[\lambda x_\beta A_\alpha]$ when $W_\pi\beta$ is 1 and $W_\pi A_\alpha$ is in $\mathfrak{D}_1$), we shall always choose b .

Reading through Section 3 with the new interpretation of Q in mind, one easily sees that Theorem 7 applies to the system which is obtained when Axiom Schema 3 is deleted from the list of axioms of Q . (Note especially that the treatment of Axiom Schema 4₄ in the proof of Theorem 7 can with a few words be modified to fit the present interpretation, so no trouble arises.) However if $W_\pi f_{11} = b$ and $W_\pi g_{11} = g$, then $W_\pi[[f_{11} \stackrel{11}{=} g_{11}] = \forall x_1[f_{11}x_1 \stackrel{1}{=} g_{11}x_1]]$ is $\dagger$, so this instance of Axiom Schema 3 cannot be derived in that system. Thus Axiom Schema 3 is independent.

Of course one might still ask whether certain instances of the Axiom Schema are derivable from the other axioms, or from the other axioms and other instances of this Axiom Schema. However for the present we shall concern ourselves only with the question of whether certain axiom schemata could be dispensed with altogether.

AXIOM SCHEMATA 4₁–4₅

It is easy to see that certain instances of Axiom Schemata 4₁–4₅ can be derived from the other schemata (without the use of any other axioms of Q), and the schemata are so closely interrelated that separate independence proofs for them seem difficult to obtain. Actually in Chapter II we shall use these schemata to prove a single theorem schema (103) which in a certain sense summarizes them, and which could be used as an axiom schema in place of them. (However to prove the schemata 4₁–4₅ from schema 103 one apparently must also use other axioms of Q.)

Hence we shall refer to Axiom Schemata 4₁–4₅ collectively as Axiom Schema 4, and simply prove the independence of Axiom Schema 4. To do this we define the λ -parity of a wff to be even if the number of occurrences of λ in the wff is even, and to be odd in the contrary case. Examination of our definitions reveals that T_0 , F_0 , $\wedge_{000}$, and $\supset_{000}$ all have even λ -parity, and that the λ -parity of $[\forall x_\beta A_0]$ is the same as that of A_0 . Hence we quickly see that the λ -parities of the axioms 1–3 and 5–10 are all even. Moreover Rule R preserves evenness of λ -parity. For if the λ -parity of $[A_\alpha \stackrel{\gamma}{=} B_\beta]$ is even, then A_α and B_β have the same λ -parity, so replacing A_α by B_β in a wff does not change the λ -parity of the wff. Thus every theorem of Q which can be derived without the use of Axiom Schema 4 has even λ -parity. However certain instances of Axiom Schema 4 have odd λ -parity (note especially 4₂), so Axiom Schema 4 is independent.

AXIOM SCHEMA 5

For each type symbol α let C_α be $[\iota_{\alpha(0\alpha)} \cdot \lambda x_\alpha T_0]$, and let $J_{\alpha(0\alpha)}$ be $[\lambda z_{0\alpha} C_\alpha]$. Note that no wff-variables occur free in these wffs.

Now consider the system Q' which is obtained when Axiom Schema 5 is deleted from the list of axioms of Q. It is easy to see by induction on the length of the proof of A_0 , that if A_0 is a theorem of Q', then B_0 is also a theorem of Q', where B_0 is the result of simultaneously replacing all occurrences of the wffs $\iota_{\alpha(0\alpha)}$ in A_0 by the wffs $J_{\alpha(0\alpha)}$ for each type symbol α . (Note that no trouble arises when A_0 is an instance of Axiom Schema 7 or 8.)

Suppose that for some α , $[\iota_{\alpha(0\alpha)} \cdot Q_{0\alpha\alpha} y_\alpha] \stackrel{\alpha}{=} y_\alpha$ were derivable in Q'.

Then $[J_{\alpha(0\alpha)} \cdot Q_{0\alpha\alpha} y_\alpha] \stackrel{\alpha}{=} y_\alpha$ would also be derivable, so by Rule R and the axiom $J_{\alpha(0\alpha)}[Q_{0\alpha\alpha} y_\alpha] \stackrel{\alpha}{=} C_\alpha$ (an instance of 4₁) we would obtain $C_\alpha \stackrel{\alpha}{=} y_\alpha$ as a theorem of Q'.

It is clear that Theorem 7 applies to Q' with respect to any principal interpretation of Q, in particular with respect to a principal interpretation in which the domain $\mathfrak{D}_1$ of the model (and hence each domain) contains at least two elements. However for such an interpretation $W_\pi[C_\alpha \stackrel{\alpha}{=} y_\alpha]$ is not always t, since we can always choose π so that $W_\pi y_\alpha$ is not the same as $W_\pi C_\alpha$. Thus we reach a contradiction. So no instance of Axiom Schema 5 is derivable in Q', and the axiom schema is independent.

It is well known that for certain types α we can introduce the wffs $\iota_{\alpha(0\alpha)}$ by definition, in the sense that we can omit mention of $\iota_{\alpha(0\alpha)}$ from our formation rules, and find a wff $L_{\alpha(0\alpha)}$ in which no wff-variable occurs free such that we can prove $[L_{\alpha(0\alpha)} \cdot Q_{0\alpha\alpha} y_\alpha] \stackrel{\alpha}{=} y_\alpha$. (See [4], p.62, and [10], p.328, for the required definitions.) The question therefore arises whether we can do this for all types α , and delete ι from our list of primitive symbols altogether. Unfortunately the answer to this question is negative, since if we deleted ι from our list of primitive symbols, for certain types α there would be no wffs $L_{\alpha(0\alpha)}$ in which no wff-variable occurs free. This we see as follows:

We define certain type symbols to be *critical*. The definition is by induction on the length of the type symbol:

- (1) The type symbol 1 is critical, but no other type symbol of length 1 is critical.
- (2) The type symbol $(\alpha\beta)$ is critical if and only if α is critical and β is not critical.

Note that 1(01) is critical.

We easily see by induction on the length of α that if $\alpha > \beta$ for some type symbol β , then α is not critical. For 2 is not critical, and if α has the form $(\delta\gamma)$ and $\alpha > \beta$, then β must have the form $(\epsilon\gamma)$, where $\delta > \epsilon$. But by inductive hypothesis δ is not critical, so α is not critical.

PROPOSITION: Let A_α be any wff of the system which is obtained when ι is deleted from the list of primitive symbols of Q. (Of course formation

rule (3) must also be dropped.) If the type α of A_α is critical, then there is some wff-variable x_β which occurs free in A_α , and whose type β is critical.

The proof is by induction on the construction of A_α .

The proposition is clearly true if A_α is of any of the forms x_α , $Q_{\alpha\beta\beta}$, $[Q\beta\gamma]$, or $[\forall aB_\alpha]$.

If A_α has the form $[B_{\alpha\beta}C_\gamma]$, where $\beta \geq \gamma$ and α is critical, we consider the following cases:

(1) β is γ . Then either γ is critical, or β is not critical so $(\alpha\beta)$ is critical.

(2) $\beta > \gamma$. Then β is not critical, so $(\alpha\beta)$ is critical.

In either case $(\alpha\beta)$ or γ must be critical, so we apply the inductive hypothesis to $B_{\alpha\beta}$ or C_γ to see that there is a wff-variable with critical type free in A_α .

If A_α has the form $[\lambda x_\beta B_\gamma]$, and α is critical, then, since α is $(\gamma\beta)$, γ must be critical and β not critical. Hence by inductive hypothesis there must be some wff-variable y_δ free in B_γ with critical type δ , and y_δ cannot be x_β , so y_δ is also free in A_α .

This completes the proof of the proposition.

AXIOM SCHEMA 6

To prove the independence of Axiom Schema 6, we use the interpretation of Q given in Section 3, with clause (10) in the definition of W_π modified as follows: $W_\pi[\forall aA_0]$ is $\mathfrak{f}$ in all cases. We easily see that under this interpretation Theorem 7 applies to the system obtained when Axiom Schema 6 is deleted from the list of axioms of Q, but of course $W_\pi[\forall aT_0]$ is $\mathfrak{f}$, so $\forall aT_0$ is not derivable in this system.

AXIOM SCHEMA 7

This time we modify clause (10) in the definition of W_π as follows: $W_\pi[\forall aA_0]$ is $\mathfrak{t}$ in all cases. Under this interpretation Theorem 7 applies when Axiom Schema 7 is dropped, but $W_\pi[[\forall aF_0] \supset F_0]$ is $\mathfrak{f}$, so Axiom Schema 7 is independent.

AXIOM SCHEMA 8

We modify clause (7) in the definition of W_π as follows: $W_\pi[Q\alpha\beta]$ is $\mathfrak{t}$ in all cases. Under this interpretation Theorem 7 applies when Axiom

Schema 8 is dropped, but if $W_{\pi}x_0$ is t and (necessarily) $W_{\pi}x_1$ is r , then $W_{\pi}[[0 = 1] \supset .x_0 \stackrel{2}{=} T_0 \stackrel{0}{=} x_1 \stackrel{2}{=} T_0]$ is f , so we see that Axiom Schema 8 is independent.

AXIOM 10

We construct a model $\mathcal{M}^{10}$ for Q as follows. Choose distinct objects t and f , and let $\mathfrak{D}_0$ and $\mathfrak{D}_1$ each be the domain which contains t and f and nothing else. Then proceed as in the case of $\mathcal{M}$. Thus the domain of individuals is taken to be the domain of truth values. The interpretation of Q in $\mathcal{M}^{10}$ is made exactly as in the case of $\mathcal{M}$, with all references to r replaced by references to f . Note that while $\mathfrak{D}_0$ and $\mathfrak{D}_1$ are now the same, the type symbols 0 and 1 in K^1 are still distinct symbols, so $W_{\pi}[Q01]$ remains f . We see that Theorem 7 applies under the present interpretation of Q to the system with Axiom 10 deleted. However if πa is 0, πb is 1, and $(\pi x)0 = f = (\pi y)1$, then $W_{\pi}x_a = f = W_{\pi}y_b$, so $W_{\pi}[[x_a \stackrel{2}{=} y_b] \supset [a = b]]$ is f . Hence Axiom 10 is independent.

Thus far independence proofs for Axioms 2 and 9 have not been found.

I.5. General comments on Q

Various extensions of the system Q are of course possible. Among the simplest of these are systems obtained by adding additional axioms to the list of axioms for Q. For example, it will for certain purposes be desirable to add one or more of the following as axioms or axiom schemata:

$$(11) \quad p_{0\alpha}x_{\alpha} \supset p_{0\alpha}[i_{\alpha(0\alpha)}p_{0\alpha}].$$

$$(11') \quad \exists i_{\alpha(0\alpha)}. \forall p_{0\alpha}. \exists x_{\alpha} p_{0\alpha}x_{\alpha} \supset p_{0\alpha}[i_{\alpha(0\alpha)}p_{0\alpha}].$$

$$(12) \quad \forall y_2 \exists a \exists x_a . y_2 \stackrel{2}{=} x_a.$$

$$(12') \quad \forall a \forall x_a p_{02}x_a \supset \forall y_2 p_{02}y_2.$$

$$(13) \quad [S_0^c A_0] \supset . [S_1^c A_0] \supset . \forall a \forall b [[S_a^c A_0] \supset . [S_b^c A_0] \supset [S_{(ab)}^c A_0]] \supset \forall c A_0,$$

provided this is a wff, a and b are distinct, and all free occurrences of c in A_0 are free for a and b .

(14) $\sim [0 = 1]$.

(15) $\sim [0 = (ab)]$.

(16) $\sim [1 = (ab)]$.

Of these (11) and (11') are formulations of the Axiom of Choice, (12) and (12') are formulations of an axiom restricting the size of $\mathfrak{D}_2$, (13) restricts the range of type variables to K^1 , and (14)–(16) express the fact that certain type symbols are distinct. (We remark that without using any of the above axioms, we can prove $\sim [(0a) = a]$ in Q; see Theorem 305 in Chapter II.) It is clear that (12)–(16) are all true in all the principal interpretations of Q mentioned in Section 3 of this chapter, and that (11) and (11') can also be shown to be true if we assume the axiom of choice in our meta-language. However, we consider it to be of interest that the Theorem of Infinity can be proved, and the semantics of finite type theory formalized, in Q without the use of any of these axioms. Therefore we have not included them in the list of axioms of Q.

If one wishes not only to formalize the semantics of finite type theory in Q, but to prove in Q the consistency of a formulation of finite type theory which has an Axiom of Infinity, then one must evidently add to Q an Axiom of Infinity for individuals of type 1. (Our Theorem of Infinity is for entities of type 2.) Alternatively, one might in some appropriate way extend Q to higher transfinite types, and then interpret the individuals of the finite type theory in $\mathfrak{D}_2$ rather than in $\mathfrak{D}_1$ when formalizing the semantics of the finite type theory.

The following lemmas about wffs will be useful:

LEMMA 9: There are no type symbols α , β and γ in K_+^2 such that $\alpha > \beta$ and $\beta > \gamma$.

The proof is by induction on the length of β .

CASE 1: β is 0, 1, 2, or a type variable a .

Since none of these has the form $(\delta\varepsilon)$, they can be involved in the relation $>$ only by virtue of clause (1) of the definition of $>$. Hence there is no type symbol α such that $\alpha > 2$ and there is no type symbol γ such that $0 > \gamma$ or $1 > \gamma$ or $a > \gamma$.

CASE 2: β has the form $(\delta\epsilon)$.

If there is a type symbol γ such that $(\delta\epsilon) > \gamma$, evidently this must be by virtue of clause (2) of the definition of $>$, so γ must have the form $(\delta'\epsilon)$, where $\delta > \delta'$.

If δ is in K_+^1 , then $2 > \delta$ and $\delta > \delta'$, which violates the inductive hypothesis applied to δ .

On the other hand if δ is not in K_+^1 , then $(\delta\epsilon)$ is not in K_+^1 , so if there is a type symbol α such that $\alpha > (\delta\epsilon)$, evidently α must have the form $(\delta''\epsilon)$, where $\delta'' > \delta$. However, this would violate the inductive hypothesis applied to δ .

LEMMA 10: If $\alpha \geq \beta$ and every free variable of B_β is a free variable of A_α , and C_γ is a wff of which A_α is a wf part, then the result of replacing one occurrence of A_α (not immediately preceded by λ) in C_γ by an occurrence of B_β is a wff D_δ , where $\gamma \geq \delta$. If β is α then δ is γ . Moreover every free variable of D_δ is a free variable of C_γ .

The proof is by induction on the length of C_γ . One examines in turn the possible forms of C_γ according to the formation rules, and in each case easily sees that the conditions of the lemma are satisfied. We leave the details to the reader.

LEMMA 11: If $\gamma \geq \alpha$ and every free occurrence of x_γ in B_β is free for all free variables of A_α , then $[S_{A_\alpha}^{x_\gamma} B_\beta]$ is a wff of some type δ , where $\beta \geq \delta$.

The proof is by induction on the length of B_β . We leave the details to the reader.

LEMMA 12: $[S_{x_\beta}^{y_\beta} [S_{y_\beta}^{x_\beta} A_\alpha]]$ is A_α , provided y_β does not occur free in A_α and all free occurrences of x_β in A_α are free for y_β .

The proof is left to the reader.

LEMMA 13: If $\alpha > \beta$ and γ is in K_+^1 , then $(S_\gamma^c \alpha) > (S_\gamma^c \beta)$.

The proof is by induction on the length of α .

LEMMA 14: If $2 > \gamma$ and all free occurrences of a in the wff A_β are free for all type variables in γ , then $S_\gamma^a A_\beta$ is a wff of type $(S_\gamma^a \beta)$.

The proof is similar to the proof of Lemma 11.

CHAPTER II

BASIC LOGIC IN Q

We shall henceforth let $\mathcal{H}$ stand for an arbitrary finite set of wffs of type 0. We do not exclude the possibility that $\mathcal{H}$ may be empty. We shall often refer to $\mathcal{H}$ as a set of *hypotheses* or *premisses*. We shall write $\mathcal{H} \vdash A_0$ as an abbreviation for the phrase “ A_0 is derivable from the set $\mathcal{H}$ of hypotheses”. We define this concept inductively as follows:

- (1) If H_0 is a member of $\mathcal{H}$, then $\mathcal{H} \vdash H_0$.
- (2) If A_0 is a theorem of Q, then $\mathcal{H} \vdash A_0$.
- (3) If $\mathcal{H} \vdash C_0$ and $\mathcal{H} \vdash A_\alpha \stackrel{\alpha}{=} B_\alpha$, then $\mathcal{H} \vdash D_0$, provided that:
 - a) D_0 is a wff of type 0 which is the result of replacing one occurrence of A_α in C_0 by an occurrence of B_α ;
 - b) the occurrence of A_α in C_0 is not an occurrence of a variable immediately preceded by λ ;
 - c) for each primitive or type variable X such that an X -variable occurs free in a member of $\mathcal{H}$ and an X -variable occurs free in A_α or B_α , the occurrence of A_α in C_0 is free for all X -variables.
- (4) No wff is derivable from $\mathcal{H}$ unless its being so follows from the above rules.

By a *proof from the hypotheses* $\mathcal{H}$ we shall mean a pair of finite sequences of wffs of type 0, such that every member of the first sequence is either an axiom of Q or is inferred from previous members of the sequence by Rule R (so every member of the first sequence is a theorem of Q), and such that every member of the second sequence is either a member of $\mathcal{H}$ or a member of the first sequence or is inferred from previous members of the sequence according to clause (3) above.

When a wff D_0 is inferred from wffs C_0 and $A_\alpha \stackrel{\alpha}{=} B_\alpha$ by clause (3) in a proof from hypotheses, we shall say that D_0 is inferred from C_0 and $A_\alpha \stackrel{\alpha}{=} B_\alpha$ by *Rule R'*. Clearly in the case that $\mathcal{H}$ is empty, Rule R' is just a special case of Rule R. Consequently it is clear that if $\mathcal{H}$ is empty, $\mathcal{H} \vdash A_0$ if and only if A_0 is a theorem of Q. Hence we shall write $\vdash A_0$ to indicate that A_0 is a theorem.

We next turn to the task of proving certain theorems in Q, and proving certain metatheorems and derived rules of inference. It will be clear that the proofs of our derived rules of inference can easily be made effective in the sense of [5].

As a convenience to the reader, after each line in a proof (or condensed proof) we shall indicate by name or number the rule (or rules) of inference used to obtain it, and then, after a colon, the numbers of the theorems or (in the case of a proof from hypotheses) lines of the proof from which the given line was inferred. In some cases the rules and wffs referred to must be used more than once. In a proof from hypotheses, to indicate that a wff is a member of the set of hypotheses, we shall simply write "hypothesis" or "premiss".

100 $\vdash A_\alpha \stackrel{\gamma}{=} A_\alpha$ for any wff A_α , provided $\gamma \geq \alpha$.

Proof:

- | | |
|--|----------------------|
| .1 $\vdash [[\lambda x_\gamma x_\gamma] A_\alpha] \stackrel{\gamma}{=} A_\alpha$ | Axiom 4 ₂ |
| .2 $\vdash [[\lambda x_\gamma x_\gamma] A_\alpha] \stackrel{\gamma}{=} A_\alpha$ | Axiom 4 ₂ |
| .3 $\vdash A_\alpha \stackrel{\gamma}{=} A_\alpha$ | Rule R: .1, .2. |

101 Equivalence Rules (E-Rules): Suppose $\gamma \geq \alpha$, $\gamma \geq \beta$, $\gamma \geq \delta$.

If $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ then $\vdash B_\beta \stackrel{\gamma}{=} A_\alpha$.

If $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ and $\vdash B_\beta \stackrel{\gamma}{=} C_\delta$ then $\vdash A_\alpha \stackrel{\gamma}{=} C_\delta$.

If $\mathcal{H} \vdash A_\alpha \stackrel{\alpha}{=} B_\alpha$ then $\mathcal{H} \vdash B_\alpha \stackrel{\alpha}{=} A_\alpha$.

If $\mathcal{H} \vdash A_\alpha \stackrel{\alpha}{=} B_\alpha$ and $\mathcal{H} \vdash B_\alpha \stackrel{\alpha}{=} C_\alpha$ then $\mathcal{H} \vdash A_\alpha \stackrel{\alpha}{=} C_\alpha$.

Proof of the first rule:

- 101.1 $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ by hypothesis
 .2 $\vdash B_\beta \stackrel{\gamma}{=} A_\alpha$ Rule R: 100,101.1.

Proof of the second rule:

- .3 $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ by hypothesis
 .4 $\vdash B_\beta \stackrel{\gamma}{=} C_\delta$ by hypothesis
 .5 $\vdash A_\alpha \stackrel{\gamma}{=} C_\delta$ Rule R: .3, .4.

The proofs of the third and fourth rules are similar, with Rule R' being used in place of Rule R.

- 102 Rule RR: If $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ or $\vdash B_\beta \stackrel{\gamma}{=} A_\alpha$, where $\gamma \geq \alpha$ and $\gamma \geq \beta$, and if $\mathcal{H} \vdash C_0$, then $\mathcal{H} \vdash D_0$, where D_0 is a wff of type 0 which is the result of replacing one occurrence of A_α (not immediately preceded by λ) in C_0 by an occurrence of B_β .

(Note that the restrictions on Rule R' do not apply to Rule RR.)

Proof:

- .1 $\vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ by hypothesis and (if necessary) E-Rules
 .2 $\vdash C_0 \stackrel{0}{=} C_0$ 100
 .3 $\vdash C_0 \stackrel{0}{=} D_0$ Rule R: .2, .1
 .4 $\mathcal{H} \vdash C_0 \stackrel{0}{=} D_0$ since this is a theorem
 .5 $\mathcal{H} \vdash C_0$ by hypothesis
 .6 $\mathcal{H} \vdash D_0$ Rule R': .5, .4.

- 103 $\vdash [[\lambda x_\gamma B_\beta] A_\alpha] \stackrel{\beta}{=} [\mathcal{S}_{\lambda_\alpha}^{\gamma} B_\beta]$, provided that $\gamma \geq \alpha$, no x -variable other than x_γ occurs free in B_β , and all free occurrences of x_γ in B_β are free for all free variables of A_α .

Note that by Lemma 11 and formation rules (5) and (6'), the above formula must be well-formed.

The proof is by induction on the length of B_β .

CASE 1: x_γ does not occur free in B_β .

Hence no x -variable occurs free in B_β , and $[S_{A_\alpha}^{x_\gamma} B_\beta]$ is B_β , so the desired theorem is Axiom 4₁.

CASE 2: B_β is x_γ .

Then the desired theorem is Axiom 4₂.

CASE 3: B_β has the form $[D_{\beta\delta} C_\epsilon]$, where $\delta \geq \epsilon$.

$$\vdash [[\lambda x_\gamma [D_{\beta\delta} C_\epsilon]] A_\alpha] \stackrel{\beta}{=} . [[\lambda x_\gamma D_{\beta\delta}] A_\alpha] [[\lambda x_\gamma C_\epsilon] A_\alpha] \quad \text{Axiom 4}_3$$

$$\vdash [[\lambda x_\gamma D_{\beta\delta}] A_\alpha] \stackrel{\beta\delta}{=} [S_{A_\alpha}^{x_\gamma} D_{\beta\delta}] \quad \text{by inductive hypothesis}$$

$$\vdash [[\lambda x_\gamma [D_{\beta\delta} C_\epsilon]] A_\alpha] \stackrel{\beta}{=} . [S_{A_\alpha}^{x_\gamma} D_{\beta\delta}] [[\lambda x_\gamma C_\epsilon] A_\alpha] \quad \text{by Rule R}$$

(Note that by Lemma 11 $[S_{A_\alpha}^{x_\gamma} D_{\beta\delta}]$ is a wff of some type $(\beta'\delta)$, where $\beta \geq \beta'$, so the above formula is well-formed.)

$$\vdash [[\lambda x_\gamma C_\epsilon] A_\alpha] \stackrel{\epsilon}{=} [S_{A_\alpha}^{x_\gamma} C_\epsilon] \quad \text{by inductive hypothesis}$$

$$\vdash [[\lambda x_\gamma [D_{\beta\delta} C_\epsilon]] A_\alpha] \stackrel{\beta}{=} . [S_{A_\alpha}^{x_\gamma} D_{\beta\delta}] [S_{A_\alpha}^{x_\gamma} C_\epsilon] \quad \text{by Rule R.}$$

(Again by Lemma 11 we see that this is a wff, since $[S_{A_\alpha}^{x_\gamma} C_\epsilon]$ is a wff of some type ϵ' , where $\epsilon \geq \epsilon'$, so $\delta \geq \epsilon'$.) Now $[S_{A_\alpha}^{x_\gamma} [D_{\beta\delta} C_\epsilon]]$ is $[S_{A_\alpha}^{x_\gamma} D_{\beta\delta}] [S_{A_\alpha}^{x_\gamma} C_\epsilon]$, so we have the desired theorem.

CASE 4: B_β is $[\lambda y_\delta C_\epsilon]$.

If x_γ has no free occurrences in B_β then this is dealt with in case 1. If x_γ does have free occurrences in B_β then y is not x , and since all free occurrences of x_γ in B_β are free for all free variables of A_α , no y -variables are free in A_α . Hence

$$\vdash [[\lambda x_\gamma . \lambda y_\delta C_\epsilon] A_\alpha] \stackrel{\epsilon\delta}{=} [\lambda y_\delta . [\lambda x_\gamma C_\epsilon] A_\alpha] \quad \text{Axiom 4}_4$$

(Note that this is indeed a wff.)

$$\vdash [\lambda x_\gamma C_\epsilon] A_\alpha \stackrel{\epsilon}{=} [S_{A_\alpha}^{x_\gamma} C_\epsilon] \quad \text{by inductive hypothesis}$$

$$\vdash [[\lambda x_\gamma . \lambda y_\delta C_\epsilon] A_\alpha] \stackrel{\epsilon\delta}{=} [\lambda y_\delta [S_{A_\alpha}^{x_\gamma} C_\epsilon]] \quad \text{by Rule R.}$$

(Since by Lemma 11 $[S_{A_\alpha}^{x_\gamma} C_\epsilon]$ is a wff of some type ϵ' , where $\epsilon \geq \epsilon'$,

and every free variable of $[S_{A_\alpha}^{x_\gamma} C_\varepsilon]$ is a free variable of $[\lambda x_\gamma C_\varepsilon] A_\alpha$, we see from Lemma 10 that this is a wff.)

Now $[S_{A_\alpha}^{x_\gamma} [\lambda y_\delta C_\varepsilon]]$ is $[\lambda y_\delta [S_{A_\alpha}^{x_\gamma} C_\varepsilon]]$, so this is the desired theorem.

CASE 5: B_β is $\forall c D_0$.

If x_γ has no free occurrences in B_β , this is dealt with in case 1.

If x_γ does have free occurrences in B_β , then c cannot occur in γ (as we see from formation rule (7)), and since all free occurrences of x_γ in B_β are free for all free variables of A_α , c cannot occur free in A_α .

$$\vdash [[\lambda x_\gamma. \forall c D_0] A_\alpha] \stackrel{0}{=} [\forall c. [\lambda x_\gamma D_0] A_\alpha] \quad \text{Axiom 4}_5$$

$$\vdash [\lambda x_\gamma D_0] A_\alpha \stackrel{0}{=} S_{A_\alpha}^{x_\gamma} D_0 \quad \text{by inductive hypothesis}$$

$$\vdash [[\lambda x_\gamma. \forall c D_0] A_\alpha] \stackrel{0}{=} [\forall c [S_{A_\alpha}^{x_\gamma} D_0]] \quad \text{by Rule R.}$$

(Again we may refer to Lemmas 10 and 11 to verify that this must be a wff.)

Now $[S_{A_\alpha}^{x_\gamma} [\forall c D_0]]$ is $[\forall c [S_{A_\alpha}^{x_\gamma} D_0]]$, so this is the desired theorem.

104 $\vdash [\lambda x_{\beta_1}^1 \lambda x_{\beta_2}^2 \dots \lambda x_{\beta_n}^n A_\alpha] x_{\beta_1}^1 x_{\beta_2}^2 \dots x_{\beta_n}^n \stackrel{\alpha}{=} A_\alpha$ ($n = 1, 2, 3, \dots$), provided that for each i ($n \geq i \geq 1$), no x^i -variable other than $x_{\beta_i}^i$ occurs free in A_α . (Note that if the primitive variables x^i and x^j are the same but the type symbols β_i and β_j are distinct, neither $x_{\beta_i}^i$ nor $x_{\beta_j}^j$ can be free in A_α .)

The proof is by induction on n .

If $n = 1$ we have

$$\vdash [\lambda x_{\beta_\alpha} A_\alpha] x_{\beta_\alpha} \stackrel{\alpha}{=} A_\alpha \quad \text{by 103, since our conditions}$$

assure that the conditions on 103 are satisfied.

If $n > 1$ by inductive hypothesis we have

$$\vdash [\lambda x_{\beta_1}^1 \lambda x_{\beta_2}^2 \dots \lambda x_{\beta_{n-1}}^{n-1} [\lambda x_{\beta_n}^n A_\alpha]] x_{\beta_1}^1 x_{\beta_2}^2 \dots x_{\beta_{n-1}}^{n-1} \stackrel{\alpha \beta_n}{=} [\lambda x_{\beta_n}^n A_\alpha]$$

$$\vdash [\lambda x_{\beta_1}^1 \lambda x_{\beta_2}^2 \dots \lambda x_{\beta_{n-1}}^{n-1} [\lambda x_{\beta_n}^n A_\alpha]] x_{\beta_1}^1 x_{\beta_2}^2 \dots x_{\beta_{n-1}}^{n-1} x_{\beta_n}^n \stackrel{\alpha}{=} [\lambda x_{\beta_n}^n A_\alpha] x_{\beta_n}^n$$

by Rule R and 100

$\vdash [\lambda x_{\beta_n}^n A_\alpha] x_{\beta_n}^n \stackrel{\alpha}{=} A_\alpha$ by inductive hypothesis

$\vdash [\lambda x_{\beta_1}^1 \lambda x_{\beta_2}^2 \cdots \lambda x_{\beta_{n-1}}^{n-1} [\lambda x_{\beta_n}^n A_\alpha]] x_{\beta_1}^1 x_{\beta_2}^2 \cdots x_{\beta_{n-1}}^{n-1} x_{\beta_n}^n \stackrel{\alpha}{=} A_\alpha$ by Rule R.

105 If $\mathcal{H} \vdash B_\beta \stackrel{\beta}{=} C_\beta$ then $\mathcal{H} \vdash \S_{A_\alpha}^{x_\gamma} [B_\beta \stackrel{\beta}{=} C_\beta]$, provided that

- (1) $\gamma \geq \alpha$ and all free occurrences of x_γ in $[B_\beta \stackrel{\beta}{=} C_\beta]$ are free for all free variables of A_α ;
- (2) x_γ is the only x -variable free in $[B_\beta \stackrel{\beta}{=} C_\beta]$;
- (3) no x -variable occurs free in a member of $\mathcal{H}$.

Proof:

- .1 $\mathcal{H} \vdash B_\beta \stackrel{\beta}{=} C_\beta$ by hypothesis
- .2 $\mathcal{H} \vdash [\lambda x_\gamma B_\beta] A_\alpha \stackrel{\beta}{=} [\lambda x_\gamma C_\beta] A_\alpha$ 100 (Using (2) we see that this is a wff.)
- .3 $\mathcal{H} \vdash [\lambda x_\gamma B_\beta] A_\alpha \stackrel{\beta}{=} [\lambda x_\gamma C_\beta] A_\alpha$ R': .2, .1 (Using (2) and (3) we see that the conditions on R' are satisfied.)
- .4 $\vdash [\lambda x_\gamma B_\beta] A_\alpha \stackrel{\beta}{=} \S_{A_\alpha}^{x_\gamma} B_\beta$ 103 (Here use (1) and (2).)
- .5 $\vdash [\lambda x_\gamma C_\beta] A_\alpha \stackrel{\beta}{=} \S_{A_\alpha}^{x_\gamma} C_\beta$ 103 (Here use (1) and (2).)
- .6 $\mathcal{H} \vdash [\S_{A_\alpha}^{x_\gamma} B_\beta] \stackrel{\beta}{=} [\S_{A_\alpha}^{x_\gamma} C_\beta]$ RR: .3, .4, .5.

This is the desired theorem.

106 Rule of Specialization (Rule S): If $\mathcal{H} \vdash \forall x_\gamma B_0$ then $\mathcal{H} \vdash \S_{A_\alpha}^{x_\gamma} B_0$, provided that $\gamma \geq \alpha$ and all free occurrences of x_γ in B_0 are free for all free variables of A_α .

Proof:

- .1 $\mathcal{H} \vdash [\lambda x_\gamma B_0] \stackrel{0\gamma}{=} [\lambda x_\gamma T_0]$ by hypothesis, def. of $\forall$
- .2 $\mathcal{H} \vdash [\lambda x_\gamma B_0] A_\alpha \stackrel{0}{=} [\lambda x_\gamma B_0] A_\alpha$ 100
- .3 $\mathcal{H} \vdash [\lambda x_\gamma T_0] A_\alpha \stackrel{0}{=} [\lambda x_\gamma B_0] A_\alpha$ R': .2, .1

- 106.4 $\mathcal{H} \vdash T_0 \stackrel{0}{=} S_{A_\alpha}^{x_\gamma} B_0$ RR: 106.3, 103
 .5 $\mathcal{H} \vdash T_0$ 100, def. of T_0
 .6 $\mathcal{H} \vdash S_{A_\alpha}^{x_\gamma} B_0$ R': .5, .4.

- 107 $\vdash [B_\beta \stackrel{\alpha}{=} B_\beta] = T_0$ provided $\alpha \geq \beta$.
 (Note the special case $\vdash [[T_0 \stackrel{0}{=} T_0] \stackrel{0}{=} T_0]$.)

Proof:

- .1 $\vdash [\lambda y_\alpha y_\alpha] \stackrel{\alpha\alpha}{=} [\lambda y_\alpha y_\alpha] = \forall x_\alpha . [\lambda y_\alpha y_\alpha] x_\alpha \stackrel{\alpha}{=} [\lambda y_\alpha y_\alpha] x_\alpha$ 105: Axiom 3
 .2 $\vdash \forall x_\alpha . [\lambda y_\alpha y_\alpha] x_\alpha \stackrel{\alpha}{=} [\lambda y_\alpha y_\alpha] x_\alpha$ R: 100, 107.1
 .3 $\vdash [\lambda y_\alpha y_\alpha] x_\alpha \stackrel{\alpha}{=} x_\alpha$ Axiom 4₂
 .4 $\vdash [\lambda x_\alpha . x_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{0\alpha}{=} [\lambda x_\alpha T_0]$ R: .2, .3; def. of $\forall$
 .5 $\vdash [\lambda x_\alpha . x_\alpha \stackrel{\alpha}{=} x_\alpha] B_\beta \stackrel{0}{=} [\lambda x_\alpha . x_\alpha \stackrel{\alpha}{=} x_\alpha] B_\beta$ 100
 .6 $\vdash [\lambda x_\alpha . x_\alpha \stackrel{\alpha}{=} x_\alpha] B_\beta \stackrel{0}{=} [\lambda x_\alpha T_0] B_\beta$ R: .5, .4
 .7 $\vdash [B_\beta \stackrel{\alpha}{=} B_\beta] \stackrel{0}{=} T_0$ R: 107.6, 103.

- 108 If $\mathcal{H} \vdash B_\beta \stackrel{\beta}{=} C_\beta$ and $\alpha \geq \beta$, then $\mathcal{H} \vdash [B_\beta \stackrel{\alpha}{=} C_\beta] \stackrel{0}{=} T_0$.

Proof: Apply R' to 107 and the hypothesis.

- 109 $\vdash [T_0 \wedge T_0] = T_0$.

Proof:

- .1 $\vdash [[\lambda y_0 T_0] T_0 \wedge [\lambda y_0 T_0] F_0] = \forall x_0 . [\lambda y_0 T_0] x_0$ 105: Axiom 1
 .2 $\vdash [T_0 \wedge T_0] = \forall x_0 T_0$ R: 109.1, 103
 .3 $\vdash [\lambda x_0 T_0] \stackrel{00}{=} [\lambda x_0 T_0] \stackrel{0}{=} T_0$ 107
 .4 $\vdash [T_0 \wedge T_0] = T_0$ E-Rules (101): .2, .3.

(Note that $\forall x_0 T_0$ is $[\lambda x_0 T_0] = [\lambda x_0 T_0]$.)

- 110 $\vdash T_0 \wedge T_0$ E-Rules (101): 109; R: 100, def. of T_0 .

- 111 If A_0 is any wff with no free x -variables other than x_0 and $\vdash [\S_{T_0}^{x_0} A_0] \stackrel{0}{=} T_0$ and $\vdash [\S_{F_0}^{x_0} A_0] \stackrel{0}{=} T_0$, then $\vdash A_0$.

Proof:

- .1 $\vdash T_0 = [\lambda x_0 A_0] T_0$ E-Rules (101): 103, hypothesis
- .2 $\vdash T_0 = [\lambda x_0 A_0] F_0$ E-Rules (101): 103, hypothesis
- .3 $\vdash [\lambda x_0 A_0] T_0 \wedge [\lambda x_0 A_0] F_0$ R: 110, 111.1, 111.2
- .4 $\vdash [[\lambda x_0 A_0] T_0 \wedge [\lambda x_0 A_0] F_0] = \forall x_0 . [\lambda x_0 A_0] x_0$ 105: Axiom 1
- .5 $\vdash \forall x_0 . [\lambda x_0 A_0] x_0$ R: .3, .4
- .6 $\vdash \forall x_0 A_0$ R: 111.5, 103
- .7 $\vdash A_0$ Rule S (106): .6.

- 112 $\vdash [T_0 \wedge F_0] = F_0$.

Proof:

- .1 $\vdash [[\lambda x_0 x_0] T_0 \wedge [\lambda x_0 x_0] F_0] = \forall x_0 . [\lambda x_0 x_0] x_0$ 105: Axiom 1
- .2 $\vdash [T_0 \wedge F_0] = \forall x_0 x_0$ R: 112.1, 103.

Now $\forall x_0 x_0$ is $[[\lambda x_0 x_0] = [\lambda x_0 T_0]]$, which is F_0 , so this is the desired theorem.

- 113 $\vdash [T_0 \wedge A_0] = A_0$.

Proof:

- .1 $\vdash [[T_0 \wedge T_0] = T_0] = T_0$ 108: 109
- .2 $\vdash [[T_0 \wedge F_0] = F_0] = T_0$ 108: 112
- .3 $\vdash [T_0 \wedge x_0] = x_0$ 111: .1, .2
- .4 $\vdash [T_0 \wedge A_0] = A_0$ 105: .3.

114 $\vdash [F_0 = T_0] = F_0$.

Proof:

- .1 $\vdash [[\lambda x_0 . x_0 = T_0]T_0 \wedge [\lambda x_0 . x_0 = T_0]F_0] = \forall x_0 . [\lambda x_0 . x_0 = T_0]x_0$
105: Axiom 1
- .2 $\vdash [[T_0 = T_0] \wedge [F_0 = T_0]] = \forall x_0 . x_0 = T_0$ R: 114.1, 103
- .3 $\vdash [T_0 \wedge [F_0 = T_0]] = \forall x_0 . x_0 = T_0$ R: 114.2, 107
- .4 $\vdash [F_0 = T_0] = \forall x_0 . x_0 = T_0$ R: 114.3, 113
- .5 $[[\lambda x_0 x_0] = [\lambda x_0 T_0]] = \forall x_0 . [\lambda x_0 x_0]x_0 = [\lambda x_0 T_0]x_0$
105: Axiom 3
- .6 $\vdash F_0 = \forall x_0 . x_0 = T_0$ R: 114.5, 103; def. of F_0
- .7 $\vdash [F_0 = T_0] = F_0$ E-Rules (101): .4, .6.

115 $\vdash [A_0 = T_0] = A_0$.

Proof:

- .1 $\vdash [[T_0 = T_0] = T_0] = T_0$ 108: 107
- .2 $\vdash [[F_0 = T_0] = F_0] = T_0$ 108: 114
- .3 $\vdash [x_0 = T_0] = x_0$ 111: .1, .2
- .4 $\vdash [A_0 = T_0] = A_0$ 105: .3.

116 Rule T: $\mathcal{H} \vdash A_0$ if and only if $\mathcal{H} \vdash A_0 = T_0$.

Proof: Use Rule R', E-Rules (101), and 115.

117 Rule of Generalization (Rule Gen):

If $\mathcal{H} \vdash A_0$ then $\mathcal{H} \vdash \forall x_\alpha A_0$, provided that

- (1) no x -variable other than x_α occurs free in A_0 ;
- (2) no x -variable occurs free in a member of $\mathcal{H}$.

Proof:

- 117.1 $\mathcal{H} \vdash A_0 = T_0$ Rule T: hypothesis
 .2 $\mathcal{H} \vdash [\lambda x_\alpha A_0] = [\lambda x_\alpha A_0]$ 100
 .3 $\mathcal{H} \vdash [\lambda x_\alpha A_0] = [\lambda x_\alpha T_0]$ R': .2, .1.

We see from the definition of $\forall$ that this is the desired theorem. The reader should note that under our conditions (1) and (2) the application of R' above is legitimate.

118 If $\mathcal{H} \vdash B_0$ then $\mathcal{H} \vdash \S_{A_\alpha}^{x_\gamma} B_0$, provided that

- (1) $\gamma \geq \alpha$ and all free occurrences of x_γ in B_0 are free for all free variables of A_α ;
- (2) x_γ is the only x -variable free in B_0 ;
- (3) no x -variable occurs free in $\mathcal{H}$.

Proof:

- .1 $\mathcal{H} \vdash \forall x_\gamma B_0$ Rule Gen: hypothesis
 .2 $\mathcal{H} \vdash \S_{A_\alpha}^{x_\gamma} B_0$ Rule S (106): .1.

119 Rule of Simultaneous Substitution (Rule Sub):

If $\mathcal{H} \vdash B_0$ then $\mathcal{H} \vdash \S_{A_{\alpha_1}^1 \dots A_{\alpha_n}^n}^{x_{\gamma_1}^1 \dots x_{\gamma_n}^n} B_0$ (where $n = 1, 2, 3, \dots$), provided that:

- (1) the primitive variables $x^1, \dots, x^n$ are distinct;
for each i ($n \geq i \geq 1$) the following conditions hold:
- (2) $\gamma_i \geq \alpha_i$ and all free occurrences of $x_{\gamma_i}^i$ in B_0 are free for all free variables of $A_{\alpha_i}^i$;
- (3) no x^i -variable other than $x_{\gamma_i}^i$ occurs free in B_0 ;
- (4) no x^i -variable occurs free in a member of $\mathcal{H}$.

(Note that 119 is the same as 118 for the case where $n = 1$.)

Proof: Let $y^1, \dots, y^n$ be distinct primitive variables which are distinct from $x^1, \dots, x^n$ and from all primitive variables occurring in B_0 , $A_{\alpha_1}^1, \dots, A_{\alpha_n}^n$, or in members of $\mathcal{H}$.

- .1 $\mathcal{H} \vdash B_0$ by hypothesis
- .2 $\mathcal{H} \vdash \mathbb{S}_{y^1_{\gamma_1} \dots y^n_{\gamma_n}}^{x^1_{\gamma_1} \dots x^n_{\gamma_n}} B_0$ 118 (applied n times): .1
- .3 $\mathcal{H} \vdash \mathbb{S}_{A_{\alpha_1}^1 \dots A_{\alpha_n}^n}^{y^1_{\gamma_1} \dots y^n_{\gamma_n}} \left[\mathbb{S}_{y^1_{\gamma_1} \dots y^n_{\gamma_n}}^{x^1_{\gamma_1} \dots x^n_{\gamma_n}} B_0 \right]$ 118 (applied n times): .2.

This is easily seen to be the desired theorem.

$120 \vdash [\lambda x_\beta A_\alpha] \stackrel{\alpha\beta}{=} [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]]$, provided that

- (1) no x -variable other than x_β is free in A_α ;
- (2) no y -variable is free in A_α ;
- (3) all free occurrences of x_β in A_α are free for y_β .

(The reader will recall that x is a particular primitive variable, while y stands for an arbitrary primitive variable.)

Proof:

- .1 $\vdash \forall x_\beta . A_\alpha \stackrel{\alpha}{=} A_\alpha$ Gen (117): 100
- .2 $\vdash [\lambda x_\beta A_\alpha] x_\beta \stackrel{\alpha}{=} A_\alpha$ 103
- .3 $\vdash [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]] x_\beta \stackrel{\alpha}{=} A_\alpha$ 103 (We here use lemma 12 to see that $[\mathbb{S}_{x_\beta}^{y_\beta} [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]]$ is A_α ; it is clear that the conditions on 103 are satisfied.)
- .4 $\vdash \forall x_\beta . [\lambda x_\beta A_\alpha] x_\beta \stackrel{\alpha}{=} [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]] x_\beta$ RR (102): .1, .2, .3
- .5 $\vdash [[\lambda x_\beta A_\alpha] \stackrel{\alpha\beta}{=} [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]]] = \forall x_\beta . [\lambda x_\beta A_\alpha] x_\beta \stackrel{\alpha}{=} [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]] x_\beta$ Sub: Axiom 3
- .6 $\vdash [\lambda x_\beta A_\alpha] \stackrel{\alpha\beta}{=} [\lambda y_\beta [\mathbb{S}_{y_\beta}^{x_\beta} A_\alpha]]$ RR: .4, .5.

121 $\vdash [f_{\alpha\beta} \stackrel{\alpha\beta}{=} g_{\alpha\beta}] = \forall x_\beta . f_{\alpha\beta} x_\beta \stackrel{\alpha}{=} g_{\alpha\beta} x_\beta$, provided f and g are distinct from x .

Proof: Let t and s be primitive variables distinct from one another and from f, g, x, f, g, x .

$$.1 \vdash [t_{\alpha\beta} \stackrel{\alpha\beta}{=} s_{\alpha\beta}] = \forall x_\beta . t_{\alpha\beta} x_\beta \stackrel{\alpha}{=} s_{\alpha\beta} x_\beta \quad \text{Sub: Axiom 3}$$

$$.2 \vdash [\lambda x_\beta . t_{\alpha\beta} x_\beta \stackrel{\alpha\beta}{=} s_{\alpha\beta} x_\beta] = [\lambda x_\beta . t_{\alpha\beta} x_\beta \stackrel{\alpha\beta}{=} s_{\alpha\beta} x_\beta] \quad 120$$

$$.3 \vdash [\lambda x_\beta T_0] = [\lambda x_\beta T_0] \quad 120$$

$$.4 \vdash [t_{\alpha\beta} \stackrel{\alpha\beta}{=} s_{\alpha\beta}] = \forall x_\beta . t_{\alpha\beta} x_\beta \stackrel{\alpha}{=} s_{\alpha\beta} x_\beta \quad \text{R: .1, .2, .3; def. of } \forall$$

$$.5 \vdash [f_{\alpha\beta} \stackrel{\alpha\beta}{=} g_{\alpha\beta}] = \forall x_\beta . f_{\alpha\beta} x_\beta \stackrel{\alpha}{=} g_{\alpha\beta} x_\beta \quad \text{Sub: .4.}$$

122 $\vdash [\lambda x_\beta A_\alpha] \stackrel{\alpha\beta}{=} [\lambda y_\beta [S_{y_\beta}^{x_\beta} A_\alpha]]$, provided that

- (1) no x -variable other than x_β is free in A_α ;
- (2) no y -variable is free in A_α ;
- (3) all free occurrences of x_β in A_α are free for y_β .

The proof is essentially the same as the proof of 120; simply replace x_β by x_β , and use 121 in place of Axiom 3.

123 $\vdash [\forall x_\beta A_0] = [\forall y_\beta [S_{y_\beta}^{x_\beta} A_0]]$, provided that

- (1) no x -variable other than x_β is free in A_0 ;
- (2) no y -variable is free in A_0 ;
- (3) all free occurrences of x_β in A_0 are free for y_β .

The proof is immediate from 100, 122, and the definition of $\forall x_\beta A_0$.

124 $\vdash [g_{00} T_0 \wedge g_{00} F_0] = \forall x_0 . g_{00} x_0$, provided g is distinct from x .

The proof is similar to the proof of 121, with Axiom 1 being used in place of Axiom 3.

125 Rule of Conjunction (Rule C):

If $\mathcal{H} \vdash A_0$ and $\mathcal{H} \vdash B_0$, then $\mathcal{H} \vdash A_0 \wedge B_0$.

Proof:

- .1 $\mathcal{H} \vdash T_0 = A_0$ Rule T (116), E-Rules (101): hypothesis
- .2 $\mathcal{H} \vdash T_0 = B_0$ Rule T (116), E-Rules (101): hypothesis
- .3 $\mathcal{H} \vdash A_0 \wedge B_0$ R': 110, 125.1, 125.2.

126 Rule of Cases: If A_0 is any wff with no free x -variables other than x_0 , and $\mathcal{H} \vdash \S_{T_0}^{x_0} A_0$ and $\mathcal{H} \vdash \S_{F_0}^{x_0} A_0$, then $\mathcal{H} \vdash A_0$.

Proof:

- .1 $\mathcal{H} \vdash [\lambda x_0 A_0] T_0$ RR (102): 103, hypothesis
- .2 $\mathcal{H} \vdash [\lambda x_0 A_0] F_0$ RR (102): 103, hypothesis
- .3 $\mathcal{H} \vdash [\lambda x_0 A_0] T_0 \wedge [\lambda x_0 A_0] F_0$ Rule C: .1, .2
- .4 $\mathcal{H} \vdash [[\lambda x_0 A_0] T_0 \wedge [\lambda x_0 A_0] F_0] = \forall x_0 . [\lambda x_0 A_0] x_0$ Sub: 124
- .5 $\mathcal{H} \vdash \forall x_0 . [\lambda x_0 A_0] x_0$ R': .3, .4
- .6 $\mathcal{H} \vdash \forall x_0 A_0$ RR: 126.5, 103
- .7 $\mathcal{H} \vdash A_0$ Rule S (106): .6.

127 $\vdash [T_0 \supset B_0] = B_0$.

Proof:

- .1 $\vdash [T_0 \supset B_0] = [[T_0 \wedge B_0] = T_0]$ R: 100, 103, def. of $\supset$
- .2 $\vdash [T_0 \supset B_0] = [B_0 = T_0]$ R: 127.1, 113
- .3 $\vdash [T_0 \supset B_0] = B_0$ R: 127.2, 115.

128 Rule of Modus Ponens (Rule MP):

If $\mathcal{H} \vdash A_0$ and $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash B_0$.

Proof:

- 128.1 $\mathcal{H} \vdash A_0$ by hypothesis
 .2 $\mathcal{H} \vdash A_0 = T_0$ Rule T (116): .1
 .3 $\mathcal{H} \vdash A_0 \supset B_0$ by hypothesis
 .4 $\mathcal{H} \vdash T_0 \supset B_0$ R': .3, .2
 .5 $\mathcal{H} \vdash B_0$ R': 128.4, 127.

129 $\vdash F_0 \supset A_0$.

Proof: Let y be distinct from all primitive variables occurring in A_0 .

- .1 $\vdash [[\lambda x_0 x_0] = [\lambda x_0 T_0]] \supset .[\lambda y_{00} . y_{00} A_0] [\lambda x_0 x_0]$
 $= [\lambda y_{00} . y_{00} A_0] [\lambda x_0 T_0]$ Sub: Axiom 2
 .2 $\vdash F_0 \supset .[\lambda x_0 x_0] A_0 = [\lambda x_0 T_0] A_0$ R: 129.1, 103, def. of F_0
 .3 $\vdash F_0 \supset .A_0 = T_0$ R: 129.2, 103
 .4 $\vdash F_0 \supset A_0$ R: 129.3, 115.

130 $\vdash [T_0 \supset T_0] = T_0; \vdash [T_0 \supset F_0] = F_0;$

$\vdash [F_0 \supset T_0] = T_0; \vdash [F_0 \supset F_0] = T_0.$

Proof: by 127, 129 and Rule T (116).

131 $\vdash [T_0 \wedge T_0] = T_0; \vdash [T_0 \wedge F_0] = F_0;$

$\vdash [F_0 \wedge T_0] = F_0; \vdash [F_0 \wedge F_0] = F_0.$

Proof: Let A_0 be any wff of type 0.

- .1 $\vdash [F_0 \supset A_0] = [[F_0 \wedge A_0] = F_0]$ R: 100, 103, def. of $\supset$

$$131.2 \vdash [F_0 \wedge A_0] = F_0$$

R: 129, 131.1.

Now 109, 112, and 131.2 are the desired theorems.

$$132 \vdash [T_0 \overset{\circ}{=} T_0] \overset{\circ}{=} T_0; [T_0 \overset{\circ}{=} F_0] \overset{\circ}{=} F_0;$$

$$\vdash [F_0 \overset{\circ}{=} T_0] \overset{\circ}{=} F_0; \vdash [F_0 \overset{\circ}{=} F_0] \overset{\circ}{=} T_0.$$

Proof:

$$.1 \vdash [T_0 \overset{\circ}{=} F_0] \supset . [\lambda x_0 . F_0 \overset{\circ}{=} x_0] T_0 \overset{\circ}{=} [\lambda x_0 . F_0 \overset{\circ}{=} x_0] F_0$$

Sub: Axiom 2

$$.2 \vdash [T_0 \overset{\circ}{=} F_0] \supset . [F_0 \overset{\circ}{=} T_0] \overset{\circ}{=} [F_0 \overset{\circ}{=} F_0]$$

R: 132.1, 103

$$.3 \vdash [T_0 \overset{\circ}{=} F_0] \supset . F_0 = T_0$$

R: 132.2, 114, 107

$$.4 \vdash [T_0 \overset{\circ}{=} F_0] \supset F_0$$

R: 132.3, 114

$$.5 \vdash [[T_0 \overset{\circ}{=} F_0] \supset F_0] \overset{\circ}{=} . [[T_0 \overset{\circ}{=} F_0] \wedge F_0] \overset{\circ}{=} [T_0 \overset{\circ}{=} F_0]$$

R: 100, 103, def. of $\supset$

$$.6 \vdash [[T_0 \overset{\circ}{=} F_0] \wedge F_0] \overset{\circ}{=} [T_0 \overset{\circ}{=} F_0]$$

R: .4, .5

$$.7 \vdash [x_0 \wedge F_0] \overset{\circ}{=} F_0$$

Rule of Cases (126): 131

$$.8 \vdash [[T_0 \overset{\circ}{=} F_0] \wedge F_0] \overset{\circ}{=} F_0$$

Sub: .7

$$.9 \vdash [T_0 \overset{\circ}{=} F_0] \overset{\circ}{=} F_0$$

E-Rules (101): .6, .8.

Now 107, 114, and 132.9 are the desired theorems.

$$133 \vdash \sim T_0 = F_0; \vdash \sim F_0 = T_0 \quad \text{by E-Rules (101): 103, 132, def. of } \sim.$$

$$134 \vdash [T_0 \vee T_0] = T_0; \vdash [T_0 \vee F_0] = T_0;$$

$$\vdash [F_0 \vee T_0] = T_0; \vdash [F_0 \vee F_0] = F_0.$$

Proof: by Rule R, 100, 103, 133, 131, def. of $\vee$.

DEFINITIONS: The class of *propositional wffs* is the smallest class of wffs of type 0 which contains T_0 , F_0 , and all wff-variables of type 0, and which, if it contains A_0 and B_0 , also contains $[A_0 \overset{0}{=} B_0]$, $[A_0 \wedge B_0]$, $[A_0 \supset B_0]$, $\sim A_0$, and $[A_0 \vee B_0]$. A propositional wff A_0 is a *tautology* if $W_\pi A_0 = t$ for every assignment π . A wff B_0 of type 0 is *tautologous* if there is a tautology A_0 such that B_0 is $S_{c_0^1 \dots c_0^n}^{p_0^1 \dots p_0^n} A_0$. ($p_0^1, \dots, p_0^n$ must of course be distinct wff-variables of type 0.)

In the light of Lemma 1 it is clear that there is an effective method for determining of any propositional wff whether or not it is a tautology. This method is the usual method using truth tables. (To show this in detail one would of course make use of Lemmas 2–5 also. Similar lemmas concerning $\sim_{00}$ and $\vee_{000}$ can easily be obtained from 133 and 134 by the use of Theorem 7.) It is also clear that there is an effective method for determining of any wff of type 0 whether or not it is tautologous.

135 If A_0 is a tautology, then $\vdash A_0$.

The proof is by induction on the number of wff-variables which have free occurrences in A_0 . (Of course all such variables must be of type 0 since A_0 is a propositional wff.)

We easily see by induction on the length of B_0 that if B_0 is any propositional wff in which no variables are free, $\vdash B_0 \overset{0}{=} T_0$ or $\vdash B_0 \overset{0}{=} F_0$. We simply use Rule R and Theorems 100, 130, 131, 132, 133 and 134.

Now suppose A_0 is a tautology without free variables. If $\vdash A_0 \overset{0}{=} F_0$, then by Theorem 7 we see that $W_\pi[A_0 \overset{0}{=} F_0] = t$ for every assignment π , so $W_\pi A_0 = W_\pi F_0 = f$, contradicting our assumption that A_0 is a tautology. Hence we must have $\vdash [A_0 \overset{0}{=} T_0]$, so by Rule T (116), $\vdash A_0$.

Next suppose that A_0 does contain free variables; let x_0 be one of the variables which occur free in A_0 . Clearly $S_{T_0}^{x_0} A_0$ and $S_{F_0}^{x_0} A_0$ are tautologies, so by inductive hypothesis, $\vdash S_{T_0}^{x_0} A_0$ and $\vdash S_{F_0}^{x_0} A_0$. Hence $\vdash A_0$ by the Rule of Cases (126). (Since A_0 is a propositional wff, it cannot contain free occurrences of x -variables other than x_0 .)

It should be noted that the process of constructing a proof for any tautology is effective.

136 Rule P: If $\mathcal{H} \vdash A_0^1, \dots, \mathcal{H} \vdash A_0^n$, and if $[A_0^1 \supset \dots \supset A_0^n \supset B_0]$ is tautologous, then $\mathcal{H} \vdash B_0$. ($n = 0, 1, 2, 3, \dots$; if $n = 0$, Rule P is to be interpreted as follows: if B_0 is tautologous, then $\mathcal{H} \vdash B_0$.)

Proof:

.1 $\mathcal{H} \vdash A_0^1 \supset \dots \supset A_0^n \supset B_0$ Rule Sub (119): 135

.2 $\mathcal{H} \vdash A_0^1 \supset \dots \supset A_0^n \supset B_0$ Theorem 136.1

.3 $\mathcal{H} \vdash B_0$ Rule MP (128) (applied n times): .2, hypotheses.

137 Rule of Generalization for Type Variables (Rule Gen_t): If $\mathcal{H} \vdash A_0$ then $\mathcal{H} \vdash \forall a A_0$, provided this is a wff and a is not free in any member of $\mathcal{H}$.

Proof:

.1 $\mathcal{H} \vdash T_0 = A_0$ Rule T (116), E-Rules (101): hypothesis

.2 $\mathcal{H} \vdash \forall a T_0$ Axiom 6

.3 $\mathcal{H} \vdash \forall a A_0$ R': .2, .1.

138 Rule of Specialization for Type Variables (Rule S_t): If $\mathcal{H} \vdash \forall a A_0$ then $\mathcal{H} \vdash \S_\gamma^a A_0$, provided that $2 > \gamma$ and all free occurrences of a in A_0 are free for all type variables in γ .

Proof:

.1 $\mathcal{H} \vdash \forall a A_0$ by hypothesis

.2 $\mathcal{H} \vdash [\forall a A_0] \supset [\S_\gamma^a A_0]$ Axiom 7 (It is clear that this is a wff, since $\forall a A_0$ is derivable from $\mathcal{H}$ and so must be a wff, and by Lemma 14, $\S_\gamma^a A_0$ must be a wff of type 0.)

.3 $\mathcal{H} \vdash \S_\gamma^a A_0$ Rule MP (128): .1, .2.

139 If $\mathcal{H} \vdash A_0$ then $\mathcal{H} \vdash \S_\gamma^a A_0$, provided that

(1) $2 > \gamma$ and all free occurrences of a in A_0 are free for all type variables in γ ;

- (2) a is not free in $\mathcal{H}$;
 (3) for each wff-variable x_β such that x_β occurs free in A_0 and a occurs in β , no x -variable occurs free in a member of $\mathcal{H}$ and no x -variable other than x_β occurs free in A_0 .

Proof: Let $x_{\beta_1}^1, \dots, x_{\beta_n}^n$ be a list of all wff-variables such that $x_{\beta_i}^i$ occurs free in A_0 and a occurs in β_i .

- .1 $\mathcal{H} \vdash A_0$ by hypothesis
 .2 $\mathcal{H} \vdash \forall x_{\beta_1}^1 \dots \forall x_{\beta_n}^n A_0$ Gen (117): .1
 .3 $\mathcal{H} \vdash \forall a \forall x_{\beta_1}^1 \dots \forall x_{\beta_n}^n A_0$ Gen_t (137): .2
 .4 $\mathcal{H} \vdash S_{\gamma}^a \forall x_{\beta_1}^1 \dots \forall x_{\beta_n}^n A_0$ Rule S_t (138): .3.
 Let δ_i be $(S_{\gamma}^a \beta_i)$, where $i = 1, \dots, n$. Then we have

$$\mathcal{H} \vdash \forall x_{\delta_1}^1 \dots \forall x_{\delta_n}^n [S_{\gamma}^a A_0]$$

 .5 $\mathcal{H} \vdash S_{\gamma}^a A_0$ Rule S (106): .4.

It is easy to check that the applications of rules of inference in this proof are legitimate under the restrictions (1), (2) and (3) of our theorem.

140 Rule of Simultaneous Substitution for Type Variables (Rule Sub_t): If

$\mathcal{H} \vdash A_0$ then $\mathcal{H} \vdash S_{\gamma_1}^{a^1} \dots S_{\gamma_n}^{a^n} A_0$ (where $n = 1, 2, 3, \dots$), provided that for each i ($n \geq i \geq 1$), the following conditions are satisfied:

- (1) $2 > \gamma_i$ and all free occurrences of a^i in A_0 are free for all type variables in γ_i ;
 (2) a^i is not free in any member of $\mathcal{H}$;
 (3) for each wff-variable x_β such that x_β occurs free in A_0 and a^i occurs in β , no x -variable occurs free in a member of $\mathcal{H}$ and no x -variable other than x_β occurs free in A_0 .

Proof: Let $b^1, \dots, b^n$ be distinct type variables which are distinct from $a^1, \dots, a^n$ and from all type variables occurring in $A_0, \gamma_1, \dots, \gamma_n$, or in members of $\mathcal{H}$.

140.1 $\mathcal{H} \vdash A_0$

by hypothesis

.2 $\mathcal{H} \vdash S_{b^1}^{a^1} \dots S_{b^n}^{a^n} A_0$

139 (applied n times): .1

.3 $\mathcal{H} \vdash S_{\gamma^1}^{b^1} \dots S_{\gamma^n}^{b^n} [S_{b^1}^{a^1} \dots S_{b^n}^{a^n} A_0]$

139 (applied n times): .2.

This is the desired theorem.

One must of course check that the conditions attached to rule 139 are satisfied each time the rule is applied in the above proof. A little reflection will convince the reader that this is so.

141 $\vdash \forall x_\alpha [A_0 \supset B_0] = .A_0 \supset \forall x_\alpha B_0$, provided that no x -variable occurs free in A_0 and no x -variable other than x_α occurs free in B_0 .

Proof: Let p be a primitive variable not occurring in B_0 and distinct from x .

.1 $\vdash \forall x_\alpha B_0 = .T_0 \supset \forall x_\alpha B_0$

Rule P (136)

.2 $\vdash B_0 = .T_0 \supset B_0$

Rule P

.3 $\vdash \forall x_\alpha T_0$ 100, def. of $\forall$.4 $\vdash \forall x_\alpha T_0 = .F_0 \supset \forall x_\alpha B_0$

Rule P: .3

.5 $\vdash T_0 = .F_0 \supset B_0$

Rule P

.6 $\vdash \forall x_\alpha [F_0 \supset B_0] = .F_0 \supset \forall x_\alpha B_0$

R: .4, .5

.7 $\vdash \forall x_\alpha [T_0 \supset B_0] = .T_0 \supset \forall x_\alpha B_0$

R: .1, .2

.8 $\vdash \forall x_\alpha [p_0 \supset B_0] = .p_0 \supset \forall x_\alpha B_0$

Rule of Cases (126): .7, .6

.9 $\vdash \forall x_\alpha [A_0 \supset B_0] = .A_0 \supset \forall x_\alpha B_0$

Sub (119): .8.

142 $\vdash \forall a [A_0 \supset B_0] = .A_0 \supset \forall a B_0$, provided that this is a wff and a is not free in A_0 .

The proof is similar to the proof of 141. In place of 141.3 one of course uses Axiom 6, which is $\forall a T_0$.

143 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash A_0 \supset \forall x_\alpha B_0$, provided that:

- (1) no x -variable occurs free in A_0 or in a member of $\mathcal{H}$;
- (2) no x -variable other than x_α occurs free in B_0 .

Proof: By Gen (117), RR (102), and 141.

144 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash A_0 \supset \forall a B_0$, provided that:

- (1) a does not occur free in A_0 or in a member of $\mathcal{H}$;
- (2) $\forall a B_0$ is a wff.

Proof: By Gen₁ (137), RR, and 142.

145 $\vdash \forall x_\alpha A_0 \supset [\S_{C_\gamma}^{x_\alpha} A_0]$, provided that $\alpha \geq \gamma$ and $\forall x_\alpha A_0$ is a wff and all free occurrences of x_α in A_0 are free for all free variables of C_γ .

Proof: Let p be a primitive variable not occurring in C_γ .

- .1 $\vdash [\lambda x_\alpha A_0] = [\lambda x_\alpha T_0] \supset .[\lambda p_{0\alpha} . p_{0\alpha} C_\gamma] [\lambda x_\alpha A_0]$
 $\quad = [\lambda p_{0\alpha} . p_{0\alpha} C_\gamma] [\lambda x_\alpha T_0]$ Sub: Axiom 2
- .2 $\vdash \forall x_\alpha A_0 \supset .[\S_{C_\gamma}^{x_\alpha} A_0] = T_0$ R: 145.1, 103; def. of $\forall$
- .3 $\vdash \forall x_\alpha A_0 \supset [\S_{C_\gamma}^{x_\alpha} A_0]$ Rule P: .2.

146 $\vdash \forall x_\alpha A_0 \supset A_0$, provided this is a wff.

Proof: By 145, with x_α as C_γ .

147 $\vdash \forall a A_0 \supset A_0$, provided this is a wff.

This is an instance of Axiom 7, with a as γ .

148 $\vdash \forall x_\alpha [A_0 \supset B_0] \supset .\forall x_\alpha A_0 \supset B_0$, provided this is a wff.

Proof:

- .1 $\vdash \forall x_\alpha [A_0 \supset B_0] \supset .A_0 \supset B_0$ 146
- .2 $\vdash \forall x_\alpha [A_0 \supset B_0] \supset .\forall x_\alpha A_0 \supset B_0$ Rule P: 148.1, 146.

149 $\vdash \forall a[A_0 \supset B_0] \supset .\forall aA_0 \supset B_0$, provided this is a wff.

The proof is similar to the proof of 148, with 147 used in place of 146.

150 $\vdash \forall x_\alpha[A_0 \supset B_0] \supset .\forall x_\alpha A_0 \supset \forall x_\alpha B_0$, provided this is a wff.

Proof:

.1 $\vdash \forall x_\alpha[A_0 \supset B_0] \supset \forall x_\alpha .\forall x_\alpha A_0 \supset B_0$ 143: 148

.2 $\vdash \forall x_\alpha[\forall x_\alpha A_0 \supset B_0] = .\forall x_\alpha A_0 \supset \forall x_\alpha B_0$ 141.

Theorem 150 is then obtained by Rule R.

151 $\vdash \forall a[A_0 \supset B_0] \supset .\forall aA_0 \supset \forall aB_0$, provided this is a wff.

The proof is similar to the proof of 150, with 144, 149 and 142 used in place of 143, 148 and 141.

152 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \forall x_\alpha A_0 \supset \forall x_\alpha B_0$, provided that this is a wff and no x -variable occurs free in $\mathcal{H}$.

Proof: Use Rule Gen (117), 150, and Rule MP (128).

153 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \forall aA_0 \supset \forall aB_0$, provided that this is a wff and a is not free in $\mathcal{H}$.

Proof: Use Rule Gen_t (137), 151, and Rule MP.

154 $\vdash [\forall x_\alpha \forall y_\beta A_0] = [\forall y_\beta \forall x_\alpha A_0]$, provided this is a wff.

Proof:

.1 $\vdash \forall x_\alpha \forall y_\beta A_0 \supset \forall x_\alpha A_0$ 152: 146

.2 $\vdash \forall x_\alpha \forall y_\beta A_0 \supset \forall y_\beta \forall x_\alpha A_0$ 143: .2

.3 $\vdash \forall y_\beta \forall x_\alpha A_0 \supset \forall x_\alpha \forall y_\beta A_0$ by Theorem Schema 154.2.

The desired theorem is obtained by Rule P from .2 and .3.

155 $\vdash \forall a \forall b A_0 = \forall b \forall a A_0$, provided this is a wff.

The proof, using 153, 147 and 144, is similar to the proof of 154.

156 $\vdash \forall a \forall x_\beta A_0 = \forall x_\beta \forall a A_0$, provided this is a wff, and a does not occur in β . (Note that if this is a wff, either a does not occur in β or x_β does not occur free in A_0 . This is clear since $\forall a A_0$ must be a wff.)

Proof:

.1 $\vdash \forall a \forall x_\beta A_0 \supset \forall a A_0$ 153: 146

.2 $\vdash \forall a \forall x_\beta A_0 \supset \forall x_\beta \forall a A_0$ 143: .2

.3 $\vdash \forall x_\beta \forall a A_0 \supset \forall a \forall x_\beta A_0$ 152, 144: 147. (Note
that in applying 144 we make use of the fact that a does not occur
in β .)

The desired theorem is obtained by Rule P from .2 and .3.

157 $\vdash \forall a A_0 = \forall b [S_b^a A_0]$, provided that

(1) $\forall a A_0$ is a wff;

(2) b does not occur free in A_0 ;

(3) all free occurrences of a in A_0 are free for b .

(With the aid of Lemma 14 it is easy to see that under these conditions $\forall b [S_b^a A_0]$ must also be a wff.)

Proof:

.1 $\vdash \forall a A_0 \supset [S_b^a A_0]$ Axiom 7

.2 $\vdash \forall a A_0 \supset \forall b [S_b^a A_0]$ 144: .1

.3 $\vdash \forall b [S_b^a A_0] \supset A_0$ Axiom 7 (Note that $S_a^b [S_b^a A_0]$ is A_0 .)

.4 $\vdash \forall b [S_b^a A_0] \supset \forall a A_0$ 144: .3.

The desired theorem follows from .2 and .4 by Rule P.

158 $\vdash \forall a[A_0 = B_0] \supset . \forall a A_0 = \forall a B_0$, provided this is a wff.

Proof:

- | | |
|--|---------|
| .1 $\vdash [A_0 = B_0] \supset . A_0 \supset B_0$ | Rule P |
| .2 $\vdash [A_0 = B_0] \supset . B_0 \supset A_0$ | Rule P |
| .3 $\vdash \forall a[A_0 = B_0] \supset \forall a . A_0 \supset B_0$ | 153: .1 |
| .4 $\vdash \forall a[A_0 = B_0] \supset \forall a . B_0 \supset A_0$ | 153: .2 |
| .5 $\vdash \forall a[A_0 \supset B_0] \supset . \forall a A_0 \supset \forall a B_0$ | 151 |
| .6 $\vdash \forall a[B_0 \supset A_0] \supset . \forall a B_0 \supset \forall a A_0$ | 151. |

The desired theorem follows from .3, .4, .5 and .6 by Rule P.

159 $\vdash [\lambda x_\gamma A_\alpha] \overset{\alpha\gamma}{=} [\lambda x_\gamma B_\alpha] = \forall x_\gamma . A_\alpha \overset{\alpha}{=} B_\alpha$, provided this is a wff.

Proof:

- | | |
|--|----------------|
| .1 $\vdash [\lambda x_\gamma A_\alpha] \overset{\alpha\gamma}{=} [\lambda x_\gamma B_\alpha] = \forall x_\gamma . [\lambda x_\gamma A_\alpha] x_\gamma \overset{\alpha}{=} [\lambda x_\gamma B_\alpha] x_\gamma$ | |
| | Sub (119): 121 |
| .2 $\vdash [\lambda x_\gamma A_\alpha] \overset{\alpha\gamma}{=} [\lambda x_\gamma B_\alpha] = \forall x_\gamma . A_\alpha \overset{\alpha}{=} B_\alpha$ | R: 159.1, 104. |

160 $\vdash [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_n}^n A_\alpha] = [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_n}^n B_\alpha] = \forall x_{\beta_1}^1 \dots \forall x_{\beta_n}^n . A_\alpha = B_\alpha$,

provided that $n \geq 1$ and

- (1) $x^1, \dots, x^n$ are distinct primitive variables;
- (2) for each i ($n \geq i \geq 1$), no x^i -variable other than $x_{\beta_1}^i$ occurs free in A_α or B_α .

The proof is by induction on n . For $n = 1$, the theorem has already been proved as Theorem 159. So suppose $n > 1$.

$$\begin{aligned}
 .1 \vdash & [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_{n-1}}^{n-1} [\lambda x_{\beta_n}^n A_\alpha]] = [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_{n-1}}^{n-1} [\lambda x_{\beta_n}^n B_\alpha]] \\
 & = \forall x_{\beta_1}^1 \dots \forall x_{\beta_{n-1}}^{n-1} . [\lambda x_{\beta_n}^n A_\alpha] = [\lambda x_{\beta_n}^n B_\alpha] \quad \text{by inductive hypothesis}
 \end{aligned}$$

$$\begin{aligned}
 160.2 \vdash [\lambda x_{\beta_1}^1 \cdots \lambda x_{\beta_n-1}^{n-1} [\lambda x_{\beta_n}^n A_\alpha]] &= [\lambda x_{\beta_1}^1 \cdots \lambda x_{\beta_n-1}^{n-1} [\lambda x_{\beta_n}^n B_\alpha]] \\
 &= \forall x_{\beta_1}^1 \cdots \forall x_{\beta_n-1}^{n-1} \forall x_{\beta_n}^n . A_\alpha = B_\alpha
 \end{aligned}$$

R: 160.1, 159.

$$161 \vdash \forall a^1 \cdots \forall a^n \forall x_{\beta_1}^1 \cdots \forall x_{\beta_m}^m [A_\alpha = B_\alpha] \supset . C_0 = D_0$$

(where $n \geq 0$ and $m \geq 0$), provided that:

- (1) A_α , B_α , C_0 , and D_0 are wffs such that C_0 contains at least one occurrence of A_α and D_0 is the result of replacing one occurrence of A_α in C_0 by an occurrence of B_α ;
- (2) $a^1, \dots, a^n, x_{\beta_1}^1, \dots, x_{\beta_m}^m$ is a complete list of distinct type variables and wff-variables which are free in A_α or B_α but for which the occurrence of A_α in C_0 is not free in C_0 .

Proof:

We first note that under our hypotheses the primitive variables $x^1, \dots, x^m$ must be distinct. For suppose not; then for some indices i and j and primitive variable x we have $x^i = x^j = x$ but $\beta_i \neq \beta_j$. Now the occurrence of A_α in C_0 is free for neither x_{β_i} nor x_{β_j} , and hence either (i) C_0 contains a wf part of the form $[\lambda x_\gamma E_\delta]$, where A_α is contained in E_δ and is free in E_δ for x_{β_i} and x_{β_j} , or (ii) C_0 contains a wf part of the form $[\forall b E_\delta]$ (where of course δ is 0), where A_α is contained in E_δ and is free in E_δ for x_{β_i} and x_{β_j} , and b occurs in β_i or β_j .

Hence D_0 must contain a corresponding wf part of the form $[\lambda x_\gamma E'_\delta]$ or $[\forall b E'_\delta]$, where E'_δ is obtained from E_δ by replacing the occurrence of A_α by an occurrence of B_α . Since x_{β_i} and x_{β_j} are each free in A_α or B_α , in each case x_{β_i} and x_{β_j} must each be free in E_δ or E'_δ . Hence in case (ii) above either $[\forall b E_\delta]$ or $[\forall b E'_\delta]$ cannot be well-formed (as we see from formation rule (7')), so either C_0 or D_0 cannot be a wff. This, however, violates hypothesis (1) of our theorem. On the other hand in case (i), β_i and β_j cannot both be the same as γ , so $[\lambda x_\gamma E_\delta]$ and $[\lambda x_\gamma E'_\delta]$ cannot both be wffs (as we see from formation rule (6')), so again hypothesis (1) is violated.

Thus the primitive variables $x^1, \dots, x^m$ must indeed be distinct.

We next observe that $\forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha]$ must be a wff under our hypotheses. To see this, note that for each j ($m \geq j \geq 1$), C_0 must contain a wf part $[\lambda x_{\beta_j}^j E_\delta]$, where A_α is contained in E_δ and is free in E_δ for $x_{\beta_j}^j$. Hence neither A_α nor B_α can have free occurrences of any x^j -variable other than $x_{\beta_j}^j$. Also for each i ($n \geq i \geq 1$), if y_γ is any free wff-variable of A_α or B_α such that a^i occurs in γ , then y_γ is contained in the list $x_{\beta_1}^1, \dots, x_{\beta_m}^m$; this is clear since A_α is not free in C_0 for a^i , and so A_α is not free in C_0 for y_γ .

It is thus clear that 161 must be a wff under the given hypotheses.

We now prove 161 for all wffs A_α, B_α, C_0 , and D_0 satisfying our hypotheses by induction on the number p of wf parts of C_0 of the form $[\forall b E_0]$, where the occurrence of A_α in question is in E_0 .

CASE 1: $p = 0$.

Clearly $n = 0$. Let t be a primitive variable not occurring in C_0 or D_0 and let G_0 be the result of replacing the given occurrence of A_α in C_0 by an occurrence of the wff $[t_{\alpha\beta_m} \dots \beta_1 x_{\beta_1}^1 \dots x_{\beta_m}^m]$. Clearly G_0 is a wff.

$$\begin{aligned}
 161.1 \vdash [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m A_\alpha] &= [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m B_\alpha] \supset \\
 &\cdot [\lambda t_{\alpha\beta_m} \dots \beta_1 G_0] [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m A_\alpha] \\
 &= [\lambda t_{\alpha\beta_m} \dots \beta_1 G_0] [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m B_\alpha] \quad \text{Sub (119): Axiom 2} \\
 .2 \vdash [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m A_\alpha] &= [\lambda x_{\beta_1}^1 \dots \lambda x_{\beta_m}^m B_\alpha] \supset .C_0 = D_0 \quad \text{R: 161.1, 103, 104} \\
 .3 \vdash \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha] &\supset .C_0 = D_0 \quad \text{R: 161.2, 160.}
 \end{aligned}$$

CASE 2: $p > 0$.

In this case C_0 must contain a wf part $[\forall b E_0]$ which is contained in no other such part, with the occurrence of A_α in E_0 . Hence D_0 contains a corresponding wf part $[\forall b E'_0]$, where E'_0 is the result of replacing the occurrence of A_α in E_0 by an occurrence of B_α .

$a^1, \dots, a^n, x_{\beta_1}^1, \dots, x_{\beta_m}^m$ is a complete list of those variables which are free in A_α or B_α but for which the occurrence of A_α in C_0 is not free

in C_0 . For some of these variables the occurrence of A_α is not free in E_0 ; let these be $a^1, \dots, a^s, x_{\beta_1}^1, \dots, x_{\beta_r}^r$, where $n \geq s \geq n-1$ and $m \geq r \geq 0$. If $s = n$ then either b is not free in A_α or B_α , or else b is already included among $a^1, \dots, a^s$, so that any free occurrences of b in A_α or B_α are not free in E_0 or E'_0 . If $s = n-1$ then a^n is b . Note that in either case b is not free in $\forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha]$.

$$161.4 \vdash \forall a^1 \dots \forall a^s \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha] \supset .E_0 = E'_0$$

by inductive hypothesis

$$.5 \vdash \forall a^1 \dots \forall a^{n-1} \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha]$$

$$\supset \forall a^1 \dots \forall a^{n-1} \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha] \quad 153: 147$$

$$.6 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha] \supset .E_0 = E'_0$$

Rule P: .4, .5

(Recall that $s = n$ or $s = n-1$.)

$$.7 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha] \supset \forall b .E_0 = E'_0 \quad 144: .6$$

($\forall b .E_0 = E'_0$ must be a wff since $\forall b E_0$ and $\forall b E'_0$ are wffs.)

$$.8 \vdash \forall b [E_0 = E'_0] \supset .[\forall b E_0] = [\forall b E'_0] \quad 158$$

$$.9 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha] \supset .[\forall b E_0] = [\forall b E'_0]$$

Rule P: .7, .8.

Note that each of $x_{\beta_{r+1}}^{r+1}, \dots, x_{\beta_m}^m$ is free in $[\forall b E_0]$ or $[\forall b E'_0]$ but that the occurrence of $[\forall b E_0]$ in C_0 is not free in C_0 for any of these variables. Let $y_{\gamma_1}^1, \dots, y_{\gamma_k}^k, x_{\beta_{r+1}}^{r+1}, \dots, x_{\beta_m}^m$ (where $k \geq 0$) be a complete list of those distinct variables which occur free in $[\forall b E_0]$ or $[\forall b E'_0]$ but for which the occurrence of $[\forall b E_0]$ is not free in C_0 . (The occurrence of $[\forall b E_0]$ is free in C_0 for all type variables.)

$$.10 \vdash \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r [A_\alpha = B_\alpha]$$

$$\supset \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m .[\forall b E_0] = [\forall b E'_0] \quad 152: .9$$

(We easily see that this is a wff.)

$$161.11 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_r}^r \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha]$$

$$\supset \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m . [\forall b E_0] = [\forall b E'_0] \quad \text{RR: 161.10, 156, 154}$$

(Since .9 is a wff and each of $x_{\beta_{r+1}}^{r+1}, \dots, x_{\beta_m}^m$ occur free in $[A_\alpha = B_\alpha]$, none of $a^1, \dots, a^n$ can occur in $\beta_{r+1}, \dots$, or β_m .)

$$.12 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha]$$

$$\supset \forall y_{\gamma_1}^1 \dots \forall y_{\gamma_k}^k \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m . [\forall b E_0] = [\forall b E'_0] \quad 143: .11$$

(Clearly no $y^1, \dots$, or y^k -variable can occur free in $\forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha]$.)

$$.13 \vdash \forall y_{\gamma_1}^1 \dots \forall y_{\gamma_k}^k \forall x_{\beta_{r+1}}^{r+1} \dots \forall x_{\beta_m}^m [[\forall b E_0] = [\forall b E'_0]] \supset .C_0 = D_0$$

by inductive hypothesis, since D_0 is obtained from C_0 by replacing the occurrence of $[\forall b E_0]$ by an occurrence of $[\forall b E'_0]$, and this occurrence of $[\forall b E_0]$ is free in C_0 for all type variables.

$$.14 \vdash \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m [A_\alpha = B_\alpha] \supset .C_0 = D_0 \quad \text{Rule P: .12, .13.}$$

This completes the proof of 161. Note that no difficulties arise in carrying out the proof if $r = m$ or $n = 0$ or $m = 0$.

162 Deduction Theorem: If $H_0^1, \dots, H_0^{k-1}, H_0^k \vdash D_0$ then $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset D_0$ ($k = 1, 2, 3, \dots$).

The proof is by induction on the length of the proof of D_0 from the hypotheses $H_0^1, \dots, H_0^k$. (By this we mean the length of the second sequence of wffs referred to in the definition of a proof from hypotheses at the beginning of this Chapter.)

CASE 1: D_0 is a theorem.

$$.1 \vdash D_0 \quad \text{by hypothesis}$$

$$.2 \vdash H_0^k \supset D_0 \quad \text{Rule P: .1}$$

$$.3 H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset D_0 \quad \text{Theorem .2.}$$

CASE 2: D_0 is one of $H_0^1, \dots, H_0^{k-1}$.

162.4 $H_0^1, \dots, H_0^{k-1} \vdash D_0$

Premiss

.5 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset D_0$

Rule P (136): .4.

CASE 3: D_0 is H_0^k .

.6 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset D_0$

Rule P (136).

CASE 4: $H_0^1, \dots, H_0^k \vdash C_0$ and $H_0^1, \dots, H_0^k \vdash [A_\alpha = B_\alpha]$ and D_0 is inferred from C_0 and $[A_\alpha = B_\alpha]$ by Rule R' in the given proof of D_0 from the hypotheses $H_0^1, \dots, H_0^k$. Thus D_0 is the result of replacing one occurrence of A_α in C_0 by an occurrence of B_α , and for each primitive or type variable X such that an X -variable occurs free in one of the hypotheses $H_0^1, \dots, H_0^k$ and an X -variable occurs free in A_α or B_α , the occurrence of A_α in C_0 is free for all X -variables.

.7 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset C_0$

by inductive hypothesis

.8 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset .A_\alpha = B_\alpha$

by inductive hypothesis.

Let $a^1, \dots, a^n, x_{\beta_1}^1, \dots, x_{\beta_m}^m$ be a complete list of all type variables and wff-variables which are free in A_α or B_α but for which the occurrence of A_α in C_0 is not free in C_0 . Then none of $a^1, \dots, a^n$ occurs free in $H_0^1, \dots$, or H_0^k , and for each j ($m \geq j \geq 1$), no x^j -variable occurs free in $H_0^1, \dots$, or H_0^k .

.9 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset \forall a^1 \dots \forall a^n \forall x_{\beta_1}^1 \dots \forall x_{\beta_m}^m . A_\alpha = B_\alpha$ 143, 144: .8

(The comments at the beginning of the proof of 161 can be applied to see that this is a wff, and hence that the restrictions on 143 and 144 are satisfied.)

.10 $H_0^1, \dots, H_0^{k-1} \vdash H_0^k \supset D_0$

Rule P: 161, 162.7, 162.9.

This completes the proof of the Deduction Theorem.

163 If $H_0^1, \dots, H_0^k \vdash D_0$ and every wff in the list $H_0^1, \dots, H_0^k$ is contained at least once in the list $G_0^1, \dots, G_0^r$, then $G_0^1, \dots, G_0^r \vdash D_0$.

Proof:

- 163.1 $H_0^1, \dots, H_0^k \vdash D_0$ by hypothesis
- .2 $\vdash H_0^1 \supset .H_0^2 \supset \dots \supset .H_0^k \supset D_0$ Deduction Theorem: .1
- .3 $\vdash G_0^1 \supset .G_0^2 \supset \dots \supset .G_0^r \supset D_0$ Rule P: .2
- .4 $G_0^1, \dots, G_0^r \vdash G_0^1$ ($i = 1, \dots, i = r$) Premiss
- .5 $G_0^1, \dots, G_0^r \vdash D_0$ Rule P: .3, .4.

164 $\vdash [\dot{S}_{C_\gamma}^{x_\alpha} A_0] \supset \exists x_\alpha A_0$, provided that

- (1) $\alpha \geq \gamma$ and all free occurrences of x_α in A_0 are free for all free variables of C_γ ;
- (2) x_α is the only x -variable free in A_0 .

Proof:

- .1 $\vdash \forall x_\alpha \sim A_0 \supset \sim [\dot{S}_{C_\gamma}^{x_\alpha} A_0]$ 145
- .2 $\vdash [\dot{S}_{C_\gamma}^{x_\alpha} A_0] \supset \exists x_\alpha A_0$ Rule P: .1; def. of $\exists$.

165 $\vdash [\dot{S}_\gamma^a A_0] \supset \exists a A_0$, provided that $2 > \gamma$ and all free occurrences of a in A_0 are free for all variables in γ and this is a wff.

The proof is similar to the proof of 164, with Axiom 7 used in place of 145.

166 Rule of Existential Generalization (Rule Exist): If $\mathcal{H} \vdash [\dot{S}_{C_\gamma}^{x_\alpha} A_0]$ then $\mathcal{H} \vdash \exists x_\alpha A_0$, provided that

- (1) $\alpha \geq \gamma$ and all free occurrences of x_α in A_0 are free for all free variables of C_γ ;
- (2) x_α is the only x -variable free in A_0 .

Proof: by Rule MP (128) and 164.

167 Rule of Existential Generalization for Type Variables (Rule Exist.):
If $\mathcal{H} \vdash [S_\gamma^a A_0]$ then $\mathcal{H} \vdash \exists a A_0$, provided that $2 > \gamma$ and all free occurrences of a in A_0 are free for all variables in γ and $\exists a A_0$ is a wff.

Proof: by Rule MP (128) and 165.

168 $\vdash \forall x_\alpha [A_0 \supset B_0] \supset \cdot \exists x_\alpha A_0 \supset B_0$, provided that

- (1) no x -variable occurs free in B_0 ;
- (2) x_α is the only x -variable free in A_0 .

Proof:

- .1 $\vdash \forall x_\alpha [A_0 \supset B_0] \supset \cdot A_0 \supset B_0$ 146
- .2 $\forall x_\alpha [A_0 \supset B_0] \vdash \forall x_\alpha [A_0 \supset B_0]$ Premiss
- .3 $\forall x_\alpha [A_0 \supset B_0] \vdash \sim B_0 \supset \sim A_0$ Rule P: .1, .2
- .4 $\forall x_\alpha [A_0 \supset B_0] \vdash \sim B_0 \supset \forall x_\alpha \sim A_0$ 143: .3
- .5 $\forall x_\alpha [A_0 \supset B_0] \vdash \exists x_\alpha A_0 \supset B_0$ Rule P: .4; def. of $\exists$.

The desired theorem follows by an application of the Deduction Theorem (162).

169 $\vdash \forall a [A_0 \supset B_0] \supset \cdot \exists a A_0 \supset B_0$, provided that

- (1) a does not occur free in B_0 ;
- (2) $\exists a A_0$ is a wff.

The proof is similar to the proof of 168. Use 147 in place of 146 and 144 in place of 143.

170 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \exists x_\alpha A_0 \supset B_0$, provided that

- (1) no x -variable occurs free in B_0 or in $\mathcal{H}$;
- (2) no x -variable other than x_α occurs free in A_0 .

Proof: by Rule Gen (117), 168, and Rule MP (128).

171 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \exists a A_0 \supset B_0$, provided that

- (1) a does not occur free in B_0 or $\mathcal{H}$;
- (2) $\exists a A_0$ is a wff.

Proof: by Rule Gen_t (137), 169, and Rule MP (128).

172 If $\mathcal{H}, H_0^1, \dots, H_0^n \vdash D_0$ then $\mathcal{H}, \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \vdash D_0$ (where $n \geq 1$), provided that

- (1) no x -variable occurs free in D_0 or in $\mathcal{H}$;
- (2) no x -variable other than x_α occurs free in $H_0^1, \dots$, or H_0^n .

Proof:

- .1 $\mathcal{H}, H_0^1, \dots, H_0^n \vdash D_0$ by hypothesis
- .2 $\mathcal{H} \vdash [H_0^1 \wedge \dots \wedge H_0^n] \supset D_0$ Deduction Theorem, Rule P: .1
- .3 $\mathcal{H} \vdash \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \supset D_0$ 170: .2
- .4 $\mathcal{H}, \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \vdash \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \supset D_0$ 163: .3
- .5 $\mathcal{H}, \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \vdash \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n]$ Premiss
- .6 $\mathcal{H}, \exists x_\alpha [H_0^1 \wedge \dots \wedge H_0^n] \vdash D_0$ MP (128): .4, .5.

173 If $\mathcal{H}, H_0^1, \dots, H_0^n \vdash D_0$ then $\mathcal{H}, \exists a [H_0^1 \wedge \dots \wedge H_0^n] \vdash D_0$ (where $n \geq 1$), provided that

- (1) a does not occur free in D_0 or in $\mathcal{H}$;
- (2) $\exists a [H_0^1 \wedge \dots \wedge H_0^n]$ is a wff.

The proof is similar to the proof of 172; use 171 in place of 170.

174 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash A_0 \supset \exists x_\alpha B_0$, provided that $\exists x_\alpha B_0$ is a wff.

Proof: $\mathcal{H} \vdash B_0 \supset \exists x_\alpha B_0$ by 164; then use Rule P.

175 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash A_0 \supset \exists a B_0$, provided that $\exists a B_0$ is a wff.

Proof: $\mathcal{H} \vdash B_0 \supset \exists a B_0$ by 165; then use Rule P.

176 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \exists x_\alpha A_0 \supset \exists x_\alpha B_0$, provided that this is a wff and no x -variable is free in $\mathcal{H}$.

Proof: by 174 and 170.

177 If $\mathcal{H} \vdash A_0 \supset B_0$ then $\mathcal{H} \vdash \exists a A_0 \supset \exists a B_0$, provided that this is a wff and a is not free in $\mathcal{H}$.

Proof: by 175 and 171.

178 $\vdash \forall x_a \exists y_2 . y_2 \stackrel{2}{=} x_a$.

Proof:

- | | |
|---|-----------------|
| .1 $\vdash x_a \stackrel{2}{=} x_a$ | 100 |
| .2 $\vdash \exists y_2 . y_2 \stackrel{2}{=} x_a$ | Exist (166): .1 |
| .3 $\vdash \forall x_a \exists y_2 . y_2 \stackrel{2}{=} x_a$ | Gen (117): .2. |

200 If $\mathcal{H} \vdash A_\alpha \stackrel{\gamma}{=} B_\beta$ or $\mathcal{H} \vdash B_\beta \stackrel{\gamma}{=} A_\alpha$ and if $\mathcal{H} \vdash \S_{A_\alpha}^{x_\gamma} C_0$, then $\mathcal{H} \vdash \S_{B_\beta}^{x_\gamma} C_0$, provided that

- (1) $\gamma \geq \alpha$ and $\gamma \geq \beta$ and all free occurrences of x_γ in C_0 are free for all free variables of A_α and B_β ;
- (2) no x -variable other than x_γ occurs free in C_0 .

(Note that under our hypotheses $\S_{B_\beta}^{x_\gamma} C_0$ must be a wff by Lemma 11.)

Proof:

- | | |
|---|-------------------------------------|
| .1 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\beta] \supset .[\lambda x_\gamma C_0] A_\alpha = [\lambda x_\gamma C_0] B_\beta$ | Sub (119): Axiom 2 |
| .2 $\vdash [B_\beta \stackrel{\gamma}{=} A_\alpha] \supset .[\lambda x_\gamma C_0] B_\beta = [\lambda x_\gamma C_0] A_\alpha$ | Sub (119): Axiom 2 |
| .3 $\vdash [\lambda x_\gamma C_0] A_\alpha \stackrel{0}{=} \S_{A_\alpha}^{x_\gamma} C_0$ | 103 |
| .4 $\vdash [\lambda x_\gamma C_0] B_\beta \stackrel{0}{=} \S_{B_\beta}^{x_\gamma} C_0$ | 103 |
| .5 $\mathcal{H} \vdash \S_{B_\beta}^{x_\gamma} C_0$ | Rule P: hypotheses, .1, .2, .3, .4. |

201 $\vdash [A_\alpha \stackrel{\alpha}{=} B_\alpha] = [A_\alpha \stackrel{\gamma}{=} B_\alpha]$, provided that $\gamma \geq \alpha$.

Proof:

- .1 $[A_\alpha \stackrel{\alpha}{=} B_\alpha] \vdash A_\alpha \stackrel{\alpha}{=} B_\alpha$ Premiss
- .2 $[A_\alpha \stackrel{\alpha}{=} B_\alpha] \vdash A_\alpha \stackrel{\gamma}{=} A_\alpha$ 100
- .3 $[A_\alpha \stackrel{\alpha}{=} B_\alpha] \vdash A_\alpha \stackrel{\gamma}{=} B_\alpha$ R': .1, .2
- .4 $\vdash [A_\alpha \stackrel{\alpha}{=} B_\alpha] \supset [A_\alpha \stackrel{\gamma}{=} B_\alpha]$ Deduction Theorem: .3
- .5 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\alpha] \supset .[\lambda x_\alpha . A_\alpha = x_\alpha] A_\alpha = [\lambda x_\alpha . A_\alpha = x_\alpha] B_\alpha$, where x is
a primitive variable not occurring in A_α . Sub (119): Axiom 2
- .6 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\alpha] \supset .[A_\alpha \stackrel{\alpha}{=} A_\alpha] = [A_\alpha \stackrel{\alpha}{=} B_\alpha]$ R: 201.5, 103
- .7 $\vdash A_\alpha \stackrel{\alpha}{=} A_\alpha$ 100
- .8 $\vdash [A_\alpha \stackrel{\alpha}{=} B_\alpha] = [A_\alpha \stackrel{\gamma}{=} B_\alpha]$ Rule P: .4, .6, .7

202 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\beta] = [B_\beta \stackrel{\gamma}{=} A_\alpha]$, provided that $\gamma \geq \alpha$ and $\gamma \geq \beta$.

Proof:

- .1 $[A_\alpha \stackrel{\gamma}{=} B_\beta] \vdash [A_\alpha \stackrel{\gamma}{=} B_\beta]$ Premiss
- .2 $[A_\alpha \stackrel{\gamma}{=} B_\beta] \vdash [B_\beta \stackrel{\gamma}{=} A_\alpha]$ 200: 202.1, 100
- .3 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\beta] \supset [B_\beta \stackrel{\gamma}{=} A_\alpha]$ Deduction Theorem: .2
- .4 $\vdash [B_\beta \stackrel{\gamma}{=} A_\alpha] \supset [A_\alpha \stackrel{\gamma}{=} B_\beta]$ Theorem Schema 202.3
- .5 $\vdash [A_\alpha \stackrel{\gamma}{=} B_\beta] = [B_\beta \stackrel{\gamma}{=} A_\alpha]$ Rule P: .3, .4.

203 $\vdash [A_\alpha \stackrel{\delta}{=} B_\beta] \supset .[A_\alpha \stackrel{\delta}{=} C_\gamma] = [B_\beta \stackrel{\delta}{=} C_\gamma]$, provided that $\delta \geq \alpha$ and
 $\delta \geq \beta$ and $\delta \geq \gamma$.

Proof:

- .1 $[A_\alpha \stackrel{\delta}{=} B_\beta] \vdash [A_\alpha \stackrel{\delta}{=} B_\beta]$ Premiss

$$203.2 [A_\alpha \stackrel{\delta}{=} B_\beta] \vdash [A_\alpha \stackrel{\delta}{=} C_\gamma] = [A_\alpha \stackrel{\delta}{=} C_\gamma] \quad 100$$

$$.3 [A_\alpha \stackrel{\delta}{=} B_\beta] \vdash [A_\alpha \stackrel{\delta}{=} C_\gamma] = [B_\beta \stackrel{\delta}{=} C_\gamma] \quad 200: .1, .2.$$

The desired theorem follows from .3 by the Deduction Theorem.

$$204 \vdash [B_\beta \stackrel{\delta}{=} A_\alpha] \supset . [C_\gamma \stackrel{\delta}{=} A_\alpha] = [B_\beta \stackrel{\delta}{=} C_\gamma], \text{ provided that } \delta \geq \alpha \text{ and } \delta \geq \beta \text{ and } \delta \geq \gamma.$$

Proof: by Rule P: 202, 203.

205 Extended Equivalence Rules (EE-Rules): Suppose $\delta \geq \alpha$ and $\delta \geq \beta$ and $\delta \geq \gamma$.

$$\mathcal{H} \vdash [A_\alpha \stackrel{\alpha}{=} B_\alpha] \text{ if and only if } \mathcal{H} \vdash [A_\alpha \stackrel{\delta}{=} B_\alpha].$$

$$\mathcal{H} \vdash [A_\alpha \stackrel{\delta}{=} B_\beta] \text{ if and only if } \mathcal{H} \vdash [B_\beta \stackrel{\delta}{=} A_\alpha].$$

$$\text{If } \mathcal{H} \vdash [A_\alpha \stackrel{\delta}{=} B_\beta] \text{ and } \mathcal{H} \vdash [B_\beta \stackrel{\delta}{=} C_\gamma], \text{ then } \mathcal{H} \vdash [A_\alpha \stackrel{\delta}{=} C_\gamma].$$

Proof: by Rule P, 201, 202, 203.

$$206 \vdash [x_\alpha \stackrel{\beta}{=} y_\alpha] \supset . h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} y_\alpha \text{ provided that } \beta \geq \alpha.$$

Proof:

$$.1 \vdash [x_\alpha \stackrel{\beta}{=} y_\alpha] \supset . [\lambda t_\alpha . h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} t_\alpha] x_\alpha = [\lambda t_\alpha . h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} t_\alpha] y_\alpha$$

Sub: Axiom 2

$$.2 \vdash [x_\alpha \stackrel{\beta}{=} y_\alpha] \supset . [h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} x_\alpha] = [h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} y_\alpha] \quad R: 206.1, 103$$

$$.3 \vdash [x_\alpha \stackrel{\beta}{=} y_\alpha] \supset . h_{\gamma\alpha} x_\alpha \stackrel{\gamma}{=} h_{\gamma\alpha} y_\alpha \quad \text{Rule P: 100, 206.2.}$$

$$207 \vdash [\lambda x_\beta . f_{\alpha\beta} x_\beta] \stackrel{\alpha\beta}{=} f_{\alpha\beta}.$$

Proof:

$$.1 \vdash [[\lambda x_\beta . f_{\alpha\beta} x_\beta] \stackrel{\alpha\beta}{=} f_{\alpha\beta}] = \forall x_\beta . [\lambda x_\beta . f_{\alpha\beta} x_\beta] x_\beta \stackrel{\alpha}{=} f_{\alpha\beta} x_\beta$$

Sub: Axiom 3

$$207.2 \vdash \forall x_\beta . [\lambda x_\beta . f_{\alpha\beta} x_\beta] x_\beta \stackrel{\alpha}{=} f_{\alpha\beta} x_\beta \quad \text{Gen (117): 104.}$$

The theorem follows from .1 and .2 by Rule P.

$$208 \vdash [\lambda x_\alpha . y_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{0\alpha}{=} Q_{0\alpha\alpha} y_\alpha \quad \text{Sub: 207; def. of =.}$$

$$209 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . y_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{\alpha}{=} y_\alpha \quad \text{RR (102): Axiom 5, 208.}$$

$$210 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . A_\alpha \stackrel{\beta}{=} x_\alpha] \stackrel{\alpha}{=} A_\alpha, \text{ provided } \beta \geq \alpha \text{ no } x\text{-variable occurs free in } A_\alpha.$$

Proof:

Let y be a primitive variable distinct from x and x .

$$.1 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . y_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{\alpha}{=} y_\alpha \quad \text{Sub (119): 209}$$

$$.2 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . y_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{\alpha}{=} y_\alpha \quad \text{R: 122, 210.1}$$

$$.3 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . A_\alpha \stackrel{\alpha}{=} x_\alpha] \stackrel{\alpha}{=} A_\alpha \quad \text{Sub: .2}$$

$$.4 \vdash \iota_{\alpha(0\alpha)} [\lambda x_\alpha . A_\alpha \stackrel{\beta}{=} x_\alpha] \stackrel{\alpha}{=} A_\alpha \quad \text{R: 210.3, 201.}$$

$$211 \vdash p_0 \supset . [\iota q_\alpha . [p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha] \stackrel{\alpha}{=} y_\alpha, \text{ provided that } \beta \geq \alpha.$$

Proof:

$$.1 \ p_0 \vdash p_0 \quad \text{Premiss}$$

$$.2 \ p_0 \vdash [y_\alpha \stackrel{\beta}{=} q_\alpha] = . [p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha \quad \text{Rule P: .1}$$

$$.3 \ p_0 \vdash [\iota q_\alpha . y_\alpha \stackrel{\beta}{=} q_\alpha] \stackrel{\alpha}{=} y_\alpha \quad 210$$

$$.4 \ p_0 \vdash [\iota q_\alpha . [p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha] \stackrel{\alpha}{=} y_\alpha \quad \text{R': .2, .3.}$$

Then use the Deduction Theorem.

212 $\vdash \sim p_0 \supset . [\iota q_\alpha . [p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha] \stackrel{\alpha}{=} z_\alpha$, provided that $\beta \geq \alpha$.

Proof:

$$.1 \vdash \sim p_0 \supset . [\iota q_\alpha . [\sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim \sim p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \stackrel{\alpha}{=} z_\alpha$$

Sub: 211

$$.2 \vdash [[\sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim \sim p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha]$$

$$= . [p_0 \wedge y_\alpha \stackrel{\beta}{=} q_\alpha] \vee . \sim p_0 \wedge z_\alpha \stackrel{\beta}{=} q_\alpha \quad \text{Rule P.}$$

The desired theorem follows by Rule R from .1 and .2.

$$213 \vdash \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset . \iota_{\alpha(0\alpha)} p_{0\alpha} \stackrel{\alpha}{=} y_\alpha.$$

Proof:

$$.1 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash \forall x_\alpha . p_{0\alpha} x_\alpha = . Q_{0\alpha\alpha} y_\alpha x_\alpha \quad \text{Premiss}$$

$$.2 \vdash [p_{0\alpha} = Q_{0\alpha\alpha} y_\alpha] = \forall x_\alpha . p_{0\alpha} x_\alpha = Q_{0\alpha\alpha} y_\alpha x_\alpha \quad \text{Sub: Axiom 3}$$

$$.3 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash p_{0\alpha} = Q_{0\alpha\alpha} y_\alpha \quad \text{RR (102): .1, .2}$$

$$.4 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash \iota_{\alpha(0\alpha)} p_{0\alpha} \stackrel{\alpha}{=} y_\alpha \quad \text{200: Axiom 5, 213.3.}$$

The desired theorem follows from .4 by the Deduction Theorem.

$$214 \vdash \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset p_{0\alpha} [\iota_{\alpha(0\alpha)} p_{0\alpha}].$$

Proof:

$$.1 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash p_{0\alpha} y_\alpha = . y_\alpha \stackrel{\alpha}{=} y_\alpha$$

Rule S (106): Premiss

$$.2 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash p_{0\alpha} y_\alpha \quad \text{Rule P: 100, 214.1}$$

$$.3 \forall x_\alpha [p_{0\alpha} x_\alpha = . y_\alpha \stackrel{\alpha}{=} x_\alpha] \vdash p_{0\alpha} [\iota_{\alpha(0\alpha)} p_{0\alpha}] \quad \text{200: 214.2, 213.4.}$$

Then use the Deduction Theorem.

$$215 \vdash p_{0\alpha}y_\alpha \supset \forall x_\alpha [p_{0\alpha}x_\alpha \supset \cdot y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset \cdot \iota_{\alpha(0\alpha)}p_{0\alpha} \stackrel{\alpha}{=} y_\alpha.$$

Proof:

$$.1 \vdash [y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset \cdot p_{0\alpha}y_\alpha = p_{0\alpha}x_\alpha \quad \text{Sub: Axiom 2}$$

$$.2 \ p_{0\alpha}y_\alpha \vdash [p_{0\alpha}x_\alpha = \cdot y_\alpha \stackrel{\alpha}{=} x_\alpha] = \cdot p_{0\alpha}x_\alpha \supset \cdot y_\alpha \stackrel{\alpha}{=} x_\alpha$$

Rule P: Premiss, .1

$$.3 \ p_{0\alpha}y_\alpha \vdash \forall x_\alpha [p_{0\alpha}x_\alpha \supset \cdot y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset \cdot \iota_{\alpha(0\alpha)}p_{0\alpha} \stackrel{\alpha}{=} y_\alpha \quad R': 213, 215.2.$$

Then use the Deduction Theorem.

$$216 \vdash p_{0\alpha}y_\alpha \supset \forall x_\alpha [p_{0\alpha}x_\alpha \supset \cdot y_\alpha \stackrel{\alpha}{=} x_\alpha] \supset \cdot p_{0\alpha}[\iota_{\alpha(0\alpha)}p_{0\alpha}].$$

Proof:

$$.1 \vdash [\iota_{\alpha(0\alpha)}p_{0\alpha} \stackrel{\alpha}{=} y_\alpha] \supset \cdot p_{0\alpha}[\iota_{\alpha(0\alpha)}p_{0\alpha}] = p_{0\alpha}y_\alpha \quad \text{Sub: Axiom 2.}$$

The desired theorem follows from 216.1 and 215 by Rule P.

300 If $\mathcal{H} \vdash \alpha = \beta$ or $\mathcal{H} \vdash \beta = \alpha$ and if $\mathcal{H} \vdash \mathcal{S}_\alpha^c A_0$, then $\mathcal{H} \vdash \mathcal{S}_\beta^c A_0$, provided that $2 > \alpha$ and $2 > \beta$ and all free occurrences of c in A_0 are free for all variables in α and β .

Proof: by Rule P and Axiom 8.

$$301 \vdash [\alpha = \beta] \supset \cdot w_\alpha \stackrel{2}{=} w_\beta, \text{ provided that } 2 > \alpha \text{ and } 2 > \beta.$$

Proof:

$$.1 \vdash [\alpha = \beta] \supset \cdot [w_\alpha \stackrel{2}{=} w_\alpha] = [w_\alpha \stackrel{2}{=} w_\beta] \quad \text{Axiom 8}$$

$$.2 \vdash [\alpha = \beta] \supset \cdot w_\alpha \stackrel{2}{=} w_\beta \quad \text{Rule P: 301.1, 100.}$$

$$302 \vdash a = a.$$

Proof:

$$.1 \vdash [x_a \stackrel{2}{=} y_a] \supset a = a \quad \text{Sub}_t (140): \text{Axiom 10}$$

$$302.2 \vdash [x_a \stackrel{2}{=} x_a] \supset a = a$$

Sub (119): .1

$$.3 \vdash a = a$$

Rule P: 302.2, 100.

$$303 \vdash [a = b] \stackrel{0}{=} [b = a].$$

Proof:

$$.1 \vdash b = a \supset .[a = b] \stackrel{0}{=} [a = a]$$

Axiom 8

$$.2 \vdash b = a \supset a = b$$

Rule P: 302, 303.1

$$.3 \vdash a = b \supset b = a$$

Sub₁ (140): .2

$$.4 \vdash a = b \stackrel{0}{=} b = a$$

Rule P: .2, .3.

$$304 \vdash a = c \supset .[b = c] = [a = b].$$

Proof:

$$.1 \vdash a = c \supset .[a = b] \stackrel{0}{=} [c = b]$$

Axiom 8

$$.2 \vdash c = b \stackrel{0}{=} b = c$$

Sub: 303

$$.3 \vdash a = c \supset .[b = c] = [a = b]$$

Rule P: .1, .2.

$$305 \vdash \sim [(0a) = a].$$

The proof of this theorem is motivated by the Russell paradox.
We first introduce two abbreviations:

R_{0a} stands for $[\lambda x_a . \exists z_{0a} . x_a \stackrel{2}{=} z_{0a} \wedge \sim z_{0a} x_a]$;

$R_{0(0a)}$ stands for $[\lambda x_{0a} . \exists z_{0(0a)} . x_{0a} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} x_{0a}]$.

$$.1 \vdash R_{0(0a)} R_{0a} \stackrel{0}{=} \exists z_{0(0a)} . R_{0a} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a} \quad 103$$

$$.2 \vdash [(0a) = a] \supset .[\exists z_{0(0a)} . R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}]$$

$$\stackrel{0}{=} [\exists z_{0(0a)} . R_{0a} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}] \quad \text{Axiom 8}$$

(Of course to verify that this is an instance of Axiom Schema 8 one must refer to the definitions of R_{0a} and $R_{0(0a)}$.)

- 305.3 $\vdash [(0a) = a] \supset . R_{0(0a)} R_{0a} \stackrel{0}{=} \exists z_{0(0a)} . R_{0(0a)} \stackrel{2}{=} z_{0(0a)}$
 $\wedge \sim z_{0(0a)} R_{0a}$ Rule P: .1, .2
- .4 $\vdash \forall z_{0(0a)} \sim [R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}] \supset \sim . R_{0(0a)} \stackrel{2}{=} R_{0(0a)}$
 $\wedge \sim R_{0(0a)} R_{0a}$ 145
- .5 $\vdash R_{0(0a)} \stackrel{2}{=} R_{0(0a)}$ 100
- .6 $\vdash \sim R_{0(0a)} R_{0a} \supset \exists z_{0(0a)} . R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}$
 Rule P: .4, .5; def. of $\exists$
- .7 $\vdash R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \supset . [\lambda t_{0(0a)} . \sim t_{0(0a)} R_{0a}] R_{0(0a)}$
 $\stackrel{0}{=} [\lambda t_{0(0a)} . \sim t_{0(0a)} R_{0a}] z_{0(0a)}$ Sub: Axiom 2
- .8 $\vdash R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \supset . \sim R_{0(0a)} R_{0a} \stackrel{0}{=} \sim z_{0(0a)} R_{0a}$ Rule R: 305.7, 103
- .9 $\vdash [R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}] \supset \sim R_{0(0a)} R_{0a}$ Rule P: .8
- .10 $\vdash \exists z_{0(0a)} [R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}] \supset \sim R_{0(0a)} R_{0a}$ 170: .9
- .11 $\vdash \exists z_{0(0a)} [R_{0(0a)} \stackrel{2}{=} z_{0(0a)} \wedge \sim z_{0(0a)} R_{0a}] \stackrel{0}{=} \sim R_{0(0a)} R_{0a}$
 Rule P: .6, .10
- .12 $\vdash [(0a) = a] \supset . R_{0(0a)} R_{0a} \stackrel{0}{=} \sim R_{0(0a)} R_{0a}$ Rule R: .3, .11
- .13 $\vdash \sim [(0a) = a]$ Rule P: .12

CHAPTER III

THE THEOREM OF INFINITY AND RELATED RESULTS

III.1. The Theorem of Infinity

We are now ready to prove the Theorem of Infinity, and to demonstrate the applications of the rules of inference derived in Chapter II in the process. We begin by introducing some further definitions:

0_2 stands for $\iota_{2(0_2)}Q_{022}[\lambda y_1 F_0]$

S_{22} stands for $[\lambda m_2 . \iota z_2 . \exists a \exists u_a . u_a \stackrel{2}{=} m_2 \wedge [Q_{0aa}u_a] \stackrel{2}{=} z_2]$

N_{02} stands for $[\lambda n_2 . \forall p_{02} . [p_{02}0_2 \wedge \forall x_2 . p_{02}x_2 \supset p_{02}[S_{22}x_2]] \supset p_{02}n_2]$.

In the following pages 0_2 will play the role of the number zero, and S_{22} will play the role of the successor function. For the purpose of understanding the proof of the Theorem of Infinity in an intuitive sense it will be helpful to note that in our intended interpretations 0_2 denotes the empty set of individuals, and S_{22} denotes that function which takes any member of $\mathfrak{D}_\alpha$, where α is in K^1 , onto the set (in $\mathfrak{D}_{0\alpha}$) which contains that member of $\mathfrak{D}_\alpha$ and nothing else. N_{02} will play the role of the set of natural numbers.

It should be noted that the wffs defined above contain no free variables. In accordance with the conventions introduced on page 7, we shall feel free to write N_{02} as N , S_{22} as S , and 0_2 as 0 . We could have defined 0 simply as $[\lambda y_1 F_0]$, of course (see Theorem 1004 below), but it is convenient to have 0 be a wff of type 2.

$$1003.8 \mathcal{H} \vdash Q_{0aa}w_a \stackrel{2}{=} z_2 \quad 300: .6, .1$$

$$.9 \mathcal{H} \vdash Q_{0aa}u_a \stackrel{2}{=} z_2 \quad 200: .7, .8$$

$$.10 u_a \stackrel{2}{=} x_2, \exists b \exists w_b [w_b \stackrel{2}{=} x_2 \wedge .Q_{0bb}w_b \stackrel{2}{=} z_2] \vdash Q_{0aa}u_a \stackrel{2}{=} z_2$$

172, 173: .9; def. of $\mathcal{H}$

$$.11 u_a \stackrel{2}{=} x_2 \vdash \exists b \exists u_b [u_b \stackrel{2}{=} x_2 \wedge .Q_{0bb}u_b \stackrel{2}{=} z_2] \supset .Q_{0aa}u_a \stackrel{2}{=} z_2$$

Deduction Theorem: .10; RR (102): 123

$$.12 u_a \stackrel{2}{=} x_2 \vdash \exists a \exists u_a [u_a \stackrel{2}{=} x_2 \wedge .Q_{0aa}u_a \stackrel{2}{=} z_2] \supset .Q_{0aa}u_a \stackrel{2}{=} z_2$$

RR: 1003.11, 157

$$.13 u_a \stackrel{2}{=} x_2 \vdash Q_{0aa}u_a \stackrel{2}{=} z_2 \supset .u_a \stackrel{2}{=} x_2 \wedge Q_{0aa}u_a \stackrel{2}{=} z_2$$

Rule P: Premiss

$$.14 u_a \stackrel{2}{=} x_2 \vdash Q_{0aa}u_a \stackrel{2}{=} z_2 \supset \exists a \exists u_a .u_a \stackrel{2}{=} x_2 \wedge Q_{0aa}u_a \stackrel{2}{=} z_2$$

174: .13; 175

$$.15 u_a \stackrel{2}{=} x_2 \vdash \exists a \exists u_a [u_a \stackrel{2}{=} x_2 \wedge Q_{0aa}u_a \stackrel{2}{=} z_2] \stackrel{0}{=} .Q_{0aa}u_a \stackrel{2}{=} z_2$$

Rule P: .12, .14

$$.16 \vdash S_{22}x_2 \stackrel{2}{=} [\iota z_2 . \exists a \exists u_a .u_a \stackrel{2}{=} x_2 \wedge Q_{0aa}u_a \stackrel{2}{=} z_2]$$

103; def. of S_{22}

$$.17 u_a \stackrel{2}{=} x_2 \vdash S_{22}x_2 \stackrel{2}{=} [\iota z_2 . Q_{0aa}u_a \stackrel{2}{=} z_2]$$

R': .15, .16

(Note that the restrictions on R' set forth at the beginning of Chapter II are satisfied.)

$$.18 \vdash [\iota z_2 . Q_{0aa}u_a \stackrel{2}{=} z_2] \stackrel{2}{=} Q_{0aa}u_a \quad \text{Sub: 209; R: 122}$$

$$.19 u_a \stackrel{2}{=} x_2 \vdash S_{22}x_2 \stackrel{2}{=} Q_{0aa}u_a \quad \text{EE-Rules (205): .17, .18.}$$

Then use the Deduction Theorem.

$$1004 \vdash O_2 \stackrel{2}{=} [\lambda y_1 F_0] \quad \text{Sub: Axiom 5; def. of } O_2.$$

$$1005 \vdash N_{02}x_2 \supset \exists a \exists u_a .u_a \stackrel{2}{=} x_2.$$

Proof:

$$.1 \vdash [\lambda y_1 F_0] \stackrel{2}{=} O_2 \quad \text{EE-Rules (205): 1004}$$

- 1005.2 $\vdash \exists u_{01} . u_{01} \stackrel{2}{=} 0_2$ Exist (166): .1
- .3 $\vdash \exists a \exists u_a . u_a \stackrel{2}{=} 0_2$ Exist_t (167): .2
- .4 $u_a \stackrel{2}{=} x_2 \vdash Q_{0aa} u_a \stackrel{2}{=} S_{22} x_2$ EE-Rules (205): 1003.19
- .5 $u_a \stackrel{2}{=} x_2 \vdash \exists u_{0a} . u_{0a} \stackrel{2}{=} S_{22} x_2$ Exist (166): .4
- .6 $u_a \stackrel{2}{=} x_2 \vdash \exists a \exists u_a . u_a \stackrel{2}{=} S_{22} x_2$ Exist_t (167): .5
- .7 $\exists a \exists u_a [u_a \stackrel{2}{=} x_2] \vdash \exists a \exists u_a . u_a \stackrel{2}{=} S_{22} x_2$ 172, 173: .6
- .8 $\vdash \forall x_2 . \exists a \exists u_a [u_a \stackrel{2}{=} x_2] \supset \exists a \exists u_a . u_a \stackrel{2}{=} S_{22} x_2$ Deduction Theorem, Gen: .7
- .9 $Nx_2 \vdash [[\lambda t_2 . \exists a \exists u_a . u_a \stackrel{2}{=} t_2] 0 \wedge \forall x_2 . [\lambda t_2 . \exists a \exists u_a . u_a \stackrel{2}{=} t_2] x_2]$
 $\supset [\lambda t_2 . \exists a \exists u_a . u_a \stackrel{2}{=} t_2] [Sx_2] \supset [\lambda t_2 . \exists a \exists u_a . u_a \stackrel{2}{=} t_2] x_2$ Rule S (106): 1001.1
- .10 $Nx \vdash [\exists a \exists u_a [u_a \stackrel{2}{=} 0] \wedge \forall x_2 . \exists a \exists u_a [u_a \stackrel{2}{=} x_2] \supset \exists a \exists u_a [u_a \stackrel{2}{=} [Sx_2]]]$
 $\supset \exists a \exists u_a . u_a \stackrel{2}{=} x_2$ RR: 1005.9, 103
- .11 $Nx_2 \vdash \exists a \exists u_a . u_a \stackrel{2}{=} x_2$ Rule P: .3, .8, .10.

Then use the Deduction Theorem.

$$1006 \vdash u_a \stackrel{2}{=} x_2 \supset \sim . 0_2 \stackrel{2}{=} S_{22} x_2 .$$

Proof: Let $\mathcal{H}$ be $[u_a \stackrel{2}{=} x_2], [0_2 \stackrel{2}{=} S_{22} x_2]$.

- .1 $\mathcal{H} \vdash 0_2 \stackrel{2}{=} S_{22} x_2$ Premiss
- .2 $\mathcal{H} \vdash S_{22} x_2 \stackrel{2}{=} Q_{0aa} u_a$ Rule P: Premiss, 1003
- .3 $\mathcal{H} \vdash [\lambda y_1 F_0] \stackrel{2}{=} Q_{0aa} u_a$ EE-Rules (205): .1, .2, 1004
- .4 $\vdash x_{01} \stackrel{2}{=} y_{0a} \supset .(01) = (0a)$ Sub_t (140): Axiom 10
- .5 $\vdash [\lambda y_1 F_0] \stackrel{2}{=} Q_{0aa} u_a \supset .(01) = (0a)$ Sub (119): .4
- .6 $\vdash [(01) = (0a)] = [0 = 0 \wedge 1 = a]$ Sub_t: Axiom 9
- .7 $\mathcal{H} \vdash 1 = a$ Rule P: .3, .5, .6

- 1006.8 $\mathcal{H} \vdash [\lambda y_1 F_0] \stackrel{01}{=} Q_{011} u_1$ 300: .7, .3; EE-Rules
 .9 $\mathcal{H} \vdash [\lambda y_1 F_0] u_1 \stackrel{0}{=} [\lambda y_1 F_0] u_1$ 100
 .10 $\mathcal{H} \vdash [\lambda y_1 F_0] u_1 \stackrel{0}{=} Q_{011} u_1 u_1$ R': .8, .9
 .11 $\mathcal{H} \vdash F_0 \stackrel{0}{=} [u_1 \stackrel{1}{=} u_1]$ RR: .10, 103; def. of =
 .12 $\mathcal{H} \vdash F_0$ Rule P: .11, 100
 .13 $\vdash u_a \stackrel{2}{=} x_2 \supset \sim .0_2 \stackrel{2}{=} S_{22} x_2$ Deduction Theorem, Rule P: .12.

$$1007 \vdash \forall x_2 . N_{02} x_2 \supset \sim .0_2 \stackrel{2}{=} S_{22} x_2.$$

Proof:

- .1 $\vdash \exists a \exists u_a [u_a \stackrel{2}{=} x_2] \supset \sim .0_2 \stackrel{2}{=} S_{22} x_2$ 170, 171: 1006
 .2 $\vdash N_{02} x_2 \supset \sim .0_2 \stackrel{2}{=} S_{22} x_2$ Rule P: 1005, 1007.1.

Then use Rule Gen.

$$1008 \vdash \forall x_2 \forall y_2 . N_{02} x_2 \wedge N_{02} y_2 \supset .S_{22} x_2 \stackrel{2}{=} S_{22} y_2 \supset .x_2 \stackrel{2}{=} y_2.$$

Proof: Let $\mathcal{H}$ be $[u_a \stackrel{2}{=} x_2], [w_b \stackrel{2}{=} y_2], [S_{22} x_2 \stackrel{2}{=} S_{22} y_2]$.

- .1 $\mathcal{H} \vdash u_a \stackrel{2}{=} x_2$ Premiss
 .2 $\mathcal{H} \vdash w_b \stackrel{2}{=} y_2$ Premiss
 .3 $\mathcal{H} \vdash S_{22} x_2 \stackrel{2}{=} S_{22} y_2$ Premiss
 .4 $\mathcal{H} \vdash S_{22} x_2 \stackrel{2}{=} Q_{0aa} u_a$ MP (128): 1003, 1008.1
 .5 $\vdash w_b \stackrel{2}{=} y_2 \supset .S_{22} y_2 \stackrel{2}{=} Q_{0bb} w_b$ Sub_t (140), Sub (119): 1003
 .6 $\mathcal{H} \vdash S_{22} y_2 \stackrel{2}{=} Q_{0bb} w_b$ MP: .2, .5
 .7 $\mathcal{H} \vdash Q_{0aa} u_a \stackrel{2}{=} Q_{0bb} w_b$ EE-Rules (205): .3, .4, .6
 .8 $\vdash x_{0a} \stackrel{2}{=} y_{0b} \supset .(0a) = (0b)$ Sub_t: Axiom 10
 .9 $\vdash Q_{0aa} u_a \stackrel{2}{=} Q_{0bb} w_b \supset .(0a) = (0b)$ Sub: .8

- 1008.10 $\vdash [(0a) = (0b)] = .0 = 0 \wedge a = b$ Sub_t: Axiom 9
- .11 $\mathcal{H} \vdash a = b$ Rule P: .7, .9, .10
- .12 $\mathcal{H} \vdash Q_{0aa}u_a \stackrel{2}{=} Q_{0aa}w_a$ 300: .11, .7
- .13 $\mathcal{H} \vdash Q_{0aa}u_a u_a$ 100
- .14 $\mathcal{H} \vdash Q_{0aa}u_a \stackrel{0a}{=} Q_{0aa}w_a$ EE-Rules (205): .12
- .15 $\mathcal{H} \vdash Q_{0aa}w_a u_a$ R': .13, .14
- .16 $\mathcal{H} \vdash w_a \stackrel{2}{=} u_a$ EE-Rules: .15
- .17 $\mathcal{H} \vdash w_b \stackrel{2}{=} u_a$ 300: .11, .16
- .18 $\mathcal{H} \vdash x_2 \stackrel{2}{=} y_2$ EE-Rules: .1, .2, .17
- .19 $[u_a \stackrel{2}{=} x_2], [w_b \stackrel{2}{=} y_2] \vdash [S_{22}x_2 \stackrel{2}{=} S_{22}y_2] \supset x_2 \stackrel{2}{=} y_2$
Deduction Theorem: .18
- .20 $\exists a \exists u [u_a \stackrel{2}{=} x_2], \exists b \exists w_b [w_b \stackrel{2}{=} y_2] \vdash [S_{22}x_2 \stackrel{2}{=} S_{22}y_2] \supset x_2 \stackrel{2}{=} y_2$
172, 173, 163: .19
- .21 $\vdash N_{02}y_2 \supset \exists b \exists w_b . w_b \stackrel{2}{=} y_2$ Sub: 1005; R: 123, 157
- .22 $\vdash N_{02}x_2 \wedge N_{02}y_2 \supset .S_{22}x_2 \stackrel{2}{=} S_{22}y_2 \supset .x_2 \stackrel{2}{=} y_2$
Deduction Theorem: 1008.20; Rule P: 1005, 1008.21
- .23 $\vdash \forall x_2 \forall y_2 . N_{02}x_2 \wedge N_{02}y_2 \supset .S_{22}x_2 \stackrel{2}{=} S_{22}y_2 \supset .x_2 \stackrel{2}{=} y_2$
Gen (117): .22.

1009 The Theorem of Infinity

- $\vdash \exists q_2 \exists s_{22} \exists n_{02} . n_{02} q_2 \wedge \forall x_2 [n_{02} x_2 \supset n_{02} [s_{22} x_2]]$
 $\wedge \forall x_2 [n_{02} x_2 \supset \sim . q_2 \stackrel{2}{=} s_{22} x_2]$
 $\wedge \forall x_2 \forall y_2 [n_{02} x_2 \wedge n_{02} y_2 \supset . s_{22} x_2 \stackrel{2}{=} s_{22} y_2 \supset . x_2 \stackrel{2}{=} y_2].$

Proof:

- .1 $\vdash N_{02} \theta_2 \wedge \forall x_2 [N_{02} x_2 \supset N_{02} [S_{22} x_2]]$
 $\wedge \forall x_2 [N_{02} x_2 \supset \sim . \theta_2 \stackrel{2}{=} S_{22} x_2] \wedge \forall x_2 \forall y_2 [N_{02} x_2 \wedge N_{02} y_2$
 $\supset . S_{22} x_2 \stackrel{2}{=} S_{22} y_2 \supset x_2 \stackrel{2}{=} y_2]$

Rule P: 1000, 1002, 1007, 1008.

Theorem 1009 then follows by three applications of Rule Exist (166).

Before closing this section we shall prove one more theorem, which will be useful to us in the following pages.

1010 The Induction Theorem

$$\begin{aligned} & \vdash p_{02}0_2 \supset .\forall x_2[N_{02}x_2 \supset .p_{02}x_2 \supset p_{02}.S_{22}x_2] \supset \forall x_2.N_{02}x_2 \\ & \supset p_{02}x_2. \end{aligned}$$

Proof: Let $\mathcal{H}$ be $p_{02}0_2, \forall x_2[N_{02}x_2 \supset .p_{02}x_2 \supset p_{02}.S_{22}x_2], N_{02}n_2$.

- .1 $\mathcal{H} \vdash N_{02}n_2$ Premiss
- .2 $\mathcal{H} \vdash \forall p_{02}.[p_{02}0_2 \wedge \forall x_2.p_{02}x_2 \supset p_{02}.S_{22}x_2] \supset p_{02}n_2$
RR: 103, .1; def. of N_{02}
- .3 $\mathcal{H} \vdash [[\lambda t_2.Nt_2 \wedge p_{02}t_2]0_2 \wedge \forall x_2.[\lambda t_2.Nt_2 \wedge p_{02}t_2]x_2$
 $\supset [\lambda t_2.Nt_2 \wedge p_{02}t_2].S_{22}x_2] \supset [\lambda t_2.Nt_2 \wedge p_{02}t_2]n_2$
Rule S (106): .2
- .4 $\mathcal{H} \vdash [N0_2 \wedge p_{02}0_2 \wedge \forall x_2.[Nx_2 \wedge p_{02}x_2] \supset .N[S_{22}x_2]$
 $\wedge p_{02}.S_{22}x_2] \supset .Nn_2 \wedge p_{02}n_2$ RR: 103, .3
- .5 $\mathcal{H} \vdash Nx_2 \supset .p_{02}x_2 \supset p_{02}.S_{22}x_2$ Rule S(106): Premiss
- .6 $\mathcal{H} \vdash [Nx_2 \wedge p_{02}x_2] \supset .N[S_{22}x_2] \wedge p_{02}.S_{22}x_2$
Rule P: .5, 1001
- .7 $\mathcal{H} \vdash \forall x_2.[Nx_2 \wedge p_{02}x_2] \supset .N[S_{22}x_2] \wedge p_{02}.S_{22}x_2$
Gen: .6
- .8 $\mathcal{H} \vdash p_{02}0_2$ Premiss
- .9 $\mathcal{H} \vdash p_{02}n_2$ Rule P: .4, .7, .8, 1000
- .10 $p_{02}0_2, \forall x_2[Nx_2 \supset .p_{02}x_2 \supset p_{02}.S_{22}x_2] \vdash N_{02}x_2 \supset p_{02}x_2$
Deduction Theorem: .9; Sub (119)
- .11 $\vdash p_{02}0_2 \supset .\forall x_2[N_{02}x_2 \supset .p_{02}x_2 \supset p_{02}.S_{22}x_2]$
 $\supset \forall x_2.N_{02}x_2 \supset p_{02}x_2$ Gen: .10; Deduction Theorem.

III.2. Numbers and primitive recursive functions

The reader has now seen the system Q in action. In the light of the theorems and metatheorems derived in Chapter II, it is clear that all the usual theorems of simple type theory (without the Axiom of Choice) can be proved in Q by the usual methods without difficulty. (As long as one is concerned only with wffs in which type variables do not occur, one can of course restrict one's choice of wff-variables so that distinct wff-variables always have distinct primitive variables in all wffs occurring in a proof. Then many of the restrictions on the rules of inference in Chapter II will be satisfied automatically.)

It is not our present purpose to carry out a full development of type-theoretic logic in Q , but to investigate the consequences of those aspects of Q which distinguish it from the usual systems of type theory with λ -conversion, namely, the presence of type variables and transfinite types. Therefore from now on we shall feel free to leave to the reader the proofs of those theorems of Q which are in no way special to Q , but might be proved in any of the standard logical systems adequate for the formalization of mathematics. Many of these theorems will indeed require rather lengthy proofs, but in each particular case the reader who is familiar with the relevant mathematics and with type-theoretic reasoning will have no doubt that the theorem can be established in Q . In this way we will shorten our work considerably, and facilitate concentration on the novel features of Q .

In general we favor using the finite cardinals for natural numbers. However, to avoid the detour which would be required to develop cardinal number theory in Q , we shall use the wffs 0_2 , $S_{22}0_2$, $S_{22}S_{22}0_2$, $S_{22}S_{22}S_{22}0_2$, etc., as notations for natural numbers in Q . Actually we need the natural numbers at this time only to serve as Gödel numbers in the formalization of the metamathematics of the system Q_0 ; for this purpose any set of objects satisfying Peano's postulates will suffice. The reader will note that Theorems 1000, 1002, 1007, 1008, and 1010 are Peano's postulates.

DEFINITION: For each natural number n we define a wff n_2 of Q , called *the representative of n in Q* , as follows by induction on n :

If n is zero, then n_2 is 0_2 .

If n is $m + 1$, then n_2 is $[S_{22}m_2]$.

Thus 1_2 is $S_{22}0_2$, 2_2 is $S_{22}S_{22}0_2$, 3_2 is $S_{22}S_{22}S_{22}0_2$, and in general n_2 is $[S_{22}S_{22} \cdots S_{22}S_{22}0_2]$ (where S_{22} occurs n times). $[1 + 3]_2$ is $SSSS0_2$, $[2 \times 3]_2$ is $SSSSSS0_2$, $[3^2]_2$ is $SSSSSSSSSS0_2$, etc.

DEFINITION: The wff A_2 *represents* the natural number n in Q if $\vdash A_2 \stackrel{2}{=} n_2$.

We shall also need wffs to represent certain primitive recursive functions. It is convenient to let these wffs be of type (22) (for the case of functions of one argument, of course; the type will be (222) for functions of two arguments, (2222) for functions of three arguments, etc.). This requires that the functions be defined not only on natural numbers, but on all members of $\mathfrak{D}_2$; however for our purposes it will not matter what the values of these functions are for arguments in $\mathfrak{D}_2$ which are not natural numbers. We shall simply define the wffs representing primitive recursive functions so that the values of the functions will be correct when the arguments are natural numbers, and let the values on other arguments come out as they may. If we are ever forced to make a decision more explicitly on this matter (for example to assure that a function is 1-1), we shall let the function be the identity function on elements of $\mathfrak{D}_2$ which are not natural numbers. Note that the wff N_{02} denotes the set of natural numbers, and the wff S_{22} denotes the successor function.

In order to facilitate the definition of functions by primitive recursion we define the following recursion operator:

DEFINITION: $R_{22(222)_2}$ stands for the wff

$$\begin{aligned} & [\lambda p_2 \lambda h_{222} \lambda n_2 . \lambda y_2 . \forall g_{022} . [\forall x_2 [\sim N_{02}x_2 \supset g_{022}x_2x_2] \wedge g_{022}0_2p_2 \\ & \wedge \forall x_2 [N_{02}x_2 \supset \forall r_2 . g_{022}x_2r_2 \supset g_{022}[S_{22}x_2] [h_{222}x_2r_2]]] \\ & \supset g_{022}n_2y_2]. \end{aligned}$$

$$1100 \vdash [Rp_2h_{222}]0_2 = p_2.$$

$$1101 \vdash N_{02}x_2 \supset .[Rp_2h_{222}] [S_{22}x_2] = h_{222}x_2[[Rp_2h_{222}]x_2].$$

$$1102 \vdash \sim N_{02}x_2 \supset .[Rp_2h_{222}]x_2 = x_2.$$

$$\begin{aligned} 1103 \vdash N_{02}p_2 \wedge \forall y_2 \forall z_2 [N_{02}y_2 \wedge N_{02}z_2 \supset N_{02}.h_{222}y_2z_2] \\ \supset .N_{02}x_2 \supset N_{02}[[Rp_2h_{222}]x_2]. \end{aligned}$$

The proofs of these theorems are left to the reader.

We now illustrate the use of the wff R by defining several primitive recursive functions.

DEFINITION: $+_{222}$ stands for $[\lambda x_2 . Rx_2[\lambda y_2 \lambda z_2 . S_{22}z_2]]$. $A_2 + B_2$ stands for $+_{222}A_2B_2$.

$$1200 \vdash x_2 + 0_2 = x_2 \quad \text{Sub: 1100; RR: 103.}$$

$$1201 \vdash N_{02}y_2 \supset .x_2 + S_{22}y_2 = S_{22}[x_2 + y_2].$$

Proof:

$$\begin{aligned} .1 \ N_{02}y_2 \supset .[Rx_2[\lambda y_2 \lambda z_2 . S_{22}z_2]] [S_{22}y_2] \\ = [\lambda y_2 \lambda z_2 . S_{22}z_2]y_2 [[Rx_2[\lambda y_2 \lambda z_2 . S_{22}z_2]]y_2] \quad \text{Sub: 1101} \end{aligned}$$

$$\begin{aligned} .2 \vdash N_{02}y_2 \supset .x_2 + S_{22}y_2 = [\lambda y_2 \lambda z_2 . S_{22}z_2]y_2[x_2 + y_2] \\ \text{RR: .1, 103} \end{aligned}$$

$$.3 \vdash N_{02}y_2 \supset .x_2 + S_{22}y_2 = S_{22}[x_2 + y_2] \quad \text{R: .2, 103.}$$

$$1202 \vdash \sim N_{02}y_2 \supset .x_2 + y_2 = y_2 \quad \text{Sub: 1102; RR: 103.}$$

$$1203 \vdash N_{02}x_2 \wedge N_{02}y_2 \supset N_{02}[x_2 + y_2].$$

Proof:

$$.1 \vdash N_{02}z_2 \supset N_{02}.[\lambda y_2 \lambda z_2 . S_{22}z_2]y_2z_2 \quad \text{Sub: 1001; RR: 103}$$

$$\begin{aligned} .2 \vdash \forall y_2 \forall z_2 . N_{02}y_2 \wedge N_{02}z_2 \supset N_{02}.[\lambda y_2 \lambda z_2 . S_{22}z_2]y_2z_2 \\ \text{Rule P: .1; Gen} \end{aligned}$$

$$\begin{aligned} .3 \vdash N_{02}x_2 \wedge N_{02}y_2 \supset N_{02}[x_2 + y_2] \\ \text{Sub: 1103; Rule P: .2; RR: 103.} \end{aligned}$$

DEFINITION: $\times_{222}$ stands for $[\lambda x_2 . R0_2[\lambda u_2 \lambda v_2 . x_2 + v_2]]$. $A_2 \times B_2$ stands for $\times_{222} A_2 B_2$.

$$1210 \vdash x_2 \times 0_2 = 0_2.$$

$$1211 \vdash N_{02} y_2 \supset . [x_2 \times S_{22} y_2] = [x_2 + [x_2 \times y_2]].$$

The proofs, which are based on Theorems 1100 and 1101 respectively, are left to the reader.

DEFINITION: $[exp]_{222}$ stands for $[\lambda x_2 . R1_2[\lambda u_2 \lambda v_2 . x_2 \times v_2]]$. $A_2^{B_2}$ stands for $[exp]_{222} A_2 B_2$.

$$1220 \vdash x_2^{0_2} = 1_2.$$

$$1221 \vdash N_{02} y_2 \supset . x_2^{[S_{22} y_2]} = x_2 \times [x_2^{y_2}].$$

We again leave the proofs to the reader.

It should be clear that $+_{222}$ represents the addition function, $\times_{222}$ represents the multiplication function, and $[exp]_{222}$ represents the exponentiation function.

We shall use p_i as a notation in our meta-language for the i -th prime. Thus $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. We define the number theoretic function $*$, which takes two arguments, as follows. If n and m are natural numbers greater than 1, let their prime factorizations be $p_1^{r_1} p_2^{r_2} \dots p_i^{r_i}$ and $p_1^{s_1} p_2^{s_2} \dots p_j^{s_j}$ respectively. ($r_1, r_2, \dots, r_i, s_1, s_2, \dots, s_j$ are of course natural numbers; some of these may be zero, but r_1 and s_1 must be positive.) Then we define $n * m$ to be $p_1^{r_1} p_2^{r_2} \dots p_i^{r_i} p_{i+1}^{s_{i+1}} p_{i+2}^{s_{i+2}} \dots p_{i+j}^{s_{i+j}}$. If either n or m is 0 or 1, we (arbitrarily) define $n * m$ to be 0. It is well known that the function $*$ is primitive recursive. We shall let $*_{222}$ stand for a wff of Q which represents this function. (We leave the details of the definition of this wff to the reader.) In addition $A_2 * B_2$ shall stand for $*_{222} A_2 B_2$.

For technical reasons it is customary to represent primitive recursive properties by functions which take as values numbers rather than truth values and then to identify these numbers with truth values. For example one might identify zero with truth and one with falsehood. However in the system Q (as in systems of type theory with descriptions generally), it

is possible to represent such properties by propositional functions (i.e., functions which take truth values as values). For it is easy to prove that the function $[Q_{022}0_2]$ maps 0_2 onto T_0 and I_2 onto F_0 , and that the function $[\lambda p_0 . q_2 . [p_0 \wedge 0_2 = q_2] \vee . \sim p_0 \wedge I_2 = q_2]$ maps T_0 onto 0_2 and F_0 onto I_2 . (In the case of the latter function apply Theorems 211 and 212.) Hence we can always get from numbers representing truth values to the actual truth values and back again.

We shall let $[Prime > 22]_{02}$ stand for a wff of Q of type (02) which represents the following primitive recursive property of the number n: "n is prime and is greater than twenty-two". We again leave the details of the definition of this wff to the reader.

CHAPTER IV

THE FORMALIZATION OF THE SEMANTICS OF THE SYSTEM Q_0

IV.1. Definitions

We are now ready to show that the semantics of the system Q_0 of finite type theory introduced in Chapter I can be formalized in Q . (Of course our methods could be applied to other formulations of finite type theory also. The system Q_0 has been chosen for convenience.)

We begin by assigning Gödel numbers to the type symbols, primitive symbols, and wffs of Q_0 .

To each type symbol α in K^1 we assign a number " α " as follows: " 0 " = 3, " 1 " = 5 " $(\alpha\beta)$ " = $3^{\alpha} \times 5^{\beta}$ ". Thus " (01) " = $3^3 \times 5^5$, " $(1(01))$ " = $3^5 \times 5^{[3^3 \times 5^5]}$, " $((0\alpha)\alpha)$ " = $3^{[3^3 \times 5^{\alpha}]} \times 5^{\alpha}$, etc.

To each primitive symbol Z of Q_0 we assign a number ' Z ' as follows: ' $\lceil$ ' is 7, ' $\rceil$ ' is 11, ' λ ' is 13, ' $\iota_{1(01)}$ ' is 17, ' $Q_{0\alpha\alpha}$ ' is 19^{α} ', ' x_{α}^1 ' is 23^{α} ', ' x_{α}^2 ' is 29^{α} ', and in general ' x_{α}^i ' is p_{i+8}^{α} .

To each wff W of Q_0 we assign a number " W " as follows: if W consists of the sequence $Z_1 Z_2 \dots Z_n$ of primitive symbols (where $n \geq 1$), then " W " is $p_1^{Z_1} \times p_2^{Z_2} \times \dots \times p_n^{Z_n}$.

We remind the reader that each of the above numbers has a representative in Q . We shall let $["\alpha"]_2$ stand for the wff of Q which is the representative of the number " α ", $["W"]_2$ stand for the representative of the number " W ", etc. Thus $["0"]_2$ is $SSS0$, and $["(00)"]_2$ is the wff 3375_2 .

We next state the fundamental definitions involved in our formalization of the semantics of Q_0 . To assist the reader in comprehending the significance of these definitions, after each definition we shall informally explain the meaning (with respect to our Gödel numbering) of the wff introduced by the definition. For convenience and brevity, we shall per-

mit ourselves to speak rather loosely in making these explanations. Theorems involving the wffs introduced below are stated in the next section of this chapter.

DEFINITIONS: $[TS]_{02}$ stands for $[\lambda x_2 . \forall p_{02} . [p_{02}3 \wedge p_{02}5 \wedge \forall y_2 \forall z_2 [p_{02}y_2 \wedge p_{02}z_2 \supset p_{02}[3^{y_2} \times 5^{z_2}]]] \supset p_{02}x_2]$.

(Of course 3 and 5 in the above wff stand for 3_2 (or $SSS0$) and 5_2 (or $SSSSS0$) respectively. As explained in Chapter I, we often leave the subscripts off abbreviations of wffs when this will cause no confusion.)

$[TS]_{02}A_2$ means that A_2 represents the number of a type symbol.

DEFINITION: $[Variable]_{022}$ stands for

$[\lambda x_2 \lambda t_2 . \forall p_{022} . \forall y_2 \forall z_2 [[Prime > 22]_{02}y_2 \wedge [TS]_{02}z_2 \supset p_{022}[y_2^{z_2}]z_2] \supset p_{022}x_2 t_2]$.

$[Variable]_{022}A_2B_2$ means that there is a type symbol α and a variable x_α^1 such that A_2 represents ' x_α^1 ' and B_2 represents " α ".

DEFINITION: $[Var]_{02}$ stands for

$[\lambda v_2 . [Variable]_{022}v_2[t_2]_{02}[Variable]_{022}v_2]$.

$[Var]_{02}A_2$ means that there is a variable x_α^1 such that A_2 represents ' x_α^1 '.

DEFINITION: $[Wff]_{022}$ stands for

$[\lambda x_2 \lambda r_2 . \forall p_{022} . [[p_{022}2^{17}[3^5 \times 5^{[3^3 \times 5^5]]}] \wedge \forall y_2 [[TS]_{02}y_2 \supset p_{022}[2^{[19^{y_2}]]}[3^{[3^3 \times 5^{y_2}]} \times 5^{y_2}]] \wedge \forall w_2 \forall t_2 [[Variable]_{022}w_2 t_2 \supset p_{022}[2^{w_2}]t_2] \wedge \forall y_2 \forall z_2 \forall t_2 \forall u_2 [[p_{022}y_2[3^{t_2} \times 5^{u_2}] \wedge p_{022}z_2 u_2 \wedge [TS]_{02}t_2] \supset p_{022}[2^7 * y_2 * z_2 * 2^{11}]t_2] \wedge \forall w_2 \forall t_2 \forall y_2 \forall u_2 [[[Variable]_{022}w_2 t_2 \wedge p_{022}y_2 u_2] \supset p_{022}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}][3^{u_2} \times 5^{t_2}]]] \supset p_{022}x_2 r_2]$.

$[Wff]_{022}A_2B_2$ means that there is a type symbol α and a wff C_α of type α such that A_2 represents " C_α " and B_2 represents " α ".

DEFINITION: $[Wf]_{02}$ stands for $[\lambda v_2 . [Wff]_{022} v_2 [\iota_{2(02)} [Wff]_{022} v_2]]$.
 $[Wf]_{02} A_2$ means that there is a wff C_α such that A_2 represents " C_α ".

DEFINITION: Φ_{022} stands for

$$\begin{aligned} & [\lambda t_2 \lambda w_2 . \forall p_{022} . [\forall z_0 p_{022} \mathcal{J}_2 z_0 \wedge \forall z_1 p_{022} \mathcal{S}_2 z_1 \\ & \wedge \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a p_{022} r_2 u_a \wedge \exists v_b p_{022} s_2 v_b] \\ & \supset \forall z_{ab} p_{022} [\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}] z_{ab}] \supset p_{022} t_2 w_2]. \end{aligned}$$

$\Phi_{022} A_2 B_\beta$ (where $2 \geq \beta$) means there is a type symbol α in K^1 such that A_2 represents " α " and the denotation of B_β is in $\mathfrak{D}_\alpha$. (Evidently if $\vdash \Phi_{022} A_2 B_\beta$, then β will be either 2 or α .)

With the definition of Φ_{022} we have moved from wffs concerned with syntactic notions to wffs concerned with semantic notions. Note the use of type variables in Φ_{022} .

DEFINITION: $[Assign]_{0(22)}$ stands for

$$[\lambda g_{22} . \forall x_2 \forall t_2 . [Variable]_{022} x_2 t_2 \supset \Phi_{022} t_2 [g_{22} x_2]].$$

$[Assign]_{0(22)} G_{22}$ means that G_{22} is an assignment of values in the domains $\mathfrak{D}_\alpha$ to the variables x_α^1 of Q_0 ; i.e., G_{22} is (more precisely, denotes) a function such that for each variable x_α^1 of Q_0 , $G_{22}['x_\alpha^1']_2$ is in $\mathfrak{D}_\alpha$.

DEFINITION: $[Maps]_{022}$ stands for

$$\begin{aligned} & [\lambda t_2 \lambda w_2 . \forall p_{022} . [p_{022} \mathcal{J}_2 T_0 \wedge p_{022} \mathcal{S}_2 [\iota_{1(01)} . \lambda x_1 T_0] \\ & \wedge \forall a \forall b \forall u_a \forall v_b \forall r_2 \forall s_2 . [p_{022} r_2 u_a \wedge p_{022} s_2 v_b] \\ & \supset p_{022} [\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}] [\lambda z_b u_a]] \supset p_{022} t_2 w_2]. \end{aligned}$$

DEFINITION: $[Map]_{22}$ stands for

$$[\lambda u_2 . \iota_{2(02)} . [Maps]_{022} [\iota_{2(02)} . [Variable]_{022} u_2]].$$

$[Maps]_{022}$ is a particular set of ordered pairs such that $[Maps]_{022} A_2 B_\beta$ implies that $\Phi_{022} A_2 B_\beta$. Moreover for each type symbol α in K^1 there is a unique B_α such that $[Maps]_{022} ["\alpha"]_2 B_\alpha$. $[Map]_{22}$ is a function such that for each variable x_α^1 of Q_0 , if $[Maps]_{022} ["\alpha"]_2 B_\alpha$, then $[Map]_{22} ['x_\alpha^1']_2$ is B_α . Thus $[Map]_{22}$ satisfies $[Assign]_{0(22)}$.

DEFINITION: $[Mod]_{2222(22)}$ stands for

$[\lambda h_{22} \lambda j_2 \lambda y_2 \lambda k_2 . \iota q_2 . [k_2 \stackrel{2}{=} j_2 \wedge y_2 \stackrel{2}{=} q_2] \vee . \sim [k_2 \stackrel{2}{=} j_2] \wedge h_{22} k_2 \stackrel{2}{=} q_2]$.
 $[Mod]_{2222(22)} H_{22} J_2 A_2$ is the modification of the function H_{22} to have for the argument J_2 the value A_2 .

DEFINITION: $[Value]_{022(22)}$ stands for

$[\lambda g_{22} \lambda n_2 \lambda x_2 . \forall p_{022(22)} . \forall h_{22} [[Assign]_{0(22)} h_{22} \supset . p_{022(22)} h_{22} [2^{17}] \iota_{1(01)}]$
 $\wedge \forall a \forall t_2 [[TS]_{02} t_2 \wedge \exists u_a \Phi_{022} t_2 u_a \supset p_{022(22)} h_{22} [2^{19^{t_2}}] Q_{0aa}]$
 $\wedge \forall v_2 [[Var]_{02} v_2 \supset p_{022(22)} h_{22} [2^{v_2}] [h_{22} v_2]]$
 $\wedge \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall y_{ab} \forall z_b [[Wff]_{022} j_2 [3^{t_2} \times 5^{u_2}]$
 $\wedge [Wff]_{022} m_2 u_2 \wedge [TS]_{02} t_2 \wedge \Phi_{022} [3^{t_2} \times 5^{u_2}] y_{ab} \wedge \Phi_{022} u_2 z_b$
 $\wedge p_{022(22)} h_{22} j_2 y_{ab} \wedge p_{022(22)} h_{22} m_2 z_b]$
 $\supset p_{022(22)} h_{22} [2^7 * j_2 * m_2 * 2^{11}] [y_{ab} z_b]]$
 $\wedge \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall f_{ab} [[Variable]_{022} j_2 t_2$
 $\wedge [Wff]_{022} m_2 u_2 \wedge \forall y_b . \Phi_{022} t_2 y_b \wedge \Phi_{022} u_2 [f_{ab} y_b]$
 $\wedge p_{022(22)} [[Mod]_{2222(22)} h_{22} j_2 y_b] m_2 [f_{ab} y_b]]$
 $\supset p_{022(22)} h_{22} [2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}] f_{ab}] \supset p_{022(22)} g_{22} n_2 x_2]$.

$[Value]_{022(22)} G_{22} A_2 B_\beta$ means that G_{22} is an assignment and there is a wff C_β of Q_0 such that A_2 represents " C_β " and the value of the wff C_β (according to the semantical rules of Q_0) with respect to the assignment G_{22} is B_β .

DEFINITION: $[Val]_{22(22)}$ stands for

$[\lambda g_{22} \lambda n_2 . \iota q_2 . [[Assign]_{0(22)} g_{22} \wedge [Wf]_{02} n_2$
 $\wedge [\iota_{2(02)} . [Value]_{022(22)} g_{22} n_2] \stackrel{2}{=} q_2] \vee . \sim [[Assign]_{0(22)} g_{22}$
 $\wedge [Wf]_{02} n_2] \wedge F_0 \stackrel{2}{=} q_2]$.

If G_{22} is an assignment and C_β is a wff of Q_0 , $[Val]_{22(22)} G_{22} ["C_\beta"]_2$ is the value of C_β with respect to the assignment G_{22} according to the semantical rules of Q_0 .

DEFINITION: $[True]_{02}$ stands for

$$[\lambda n_2 . \forall g_{22} . [Assign]_{0(22)} g_{22} \supset . [Val]_{22(22)} g_{22} n_2 \stackrel{2}{=} T_0].$$

$[True]_{02}["C_0"]_2$ means that C_0 is true according to the semantical rules of Q_0 .

IV.2. Theorems

In order to satisfy ourselves that the system Q is sufficiently strong to serve as a meta-language for Q_0 , and that the wffs defined in the preceding section adequately express the intuitive concepts they are intended to express, we next establish certain theorems involving these wffs. The reader should note that Theorems 2705–2709 below express the semantical rules of Q_0 . These theorems will then be used to prove a general metatheorem (Theorem 2710) which guarantees that for every assignment G_{22} and for every wff A_α of Q_0 , $[Val]_{22(22)} G_{22} ["A_\alpha"]_2$ is what it ought to be. As a corollary (Theorem 2711) we shall see that the wff $[True]_{02}$ satisfies Tarski's criterion for an adequate definition of truth¹²).

The proofs of some of these theorems are very long, and it may help the reader to have a general view of what is going on before getting involved in the proofs. Therefore in this section we shall simply list the theorems. Proofs will be found in the next section of this chapter.

2000 Let x , y , and z be distinct primitive variables, and let P_0 be a wff in which no y -variables and no z -variables occur free, no x -variables other than x_2 occur free, and all free occurrences of x_2 are free for y_2 and z_2 . If $\mathcal{H} \vdash \mathcal{S}_3^{x_2} P_0$, $\mathcal{H} \vdash \mathcal{S}_5^{x_2} P_0$, and $\mathcal{H} \vdash \forall y_2 \forall z_2 . [[\mathcal{S}_{y_2}^{x_2} P_0] \wedge [\mathcal{S}_{z_2}^{x_2} P_0]] \supset \mathcal{S}_{[3y_2 \times 5z_2]}^{x_2} P_0$, then $\mathcal{H} \vdash [TS]_{02} x_2 \supset P_0$.

2001 $\vdash [TS]_{02} x_2 \supset N_{02} x_2$.

2002 $\vdash [TS]_{02} 3$.

2003 $\vdash [TS]_{02} 5$.

¹²) See [17], pp.187-188.

2004 $\vdash [TS]_{02}y \wedge [TS]_{02}z \supset [TS]_{02}[3^{y_2} \times 5^{z_2}]$.

2005 If α is any type symbol in K^1 , then $\vdash [TS]_{02}["\alpha"]_2$.

2006 $\vdash \sim[TS]_{02}0 \wedge \sim[TS]_{02}1$.

2100 Let y and z be distinct primitive variables and let P_0 be a wff in which no y -variables other than y_2 , and no z -variables other than z_2 , occur free, and in which all free occurrences of y_2 are free for z_2 . If $\mathcal{H} \vdash \forall y_2 \forall z_2 . [Prime > 22]_{02}y_2 \wedge [TS]_{02}z_2 \supset S_{y_2 z_2}^{y_2} P_0$, then $\mathcal{H} \vdash [Variable]_{022}y_2 z_2 \supset P_0$.

2101 $\vdash [Prime > 22]_{02}y_2 \wedge [TS]_{02}z_2 \supset [Variable]_{022}[y_2^{z_2}]z_2$.

2102 If x_α^1 is any variable of Q_0 , then $\vdash [Variable]_{022}[x_\alpha^1]_2 ["\alpha"]_2$.

2103 $\vdash [Prime > 22]_{02}y_2 \supset N_{02}y_2$.

2104 $\vdash [Variable]_{022}x_2 t_2 \supset .N_{02}x_2 \wedge [TS]_{02}t_2$.

2105 $\vdash [Variable]_{022}x_2 t_2 \supset .[Variable]_{022}x_2 s_2 \supset t_2 = s_2$.

2106 $\vdash [Variable]_{022}x_2 t_2 \supset [l_{2(02)}[Variable]_{022}x_2] = t_2$.

2107 $\vdash [Variable]_{022}x_2 t_2 \supset [Var]_{02}x_2$.

2108 $\vdash [Var]_{02}x_2 \supset N_{02}x_2$.

2200 Let P_0 be a wff and let $i, j, k, q, r, s, t, u, v, w, x, y, z$ be primitive variables such that:

- 1) no r -variable except r_2 , and no x -variable except x_2 , occurs free in P_0 ;
- 2) no i -, j -, k -, q -, s -, t -, u -, v -, w -, y -, or z -variable occurs free in P_0 ;
- 3) r and x are distinct; j and v are distinct; k, s, y , and z are distinct; t, u, q , and w are distinct;
- 4) all free occurrences of r_2 in P_0 are free for i_2, j_2, k_2, s_2, t_2 , and u_2 ;

all free occurrences of x_2 in P_0 are free for i_2, v_2, y_2, z_2, w_2 , and q_2 .

For notational convenience we shall temporarily use $P(A_2; B_2)$ as an abbreviation for $[S_{A_2 B_2}^{x_2 y_2} P_0]$. If

- (1) $\mathcal{H} \vdash P(2^{17}; 3^5 \times 5^{[3^3 \times 5^5]}),$
- (2) $\mathcal{H} \vdash \forall i_2. [TS]i_2 \supset P(2^{[19^{i_2}]}; 3^{[3^3 \times 5^{i_2}]} \times 5^{i_2}),$
- (3) $\mathcal{H} \vdash \forall v_2 \forall j_2. [Variable]v_2 j_2 \supset P(2^{v_2}; j_2),$
- (4) $\mathcal{H} \vdash \forall y_2 \forall z_2 \forall k_2 \forall s_2. [P(y_2; 3^{k_2} \times 5^{s_2}) \wedge P(z_2; s_2) \wedge [TS]k_2]$
 $\supset P(2^7 * y_2 * z_2 * 2^{11}; k_2),$ and
- (5) $\mathcal{H} \vdash \forall w_2 \forall t_2 \forall q_2 \forall u_2. [[Variable]w_2 t_2 \wedge P(q_2; u_2)]$
 $\supset P(2^7 * 2^{13} * 2^{w_2} * q_2 * 2^{11}; 3^{u_2} \times 5^{t_2}),$

then $\mathcal{H} \vdash [Wff]_{022} x_2 r_2 \supset P_0$.

$$2201 \vdash [Wff]_{022} w_2 t_2 \supset .N_{02} w_2 \wedge [TS]_{02} t_2.$$

$$2202 \vdash [Wff]_{022} [2^{17}] [3^5 \times 5^{[3^3 \times 5^5]}].$$

$$2203 \vdash [TS]_{02} y_2 \supset [Wff]_{022} [2^{19^{y_2}}] [3^{[3^3 \times 5^{y_2}]} \times 5^{y_2}].$$

$$2204 \vdash [Variable]_{022} w_2 t_2 \supset [Wff]_{022} [2^{w_2}] t_2.$$

$$2205 \vdash [[Wff]_{022} y_2 [3^{t_2} \times 5^{u_2}] \wedge [Wff]_{022} z_2 u_2 \wedge [TS]_{02} t_2]$$

 $\supset [Wff]_{022} [2^7 * y_2 * z_2 * 2^{11}] t_2.$

$$2206 \vdash [[Variable]_{022} w_2 t_2 \wedge [Wff]_{022} y_2 u_2]$$

 $\supset [Wff]_{022} [2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}] [3^{u_2} \times 5^{t_2}].$

$$2207 \text{ If } A_\alpha \text{ is any wff of } Q_0, \vdash [Wff]_{022} ["A_\alpha"]_2 ["\alpha"]_2.$$

$$2208 \vdash [Wff]_{022} w_2 t_2 \supset .[Wff]_{022} w_2 s_2 \supset t_2 = s_2.$$

$$2209 \vdash [Wff]_{022} w_2 t_2 \supset [Wf]_{02} w_2.$$

2300 Let P_{022} be a wff, let a and b be type variables, and let $r, s, t, u, v, w, x, y, z$ be primitive variables such that:

- 1) neither a nor b occurs free in P_{022} ;
no r -, s -, t -, u -, v -, w -, x -, y -, or z -variable occurs free in P_{022} ;
- 2) a and b are distinct;
 r is distinct from s, u , and z ;
 s is distinct from r, v , and z .

If $\mathcal{H} \vdash \forall x_0 P_{022} \beta_2 x_0$, $\mathcal{H} \vdash \forall y_1 P_{022} \gamma_2 y_1$ and

$$\begin{aligned} \mathcal{H} \vdash \forall r_2 \forall s_2 \forall a \forall b. [\exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b] \\ \supset \forall z_{ab} P_{022} [\beta^{r_2} \times \gamma^{s_2}] z_{ab}, \end{aligned}$$

then $\mathcal{H} \vdash \Phi_{022} t_2 w_2 \supset P_{022} t_2 w_2$.

$$2301 \vdash \Phi_{022} \beta q_0.$$

$$2302 \vdash \Phi_{022} \gamma q_1.$$

$$2303 \vdash [\Phi_{022} r_2 u_c \wedge \Phi_{022} s_2 v_d] \supset \Phi_{022} [\beta^{r_2} \times \gamma^{s_2}] w_{cd}.$$

$$2304 \vdash [\exists u_c \Phi_{022} r_2 u_c \wedge \exists v_d \Phi_{022} s_2 v_d] \supset \Phi_{022} [\beta^{r_2} \times \gamma^{s_2}] w_{cd}.$$

$$2305 \text{ If } \alpha \text{ is any type symbol in } K^1, \text{ then } \vdash \Phi_{022} ["\alpha"]_2 x_\alpha.$$

$$2306 \vdash \exists u_a \Phi_{022} t_2 u_a \supset \Phi_{022} [\beta^{[3^3 \times 5^{t_2}]} \times \gamma^{t_2}] q_{0aa}.$$

$$2307 \vdash \Phi_{022} t_2 w_2 \supset .[TS]_{02} t_2 \wedge \exists c \exists x_c . w_2 \stackrel{2}{=} x_c.$$

$$2308 \vdash [TS]_{02} t_2 \supset \exists c \exists x_c \Phi_{022} t_2 x_c.$$

$$\begin{aligned} 2309 \vdash [[TS]_{02} y_2 \wedge [TS]_{02} z_2 \wedge \Phi_{022} [\beta^{y_2} \times \gamma^{z_2}] x_{ab}] \\ \supset . \exists m_a \Phi_{022} y_2 m_a \wedge \exists n_b \Phi_{022} z_2 n_b. \end{aligned}$$

$$2310 \vdash [[TS]_{02} y_2 \wedge [TS]_{02} z_2 \wedge \Phi_{022} [\beta^{y_2} \times \gamma^{z_2}] w_2] \supset \exists a \exists b \exists z_{ab} . w_2 \stackrel{2}{=} z_{ab}.$$

$$2311 \vdash \Phi_{022} \beta_2 q_a \supset a = 0.$$

$$2312 \vdash \Phi_{022} \mathcal{J}_2 q_a \supset a = 1.$$

$$2313 \vdash [\Phi_{022} t_2 u_a \wedge \Phi_{022} t_2 v_b] \supset a = b.$$

$$2314 \vdash [\exists u_a \Phi_{022} t_2 u_a \wedge \exists v_b \Phi_{022} t_2 v_b] \supset a = b.$$

$$2315 \vdash \Phi_{022} t_2 u_a \supset \Phi_{022} t_2 v_a.$$

$$2316 \vdash \exists m_a \Phi_{022} t_2 m_a \supset \Phi_{022} t_2 v_a.$$

$$2317 \vdash \exists m_a \Phi_{022} t_2 m_a \supset \forall v_a \Phi_{022} t_2 v_a.$$

$$2318 \vdash [TS]_{02} t_2 \supset \exists c \forall x_c \Phi_{022} t_2 x_c.$$

$$2319 \vdash [TS]_{02} t_2 \stackrel{0}{=} \exists a \forall b \forall u_b . \Phi_{022} t_2 u_b \stackrel{0}{=} . a = b.$$

$$2320 \vdash \exists x_a \Phi_{022} t_2 x_a \supset . \Phi_{022} t_2 w_2 \supset \exists y_a . w_2 \stackrel{2}{=} y_a.$$

$$2321 \vdash [\Phi_{022} [3^{t_2} \times 5^{u_2}] m_2 \wedge \Phi_{022} u_2 n_2 \wedge [TS]_{02} t_2] \\ \supset \exists a \exists b \exists y_{ab} \exists z_b . m_2 \stackrel{2}{=} y_{ab} \wedge n_2 \stackrel{2}{=} z_b.$$

2400 Let P_{022} be a wff, let a and b be distinct type variables, and let r, s, t, u, v, w, z be primitive variables such that:

- 1) neither a nor b occurs free in P_{022} ;
 no r -, s -, t -, u -, v -, w -, or z -variable occurs free in P_{022} ;
 u, v, r , and s are distinct; u and z are distinct.

If $\mathcal{H} \vdash P_{022} \mathcal{J}_2 T_0$, $\mathcal{H} \vdash P_{022} \mathcal{J}_2 [t_{1(01)} . \lambda x_1 T_0]$ and

$$\mathcal{H} \vdash \forall a \forall b \forall u_a \forall v_b \forall r_2 \forall s_2 . [P_{022} r_2 u_a \wedge P_{022} s_2 v_b] \\ \supset P_{022} [3^{r_2} \times 5^{s_2}] [\lambda z_b u_a],$$

then $\mathcal{H} \vdash [Maps]_{022} t_2 w_2 \supset P_{022} t_2 w_2$.

$$2401 \vdash [Maps]_{022} \mathcal{J}_2 T_0.$$

$$2402 \vdash [Maps]_{022} \mathcal{J}_2 [t_{1(01)} . \lambda x_1 T_0].$$

$$2403 \vdash [[Maps]_{022}r_2u_a \wedge [Maps]_{022}s_2v_b] \supset [Maps]_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}] [\lambda z_b u_a].$$

$$2404 \vdash [Maps]_{022}t_2w_2 \supset \Phi_{022}t_2w_2.$$

$$2405 \vdash [Maps]_{022}t_2w_2 \supset .[TS]_{02}t_2 \wedge \exists c \exists x_c . w_2 \stackrel{2}{=} x_c.$$

$$\begin{aligned} 2406 \vdash & [[TS]_{02}r_2 \wedge [TS]_{02}s_2 \wedge [Maps]_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]w_2] \\ & \supset \exists a \exists b \exists u_a \exists v_b . [Maps]_{022}r_2u_a \wedge [Maps]_{022}s_2v_b \wedge w_2 \stackrel{2}{=} [\lambda z_b u_a]. \end{aligned}$$

$$\begin{aligned} 2407 \vdash & [TS]_{02}t_2 \supset \exists g_2 . [Maps]_{022}t_2g_2 \wedge \forall w_2 . [Maps]_{022}t_2w_2 \\ & \supset .g_2 \stackrel{2}{=} w_2. \end{aligned}$$

$$2408 \vdash [Variable]_{022}u_2t_2 \supset [Maps]_{022}t_2[[Map]_{22}u_2].$$

$$2409 \vdash [Assign]_{0(22)}[Map]_{22}.$$

$$2410 \vdash \exists g_{22} . [Assign]_{0(22)}g_{22}.$$

$$2500 \vdash x_2 \stackrel{2}{=} j_2 \supset .[Mod]_{2222(22)}h_{22}j_2y_2x_2 \stackrel{2}{=} y_2.$$

$$2501 \vdash [Mod]_{2222(22)}h_{22}j_2y_2j_2 \stackrel{2}{=} y_2.$$

$$2502 \vdash \sim[x_2 \stackrel{2}{=} j_2] \supset .[Mod]_{2222(22)}h_{22}j_2y_2x_2 \stackrel{2}{=} h_{22}x_2.$$

$$\begin{aligned} 2503 \vdash & [[Variable]_{022}j_2s_2 \wedge \Phi_{022}s_2y_2] \supset .[Assign]_{0(22)}h_{22} \\ & \supset [Assign]_{0(22)}[[Mod]_{2222(22)}h_{22}j_2y_2]. \end{aligned}$$

2600 Let $P_{022(22)}$ be a wff in which no type variables or wff-variables occur free, and let $\mathcal{H}$ be a set of hypotheses (possibly the empty set) in which no h -variables occur free. If

$$(1) \mathcal{H}, [Assign]h_{22} \vdash P_{022(22)}h_{22}[\mathcal{J}^{I^7}]_{1(01)};$$

$$\begin{aligned} (2) \mathcal{H}, [Assign]h_{22} \vdash & \forall a \forall t_2 . [[TS]t_2 \wedge \exists u_a \Phi_{022}t_2u_a] \\ & \supset P_{022(22)}h_{22}[\mathcal{J}^{I^9t_2}]Q_{0aa}; \end{aligned}$$

$$(3) \mathcal{H}, [Assign]h_{22} \vdash \forall v_2 . [Var]v_2 \supset P_{022(22)}h_{22}[\mathcal{J}^{v_2}] [h_{22}v_2];$$

$$\begin{aligned}
(4) \mathcal{H}, [Assign]_{h_{22}} \vdash & \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall y_{ab} \forall z_b . [[Wff]_{j_2} [3^{t_2} \times 5^{u_2}] \\
& \wedge [Wff]_{m_2 u_2} \wedge [TS]_{t_2} \wedge \Phi_{022} [3^{t_2} \times 5^{u_2}]_{y_{ab}} \wedge \Phi_{022} u_2 z_b \\
& \wedge P_{022(22)} h_{22} j_2 y_{ab} \wedge P_{022(22)} h_{22} m_2 z_b] \\
& \supset P_{022(22)} h_{22} [2^7 * j_2 * m_2 * 2^{11}] [y_{ab} z_b];
\end{aligned}$$

$$\begin{aligned}
(5) \mathcal{H}, [Assign]_{h_{22}} \vdash & \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall f_{ab} . [[Variable]_{j_2 t_2} \\
& \wedge [Wff]_{m_2 u_2} \wedge \forall y_b . \Phi_{022} t_2 y_b \wedge \Phi_{022} u_2 [f_{ab} y_b] \\
& \wedge P_{022(22)} [[Mod]_{h_{22} j_2 y_b} m_2 [f_{ab} y_b]] \\
& \supset P_{022(22)} h_{22} [2^7 * 2^{13} * 2^{12} * m_2 * 2^{11}] f_{ab};
\end{aligned}$$

$$\text{then } \mathcal{H} \vdash [Value]_{022(22)} g_{22} n_2 x_2 \supset P_{022(22)} g_{22} n_2 x_2.$$

$$\begin{aligned}
2601 \vdash & [Value]_{022(22)} g_{22} n_2 x_2 \supset . [Assign]_{0(22)} g_{22} \\
& \wedge \exists s_2 . [Wff]_{022} n_2 s_2 \wedge \Phi_{022} s_2 x_2.
\end{aligned}$$

$$2602 \vdash [Value]_{022(22)} g_{22} n_2 y_2 \supset . [Assign]_{0(22)} g_{22} \wedge [Wf]_{02} n_2.$$

$$2603 \vdash [[Value]_{022(22)} g_{22} n_2 x_2 \wedge [Wff]_{022} n_2 t_2] \supset \Phi_{022} t_2 x_2.$$

$$2604 \vdash [Assign]_{0(22)} h_{22} \supset [Value]_{022(22)} h_{22} [2^{17}]_{t_1(01)}.$$

$$\begin{aligned}
2605 \vdash & [Assign]_{0(22)} h_{22} \supset . [[TS]_{02} t_2 \wedge \exists u_a . \Phi_{022} t_2 u_a] \\
& \supset [Value]_{022(22)} h_{22} [2^{19^{t_2}}] Q_{0aa}.
\end{aligned}$$

$$\begin{aligned}
2606 \vdash & [Assign]_{0(22)} h_{22} \supset . [Var]_{02} v_2 \\
& \supset [Value]_{022(22)} h_{22} [2^{v_2}] [h_{22} v_2].
\end{aligned}$$

$$\begin{aligned}
2607 \vdash & [Assign]_{0(22)} h_{22} \supset . [[Wff]_{022} j_2 [3^{t_2} \times 5^{u_2}] \wedge [Wff]_{022} m_2 u_2 \\
& \wedge [TS]_{02} t_2 \wedge \Phi_{022} [3^{t_2} \times 5^{u_2}]_{y_{ab}} \wedge \Phi_{022} u_2 z_b \\
& \wedge [Value]_{022(22)} h_{22} j_2 y_{ab} \wedge [Value]_{022(22)} h_{22} m_2 z_b] \\
& \supset [Value]_{022(22)} h_{22} [2^7 * j_2 * m_2 * 2^{11}] [y_{ab} z_b].
\end{aligned}$$

$$\begin{aligned}
2608 \vdash & [Assign]_{0(22)} h_{22} \supset . [[Variable]_{022} j_2 t_2 \wedge [Wff]_{022} m_2 u_2 \\
& \wedge \forall y_b . \Phi_{022} t_2 y_b \wedge \Phi_{022} u_2 [f_{ab} y_b] \\
& \wedge [Value]_{022(22)} [[Mod]_{2222(22)} h_{22} j_2 y_b] m_2 [f_{ab} y_b]] \\
& \supset [Value]_{022(22)} h_{22} [2^7 * 2^{13} * 2^{12} * m_2 * 2^{11}] f_{ab}.
\end{aligned}$$

$$2609 \vdash [Value]_{022(22)} g_{22} [2^{17}] j_2 \supset . j_2 \stackrel{2}{=} \iota_{1(01)}.$$

$$\begin{aligned}
2610 \vdash & [Value]_{022(22)} g_{22} [2^{19t_2}] j_2 \supset . [TS]_{02} t_2 \supset \exists a . \exists u_a \Phi_{022} t_2 u_a \\
& \wedge . j_2 \stackrel{2}{=} Q_{0aa}.
\end{aligned}$$

$$2611 \vdash [Value]_{022(22)} g_{22} [2^{v_2}] j_2 \supset . [Variable]_{022} v_2 t_2 \supset . j_2 \stackrel{2}{=} g_{22} v_2.$$

$$\begin{aligned}
2612 \vdash & [Value]_{022(22)} g_{22} [2^7 * k_2 * m_2 * 2^{11}] j_2 \\
& \supset . [[Wff]_{022} k_2 [3^{t_2} \times 5^{u_2}] \wedge [Wff]_{022} m_2 u_2 \wedge [TS]_{02} t_2] \\
& \supset \exists a \exists b \exists y_{ab} \exists z_b . \Phi_{022} [3^{t_2} \times 5^{u_2}] y_{ab} \wedge \Phi_{022} u_2 z_b \\
& \wedge [Value]_{022(22)} g_{22} k_2 y_{ab} \wedge [Value]_{022(22)} g_{22} m_2 z_b \wedge . j_2 \\
& \stackrel{2}{=} [y_{ab} z_b].
\end{aligned}$$

$$\begin{aligned}
2613 \vdash & [Value]_{022(22)} g_{22} [2^7 * 2^{13} * 2^{k_2} * m_2 * 2^{11}] j_2 \\
& \supset . [[Variable]_{022} k_2 t_2 \wedge [Wff]_{022} m_2 u_2] \supset \exists a \exists b \exists f_{ab} . j_2 \\
& \stackrel{2}{=} f_{ab} \wedge \forall y_b . \Phi_{022} t_2 y_b \wedge \Phi_{022} u_2 [f_{ab} y_b] \\
& \wedge [Value]_{022(22)} [[Mod]_{2222(22)} g_{22} k_2 y_b] m_2 [f_{ab} y_b].
\end{aligned}$$

$$\begin{aligned}
2614 \vdash & [[Variable]_{022} w_2 t_2 \wedge [Wff]_{022} y_2 u_2 \wedge \forall g_{22} [[Assign]_{0(22)} g_{22} \\
& \supset \exists i_a . [Value]_{022(22)} g_{22} y_2 i_a \wedge \forall j_2 . [Value]_{022(22)} g_{22} y_2 j_2 \\
& \supset . j_2 \stackrel{2}{=} i_a] \wedge \exists m_b \Phi_{022} t_2 m_b] \supset . [Assign]_{0(22)} h_{22} \\
& \supset . [Value]_{022(22)} h_{22} [2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}] \\
& [\lambda z_b . \iota x_a . [Value]_{022(22)} [[Mod]_{2222(22)} h_{22} w_2 z_b] y_2 x_a].
\end{aligned}$$

$$2615 \vdash [[Wff]_{022}x_2r_2 \supset \exists a \forall g_{22} . [Assign]_{0(22)}g_{22}$$

$$\supset \exists i_a . [Value]_{022(22)}g_{22}x_2i_a \wedge \forall j_2 . [Value]_{022(22)}g_{22}x_2j_2$$

$$\supset .j_2 \stackrel{2}{=} i_a.$$

$$2616 \vdash [[Assign]_{0(22)}h_{22} \wedge [Variable]_{022}w_2t_2 \wedge [Wff]_{022}y_2u_2$$

$$\wedge \exists m_b \Phi_{022}t_2m_b \wedge \exists n_a \Phi_{022}u_2n_a]$$

$$\supset .[Value]_{022(22)}h_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]$$

$$[\lambda z_b . \iota x_a . [Value]_{022(22)}[[Mod]_{2222(22)}h_{22}w_2z_b]y_2x_a].$$

$$2617 \vdash [[Wff]_{022}x_2r_2 \wedge [Assign]_{0(22)}g_{22}]$$

$$\supset \exists i_2 . [Value]_{022(22)}g_{22}x_2i_2 \wedge \forall j_2 . [Value]_{022(22)}g_{22}x_2j_2$$

$$\supset .i_2 \stackrel{2}{=} j_2.$$

$$2618 \vdash [[Wff]_{022}x_2r_2 \wedge [Assign]_{0(22)}g_{22}]$$

$$\supset .[Value]_{022(22)}g_{22}x_2[\iota_{2(02)} . [Value]_{022(22)}g_{22}x_2].$$

$$2619 \vdash [[Wff]_{022}x_2r_2 \wedge [Assign]_{0(22)}g_{22} \wedge [Value]_{022(22)}g_{22}x_2y_2]$$

$$\supset .[\iota_{2(02)} . [Value]_{022(22)}g_{22}x_2] \stackrel{2}{=} y_2.$$

$$2620 \vdash [[Wff]_{022}x_2r_2 \wedge [Assign]_{0(22)}g_{22}]$$

$$\supset .[Value]_{022(22)}g_{22}x_2y_2 \stackrel{0}{=} .[\iota_{2(02)} . [Value]_{022(22)}g_{22}x_2] \stackrel{2}{=} y_2.$$

$$2700 \vdash \sim [[Assign]_{0(22)}g_{22} \wedge [Wf]_{02}n_2] \supset .[Val]_{22(22)}g_{22}n_2 \stackrel{2}{=} F_0.$$

$$2701 \vdash [[Assign]_{0(22)}g_{22} \wedge [Wf]_{02}n_2]$$

$$\supset .[Val]_{22(22)}g_{22}n_2 \stackrel{2}{=} [\iota_{2(02)} . [Value]_{022(22)}g_{22}n_2].$$

$$2702 \vdash [[Assign]_{0(22)}g_{22} \wedge [Wf]_{02}n_2]$$

$$\supset [Value]_{022(22)}g_{22}n_2[[Val]_{22(22)}g_{22}n_2].$$

$$2703 \vdash [[Assign]_{0(22)}g_{22} \wedge [Wf]_{02}n_2 \supset . [Value]_{022(22)}g_{22}n_2y_2 \\ \stackrel{0}{=} . [[Val]_{22(22)}g_{22}n_2] \stackrel{2}{=} y_2.$$

$$2704 \vdash [Value]_{022(22)}g_{22}n_2y_2 \supset . [[Val]_{22(22)}g_{22}n_2] \stackrel{2}{=} y_2.$$

$$2705 \vdash [Assign]_{0(22)}g_{22} \supset . [[Val]_{22(22)}g_{22}[2^{17}]] \stackrel{2}{=} \iota_{1(01)}.$$

$$2706 \vdash [Assign]_{0(22)}g_{22} \supset . [[TS]_{02}t_2 \wedge \exists u_a \Phi_{022}t_2u_a] \\ \supset . [[Val]_{22(22)}g_{22}[2^{19}t_2]] \stackrel{2}{=} Q_{0aa}.$$

$$2707 \vdash [Assign]_{0(22)}g_{22} \supset . [Var]_{02}v_2 \supset . [[Val]_{22(22)}g_{22}[2^{v_2}]] \stackrel{2}{=} [g_{22}v_2].$$

$$2708 \vdash [[Assign]_{0(22)}g_{22} \wedge [Wff]_{022}j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]_{022}m_2u_2 \\ \wedge [TS]_{02}t_2 \wedge [[Val]_{22(22)}g_{22}j_2 \stackrel{2}{=} y_{ab}] \wedge . [Val]_{22(22)}g_{22}m_2 \stackrel{2}{=} z_b] \\ \supset . [[Val]_{22(22)}g_{22}[2^7 * j_2 * m_2 * 2^{11}]] \stackrel{2}{=} [y_{ab}z_b].$$

$$2709 \vdash [[Assign]_{0(22)}g_{22} \wedge [Variable]_{022}w_2t_2 \wedge [Wff]_{022}y_2u_2 \\ \wedge \exists m_b \Phi_{022}t_2m_b \wedge \exists n_a \Phi_{022}u_2n_a] \\ \supset . [Val]_{22(22)}g_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}] \\ \stackrel{2}{=} [\lambda z_b . \iota x_a . [Val]_{22(22)}[[Mod]_{2222(22)}g_{22}w_2z_b]y_2 \stackrel{2}{=} x_a].$$

We are now ready for the main theorem of this section. We begin by defining a map Ψ from wffs of Q_0 to wffs of Q .

DEFINITION: If A_α is any wff of Q_0 , $[\Psi A_\alpha]$ shall be the result of replacing each occurrence of a variable of Q_0 in A_α by an occurrence of a wff of Q in the following manner:

- (1) If x_β^1 occurs bound in A_α , it shall be replaced at each of its bound occurrences by the variable $\overset{n}{z}_\beta$ of Q , where n is the number ' x_β^1 '.
- (2) If x_β^1 occurs free in A_α , it shall be replaced at each of its free occurrences by the wff $[\iota y_\beta . g_{22}[\overset{1}{x}_\beta^1]_2 \stackrel{2}{=} y_\beta]$.

These replacements are to be carried out for all variables x_β^1 occurring in A_α . Note that distinct variables of Q_0 are replaced by distinct variables and wffs of Q .

We readily verify that if A_α is any wff of Q_0 , $[\Psi A_\alpha]$ is a wff of Q of the same type with no free variables other than g_{22} . $[\Psi A_\alpha]$ will be called *the interpretation of A_α in Q with respect to g_{22}* .

2710 $\vdash [\text{Assign}]_{0(22)} g_{22} \supset . [\text{Val}]_{22(22)} g_{22} [“A_\alpha”]_2 \stackrel{2}{=} [\Psi A_\alpha]$ for each wff A_α of Q_0 .

We shall give the proof of this metatheorem immediately. In this proof, the notation $[S_W^X C]$ stands for the result of substituting the formula W for the variable X at all occurrences of X in the formula C .

The proof is by induction on the construction of A_α .

CASE 1: A_α is a variable x_α^1 .

Then $[\Psi A_\alpha]$ is $[\iota y_\alpha . g_{22} [“x_\alpha^1”]_2 \stackrel{2}{=} y_\alpha]$, and $“A_\alpha”$ is $2^{“x_\alpha^1”}$.

2710.01 $\vdash 2_2^{[“x_\alpha^1”]_2} \stackrel{2}{=} [“A_\alpha”]_2$.

The proof by recursive arithmetic is left to the reader. Use theorems 1200–1221, etc.

.02 $\vdash [\text{Variable}] [“x_\alpha^1”]_2 [“\alpha”]_2$. 2102

.03 $\vdash [\text{Var}] [“x_\alpha^1”]_2$ Sub: 2107; MP: .02

.04 $\vdash [\text{Assign}]_{g_{22}} \supset . [\text{Var}] [“x_\alpha^1”]_2 \supset . [\text{Val}]_{g_{22}} [2_2^{[“x_\alpha^1”]_2}] \stackrel{2}{=} g_{22} [“x_\alpha^1”]_2$
Sub: 2707.

Let $\mathcal{H}$ be $[\text{Assign}]_{g_{22}}, g_{22} [“x_\alpha^1”]_2 \stackrel{2}{=} u_\alpha$.

.05 $\mathcal{H} \vdash [\text{Assign}]_{g_{22}}$ Premiss

.06 $\mathcal{H} \vdash [\text{Val}]_{g_{22}} [“A_\alpha”]_2 \stackrel{2}{=} g_{22} [“x_\alpha^1”]_2$ Rule P: .03, .04, .05

Rule R': .01

.07 $\mathcal{H} \vdash g_{22} [“x_\alpha^1”]_2 \stackrel{2}{=} u_\alpha$ Premiss

.08 $\vdash [\iota y_\alpha . u_\alpha \stackrel{2}{=} y_\alpha] \stackrel{2}{=} u_\alpha$ EE-Rules (205): 210

$$2710.09 \mathcal{H} \vdash [\iota y_\alpha \cdot g_{22}[\iota x_\alpha^1]_2 \stackrel{2}{=} y_\alpha] \stackrel{2}{=} g_{22}[\iota x_\alpha^1]_2 \quad 200: .07, .08$$

$$.10 \mathcal{H} \vdash [Val]_{g_{22}}[“A_\alpha”]_2 \stackrel{2}{=} [\iota y_\alpha \cdot g_{22}[\iota x_\alpha^1]_2 \stackrel{2}{=} y_\alpha] \\ \text{EE-Rules (205): .06, .09}$$

$$.11 [Assign]_{g_{22}}, \exists u_\alpha [g_{22}[\iota x_\alpha^1]_2 \stackrel{2}{=} u_\alpha] \vdash [Val]_{g_{22}}[“A_\alpha”]_2 \stackrel{2}{=} [\Psi A_\alpha] \\ 172: .10$$

$$.12 \vdash \exists x_\alpha \Phi_{022}[“\alpha”]_2 x_\alpha \quad \text{Exist (166): 2305}$$

$$.13 \vdash \exists x_\alpha \Phi_{022}[“\alpha”]_2 x_\alpha \supset .\Phi_{022}[“\alpha”]_2 [g_{22}[\iota x_\alpha^1]_2] \supset \exists u_\alpha \cdot g_{22}[\iota x_\alpha^1]_2 \\ \stackrel{2}{=} u_\alpha \quad \text{R: 123, 2320; Sub}_t, \text{Sub}$$

$$.14 [Assign]_{g_{22}} \vdash [Variable] [\iota x_\alpha^1]_2 [“\alpha”]_2 \supset \Phi_{022}[“\alpha”]_2 [g_{22}[\iota x_\alpha^1]_2] \\ \text{R': 103, premiss; Rule S (106); (Definition of [Assign])}$$

$$.15 \vdash [Assign]_{g_{22}} \supset .[Val]_{g_{22}}[“A_\alpha”]_2 \stackrel{2}{=} [\Psi A_\alpha] \\ \text{Deduction Theorem: .11, .14; Rule P: .02, .12, .13.}$$

CASE 2: A_α is $\iota_{1(01)}$.

Then $[\Psi A_\alpha]$ is $\iota_{1(01)}$ and “ A_α ” is 2^{17} .

$$.20 \vdash 2_2^{172} \stackrel{2}{=} [2^{17}]_2 \quad \text{The proof by recursive arithmetic is left to the reader.}$$

$$.21 \vdash [Assign]_{g_{22}} \supset .[Val]_{g_{22}}[2^{17}]_2 \stackrel{2}{=} \iota_{1(01)} \quad \text{R: .20, 2705.}$$

CASE 3: A_α is $Q_{0\beta\beta}$.

Then $[\Psi A_\alpha]$ is $Q_{0\beta\beta}$ and “ A_α ” is $2^{[19^{“\beta”}]}$.

$$.30 \vdash [TS] [“\beta”]_2 \quad 2005$$

$$.31 \vdash \exists u_\beta \Phi_{022}[“\beta”]_2 u_\beta \quad \text{Exist: 2305}$$

$$.32 \vdash [Assign]_{g_{22}} \supset .[[TS] [“\beta”]_2 \wedge \exists u_\beta \Phi_{022}[“\beta”]_2 u_\beta] \\ \supset .[Val]_{g_{22}}[2_2^{[19_2^{[“\beta”]_2}]}] \stackrel{2}{=} Q_{0\beta\beta} \quad \text{Sub}_t, \text{Sub: 2706}$$

$$.33 \vdash 2_2^{[19_2^{[“\beta”]_2}]} \stackrel{2}{=} [“A_\alpha”]_2 \quad \text{The proof is left to the reader.}$$

$$2710.34 \vdash [Assign]g_{22} \supset .[Val]g_{22}["A_\alpha"] \stackrel{2}{=} Q_{0\beta\beta}$$

Rule P: .30, .31, .32; R: .33.

CASE 4: A_α has the form $[B_{\alpha\beta}C_\beta]$.

Then $[\Psi A_\alpha]$ is $[[\Psi B_{\alpha\beta}][\Psi C_\beta]]$ and " A_α " is the number $2^7 * "B_{\alpha\beta}" * "C_\beta" * 2^{11}$. Also " $(\alpha\beta)$ " is $3^{''\alpha''} \times 5^{''\beta''}$.

$$.40 \vdash [2_2^{72} * ["B_{\alpha\beta}"]_2 * ["C_\beta"]_2 * 2_2^{112}] \stackrel{2}{=} ["A_\alpha"]_2.$$

The proof is left to the reader.

$$.41 \vdash [Wff] ["B_{\alpha\beta}"]_2 ["(\alpha\beta)"]_2 \quad 2207$$

$$.42 \vdash [3_2^{''\alpha''1_2} \times 5_2^{''\beta''1_2}] \stackrel{2}{=} ["(\alpha\beta)"]_2. \text{ The proof is left to the reader.}$$

$$.43 \vdash [Wff] ["B_{\alpha\beta}"]_2 [3_2^{''\alpha''1_2} \times 5_2^{''\beta''1_2}] \quad \text{RR: .41, .42}$$

$$.44 \vdash [Wff] ["C_\beta"]_2 ["\beta"]_2 \quad 2207$$

$$.45 \vdash [TS] ["\alpha"]_2 \quad 2005$$

$$.46 \vdash [Assign]g_{22} \supset .[Val]g_{22}["B_{\alpha\beta}"]_2 \stackrel{2}{=} [\Psi B_{\alpha\beta}]$$

by inductive hypothesis

$$.47 \vdash [Assign]g_{22} \supset .[Val]g_{22}["C_\beta"]_2 \stackrel{2}{=} [\Psi C_\beta]$$

by inductive hypothesis

$$.48 \vdash [[Assign]g_{22} \wedge [Wff] ["B_{\alpha\beta}"]_2 [3_2^{''\alpha''1_2} \times 5_2^{''\beta''1_2}]$$

$$\wedge [Wff] ["C_\beta"]_2 ["\beta"]_2 \wedge [TS] ["\alpha"]_2$$

$$\wedge [[Val]g_{22}["B_{\alpha\beta}"]_2 \stackrel{2}{=} [\Psi B_{\alpha\beta}]] \wedge .[Val]g_{22}["C_\beta"]_2 \stackrel{2}{=} [\Psi C_\beta]]$$

$$\supset .[Val]g_{22}[2_2^{72} * ["B_{\alpha\beta}"]_2 * ["C_\beta"]_2 * 2_2^{112}] \stackrel{2}{=} [[\Psi B_{\alpha\beta}][\Psi C_\beta]]$$

Sub_t, Sub: 2708

$$.49 \vdash [Assign]g_{22} \supset .[Val]g_{22}["A_\alpha"]_2 \stackrel{2}{=} [[\Psi B_{\alpha\beta}][\Psi C_\beta]]$$

Rule P: .43–.48; R: .40.

CASE 5: A_α has the form $[\lambda x_\beta^j B_\gamma]$.

Then " A_α " is the number $2^7 * 2^{13} * 2^{x_\beta^j} * "B_\gamma" * 2^{11}$.

Let C_γ be that formula (incidentally, not a wff of Q_0 or of Q , in general) obtained from B_γ by replacing each occurrence of each variable of B_γ

other than free occurrences of x_β^j by a wff of Q as though one were forming $[\Psi B_\gamma]$. Thus $[\Psi B_\gamma]$ is $\left[S_{[\gamma_\beta \cdot g_{22}[\cdot x_\beta^j \cdot]_2 \stackrel{2}{=} \gamma_\beta]}^{x_\beta^j} C_\gamma \right]$. Let k be the number ' x_β^j '. Then $[\Psi A_\alpha]$ is $\left[\lambda z_\beta \left[S_{z_\beta}^{x_\beta^j} C_\gamma \right] \right]$.

$$2710.50 \vdash [2_2^{7_2} * 2_2^{13_2} * 2_2^{[x_\beta^j]_2} * ["B_\gamma"]_2 * 2_2^{11_2}] \stackrel{2}{=} ["A_\alpha"]_2.$$

The proof is left to the reader.

$$.51 \vdash [Variable] [x_\beta^j]_2 ["\beta"]_2 \quad 2102$$

$$.52 \vdash [Wff] ["B_\gamma"]_2 ["\gamma"]_2 \quad 2207$$

$$.53 \vdash \exists m_\beta \Phi_{022} ["\beta"]_2 m_\beta \quad \text{Exist: } 2305$$

$$.54 \vdash \exists n_\gamma \Phi_{022} ["\gamma"]_2 n_\gamma \quad \text{Exist: } 2305$$

$$.55 \vdash [[Assign]g_{22} \wedge [Variable] [x_\beta^j]_2 ["\beta"]_2 \wedge [Wff] ["B_\gamma"]_2 ["\gamma"]_2 \\ \wedge \exists m_\beta \Phi_{022} ["\beta"]_2 m_\beta \wedge \exists n_\gamma \Phi_{022} ["\gamma"]_2 n_\gamma]$$

$$\supset . [Val]g_{22} [2_2^{7_2} * 2_2^{13_2} * 2_2^{[x_\beta^j]_2} * ["B_\gamma"]_2 * 2_2^{11_2}] \\ \stackrel{2}{=} [\lambda z_\beta . \lambda x_\gamma . [Val] [[Mod]g_{22} [x_\beta^j]_2 z_\beta] ["B_\gamma"]_2 \stackrel{2}{=} x_\gamma]$$

Sub₁, Sub: 2709

$$.56 \vdash [Assign]g_{22} \supset . [Val]g_{22} ["A_\alpha"]_2$$

$$\stackrel{2}{=} [\lambda z_\beta . \lambda x_\gamma . [Val] [[Mod]g_{22} [x_\beta^j]_2 z_\beta] ["B_\gamma"]_2 \stackrel{2}{=} x_\gamma]$$

Rule P: .51-.55; R: .50

$$.57 \vdash [Assign]g_{22} \supset . [Val]g_{22} ["B_\gamma"]_2 \stackrel{2}{=} [\Psi B_\gamma]$$

by inductive hypothesis

$$.58 \vdash [Assign] [[Mod]g_{22} [x_\beta^j]_2 z_\beta]$$

$$\supset . [Val] [[Mod]g_{22} [x_\beta^j]_2 z_\beta] ["B_\gamma"]_2 \stackrel{2}{=} [S_{[Mod]g_{22} [x_\beta^j]_2 z_\beta}^{g_{22}} [\Psi B_\gamma]]$$

Sub (119): .57 (Note that the restrictions on Rule Sub are satisfied, since no g-variable occurs bound in $[\Psi B_\gamma]$ and no z-variable occurs in $[\Psi B_\gamma]$.)

$$\begin{aligned}
 2710.59 \vdash & [[Variable] ['x_\beta^j']_2 ['\beta']_2 \wedge \Phi_{022}['\beta']_2 z_\beta] \supset .[Assign]g_{22} \\
 & \supset [Assign] [[Mod]g_{22}['x_\beta^j']_2 z_\beta] \quad \text{Sub: 2503}
 \end{aligned}$$

$$\begin{aligned}
 .60 [Assign]g_{22} \vdash & [Val] [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['B_\gamma']_2 \\
 & \stackrel{2}{=} \left[S_{[[Mod]g_{22}['x_\beta^j']_2 z_\beta]}^{g_{22}} [\Psi B_\gamma] \right] \quad \text{Rule P: .59, .51, 2305, Premiss, .58.}
 \end{aligned}$$

Let x_δ^j be any free variable of B_γ distinct from x_β^j . (If there are no such variables, lines .61–.63 of the proof may be disregarded.) Note that ' x_δ^j ' $\neq$ ' x_β^j '.

$$.61 \vdash \sim .['x_\delta^j']_2 \stackrel{2}{=} ['x_\beta^j']_2. \quad \text{The proof is left to the reader.}$$

$$\begin{aligned}
 .62 \vdash \sim & [[['x_\delta^j']_2 \stackrel{2}{=} ['x_\beta^j']_2] \supset .[[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['x_\delta^j']_2 \\
 & \stackrel{2}{=} g_{22}['x_\delta^j']_2 \quad \text{Sub: 2502}
 \end{aligned}$$

$$.63 \vdash [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['x_\delta^j']_2 \stackrel{2}{=} g_{22}['x_\delta^j']_2 \quad \text{MP: .61, .62}$$

$$\begin{aligned}
 .64 [Assign]g_{22} \vdash & [Val] [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['B_\gamma']_2 \\
 & \stackrel{2}{=} \left[S_{[\imath y_\beta . [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['x_\beta^j']_2 \stackrel{2}{=} y_\beta]}^{x_\beta^j} C_\gamma \right] \quad \text{R': .63, .60.}
 \end{aligned}$$

To see that .64 does indeed follow from .63 and .60 by applications of Rule R', we must consider the structure of the wff $\left[S_{[[Mod]g_{22}['x_\beta^j']_2 z_\beta]}^{g_{22}} [\Psi B_\gamma] \right]$ occurring in .60. Note first that g_{22} occurs in $[\Psi B_\gamma]$ only in wf parts of the form $[\imath y_\delta . g_{22}['x_\delta^j']_2 \stackrel{2}{=} y_\delta]$, where $x_\delta^j \neq x_\beta^j$, or of the form $[\imath y_\beta . g_{22}['x_\beta^j']_2 \stackrel{2}{=} y_\beta]$, and that these wf parts occur only where they have replaced free occurrences of variables x_δ^j or x_β^j , respectively, in forming $[\Psi B_\gamma]$ from B_γ . Hence $\left[S_{[[Mod]g_{22}['x_\beta^j']_2 z_\beta]}^{g_{22}} [\Psi B_\gamma] \right]$ contains in these places respectively wf parts of the form $[\imath y_\delta . [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['x_\delta^j']_2 \stackrel{2}{=} y_\delta]$ and $[\imath y_\beta . [[Mod]g_{22}['x_\beta^j']_2 z_\beta] ['x_\beta^j']_2 \stackrel{2}{=} y_\beta]$, since all occurrences of g_{22} in $[\Psi B_\gamma]$ are free.

Now by Rule R' and theorem schema .63 we may replace each

wf part of $\left[S_{[Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}}^{\sigma_{22}} [\Psi B_{\gamma}] \right]$ of the form
 $[\iota y_{\delta} \cdot [[Mod] g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}] [\cdot x_{\delta}^j]_2 \stackrel{2}{=} y_{\delta}]$, where $x_{\delta}^j \neq x_{\beta}^j$, by
 $[\iota y_{\delta} \cdot g_{22}[\cdot x_{\delta}^j]_2 \stackrel{2}{=} y_{\delta}]$. The result is clearly
 $\left[S_{[\iota y_{\beta} \cdot [Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}][\cdot x_{\beta}^j]_2 \stackrel{2}{=} y_{\beta}}^{\sigma_{22}} C_{\gamma} \right]$.

$$2710.65 \vdash [[Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}] [\cdot x_{\beta}^j]_2 \stackrel{2}{=} z_{\beta} \quad \text{Sub: 2501}$$

$$.66 \vdash [\iota y_{\beta} \cdot z_{\beta} \stackrel{2}{=} y_{\beta}] \stackrel{\beta}{=} z_{\beta} \quad 210$$

$$.67 \vdash [\iota y_{\beta} \cdot [[Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}] [\cdot x_{\beta}^j]_2 \stackrel{2}{=} y_{\beta}] \stackrel{\beta}{=} z_{\beta} \quad \text{RR: .65, .66}$$

$$.68 [Assign]g_{22} \vdash [Val] [[Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}] [\cdot B_{\gamma}]_2 \stackrel{2}{=} \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \quad \text{R': .64, .67}$$

$$.69 \vdash [\iota x_{\gamma} \cdot \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \stackrel{2}{=} x_{\gamma}] \stackrel{\gamma}{=} \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \quad 210$$

(Note that $\left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right]$ is a wff of Q of type γ in which no x-variable occurs free.)

$$.70 [Assign]g_{22} \vdash [\iota x_{\gamma} \cdot [Val] [[Mod]g_{22}[\cdot x_{\beta}^j]_2 z_{\beta}] [\cdot B_{\gamma}]_2 \stackrel{2}{=} x_{\gamma}] \stackrel{\gamma}{=} \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \quad 200: .68, .69$$

$$.71 [Assign]g_{22} \vdash [Val]g_{22}[\cdot A_{\alpha}]_2 \stackrel{2}{=} \left[\lambda z_{\beta} \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \right] \quad \text{MP: Premiss, .56; R': .70}$$

$$.72 \vdash \left[\lambda z_{\beta} \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \right] \stackrel{\gamma\beta}{=} \left[\lambda z_{\beta}^k \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \right]$$

122 (Note that the restrictions on 122 are satisfied.)

$$.73 \vdash [Assign]g_{22} \supset .[Val]g_{22}[\cdot A_{\alpha}]_2 \stackrel{2}{=} \left[\lambda z_{\beta}^k \left[S_{z_{\beta}}^{\sigma_{22}} C_{\gamma} \right] \right]$$

Deduction Theorem: .71; Rule R: .72.

This completes Case 5, and hence the proof of 2710.

As mentioned earlier, the following theorem, which is simply a corollary of 2710, shows that the wff $[True]_{02}$ satisfies Tarski's criterion for an adequate definition of truth¹²).

2711 Let A_0 be any wff of Q_0 of type 0 with no free variables. Then

$$\vdash [True]_{02}["A_0"]_2 \stackrel{0}{=} [\Psi A_0].$$

Note that $[\Psi A_0]$ is a wff of Q of type 0 with no free variables. Indeed, $[\Psi A_0]$ is simply the result of translating A_0 into Q by replacing the variables in A_0 by wff-variables of Q in an appropriate manner.

Proof of 2711:

Let $\mathcal{H}^1$ be $[Assign]_{g_{22}}, [True] ["A_0"]_2$.

$$.01 \mathcal{H}^1 \vdash [Val]_{g_{22}}["A_0"]_2 \stackrel{2}{=} [\Psi A_0] \quad \text{MP: Premiss, 2710}$$

$$.02 \mathcal{H}^1 \vdash \forall_{g_{22}} . [Assign]_{g_{22}} \supset . [Val]_{g_{22}}["A_0"]_2 \stackrel{2}{=} T_0$$

$$.03 \mathcal{H}^1 \vdash [Val]_{g_{22}}["A_0"]_2 \stackrel{2}{=} T_0 \quad \text{RR: 103, premiss, (definition of } [True]) \text{ Rule S: .02; MP: Premiss}$$

$$.04 \mathcal{H}^1 \vdash [\Psi A_0] \stackrel{0}{=} T_0 \quad \text{EE-Rules (205): .01, .03}$$

$$.05 \mathcal{H}^1 \vdash [\Psi A_0] \quad \text{Rule T (116): .04}$$

$$.06 \vdash [Assign]_{g_{22}} \supset . [True] ["A_0"]_2 \supset [\Psi A_0] \quad \text{Deduction Theorem: .05}$$

$$.07 \vdash \exists_{g_{22}} [Assign]_{g_{22}} \supset . [True] ["A_0"]_2 \supset [\Psi A_0] \quad 170: .06$$

(Here we have used the fact that A_0 contains no free variables, and hence that no g -variable is free in $[\Psi A_0]$.)

$$.08 \vdash [True] ["A_0"]_2 \supset [\Psi A_0] \quad \text{MP: .07, 2410.}$$

Let $\mathcal{H}^2$ be $[\Psi A_0], [Assign]_{g_{22}}$.

$$.09 \mathcal{H}^2 \vdash [\Psi A_0] \stackrel{0}{=} T_0 \quad \text{Rule T: Premiss}$$

¹²) See [17], pp.187–188.

- 2711.10 $\mathcal{H}^2 \vdash [Val]_{g_{22}}[“A_0”]_2 \stackrel{2}{=} [\Psi A_0]$ MP: Premiss, 2710
- .11 $\mathcal{H}^2 \vdash [Val]_{g_{22}}[“A_0”]_2 \stackrel{2}{=} T_0$ R': .09, .10
- .12 $[\Psi A_0] \vdash \forall g_{22} . [Assign]_{g_{22}} \supset . [Val]_{g_{22}}[“A_0”]_2 \stackrel{2}{=} T_0$
Deduction Theorem, Gen: .11
- .13 $[\Psi A_0] \vdash [True] [“A_0”]_2$ RR: 103, .12, (definition of $[True]$)
- .14 $\vdash [True] [“A_0”]_2 \stackrel{0}{=} [\Psi A_0]$ Deduction Theorem: .13;
Rule P: .08.

IV.3. Proofs

We must now indicate how the theorems stated in the preceding section may be proved. Since Theorems 2000–2209 involve nothing which is special to transfinite type theory, and follow rather directly from the definitions of the wffs involved in them, we shall leave the proofs of these theorems to the reader. It may be helpful to note that since most of the definitions introduced in section 1 are of the same general character, the proofs of certain basic theorems involving the wffs defined in section 1 will be similar in character. Thus, for example, the proofs of 2000, 2100, 2200, 2300, etc., are similar; the proofs of 2002, 2003, 2101, 2202, 2203, 2204, 2301, and 2302 are similar; the proofs of 2004, 2205, 2206, and 2303 are similar; and the proofs of 2005, 2102, 2207, and 2305 are similar.

Proof of 2300:

Let a and b be type variables, and $r, s, t, u, v, w, x, y, z$ be primitive variables satisfying the hypotheses of Theorem 2300. Also let p, m , and n be primitive variables which are distinct from one another, from the primitive variables mentioned above, and from all primitive variables occurring in P_{022} or in Φ_{022} .

- 2300.1 $\Phi_{022} t_2 w_2 \vdash [\lambda m_2 \lambda n_2 . \forall p_{022} . [\forall x_0 p_{022} \beta_2 x_0 \wedge \forall y_1 p_{022} \delta_2 y_1$
 $\wedge \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a p_{022} r_2 u_a \wedge \exists v_b p_{022} s_2 v_b]$
 $\supset \forall z_{ab} p_{022} [\beta^{r_2} \times \delta^{s_2}] z_{ab}] \supset p_{022} m_2 n_2] t_2 w_2$
 RR (102): Premiss, 122, 123, 157; def. of Φ_{022}

$$\begin{aligned}
2300.2 \quad & \Phi_{022} t_2 w_2 \vdash [\forall x_0 P_{022} \beta_2 x_0 \wedge \forall y_1 P_{022} \delta_2 y_1 \\
& \wedge \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b] \\
& \supset \forall z_{ab} P_{022} [\beta^{r_2} \times \delta^{s_2}] z_{ab}] \supset P_{022} t_2 w_2
\end{aligned}$$

RR: 103, .1; Rule S (106).

Theorem 2300 then follows by applying the Deduction Theorem to 2300.2, and using Rule P with the hypotheses of Theorem 2300.

Proof of 2301:

$$\begin{aligned}
2301.1 \quad & \vdash \forall z_0 p_{022} \beta_2 z_0 \supset p_{022} \beta_2 q_0 & 145 \\
.2 \quad & \vdash \forall p_{022} . [\forall z_0 p_{022} \beta_2 z_0 \wedge \forall z_1 p_{022} \delta_2 z_1 \\
& \wedge \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a p_{022} r_2 u_a \wedge \exists v_b p_{022} s_2 v_b] \\
& \supset \forall z_{ab} p_{022} [\beta^{r_2} \times \delta^{s_2}] z_{ab}] \supset p_{022} \beta_2 q_0
\end{aligned}$$

Rule P, Gen: .1

$$.3 \vdash \Phi_{022} \beta_2 q_0 \quad \text{RR: 103, .2; def. of } \Phi_{022}.$$

The proof of 2302 is similar to the proof of 2301.

Proof of 2303:

For the sake of brevity let $\mathcal{H}$ be $[\Phi_{022} r_2 u_c \wedge \Phi_{022} s_2 v_d]$ and let B_0 be that wff such that Φ_{022} is $[\lambda t_2 \lambda w_2 . \forall p_{022} . B_0 \supset p_{022} t_2 w_2]$. Note that p_{022} is the only variable free in B_0 .

$$\begin{aligned}
2303.1 \quad & \mathcal{H} \vdash \Phi_{022} r_2 u_c & \text{Rule P: Premiss} \\
.2 \quad & \mathcal{H} \vdash \Phi_{022} s_2 v_d & \text{Rule P: Premiss} \\
.3 \quad & \mathcal{H} \vdash B_0 \supset p_{022} r_2 u_c & \text{RR: 103, .1; Rule S} \\
.4 \quad & \mathcal{H} \vdash B_0 \supset p_{022} s_2 v_d & \text{RR: 103, .2; Rule S} \\
.5 \quad & \mathcal{H} \vdash B_0 \supset \exists u_c p_{022} r_2 u_c & 174: .3 \\
.6 \quad & \mathcal{H} \vdash B_0 \supset \exists v_d p_{022} s_2 v_d & 174: .4 \\
.7 \quad & \vdash B_0 \supset .[\exists u_c p_{022} r_2 u_c \wedge \exists v_d p_{022} s_2 v_d] \supset \forall z_{cd} p_{022} [\beta^{r_2} \times \delta^{s_2}] z_{cd}
\end{aligned}$$

Rule P: Axiom 7, 146; definition of B_0

2303.8 $\vdash \forall z_{cd} p_{022}[3^{r_2} \times 5^{s_2}]z_{cd} \supset p_{022}[3^{r_2} \times 5^{s_2}]w_{cd}$ 145

.9 $\mathcal{H} \vdash B_0 \supset p_{022}[3^{r_2} \times 5^{s_2}]w_{cd}$ Rule P: .5-.8

.10 $\mathcal{H} \vdash \Phi_{022}[3^{r_2} \times 5^{s_2}]w_{cd}$ Gen: .9; RR: 103.

Then use the Deduction Theorem.

2304 follows from 2303 by 170 and Rule P.

Proof of 2305:

The proof is by induction on the construction of α .

CASE 1: α is 0. Then " α " is 3.

2305.1 $\vdash \Phi_{022}3_2x_0$ Sub: 2301.

CASE 2: α is 1. Then "1" is 5.

.2 $\vdash \Phi_{022}5_2x_1$ Sub: 2302.

CASE 3: α has the form $(\beta\gamma)$. Then " α " is $3^{l^{\beta}} \times 5^{l^{\gamma}}$.

.3 $\vdash [3^{l^{\beta}}]_2 \times [5^{l^{\gamma}}]_2 \stackrel{2}{=} [^{\alpha}]_2$. The proof by recursive arithmetic is left to the reader.

.4 $\vdash \Phi_{022}[^{\beta}]_2u_{\beta}$ by inductive hypothesis

.5 $\vdash \Phi_{022}[^{\gamma}]_2v_{\gamma}$ by inductive hypothesis

.6 $\vdash [\Phi_{022}[^{\beta}]_2u_{\beta} \wedge \Phi_{022}[^{\gamma}]_2v_{\gamma}] \supset \Phi_{022}[3^{l^{\beta}}]_2 \times [5^{l^{\gamma}}]_2w_{\beta\gamma}$
Sub (119), Sub_t (140): 2303

.7 $\vdash \Phi_{022}[^{\alpha}]_2w_{\alpha}$ Rule P: .4, .5, .6; R: .3

.8 $\vdash \Phi_{022}[^{\alpha}]_2x_{\alpha}$ Sub: .7.

This completes the proof of 2305.

The method of proof to be used for theorem 2306 is similar to that used in 2305. We leave the simple details to the reader.

Proof of 2307:

Let P_{022} be $[\lambda t_2 \lambda w_2 . [TS]t_2 \wedge \exists c \exists x_c . w_2 \stackrel{2}{=} x_c]$.

2307.1 $\vdash z_0 \stackrel{2}{=} z_0$ 100

.2 $\vdash \exists x_0 . z_0 \stackrel{2}{=} x_0$ Exist (166): .1

.3 $\vdash \exists c \exists x_c . z_0 \stackrel{2}{=} x_c$ Exist_t (167): .2

.4 $\vdash [TS]_{02} 3 \wedge \exists c \exists x_c . z_0 \stackrel{2}{=} x_c$ Rule P: .3, 2002

.5 $\vdash \forall z_0 . P_{022} 3 z_0$ RR: 103, .4; Gen

.6 $\vdash \forall z_1 . P_{022} 5 z_1$. The proof is similar to the proof if .5, with 2003 used in place of 2002.

.7 $\vdash \exists c \exists x_c . z_{ab} \stackrel{2}{=} x_c$. The proof is similar to the proof of .3.

.8 $\vdash [TS]_{r_2} \wedge [TS]_{s_2} \supset [TS] [3^{r_2} \times 5^{s_2}]$ Sub: 2004

.9 $\vdash [TS]_{r_2} \wedge [TS]_{s_2} \supset P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$
Rule P: .7, .8; RR: 103

.10 $\vdash ([TS]_{r_2} \wedge \exists c \exists x_c [u_a \stackrel{2}{=} x_c]) \wedge ([TS]_{s_2} \wedge \exists c \exists x_c [v_b \stackrel{2}{=} x_c])$
 $\supset \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$ 143, Rule P: .9

.11 $\vdash P_{022} r_2 u_a \wedge P_{022} s_2 v_b \supset \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$
RR: 103, .10

.12 $\vdash \exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b \supset \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$
Rule P, 170: .11

.13 $\vdash \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b]$
 $\supset \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$ Gen_t, Gen: .12

.14 $\vdash \Phi_{022} t_2 w_2 \supset P_{022} t_2 w_2$ 2300: .5, .6, .13.

Theorem 2307 then follows by Rule R and 103.

Proof of 2308:

Let P_0 be $\exists c \exists x_c \Phi_{022} t_2 x_c$. Note that t_2 is the only variable free in P_0 .

2308.1 $\vdash \exists c \exists x_c \Phi_{022} 3 x_c$ Exist (166), Exist_t (167): 2301

.2 $\vdash \exists c \exists x_c \Phi_{022} 5 x_c$ Exist, Exist_t: 2302

$$2308.3 \exists u_c \Phi_{022} r_2 u_c, \exists v_d \Phi_{022} s_2 v_d \vdash \exists c \exists x_c \Phi_{022} [3^{r_2} \times 5^{s_2}] x_c$$

Rule P: Premiss, 2304; Exist, Exist,

$$.4 \exists c \exists u_c \Phi_{022} r_2 u_c, \exists d \exists v_d \Phi_{022} s_2 v_d \vdash \exists c \exists x_c \Phi_{022} [3^{r_2} \times 5^{s_2}] x_c \quad 173, 163: .3$$

$$.5 \vdash \forall r_2 \forall s_2 . [\exists c \exists x_c \Phi_{022} r_2 x_c \wedge \exists c \exists x_c \Phi_{022} s_2 x_c] \\ \supset \exists c \exists x_c \Phi_{022} [3^{r_2} \times 5^{s_2}] x_c$$

Deduction Theorem: .4; Rule R: 123, 157; Gen, Rule P

$$.6 \vdash [TS] t_2 \supset \exists c \exists x_c \Phi_{022} t_2 x_c \quad 2000: .1, .2, .5.$$

Proof of 2309:

Let P_{022} be

$$[\lambda t_2 \lambda w_2 . \Phi_{022} t_2 w_2 \wedge \forall k_2 \forall j_2 . [t_2 \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS] k_2 \\ \wedge [TS] j_2] \supset \forall c \forall d \forall q_{cd} . w_2 \stackrel{2}{=} q_{cd} \supset . \exists m_c \Phi_{022} k_2 m_c \\ \wedge \exists n_d \Phi_{022} j_2 n_d].$$

$$2309.1 \vdash \sim .3 \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS] k_2 \wedge [TS] j_2.$$

The proof is left to the reader. Use 2001, 2006, and some number theory.

$$.2 \vdash \Phi_{022} 3x_0 \quad \text{Sub: 2301}$$

$$.3 \vdash \Phi_{022} 3x_0 \wedge \forall k_2 \forall j_2 . [3 \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS] k_2 \\ \wedge [TS] j_2] \supset \forall c \forall d \forall q_{cd} . x_0 \stackrel{2}{=} q_{cd} \supset . \exists m_c \Phi_{022} k_2 m_c \\ \wedge \exists n_d \Phi_{022} j_2 n_d \quad \text{Rule P, Gen: .1; Rule P: .2}$$

$$.4 \vdash \forall x_0 P_{022} 3x_0 \quad \text{RR: 103, .3; Gen}$$

$$.5 \vdash \forall x_1 P_{022} 5x_1 \quad \text{The proof is similar to the proof of .4.}$$

$$\text{Let } \mathcal{H}^1 \text{ be } P_{022} r_2 u_a, P_{022} s_2 v_b; \text{ let } \mathcal{H}^2 \text{ be } [3^{r_2} \times 5^{s_2}] \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \\ \wedge [TS] k_2 \wedge [TS] j_2, z_{ab} \stackrel{2}{=} q_{cd}.$$

- 2309.6 $\mathcal{H}^1 \vdash \Phi_{022} r_2 u_a$ RR: 103, Premiss; Rule P
- .7 $\mathcal{H}^1 \vdash \Phi_{022} s_2 v_b$ RR: 103, Premiss; Rule P
- .8 $\mathcal{H}^1 \vdash \Phi_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$ Sub, Sub_t: 2303; Rule P: .6, .7
- .9 $\mathcal{H}^1 \vdash [TS] r_2$ Sub: 2307; Rule P: .6
- .10 $\mathcal{H}^1 \vdash [TS] s_2$ Sub: 2307; Rule P: .7
- .11 $\mathcal{H}^1, \mathcal{H}^2 \vdash r_2 \stackrel{2}{=} k_2$ The proofs of .11 and .12 are left
- .12 $\mathcal{H}^1, \mathcal{H}^2 \vdash s_2 \stackrel{2}{=} j_2$ to the reader. Use .9, .10, $\mathcal{H}^2$, 2001,
and some number theory.
- .13 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022} k_2 u_a$ R', 163: .6, .11
- .14 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022} j_2 v_b$ R', 163: .7, .12
- .15 $\vdash [z_{ab} \stackrel{2}{=} q_{cd}] \supset [(ab) = (cd)]$ Sub, Sub_t: Axiom 10
- .16 $\mathcal{H}^1, \mathcal{H}^2 \vdash a = c$ Rule P: Premiss, .15, Axiom 9
- .17 $\mathcal{H}^1, \mathcal{H}^2 \vdash b = d$ Rule P: Premiss, .15, Axiom 9
- .18 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022} k_2 u_c$ 300: .16, .13
- .19 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022} j_2 v_d$ 300: .17, .14
- .20 $\mathcal{H}^1, \mathcal{H}^2 \vdash \exists m_c \Phi_{022} k_2 m_c \wedge \exists n_d \Phi_{022} j_2 n_d$
Exist (166), Rule P: .18, .19
- .21 $\mathcal{H}^1 \vdash \forall k_2 \forall j_2 . [[3^{r_2} \times 5^{s_2}] \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS] k_2 \wedge [TS] j_2]$
 $\supset \forall c \forall d \forall q_{cd} . z_{ab} \stackrel{2}{=} q_{cd} \supset . \exists m_c \Phi_{022} k_2 m_c \wedge \exists n_d \Phi_{022} j_2 n_d$
Deduction Theorem, Gen (117), Gen_t (137): .20
- .22 $\mathcal{H}^1 \vdash P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$ Rule P: .8, .21; RR: 103
- .23 $\exists u_a P_{022} r_2 u_a, \exists v_b P_{022} s_2 v_b \vdash \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$
Gen: .22; 172, 163

2309.24 $\vdash \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b]$

$$\supset \forall z_{ab} P_{022} [3^{r_2} \times 5^{s_2}] z_{ab}$$

Deduction Theorem, Rule P, Gen_i, Gen: .23

$$.25 \vdash \Phi_{022} t_2 w_2 \supset P_{022} t_2 w_2 \quad 2300: .4, .5, .24$$

$$.26 \vdash \Phi_{022} [3^{y_2} \times 5^{z_2}] x_{ab} \supset . \Phi_{022} [3^{y_2} \times 5^{z_2}] x_{ab} \wedge \forall k_2 \forall j_2 . [[3^{y_2} \times 5^{z_2}] \\ \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS] k_2 \wedge [TS] j_2] \supset \forall c \forall d \forall q_{cd} . x_{ab} \stackrel{2}{=} q_{cd}$$

$$\supset . \exists m_c \Phi_{022} k_2 m_c \wedge \exists n_d \Phi_{022} j_2 n_d \quad \text{Sub: .25; R: 103}$$

$$.27 \vdash \Phi_{022} [3^{y_2} \times 5^{z_2}] x_{ab} \supset . [[3^{y_2} \times 5^{z_2}] \stackrel{2}{=} [3^{y_2} \times 5^{z_2}] \wedge [TS] y_2 \\ \wedge [TS] z_2] \supset . x_{ab} \stackrel{2}{=} x_{ab} \supset . \exists m_a \Phi_{022} y_2 m_a \wedge \exists n_b \Phi_{022} z_2 n_b$$

Rule P: .26, 145, Axiom 7.

Theorem 2309 then follows from .27 and 100 by Rule P.

Proof of 2310:

Let P_{022} be

$$[\lambda t_2 \lambda w_2 . \forall m_2 \forall n_2 . [[TS] m_2 \wedge [TS] n_2 \wedge t_2 \stackrel{2}{=} [3^{m_2} \times 5^{n_2}]]$$

$$\supset \exists a \exists b \exists z_{ab} . w_2 \stackrel{2}{=} z_{ab}].$$

$$2310.1 \vdash \sim . 3 \stackrel{2}{=} [3^{m_2} \times 5^{n_2}] \wedge [TS] m_2 \wedge [TS] n_2 \quad \text{Sub: 2309.1}$$

$$.2 \vdash \forall x_0 P_{022} 3 x_0 \quad \text{Rule P, Gen, Rule RR: .1, 103}$$

$$.3 \vdash \forall x_1 P_{022} 5 x_1 \quad \text{Proof similar to the proof of .2}$$

$$.4 \vdash \exists a \exists b \exists z_{ab} . x_{cd} \stackrel{2}{=} z_{ab} \quad \text{Exist (166), Exist_i (167): 100}$$

$$.5 \vdash \forall x_{cd} P_{022} [3^{r_2} \times 5^{s_2}] x_{cd} \quad \text{Rule P, Gen, RR: .4, 103}$$

$$.6 \vdash \forall r_2 \forall s_2 \forall c \forall d . [\exists u_c P_{022} r_2 u_c \wedge \exists v_d P_{022} s_2 v_d]$$

$$\supset \forall x_{cd} P_{022} [3^{r_2} \times 5^{s_2}] x_{cd} \quad \text{Rule P, Gen: .5}$$

$$.7 \vdash \Phi_{022} t_2 w_2 \supset P_{022} t_2 w_2 \quad 2300: .2, .3, 6$$

- 2310.8 $\vdash \Phi_{022}[3^{y_2} \times 5^{z_2}]_{w_2} \supset \forall m_2 \forall n_2 . [[TS]m_2 \wedge [TS]n_2$
 $\wedge [3^{y_2} \times 5^{z_2}] \stackrel{2}{=} [3^{m_2} \times 5^{n_2}]] \supset \exists a \exists b \exists z_{ab} . w_2 \stackrel{2}{=} z_{ab}$
 Sub: .7; R: 103
- .9 $\vdash \Phi_{022}[3^{y_2} \times 5^{z_2}]_{w_2} \supset . [[TS]y_2 \wedge [TS]z_2 \wedge [3^{y_2} \times 5^{z_2}]$
 $\stackrel{2}{=} [3^{y_2} \times 5^{z_2}]] \supset \exists a \exists b \exists z_{ab} . w_2 \stackrel{2}{=} z_{ab}$ Rule P: .8, 145.
- Theorem 2310 then follows from .9 and 100 by Rule P.

Proof of 2311:

- Let P_{022} be $[\lambda t_2 \lambda w_2 . [TS]t_2 \wedge . t_2 = 3 \supset \exists z_0 . w_2 \stackrel{2}{=} z_0]$.
- 2311.1 $\vdash \exists z_0 . x_0 \stackrel{2}{=} z_0$ Exist (166): 100
- .2 $\vdash [TS]3 \wedge . 3 = 3 \supset \exists z_0 . x_0 \stackrel{2}{=} z_0$ Rule P: .1, 2002
- .3 $\vdash \forall x_0 P_{022} 3 x_0$ RR: 103, .2; Gen
- .4 $\vdash \sim . 5 = 3$. Proof left to the reader.
 Use 1000, 1001, 1007, 1008.
- .5 $\vdash [TS]5 \wedge . 5 = 3 \supset \exists z_0 . x_1 \stackrel{2}{=} z_0$ Rule P: .4, 2003
- .6 $\vdash \forall x_1 P_{022} 5 x_1$ RR: 103, .5; Gen
- .7 $\vdash [TS]r_2 \wedge [TS]s_2 \supset [TS][3^{r_2} \times 5^{s_2}]$ Sub: 2004
- .8 $\vdash [TS]r_2 \wedge [TS]s_2 \supset \sim . [3^{r_2} \times 5^{s_2}] = 3$
 Sub: 2309.1; R: 202; Rule P
- .9 $\vdash [TS]r_2 \wedge [TS]s_2 \supset P_{022}[3^{r_2} \times 5^{s_2}]x_{ab}$ Rule P: .7 .8; RR: 103
- .10 $\vdash P_{022}r_2 u_a \wedge P_{022}s_2 v_b \supset P_{022}[3^{r_2} \times 5^{s_2}]x_{ab}$ R, Rule P: 103, .9
- .11 $\vdash [\exists u_a P_{022}r_2 u_a \wedge \exists v_b P_{022}s_2 v_b] \supset \forall x_{ab} P_{022}[3^{r_2} \times 5^{s_2}]x_{ab}$
 143, Rule P, 170: .10
- .12 $\vdash \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a P_{022}r_2 u_a \wedge \exists v_b P_{022}s_2 v_b]$
 $\supset \forall x_{ab} P_{022}[3^{r_2} \times 5^{s_2}]x_{ab}$ Gen, Gen_i: .11
- .13 $\vdash \Phi_{022}t_2 w_2 \supset P_{022}t_2 w_2$ 2300: .3, .6, .12

- 2311.14 $\vdash \Phi_{022}3q_a \supset .[TS]3 \wedge .3 = 3 \supset \exists z_0 . q_a \stackrel{2}{=} z_0$ Sub: .13; R:103
 .15 $\vdash q_a \stackrel{2}{=} z_0 \supset a = 0$ Sub, Sub_t: Axiom 10
 .16 $\vdash \exists z_0 [q_a \stackrel{2}{=} z_0] \supset a = 0$ 170: .15
 .17 $\vdash \Phi_{022}3q_a \supset a = 0$ Rule P: .14 .16, 100.

The proof of 2312 closely parallels the proof of 2311.

Proof of 2313:

- 2313.1 $\vdash \Phi_{022}3u_a \supset a = 0$ Sub: 2311
 .2 $\vdash \Phi_{022}3v_b \supset b = 0$ Sub, Sub_t: 2311
 .3 $\vdash a = 0 \supset .[b = 0] \stackrel{0}{=} [a = b]$ Sub_t: 304
 .4 $\vdash [TS]3 \wedge .[\Phi_{022}3u_a \wedge \Phi_{022}3v_b] \supset a = b$ Rule P: .1, .2, .3, 2002
 .5 $\vdash \forall a \forall b \forall u_a \forall v_b . [TS]3 \wedge .[\Phi_{022}3u_a \wedge \Phi_{022}3v_b] \supset a = b$ Gen, Gen_t: .4
 .6 $\vdash \forall a \forall b \forall u_a \forall v_b . [TS]5 \wedge .[\Phi_{022}5u_a \wedge \Phi_{022}5v_b] \supset a = b.$

The proof of .6 is similar to the proof of .5, with 2312 used in place of 2311 and 2003 used in place of 2002.

Let $\mathcal{H}^1$ be

- $\forall a \forall b \forall u_a \forall v_b . [[TS]y_2 \wedge .[\Phi_{022}y_2u_a \wedge \Phi_{022}y_2v_b] \supset a = b],$
 $\forall a \forall b \forall u_a \forall v_b . [[TS]z_2 \wedge .[\Phi_{022}z_2u_a \wedge \Phi_{022}z_2v_b] \supset a = b];$ let $\mathcal{H}^2$ be
 $\Phi_{022}[3^{y_2} \times 5^{z_2}]u_a \wedge \Phi_{022}[3^{y_2} \times 5^{z_2}]v_b;$ let $\mathcal{H}^3$ be $u_a \stackrel{2}{=} s_{(cd)}, v_b \stackrel{2}{=} t(\bar{c}\bar{d}).$

- .7 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash \Phi_{022} [3^{y_2} \times 5^{z_2}]s_{(cd)} \wedge \Phi_{022} [3^{y_2} \times 5^{z_2}]t(\bar{c}\bar{d})$
 200: Premiss

- .8 $\vdash [[TS]y_2 \wedge [TS]z_2 \wedge \Phi_{022}[3^{y_2} \times 5^{z_2}]s_{(cd)}] \supset .\exists m_c \Phi_{022}y_2m_c$
 $\wedge \exists n_d \Phi_{022}z_2n_d$ Sub, Sub_t: 2309

- 2313.9 $\vdash [[TS]y_2 \wedge [TS]z_2 \wedge \Phi_{022}[3^{y_2} \times 5^{z_2}]t(\bar{c}\bar{d})] \supset \exists m_{\bar{c}}\Phi_{022}y_2m_{\bar{c}}$
 $\wedge \exists n_{\bar{d}}\Phi_{022}z_2n_{\bar{d}}$ Sub; Sub_t: 2309
- .10 $\mathcal{H}^1 \vdash [TS]y_2 \wedge .[\Phi_{022}y_2m_c \wedge \Phi_{022}y_2\bar{m}_{\bar{c}}] \supset c = \bar{c}$
 Rule S_t (138), Rule S (106): Premiss
- .11 $\mathcal{H}^1 \vdash [TS]z_2 \wedge .[\Phi_{022}z_2n_d \wedge \Phi_{022}z_2\bar{n}_{\bar{d}}] \supset d = \bar{d}$
 Rule S_t, Rule S: Premiss
- .12 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\exists m_c\Phi_{022}y_2m_c \wedge \exists \bar{m}_{\bar{c}}\Phi_{022}y_2\bar{m}_{\bar{c}}] \supset c = \bar{c}$
 Rule P, 170: .10; 163
- .13 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\exists m_c\Phi_{022}y_2m_c \wedge \exists m_{\bar{c}}\Phi_{022}y_2m_{\bar{c}}] \supset c = \bar{c}$
 RR: 123, .12
- .14 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\exists n_d\Phi_{022}z_2n_d \wedge \exists n_{\bar{d}}\Phi_{022}z_2n_{\bar{d}}] \supset d = \bar{d}$
 Rule P: 170, .11; RR: 123; 163
- .15 $\vdash [(cd) = (\bar{c}\bar{d})] \stackrel{0}{=} [c = \bar{c} \wedge d = \bar{d}]$ Sub_t: Axiom 9
- .16 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash (cd) = (\bar{c}\bar{d})$ Rule P: .7-.11, .13-.15
- .17 $\vdash [u_a \stackrel{2}{=} s_{(cd)}] \supset .a = (cd)$ Sub, Sub_t: Axiom 10
- .18 $\vdash [v_b \stackrel{2}{=} t_{(\bar{c}\bar{d})}] \supset .b = (\bar{c}\bar{d})$ Sub, Sub_t: Axiom 10
- .19 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash a = (cd)$ MP: Premiss, .17
- .20 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash b = (\bar{c}\bar{d})$ MP: Premiss, .18
- .21 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash a = b$ 300: .16, .19, .20
- .22 $\mathcal{H}^1, \mathcal{H}^2, \exists c\exists d\exists s_{(cd)}[u_a \stackrel{2}{=} s_{(cd)}],$
 $\exists \bar{c}\exists \bar{d}\exists t_{(\bar{c}\bar{d})}[v_b \stackrel{2}{=} t_{(\bar{c}\bar{d})}] \vdash a = b$ 172, 173, 163: .21
- .23 $\vdash [[TS]y_2 \wedge [TS]z_2 \wedge \Phi_{022}[3^{y_2} \times 5^{z_2}]u_a] \supset \exists c\exists d\exists s_{(cd)}.u_a \stackrel{2}{=} s_{(cd)}$
 R: 2310, 123, 157; Sub
- .24 $\vdash [[TS]y_2 \wedge [TS]z_2 \wedge \Phi_{022}[3^{y_2} \times 5^{z_2}]v_b]$
 $\supset \exists \bar{c}\exists \bar{d}\exists t_{(\bar{c}\bar{d})}.v_b \stackrel{2}{=} t_{(\bar{c}\bar{d})}$ R: 2310, 123, 157; Sub

2313.25 $\mathcal{H}^1, \mathcal{H}^2 \vdash a = b$

Deduction Theorem: .22; Rule P: .23, .24, .10, .11, Premiss

.26 $\mathcal{H}^1 \vdash [TS] [3^{y_2} \times 5^{z_2}]$ Rule P: .10, .11, 2004

.27 $\mathcal{H}^1 \vdash \forall a \forall b \forall u_a \forall v_b . [TS] [3^{y_2} \times 5^{z_2}] \wedge . [\Phi_{022} [3^{y_2} \times 5^{z_2}] u_a$

$\wedge \Phi_{022} [3^{y_2} \times 5^{z_2}] v_b] \supset a = b$

Deduction Theorem: .25; Rule P: .26; Gen, Gen_t

.28 $\vdash \forall y_2 \forall z_2 . [\forall a \forall b \forall u_a \forall v_b . [TS] y_2 \wedge . [\Phi_{022} y_2 u_a \wedge \Phi_{022} y_2 v_b]$

$\supset a = b] \wedge \forall a \forall b \forall u_a \forall v_b . [TS] z_2 \wedge . [\Phi_{022} z_2 u_a \wedge \Phi_{022} z_2 v_b]$

$\supset a = b]] \supset \forall a \forall b \forall u_a \forall v_b . [TS] [3^{y_2} \times 5^{z_2}] \wedge . [\Phi_{022} [3^{y_2} \times 5^{z_2}] u_a$

$\wedge \Phi_{022} [3^{y_2} \times 5^{z_2}] v_b] \supset a = b$

Deduction Theorem, Rule P, Gen: .27

.29 $\vdash [TS] t_2 \supset \forall a \forall b \forall u_a \forall v_b . [TS] t_2 \wedge . [\Phi_{022} t_2 u_a \wedge \Phi_{022} t_2 v_b]$

$\supset a = b$

2000: .5, .6, .28

.30 $\vdash [TS] t_2 \supset . [TS] t_2 \wedge . [\Phi_{022} t_2 u_a \wedge \Phi_{022} t_2 v_b] \supset a = b$

Rule P: .29, 146, 147

.31 $\vdash \Phi_{022} t_2 u_a \supset [TS] t_2$

Rule P, Sub: 2307

.32 $\vdash [\Phi_{022} t_2 u_a \wedge \Phi_{022} t_2 v_b] \supset a = b$

Rule P: .30, .31.

Theorem 2314 is obtained from 2313 by the use of 170 and Rule P.

Proof of 2315:

Let P_{022} be $[\lambda t_2 \lambda w_2 . \Phi_{022} t_2 w_2 \wedge \forall c \forall x_c \forall q_c . \Phi_{022} t_2 x_c \supset \Phi_{022} t_2 q_c]$.

2315.1 $\vdash \Phi_{022} 3x_c \supset c = 0$

Sub, Sub_t: 2311

.2 $\Phi_{022} 3x_c \vdash c = 0$

MP: Premiss, .1

.3 $\Phi_{022} 3x_c \vdash \Phi_{022} 3q_c$

300: .2, 2301

.4 $\vdash \forall c \forall x_c \forall q_c . \Phi_{022} 3x_c \supset \Phi_{022} 3q_c$

Deduction Theorem, Gen, Gen_t: .3

2315.5 $\vdash \forall q_0 P_{022} \exists q_0$ Rule P: 2301, .4; RR: 103, Gen

.6 $\vdash \forall q_1 P_{022} \exists q_1$. The proof of .6 is similar to the proof of .5.

Use 2312 and 2302 in place of 2311 and 2301.

Let $\mathcal{H}^1$ be $P_{022} r_2 u_a, P_{022} s_2 v_b$; let $\mathcal{H}^2$ be $\Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]x_c$.

2315.7 $\mathcal{H}^1 \vdash \Phi_{022} r_2 u_a$ RR: Premiss, 103; Rule P

.8 $\mathcal{H}^1 \vdash \Phi_{022} s_2 v_b$ RR: Premiss, 103; Rule P

.9 $\mathcal{H}^1 \vdash \Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]q_{ab}$ Sub, Sub_t: 2303; Rule P: .7, .8

.10 $\mathcal{H}^1 \vdash \Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]z_{ab}$ Sub: .9

.11 $\mathcal{H}^1, \mathcal{H}^2 \vdash (ab) = c$ Sub, Sub_t: 2313; Rule P: .9, Premiss

.12 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]q_c$ 300: .9, .11

.13 $\mathcal{H}^1 \vdash \forall c \forall x_c \forall q_c . \Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]x_c \supset \Phi_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]q_c$
Deduction Theorem, Gen, Gen_t: .12

.14 $\mathcal{H}^1 \vdash \forall z_{ab} P_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]z_{ab}$
Rule P: .10, .13; RR: 103; Gen

.15 $\vdash \forall r_2 \forall s_2 \forall a \forall b . [\exists u_a P_{022} r_2 u_a \wedge \exists v_b P_{022} s_2 v_b]$

$\supset \forall z_{ab} P_{022}[\mathcal{J}^{r_2} \times \mathcal{S}^{s_2}]z_{ab}$

172, Deduction Theorem, Rule P, Gen_t, Gen: .14

.16 $\vdash \Phi_{022} t_2 w_2 \supset P_{022} t_2 w_2$ 2300: .5, .6, .15

.17 $\vdash \Phi_{022} t_2 u_a \supset . \Phi_{022} t_2 u_a \wedge \forall c \forall x_c \forall q_c . \Phi_{022} t_2 x_c \supset \Phi_{022} t_2 q_c$
Sub: .16; R: 103

.18 $\vdash \forall c \forall x_c \forall q_c [\Phi_{022} t_2 x_c \supset \Phi_{022} t_2 q_c] \supset . \Phi_{022} t_2 u_a \supset \Phi_{022} t_2 v_a$
Rule P: Axiom 7, 145

.19 $\vdash \Phi_{022} t_2 u_a \supset \Phi_{022} t_2 v_a$ Rule P: .17, .18.

Theorems 2316 and 2317 follow easily from 2315 by the use of 170, 123, and 143.

Proof of 2318:

2318.1 $\vdash \exists x_c \Phi_{022} t_2 x_c \supset \forall x_c \Phi_{022} t_2 x_c$ Sub_t: 2317; R: 123

.2 $\vdash \exists c \exists x_c \Phi_{022} t_2 x_c \supset \exists c \forall x_c \Phi_{022} t_2 x_c$ 177: .1.

Theorem 2318 then follows by Rule P from .2 and 2308.

Proof of 2319:

2319.1 $\vdash [TS]t_2 \supset \exists a \forall x_a \Phi_{022} t_2 x_a$ R: 157, 2318

.2 $\vdash \forall x_a \Phi_{022} t_2 x_a \supset \Phi_{022} t_2 x_a$ 146

.3 $\vdash [\Phi_{022} t_2 x_a \wedge \Phi_{022} t_2 u_b] \supset a = b$ Sub: 2313

.4 $\vdash a = b \supset \Phi_{022} t_2 x_a \stackrel{0}{=} \Phi_{022} t_2 x_b$ Axiom 8

.5 $\vdash \Phi_{022} t_2 x_b \supset \Phi_{022} t_2 u_b$ Sub, Sub_t: 2315

.6 $\vdash \forall x_a \Phi_{022} t_2 x_a \supset \Phi_{022} t_2 u_b \stackrel{0}{=} a = b$ Rule P: .2-.5

.7 $\vdash \exists a \forall x_a \Phi_{022} t_2 x_a \supset \exists a \forall b \forall u_b \Phi_{022} t_2 u_b \stackrel{0}{=} a = b$
143, 144, 177: .6

.8 $\vdash \forall b \forall u_b [\Phi_{022} t_2 u_b \stackrel{0}{=} a = b] \supset \Phi_{022} t_2 u_a \stackrel{0}{=} a = a$
Rule P: Axiom 7, 146

.9 $\vdash \Phi_{022} t_2 u_a \supset [TS]t_2$ Rule P, Sub: 2307

.10 $\vdash \exists a \forall b \forall u_b [\Phi_{022} t_2 u_b \stackrel{0}{=} a = b] \supset [TS]t_2$ Rule P: .8, .9, 302; 171

.11 $\vdash [TS]t_2 \stackrel{0}{=} \exists a \forall b \forall u_b \Phi_{022} t_2 u_b \stackrel{0}{=} a = b$ Rule P: .1, .7, .10.

Proof of 2320:

2320.1 $\vdash \Phi_{022} t_2 w_2 \supset [TS]t_2 \wedge \exists c \exists y_c . w_2 \stackrel{2}{=} y_c$ R: 123, 2307.

Let $\mathcal{H}$ be $\Phi_{022} t_2 x_a, \Phi_{022} t_2 w_2, w_2 \stackrel{2}{=} y_c$.

.2 $\mathcal{H} \vdash \Phi_{022} t_2 y_c$ 200: Premiss

.3 $\mathcal{H} \vdash \Phi_{022} t_2 x_a \wedge \Phi_{022} t_2 y_c \supset a = c$ Sub, Sub_t: 2313

.4 $\mathcal{H} \vdash a = c$ Rule P: Premiss, .2, .3

2320.5 $\mathcal{H} \vdash w_2 \stackrel{2}{=} y_a$ 300: Premiss, .4

.6 $\mathcal{H} \vdash \exists y_a . w_2 \stackrel{2}{=} y_a$ Exist (166): .5

.7 $\exists x_a \Phi_{022} t_2 x_a, \Phi_{022} t_2 w_2, \exists c \exists y_c . w_2 \stackrel{2}{=} y_c \vdash \exists y_a . w_2 \stackrel{2}{=} y_a$
172, 173, 163: .6.

Theorem 2320 then follows from .1 and .7 by the Deduction Theorem and Rule P.

Proof of 2321:

Let $\mathcal{H}^1$ be $\Phi_{022}[3^{t_2} \times 5^{u_2}]m_2 \wedge \Phi_{022}u_2n_2 \wedge [TS]t_2$.

Let $\mathcal{H}^2$ be $[m_2 \stackrel{2}{=} y_{ac}], [n_2 \stackrel{2}{=} z_b]$.

2321.1 $\mathcal{H}^1 \vdash \Phi_{022}[3^{t_2} \times 5^{u_2}]m_2 \wedge \Phi_{022}u_2n_2$ Rule P: Premiss

.2 $\mathcal{H}^2 \vdash m_2 \stackrel{2}{=} y_{ac}$ Premiss

.3 $\mathcal{H}^2 \vdash n_2 \stackrel{2}{=} z_b$ Premiss

.4 $\mathcal{H}^1, \mathcal{H}^2 \vdash \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ac} \wedge \Phi_{022}u_2z_b$ 200: .1, .2, .3

.5 $\mathcal{H}^1 \vdash [TS]u_2 \wedge \exists b \exists z_b . n_2 \stackrel{2}{=} z_b$
RR: 123, 157, 2307; Sub; Rule P: .1

.6 $\vdash [[TS]t_2 \wedge [TS]u_2 \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ac}] \supset . \exists m_a \Phi_{022}t_2m_a$
 $\wedge \exists n_c \Phi_{022}u_2n_c$ Sub, Sub_t: 2309

.7 $\mathcal{H}^1, \mathcal{H}^2 \vdash \exists v_b \Phi_{022}u_2v_b$ Rule P, Exist (166): .4

.8 $\vdash [\exists n_c \Phi_{022}u_2n_c \wedge \exists v_b \Phi_{022}u_2v_b] \supset c = b$
R: 123, 2314; Sub, Sub_t

.9 $\mathcal{H}^1, \mathcal{H}^2 \vdash c = b$ Rule P: Premiss .4, .5, .6, .7, .8

.10 $\mathcal{H}^1, \mathcal{H}^2 \vdash m_2 \stackrel{2}{=} y_{ab}$ 300: .2, .9

.11 $\mathcal{H}^1, \mathcal{H}^2 \vdash \exists a \exists b \exists y_{ab} \exists z_b . m_2 \stackrel{2}{=} y_{ab} \wedge n_2 = z_b$
Rule P, Exist (166), Exist_t (167): .3, .10

$$\begin{aligned}
2321.12 \quad & \mathcal{H}^1, \exists a \exists c \exists y_{ac} [m_2 \stackrel{2}{=} y_{ac}], \exists b \exists z_b [n_2 \stackrel{2}{=} z_b] \\
& \vdash \exists a \exists b \exists y_{ab} \exists z_b . m_2 \stackrel{2}{=} y_{ab} \wedge n_2 \stackrel{2}{=} z_b \quad 172, 173: .11 \\
.13 \vdash & [[TS]t_2 \wedge [TS]u_2 \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]m_2] \supset \exists a \exists c \exists y_{ac} . m_2 \stackrel{2}{=} y_{ac} \\
& \text{Sub: 2310; R: 123, 157.}
\end{aligned}$$

Theorem 2321 now follows from .5, .12, and .13 by the Deduction Theorem and Rule P.

The proofs of 2400, 2401, 2402, and 2403 are very similar in style to the proofs of 2300, 2301, 2302, and 2303, respectively. We therefore leave the details of these proofs to the reader.

Proof of 2404:

$$\begin{aligned}
.1 \vdash & \Phi_{022}3_2T_0 \quad \text{Sub: 2301} \\
.2 \vdash & \Phi_{022}5_2[l_{1(01)} . \lambda x_1 T_0] \quad \text{Sub: 2302} \\
.3 \vdash & \forall a \forall b \forall u_a \forall v_b \forall r_2 \forall s_2 . [\Phi_{022}r_2 u_a \wedge \Phi_{022}s_2 v_b] \\
& \supset \Phi_{022}[3^{r_2} \times 5^{s_2}] [\lambda z_b u_a] \quad \text{Sub}_t, \text{Sub, Gen, Gen}_t: 2303 \\
.4 \vdash & [Maps]t_2 w_2 \supset \Phi_{022}t_2 w_2 \quad 2400: .1, .2, .3.
\end{aligned}$$

Theorem 2405 follows from 2404 and 2307 by Rule P.

Proof of 2406:

Let P_{022} be

$$\begin{aligned}
& [\lambda t_2 \lambda w_2 . [Maps]t_2 w_2 \wedge \forall k_2 \forall j_2 . [t_2 \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS]k_2 \wedge [TS]j_2] \\
& \supset \exists c \exists d \exists q_c \exists v_d . [Maps]k_2 q_c \wedge [Maps]j_2 v_d \wedge w_2 \stackrel{2}{=} [\lambda z_d q_c]].
\end{aligned}$$

$$2406.1 \vdash P_{022}3T_0 \quad \text{Rule P, Gen: 2309.1, 2401; RR: 103}$$

$$.2 \vdash P_{022}5[l_{1(01)} . \lambda x_1 T_0].$$

The proof is similar to the proof of 2406.1.

Let $\mathcal{H}^1$ be $P_{022}r_2 u_a, P_{022}s_2 v_b$.

Let $\mathcal{H}^2$ be $[3^{r_2} \times 5^{s_2}] \stackrel{2}{=} [3^{k_2} \times 5^{j_2}] \wedge [TS]k_2 \wedge [TS]j_2$.

2406.3 $\mathcal{H}^1 \vdash [Maps]r_2u_a$

RR: Premiss, 103; Rule P

.4 $\mathcal{H}^1 \vdash [Maps]s_2v_b$

RR: Premiss, 103; Rule P

.5 $\mathcal{H}^1 \vdash [Maps] [3^{r_2} \times 5^{s_2}] [\lambda z_b u_a]$

Rule P: 2403, .3, .4

.6 $\mathcal{H}^1 \vdash [TS]r_2 \wedge [TS]s_2$

Sub, Rule P: 2405, .3, .4

.7 $\mathcal{H}^1, \mathcal{H}^2 \vdash r_2 \stackrel{2}{=} k_2$. The proofs of .7 and .8 are left to the reader.

.8 $\mathcal{H}^1, \mathcal{H}^2 \vdash s_2 \stackrel{2}{=} j_2$.

Use the hypothesis $\mathcal{H}^2$ and .6. Note that the proofs of 2309.11 and 2309.12 can be used here with virtually no changes.

.9 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]k_2u_a$

R': .3, .7

.10 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]j_2v_b$

R': .4, .8

.12 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]k_2u_a \wedge [Maps]j_2v_b \wedge [\lambda z_b u_a] \stackrel{2}{=} [\lambda z_b u_a]$

Rule P: .9, .10, 100

.13 $\mathcal{H}^1, \mathcal{H}^2 \vdash \exists c \exists d \exists q_c \exists v_d . [Maps]k_2q_c \wedge [Maps]j_2v_d \wedge [\lambda z_d u_c]$

$\stackrel{2}{=} [\lambda z_d q_c]$

Exist (166), Exist_i (167): .12

.14 $\mathcal{H}^1 \vdash P_{022}[3^{r_2} \times 5^{s_2}] [\lambda z_b u_a]$

Deduction Theorem, Gen: .13; Rule P: .5; RR: 103

.15 $\vdash \forall a \forall b \forall u_a \forall v_b \forall r_2 \forall s_2 . [P_{022}r_2u_a \wedge P_{022}s_2v_b]$

$\supset P_{022}[3^{r_2} \times 5^{s_2}] [\lambda z_b u_a]$

Deduction Theorem, Rule P, Gen, Gen_i: .14

.16 $\vdash [Maps]t_2w_2 \supset P_{022}t_2w_2$

2400: .1, .2, .15

.17 $\vdash [Maps] [3^{r_2} \times 5^{s_2}]w_2 \supset . [Maps] [3^{r_2} \times 5^{s_2}]w_2$

$\wedge \forall k_2 \forall j_2 . [[3^{r_2} \times 5^{s_2}] = [3^{k_2} \times 5^{j_2}] \wedge [TS]k_2 \wedge [TS]j_2]$

$\supset \exists c \exists d \exists q_c \exists v_d . [Maps]k_2q_c \wedge [Maps]j_2v_d \wedge w_2 \stackrel{2}{=} [\lambda z_d q_c]$

Sub: .16; RR: 103

$$\begin{aligned}
2406.18 \vdash [Maps] [3^{r_2} \times 5^{s_2}] w_2 \supset .[[3^{r_2} \times 5^{s_2}] = [3^{r_2} \times 5^{s_2}] \wedge [TS]r_2 \\
\wedge [TS]s_2] \supset \exists a \exists b \exists u_a \exists v_b . [Maps]r_2 u_a \wedge [Maps]s_2 v_b \\
\wedge w_2 \stackrel{2}{=} [\lambda z_b u_a] \quad \text{Rule P: .17, 145; R: 123, 157.}
\end{aligned}$$

Theorem 2406 follows from .18 and 100 by Rule P.

Proof of 2407:

Let B_{022} be $[\lambda i_2 \lambda j_2 . [TS]i_2 \wedge .i_2 \stackrel{2}{=} 3 \supset .T_0 \stackrel{2}{=} j_2]$.

$$\begin{aligned}
2407.1 \vdash [TS]3 \wedge .3 \stackrel{2}{=} 3 \supset .T_0 \stackrel{2}{=} T_0 & \quad \text{Rule P: 2002, 100} \\
.2 \vdash B_{022}3T_0 & \quad \text{RR: .1, 103} \\
.3 \vdash [TS]5 \wedge .5 \stackrel{2}{=} 3 \supset .T_0 \stackrel{2}{=} [\iota_{1(01)} . \lambda x_1 T_0] & \quad \text{Rule P: 2003, 2311.4} \\
.4 \vdash B_{022}5[\iota_{1(01)} . \lambda x_1 T_0] & \quad \text{RR: .3, 103} \\
.5 \vdash B_{022}r_2 u_a \wedge B_{022}s_2 v_b \supset .[TS]r_2 \wedge [TS]s_2 & \quad \text{Rule P, RR: 103} \\
.6 \vdash [TS]r_2 \wedge [TS]s_2 \supset [TS] [3^{r_2} \times 5^{s_2}] & \quad \text{Sub: 2004} \\
.7 \vdash \sim .3 = [3^{r_2} \times 5^{s_2}] \wedge [TS]r_2 \wedge [TS]s_2 & \quad \text{Sub: 2309.1} \\
.8 \vdash B_{022}r_2 u_a \wedge B_{022}s_2 v_b \supset B_{022}[3^{r_2} \times 5^{s_2}] [\lambda z_b u_a] & \\
& \quad \text{Rule P: .5, .6, .7; RR: 202, 103} \\
.9 \vdash [Maps]i_2 j_2 \supset B_{022}i_2 j_2 & \quad \text{Gen, Gen}_1: .8; 2400: .2, .4 \\
.10 \vdash [Maps]3w_2 \supset .[TS]3 \wedge .3 \stackrel{2}{=} 3 \supset .T_0 \stackrel{2}{=} w_2 & \\
& \quad \text{Sub: .9; R: 103} \\
.11 \vdash \forall w_2 . [Maps]3w_2 \supset .T_0 \stackrel{2}{=} w_2 & \quad \text{Rule P: .10, 100; Gen} \\
.12 \vdash [Maps]3T_0 \wedge \forall w_2 . [Maps]3w_2 \supset .T_0 \stackrel{2}{=} w_2 & \\
& \quad \text{Rule P: 2401, .11} \\
.13 \vdash \exists g_2 . [Maps]3g_2 \wedge \forall w_2 . [Maps]3w_2 \supset .g_2 \stackrel{2}{=} w_2 & \\
& \quad \text{Exist (166): .12} \\
.14 \vdash \exists g_2 . [Maps]5g_2 \wedge \forall w_2 . [Maps]5w_2 \supset .g_2 \stackrel{2}{=} w_2. &
\end{aligned}$$

The proof of .14 closely parallels the proof of .13. Simply interchange the roles of 3 and 5, and T_0 and $[\iota_{1(01)} . \lambda x_1 T_0]$.

Let $\mathcal{H}^1$ be $[[Maps]_{y_2} g_2 \wedge \forall w_2 . [Maps]_{y_2} w_2 \supset .g_2 \stackrel{2}{=} w_2]$,
 $[[Maps]_{z_2} h_2 \wedge \forall w_2 . [Maps]_{z_2} w_2 \supset .h_2 \stackrel{2}{=} w_2]$; let $\mathcal{H}^2$ be $[g_2 \stackrel{2}{=} m_c]$,
 $[h_2 \stackrel{2}{=} n_d]$; let $\mathcal{H}^3$ be $[Maps]_{y_2} u_a \wedge [Maps]_{z_2} v_b \wedge w_2 \stackrel{2}{=} [\lambda z_b u_a]$.

2407.15 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]_{y_2} m_c \wedge \forall w_2 . [Maps]_{y_2} w_2 \supset .m_c \stackrel{2}{=} w_2$
 200: Premiss

.16 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]_{z_2} n_d \wedge \forall w_2 . [Maps]_{z_2} w_2 \supset .n_d \stackrel{2}{=} w_2$
 200: Premiss

.17 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]_{y_2} u_a \supset m_c \stackrel{2}{=} u_a$ Rule P, Rule S: .15

.18 $\mathcal{H}^1, \mathcal{H}^2 \vdash [Maps]_{z_2} v_b \supset n_d \stackrel{2}{=} v_b$ Rule P, Rule S: .16

.19 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash m_c \stackrel{2}{=} u_a$ Rule P: Premiss, .17

.20 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash n_d \stackrel{2}{=} v_b$ Rule P: Premiss, .18

.21 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash c = a$ Sub, Sub₁: Axiom 10; MP: .19

.22 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash d = b$ Sub, Sub₁: Axiom 10; MP: .20

.23 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\lambda z_b u_a] \stackrel{2}{=} w_2$ Rule P, EE-Rules (205): Premiss

.24 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\lambda z_d u_c] \stackrel{2}{=} w_2$ 300: .21, .22, .23

.25 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash u_c \stackrel{c}{=} m_c$ 300: .19; EE-Rules (205)

.26 $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3 \vdash [\lambda z_d m_c] \stackrel{2}{=} w_2$ R': .24, .25

.27 $\mathcal{H}^1, \mathcal{H}^2 \vdash \exists a \exists b \exists u_a \exists v_b [[Maps]_{y_2} u_a \wedge [Maps]_{z_2} v_b \wedge w_2$
 $\stackrel{2}{=} [\lambda z_b u_a]] \supset . [\lambda z_d m_c] \stackrel{2}{=} w_2$
 Deduction Theorem, 170, 171: .26

.28 $\vdash [[TS]_{y_2} \wedge [TS]_{z_2} \wedge [Maps] [3^{y_2} \times 5^{z_2}]_{w_2}]$
 $\supset \exists a \exists b \exists u_a \exists v_b . [Maps]_{y_2} u_a \wedge [Maps]_{z_2} v_b \wedge w_2 \stackrel{2}{=} [\lambda z_b u_a]$
 Sub: 2406

.29 $\mathcal{H}^1 \vdash [TS]_{y_2} \wedge \exists c \exists x_c . g_2 \stackrel{2}{=} x_c$ Sub: 2405; Rule P: Premiss

.30 $\mathcal{H}^1 \vdash [TS]_{z_2} \wedge \exists c \exists x_c . h_2 \stackrel{2}{=} x_c$ Sub: 2405; Rule P: Premiss

$$2407.31 \mathcal{H}^1, \mathcal{H}^2 \vdash \forall w_2 . [Maps] [3^{y_2} \times 5^{z_2}] w_2 \supset .[\lambda z_d m_c] \stackrel{2}{=} w_2$$

Rule P: .27–.30; Gen

$$.32 \vdash [Maps] y_2 m_c \wedge [Maps] z_2 n_d \supset [Maps] [3^{y_2} \times 5^{z_2}] [\lambda z_d m_c]$$

Sub, Sub₁: 2403

$$.33 \mathcal{H}^1, \mathcal{H}^2 \vdash [Maps] [3^{y_2} \times 5^{z_2}] [\lambda z_d m_c]$$

$$\wedge \forall w_2 . [Maps] [3^{y_2} \times 5^{z_2}] w_2 \supset .[\lambda z_d m_c] \stackrel{2}{=} w_2$$

Rule P: .15, .16, .31, .32

$$.34 \mathcal{H}^1, \mathcal{H}^2 \vdash \exists g_2 . [Maps] [3^{y_2} \times 5^{z_2}] g_2$$

$$\forall w_2 . [Maps] [3^{y_2} \times 5^{z_2}] w_2 \supset .g_2 = w_2 \quad \text{Exist (166): .33}$$

$$.35 \mathcal{H}^1, \exists c \exists m_c [g_2 \stackrel{2}{=} m_c], \exists d \exists n_d [h_2 \stackrel{2}{=} n_d] \vdash \exists g_2 . [Maps] [3^{y_2} \times 5^{z_2}] g_2$$

$$\wedge \forall w_2 . [Maps] [3^{y_2} \times 5^{z_2}] w_2 \supset .g_2 \stackrel{2}{=} w_2 \quad 172, 173: .34$$

$$.36 \mathcal{H}^1 \vdash \exists c \exists m_c [g_2 \stackrel{2}{=} m_c] \wedge \exists d \exists n_d [h_2 \stackrel{2}{=} n_d]$$

RR: .29, .30, 123, 157; Rule P

$$.37 \vdash \forall y_2 \forall z_2 . [\exists g_2 [[Maps] y_2 g_2 \wedge \forall w_2 . [Maps] y_2 w_2 \supset .g_2 \stackrel{2}{=} w_2]$$

$$\wedge \exists g_2 [[Maps] z_2 g_2 \wedge \forall w_2 . [Maps] z_2 w_2 \supset .g_2 \stackrel{2}{=} w_2]]$$

$$\supset \exists g_2 . [Maps] [3^{y_2} \times 5^{z_2}] g_2 \wedge \forall w_2 . [Maps] [3^{y_2} \times 5^{z_2}] w_2$$

$$\supset .g_2 \stackrel{2}{=} w_2$$

Deduction Theorem, Rule P: .35, .36; 172, Deduction Theorem

Rule P: 123; Gen

$$.38 \vdash [TS] t_2 \supset \exists g_2 . [Maps] t_2 g_2 \wedge \forall w_2 . [Maps] t_2 w_2 \supset .g_2 \stackrel{2}{=} w_2$$

2000: .13, .14, .37.

Proof of 2408:

$$2408.1 \vdash [Map] u_2 \stackrel{2}{=} [\iota_{2(02)} . [Maps] [\iota_{2(02)} . [Variable] u_2]]$$

103, def. of [Map]

$$.2 [Variable] u_2 t_2 \vdash [\iota_{2(02)} [Variable] u_2] \stackrel{2}{=} t_2$$

Sub: 2106; MP: Premiss

2408.3 $[Variable]u_2t_2 \vdash [Map]u_2 \stackrel{2}{=} [\iota_{2(02)}[Maps]t_2]$ R': .1, .2

.4 $[Variable]u_2t_2 \vdash [TS]t_2$ Sub: 2104; Rule P: Premiss

.5 $\vdash \exists g_2 [[Maps]t_2g_2 \wedge \forall w_2 . [Maps]t_2w_2 \supset .g_2 \stackrel{2}{=} w_2]$
 $\supset [Maps]t_2[\iota_{2(02)}[Maps]t_2]$ Sub: 216; Rule P: 123; 170

.6 $[Variable]u_2t_2 \vdash [Maps]t_2[\iota_{2(02)}[Maps]t_2]$ Rule P: .4, .5, 2407

.7 $[Variable]u_2t_2 \vdash [Maps]t_2[[Map]u_2]$ EE-Rules, R': .3, .6.

Theorem 2408 then follows by the Deduction Theorem.

Proof of 2409:

2409.1 $\vdash [Maps]t_2[[Map]u_2] \supset \Phi_{022}t_2[[Map]u_2]$ Sub: 2404

.2 $\vdash [Variable]u_2t_2 \supset \Phi_{022}t_2[[Map]u_2]$ Rule P: .1, 2408

.3 $\vdash \forall x_2 \forall t_2 . [Variable]x_2t_2 \supset \Phi_{022}t_2[[Map]x_2]$ Sub, Gen: .2

.4 $\vdash [Assign] [Map]$ RR: .3, 103; def. of $[Assign]$.

Theorem 2410 follows from 2409 by Rule Exist (166).

Proof of 2500:

.1 $\vdash [x_2 = j_2] \supset .[\iota q_2 . [[x_2 = j_2] \wedge y_2 = q_2] \vee .\sim[x_2 = j_2]$
 $\wedge h_{22}x_2 = q_2] = y_2$ Sub: 211

.2 $\vdash [x_2 = j_2] \supset .[Mod]h_{22}j_2y_2x_2 = y_2$
RR: 103, .1; def. of $[Mod]$.

Theorem 2501 follows from 2500 by Rule Sub, 100, and Rule MP.

Proof of 2502:

.1 $\vdash \sim[x_2 = j_2] \supset .[\iota q_2 . [[x_2 = j_2] \wedge y_2 = q_2] \vee .\sim[x_2 = j_2]$
 $\wedge h_{22}x_2 = q_2] = h_{22}x_2$ Sub: 212

.2 $\vdash \sim[x_2 = j_2] \supset .[Mod]h_{22}j_2y_2x_2 = h_{22}x_2$ RR: 103, .1.

Proof of 2503:

Let $\mathcal{H}$ be $[Variable]_{j_2s_2}, \Phi_{022}s_2y_2, [Assign]_{h_{22}}, [Variable]_{x_2t_2}$.

- 2503.1 $\mathcal{H}, [x_2 = j_2] \vdash [Variable]_{j_2t_2}$ R': Premiss
- .2 $\mathcal{H}, [x_2 = j_2] \vdash s_2 = t_2$ Sub: 2105; Rule P: .1, Premiss
- .3 $\mathcal{H}, [x_2 = j_2] \vdash \Phi_{022}t_2y_2$ R': .2, Premiss
- .4 $\mathcal{H}, [x_2 = j_2] \vdash [Mod]_{h_{22}j_2y_2x_2} = y_2$ MP: Premiss, 2500
- .5 $\mathcal{H}, [x_2 = j_2] \vdash \Phi_{022}t_2[[Mod]_{h_{22}j_2y_2x_2}]$ E-Rules, R': .3, .4
- .6 $\vdash [Assign]_{h_{22}} \supset .[Variable]_{x_2t_2} \supset \Phi_{022}t_2[h_{22}x_2]$
Rule P: 146; RR: 103; def. of $[Assign]$
- .7 $\mathcal{H}, \sim[x_2 = j_2] \vdash [Mod]_{h_{22}j_2y_2x_2} = h_{22}x_2$ MP: Premiss, 2502
- .8 $\mathcal{H}, \sim[x_2 = j_2] \vdash \Phi_{022}t_2[h_{22}x_2]$ Rule P: Premiss, .6
- .9 $\mathcal{H}, \sim[x_2 = j_2] \vdash \Phi_{022}t_2[[Mod]_{h_{22}j_2y_2x_2}]$ E-Rules, R': .7, .8
- .10 $[Variable]_{j_2s_2}, \Phi_{022}s_2y_2, [Assign]_{h_{22}} \vdash [Variable]_{x_2t_2}$
 $\supset \Phi_{022}t_2[[Mod]_{h_{22}j_2y_2x_2}]$ Deduction Theorem, Rule P: .5, .9
- .11 $[Variable]_{j_2s_2}, \Phi_{022}s_2y_2, [Assign]_{h_{22}} \vdash [Assign] [[Mod]_{h_{22}j_2y_2}]$
Gen: .10; RR: 103; def. of $[Assign]$.

Theorem 2503 now follows by the Deduction Theorem and Rule P.

Theorem 2600 follows directly from the definition of $[Value]$. The proof is similar in style to the proof of 2300. We leave the details to the reader.

Since the reader who has come this far must by now be quite familiar with the workings of the system Q , we shall henceforth abbreviate our proofs even more than has been our custom so far. In particular, after a line in a proof we shall often merely indicate the principal lines of the proof or theorems from which it is inferred, and leave it to the reader to supply the particular rules of inference or theorems from Chapter II required for the intermediate steps. In this way we shall avoid dwelling on details which by now have become routine, yet we shall supply outlines from which complete proofs can easily be constructed, if desired.

Proof of 2601:

The proof is based on Theorem 2600. Let $P_{022(22)}$ be

$$[\lambda g_{22} \lambda n_2 \lambda x_2 \cdot [Assign]g_{22} \wedge \exists s_2 \cdot [Wff]n_2 s_2 \wedge \Phi_{022s_2 x_2}].$$

$$\begin{aligned}
 2601.1 &\vdash \Phi_{022}[3^5 \times 5^{[3^3 \times 5^5]}]_{l_1(01)} & 2305 \\
 .2 &\vdash \exists s_2 \cdot [Wff][2^{17}]_{s_2} \wedge \Phi_{022s_2 l_1(01)} & .1, 2202 \\
 .3 &[Assign]h_{22} \vdash P_{022(22)}h_{22}[2^{17}]_{l_1(01)} & .2 \\
 .4 &\vdash [[TS]t_2 \wedge \exists u_a \Phi_{022t_2 u_a} \supset \exists s_2 \cdot [Wff][2^{[19^{t_2}]}]_{s_2} \\
 &\quad \wedge \Phi_{022s_2 Q_{0aa}} & 2203, 2306 \\
 .5 &[Assign]h_{22} \vdash \forall a \forall t_2 \cdot [[TS]t_2 \wedge \exists u_a \Phi_{022t_2 u_a} \\
 &\quad \supset P_{022(22)}h_{22}[2^{[19^{t_2}]}]_{Q_{0aa}} & .4 \\
 .6 &\vdash [Var]v_2 \stackrel{0}{=} [Variable]v_2[l_2(02)][Variable]v_2 & \text{def. of } [Var] \\
 .7 &\vdash [Variable]v_2[l_2(02)][Variable]v_2 \\
 &\quad \supset [Wff][2^{v_2}]_{[l_2(02)][Variable]v_2} & 2204 \\
 .8 &\vdash [Assign]h_{22} \supset \cdot [Variable]v_2[l_2(02)][Variable]v_2 \\
 &\quad \supset \Phi_{022[l_2(02)][Variable]v_2}[h_{22}v_2] & \text{def. of } [Assign] \\
 .9 &[Assign]h_{22} \vdash \forall v_2 \cdot [Var]v_2 \supset P_{022(22)}h_{22}[2^{v_2}][h_{22}v_2] & .6, .7, .8 \\
 .10 &\vdash [[Wff]j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]m_2 u_2 \wedge [TS]t_2] \\
 &\quad \supset [Wff][2^7 * j_2 * m_2 * 2^{11}]_{t_2} & 2205 \\
 .11 &\vdash [Wff]m_2 u_2 \supset [TS]u_2 & 2201 \\
 .12 &\vdash [[TS]t_2 \wedge [TS]u_2 \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]_{y_{ab}} \supset \Phi_{022t_2}[y_{ab}z_b] \\
 & & 2309, 2316 \\
 .13 &[Assign]h_{22} \vdash [[Wff]j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]m_2 u_2 \wedge [TS]t_2 \\
 &\quad \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]_{y_{ab}} \supset P_{022(22)}h_{22}[2^7 * j_2 * m_2 * 2^{11}][y_{ab}z_b] \\
 & & .10, .11, .12
 \end{aligned}$$

$$\begin{aligned}
2601.14 \quad & [Assign]h_{22} \vdash \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall y_{ab} \forall z_b . [[Wff]j_2[3^{t_2} \times 5^{u_2}]] \\
& \wedge [Wff]m_2u_2 \wedge [TS]t_2 \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ab} \wedge \Phi_{022}u_2z_b \\
& \wedge P_{022(22)}h_{22}j_2y_{ab} \wedge P_{022(22)}h_{22}m_2z_b \supset \\
& \supset P_{022(22)}h_{22}[2^7 * j_2 * m_2 * 2^{11}] [y_{ab}z_b] \quad .13 \\
.15 \quad & \vdash [[Variable]j_2t_2 \wedge [Wff]m_2u_2] \\
& \supset [Wff][2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}][3^{u_2} \times 5^{t_2}] \quad 2206 \\
.16 \quad & \vdash [\Phi_{022}u_2[f_{ab}y_b] \wedge \Phi_{022}t_2y_b] \supset \Phi_{022}[3^{u_2} \times 5^{t_2}]f_{ab} \quad 2303 \\
.17 \quad & [Assign]h_{22} \vdash \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall f_{ab} . [[Variable]j_2t_2 \\
& \wedge [Wff]m_2u_2 \wedge \forall y_b . \Phi_{022}t_2y_b \wedge \Phi_{022}u_2[f_{ab}y_b] \\
& \wedge P_{022(22)}[[Mod]h_{22}j_2y_b]m_2[f_{ab}y_b]] \\
& \supset P_{022(22)}h_{22}[2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}]f_{ab} \quad .15, .16.
\end{aligned}$$

Theorem 2601 now follows by 2600 from .3, .5, .9, .14 and .17.

Theorem 2602 follows easily from 2601 and 2209.

Proof of 2603:

$$\begin{aligned}
2603.1 \quad & [Wff]n_2s_2, \Phi_{022}s_2x_2, [Wff]n_2t_2 \vdash s_2 = t_2 \quad 2208 \\
.2 \quad & [Wff]n_2s_2, \Phi_{022}s_2x_2, [Wff]n_2t_2 \vdash \Phi_{022}t_2x_2 \quad R': \text{Premiss, .1} \\
.3 \quad & \exists s_2[[Wff]n_2s_2 \wedge \Phi_{022}s_2x_2], [Wff]n_2t_2 \vdash \Phi_{022}t_2x_2 \quad 172: .2.
\end{aligned}$$

Theorem 2603 follows from .3 and 2601 by the Deduction Theorem and Rule P.

Proof of 2604:

To avoid excessive writing we temporarily define A_0 , B_0 , C_0 , D_0 , and E_0 to be those wffs such that $[Value]$ is

$$\begin{aligned}
& [\lambda g_{22} \lambda n_2 \lambda x_2 . \forall p_{022(22)} . \forall h_{22} [[Assign]h_{22} \supset . A_0 \wedge B_0 \wedge C_0 \wedge D_0 \wedge E_0] \\
& \supset p_{022(22)}g_{22}n_2x_2].
\end{aligned}$$

Note that $p_{022(22)}$ and h_{22} are the only variables free in the wffs A_0 , B_0 , C_0 , D_0 , E_0 .

2604.1 $\vdash \forall h_{22} [[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0 \wedge D_0 \wedge E_0]$

$\supset .[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0 \wedge D_0 \wedge E_0$ 146

.2 $\vdash A_0 \supset p_{022(22)}h_{22}[2^{17}]_{l_1(01)}$ Rule P; def. of A_0

.3 $\vdash [Assign]h_{22} \supset .\forall h_{22} [[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0 \wedge D_0 \wedge E_0]$

$\supset p_{022(22)}h_{22}[2^{17}]_{l_1(01)}$ Rule P: .1, .2

.4 $\vdash [Assign]h_{22} \supset [Value]h_{22}[2^{17}]_{l_1(01)}$ 143, RR: 103, .3.

The proofs of 2605 and 2606 are similar in style to the proof of 2604. We leave the simple details to the reader.

Proof of 2607:

Let A_0 , B_0 , C_0 , D_0 and E_0 be defined as in the proof of 2604.

.1 $\vdash [Value]h_{22}j_2y_{ab} \supset .\forall h_{22} [[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0$

$\wedge D_0 \wedge E_0] \supset p_{022(22)}h_{22}j_2y_{ab}$ 146; RR: 103; def. of $[Value]$

.2 $\vdash [Value]h_{22}m_2z_b \supset .\forall h_{22} [[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0$

$\wedge D_0 \wedge E_0] \supset p_{022(22)}h_{22}m_2z_b$ 146; RR: 103

.3 $\vdash D_0 \supset .[[Wff]j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]m_2u_2 \wedge [TS]t_2$

$\wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ab} \wedge \Phi_{022}u_2z_b \wedge p_{022(22)}h_{22}j_2y_{ab}$

$\wedge p_{022(22)}h_{22}m_2z_b] \supset p_{022(22)}h_{22}[2^7 * j_2 * m_2 * 2^{11}] [y_{ab}z_b]$

Rule P: 146, 147; definition of D_0

.4 $\vdash [[Assign]h_{22} \wedge [Wff]j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]m_2u_2 \wedge [TS]t_2$

$\wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ab} \wedge \Phi_{022}u_2z_b \wedge [Value]h_{22}j_2y_{ab}$

$\wedge [Value]h_{22}m_2z_b] \supset .\forall h_{22} [[Assign]h_{22} \supset .A_0 \wedge B_0 \wedge C_0$

$\wedge D_0 \wedge E_0] \supset p_{022(22)}h_{22}[2^7 * j_2 * m_2 * 2^{11}] [y_{ab}z_b]$

Rule P: .1, .2, .3, 2604.1.

Theorem 2607 now follows from .4 by 143, RR, 103, Rule P, and the definition of $[Value]$.

The proof of 2608 is similar in style to the proof of 2607. We leave the details to the reader.

The proofs of theorems 2609, 2610, 2611, 2612 and 2613 are similar in style, and are straightforward though somewhat lengthy. We shall prove 2613 as an example, but leave the others to the reader. In each case the proof is based upon 2600. For the proof of 2609 let $P_{022(22)}$ be

$$[\lambda g_{22} \lambda n_2 \lambda x_2 . n_2 = 2^{17} \supset x_2 \stackrel{2}{=} \iota_{1(01)}].$$

For the proof of 2610 let $P_{022(22)}$ be

$$[\lambda g_{22} \lambda n_2 \lambda x_2 . \forall s_2 . [[TS]_{s_2} \wedge n_2 = [2^{[19^{s_2}]}]] \supset \exists c \exists u_c . \Phi_{022} s_2 u_c \wedge x_2 \stackrel{2}{=} Q_{occ}].$$

For the proof of 2611 let $P_{022(22)}$ be

$$[\lambda g_{22} \lambda n_2 \lambda x_2 . \forall w_2 \forall s_2 . n_2 = 2^{w_2} \supset . [Variable] w_2 s_2 \supset x_2 \stackrel{2}{=} g_{22} w_2].$$

For the proof of 2612 let $P_{022(22)}$ be

$$\begin{aligned} & [\lambda g_{22} \lambda n_2 \lambda x_2 . [Value]_{g_{22} n_2 x_2} \wedge \forall \bar{k}_2 \forall \bar{m}_2 \forall \bar{i}_2 \forall \bar{u}_2 . [[Wff] \bar{k}_2 [3^{\bar{i}_2} \times 5^{\bar{u}_2}] \\ & \wedge [Wff] \bar{m}_2 \bar{u}_2 \wedge [TS] \bar{i}_2 \wedge n_2 = [2^7 * \bar{k}_2 * \bar{m}_2 * 2^{11}]] \\ & \supset \exists \bar{a} \exists \bar{b} \exists \bar{y}_{\bar{a}\bar{b}} \exists \bar{z}_{\bar{b}} . \Phi_{022} [3^{\bar{i}_2} \times 5^{\bar{u}_2}] \bar{y}_{\bar{a}\bar{b}} \wedge \Phi_{022} \bar{u}_2 \bar{z}_{\bar{b}} \wedge [Value]_{g_{22} \bar{k}_2 \bar{y}_{\bar{a}\bar{b}}} \\ & \wedge [Value]_{g_{22} \bar{m}_2 \bar{z}_{\bar{b}}} \wedge x_2 \stackrel{2}{=} [\bar{y}_{\bar{a}\bar{b}} \bar{z}_{\bar{b}}]]. \end{aligned}$$

Proof of 2613:

Let $P_{022(22)}$ be

$$\begin{aligned} & [\lambda g_{22} \lambda n_2 \lambda x_2 . [Value]_{g_{22} n_2 x_2} \wedge \forall \bar{k}_2 \forall \bar{m}_2 \forall \bar{i}_2 \forall \bar{u}_2 . [[Variable] \bar{k}_2 \bar{i}_2 \\ & \wedge [Wff] \bar{m}_2 \bar{u}_2 \wedge n_2 \stackrel{2}{=} [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]] \supset \exists \bar{a} \exists \bar{b} \exists \bar{f}_{\bar{a}\bar{b}} . x_2 \\ & \stackrel{2}{=} \bar{f}_{\bar{a}\bar{b}} \wedge \forall \bar{y}_{\bar{b}} . \Phi_{022} \bar{i}_2 \bar{y}_{\bar{b}} \wedge \Phi_{022} \bar{u}_2 [\bar{f}_{\bar{a}\bar{b}} \bar{y}_{\bar{b}}] \\ & \wedge [Value] [[Mod]_{g_{22} \bar{k}_2 \bar{y}_{\bar{b}}} \bar{m}_2 [\bar{f}_{\bar{a}\bar{b}} \bar{y}_{\bar{b}}]]. \end{aligned}$$

2613.1 $[Assign]h_{22} \vdash [Value]h_{22}[2^{17}]_{l_1(01)}$ 2604

$$\begin{aligned} .2 \vdash \sim.[Variable]\bar{k}_2\bar{t}_2 \wedge [Wff]\bar{m}_2\bar{u}_2 \wedge 2^{17} \\ = [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]. \end{aligned}$$

The proof is left to the reader.

.3 $[Assign]h_{22} \vdash P_{022(22)}h_{22}[2^{17}]_{l_1(01)}$.1, .2

$$\begin{aligned} .4 \vdash [[TS]t_2 \wedge \exists u_a \Phi_{022}t_2u_a] \supset \sim.[Variable]\bar{k}_2\bar{t}_2 \wedge [Wff]\bar{m}_2\bar{u}_2 \\ \wedge 2^{[19^{t_2}]} = [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]. \end{aligned}$$

The proof is left to the reader.

.5 $[Assign]h_{22} \vdash \forall a \forall t_2 . [[TS]t_2 \wedge \exists u_a \Phi_{022}t_2u_a] \\ \supset P_{022(22)}h_{22}[2^{[19^{t_2}]}]Q_{0aa}$.4, 2605

$$\begin{aligned} .6 \vdash [Var]v_2 \supset \sim.[Variable]\bar{k}_2\bar{t}_2 \wedge [Wff]\bar{m}_2\bar{u}_2 \wedge 2^{v_2} \\ = [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]. \end{aligned}$$

The proof is left to the reader.

.7 $[Assign]h_{22} \vdash \forall v_2 . [Var]v_2 \supset P_{022(22)}h_{22}[2^{v_2}] [h_{22}v_2]$.6, 2606

$$\begin{aligned} .8 \vdash P_{022(22)}h_{22}j_2y_{ab} \wedge P_{022(22)}h_{22}m_2z_b \supset .[Value]h_{22}j_2y_{ab} \\ \wedge [Value]h_{22}m_2z_b \quad \text{def. of } P_{022(22)} \end{aligned}$$

$$\begin{aligned} .9 \vdash [Wff]j_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]m_2u_2 \supset \sim.[Variable]\bar{k}_2\bar{t}_2 \\ \wedge [Wff]\bar{m}_2\bar{u}_2 \wedge [2^7 * j_2 * m_2 * 2^{11}] = [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]. \end{aligned}$$

The proof is left to the reader. (Note that 13 is ' λ ', but that λ cannot be the first symbol in a wff.)

.10 $[Assign]h_{22} \vdash \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall y_{ab} \forall z_b . [[Wff]j_2[3^{t_2} \times 5^{u_2}] \\ \wedge [Wff]m_2u_2 \wedge [TS]t_2 \wedge \Phi_{022}[3^{t_2} \times 5^{u_2}]y_{ab} \wedge \Phi_{022}u_2z_b \\ \wedge P_{022(22)}h_{22}j_2y_{ab} \wedge P_{022(22)}h_{22}m_2z_b] \\ \supset P_{022(22)}h_{22}[2^7 * j_2 * m_2 * 2^{11}] [y_{ab}z_b]$.8, .9, 2607.

Let $\mathcal{H}^1$ be $[Assign]h_{22}$,

$$[[Variable]j_2t_2 \wedge [Wff]m_2u_2 \wedge \forall y_b. \Phi_{022}t_2y_b \wedge \Phi_{022}u_2[f_{ab}y_b] \\ \wedge P_{022(22)}[[Mod]h_{22}j_2y_b]m_2[f_{ab}y_b]];$$

let $\mathcal{H}^2$ be

$$[[Variable]\bar{k}_2\bar{t}_2 \wedge [Wff]\bar{m}_2\bar{u}_2 \wedge [2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}] \\ = [2^7 * 2^{13} * 2^{\bar{k}_2} * \bar{m}_2 * 2^{11}]].$$

$$2613.11 \vdash P_{022(22)}[[Mod]h_{22}j_2y_b]m_2[f_{ab}y_b]$$

$$\supset [Value] [[Mod]h_{22}j_2y_b]m_2[f_{ab}y_b] \quad \text{def. of } P_{022(22)}$$

$$.12 \mathcal{H}^1 \vdash \forall y_b. \Phi_{022}t_2y_b \wedge \Phi_{022}u_2[f_{ab}y_b]$$

$$\wedge [Value] [[Mod]h_{22}j_2y_b]m_2[f_{ab}y_b] \quad \text{Premiss, .11}$$

$$.13 \mathcal{H}^1 \vdash [Value]h_{22}[2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}]f_{ab} \quad \text{Premiss, .12, 2608}$$

$$.14 [Variable]j_2t_2, [Wff]m_2u_2, \mathcal{H}^2 \vdash j_2 = \bar{k}_2 \wedge m_2 = \bar{m}_2.$$

The proof is left to the reader.

$$.15 \mathcal{H}^1, \mathcal{H}^2 \vdash j_2 = \bar{k}_2 \quad .14$$

$$.16 \mathcal{H}^1, \mathcal{H}^2 \vdash m_2 = \bar{m}_2 \quad .14$$

$$.17 \mathcal{H}^1, \mathcal{H}^2 \vdash [Variable]\bar{k}_2t_2 \quad \text{Rule P, R': Premiss, .15}$$

$$.18 \mathcal{H}^1, \mathcal{H}^2 \vdash [Wff]\bar{m}_2u_2 \quad \text{Rule P, R': Premiss, .16}$$

$$.19 \vdash [Variable]\bar{k}_2t_2 \supset .[Variable]\bar{k}_2\bar{t}_2 \supset t_2 = \bar{t}_2 \quad 2105$$

$$.20 \vdash [Wff]\bar{m}_2u_2 \supset .[Wff]\bar{m}_2\bar{u}_2 \supset u_2 = \bar{u}_2 \quad 2208$$

$$.21 \mathcal{H}^1, \mathcal{H}^2 \vdash t_2 = \bar{t}_2 \quad \text{Premiss, .17, .19}$$

$$.22 \mathcal{H}^1, \mathcal{H}^2 \vdash u_2 = \bar{u}_2 \quad \text{Premiss, .18, .20}$$

$$.23 \mathcal{H}^1, \mathcal{H}^2 \vdash f_{ab} \stackrel{2}{=} f_{ab} \wedge \forall y_b. \Phi_{022}\bar{t}_2y_b \wedge \Phi_{022}\bar{u}_2[f_{ab}y_b]$$

$$\wedge [Value] [[Mod]h_{22}\bar{k}_2y_b]\bar{m}_2[f_{ab}y_b]$$

$$\text{R': .12, .15, .16, .21, .22; Rule P: 100}$$

$$2613.24 \mathcal{H}^1, \mathcal{H}^2 \vdash \exists \bar{a} \exists \bar{b} \exists \bar{f}_{\bar{a}\bar{b}} . f_{ab} \stackrel{2}{=} \bar{f}_{\bar{a}\bar{b}} \wedge \forall \bar{y}_{\bar{b}} . \Phi_{022} \bar{t}_2 \bar{y}_{\bar{b}} \wedge \Phi_{022} \bar{u}_2 [\bar{f}_{\bar{a}\bar{b}} \bar{y}_{\bar{b}}]$$

$$\wedge [Value] [[Mod] h_{22} \bar{k}_2 \bar{y}_{\bar{b}}] \bar{m}_2 [\bar{f}_{\bar{a}\bar{b}} \bar{y}_{\bar{b}}]$$

$$R': .23, 123; \text{Exist} (166), \text{Exist}_t (167)$$

$$.25 \mathcal{H}^1 \vdash P_{022(22)} h_{22} [2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}] f_{ab} \quad .24, .13$$

$$.26 [Assign] h_{22} \vdash \forall j_2 \forall m_2 \forall t_2 \forall u_2 \forall a \forall b \forall f_{ab} . [[Variable] j_2 t_2$$

$$\wedge [Wff] m_2 u_2 \wedge \forall y_b . \Phi_{022} t_2 y_b \wedge \Phi_{022} u_2 [f_{ab} y_b]$$

$$\wedge P_{022(22)} [[Mod] h_{22} j_2 y_b] m_2 [f_{ab} y_b]$$

$$\supset P_{022(22)} h_{22} [2^7 * 2^{13} * 2^{j_2} * m_2 * 2^{11}] f_{ab} \quad .25$$

$$.27 \vdash [Value] g_{22} [2^7 * 2^{13} * 2^{k_2} * m_2 * 2^{11}] j_2$$

$$\supset P_{022(22)} g_{22} [2^7 * 2^{13} * 2^{k_2} * m_2 * 2^{11}] j_2$$

$$2600: .3, .5, .7, .10, .26; \text{Sub.}$$

Theorem 2613 follows easily from .27 and the definition of $P_{022(22)}$.

Proof of 2614:

Let $\mathcal{H}$ be

$$[Variable] w_2 t_2, [Wff] y_2 u_2, \forall g_{22} [[Assign] g_{22} \supset \exists i_a . [Value] g_{22} y_2 i_a$$

$$\wedge \forall j_2 . [Value] g_{22} y_2 j_2 \supset . j_2 \stackrel{2}{=} i_a], \exists m_b \Phi_{022} t_2 m_b, [Assign] h_{22} .$$

$$2614.1 \vdash [Assign] h_{22} \supset . [[Variable] w_2 t_2 \wedge [Wff] y_2 u_2 \wedge \forall z_b . \Phi_{022} t_2 z_b$$

$$\wedge \Phi_{022} u_2 [[\lambda z_b . \iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a] z_b]$$

$$\wedge [Value] [[Mod] h_{22} w_2 z_b] y_2 [[\lambda z_b . \iota x_a$$

$$.[Value] [[Mod] h_{22} w_2 z_b] y_2 x_a] z_b]]$$

$$\supset . [Value] h_{22} [2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]$$

$$[\lambda z_b . \iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a]$$

$$R: 2608, 123; \text{Sub}$$

- 2614.2 $\mathcal{H} \vdash \forall z_b [\Phi_{022} t_2 z_b \wedge \Phi_{022} u_2 [\iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a]$
 $\wedge [Value] [[Mod] h_{22} w_2 z_b] y_2 [\iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a]$
 $\supset . [Value] h_{22} [2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]$
 $[\iota z_b . \iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a]$
R: 103, .1; Rule P: Premiss
- .3 $\vdash \exists i_a [p_{0a} i_a \wedge \forall j_a . p_{0a} j_a \supset j_a \stackrel{a}{=} i_a] \supset p_{0a} [\iota_{a(0a)} p_{0a}]$
R: 216, 123, 202; Sub, Rule P, 170
- .4 $\vdash \exists i_a [\lambda x_a . [Value] g_{22} y_2 x_a] i_a \wedge \forall j_a . [\lambda x_a . [Value] g_{22} y_2 x_a] j_a \supset j_a$
 $\stackrel{a}{=} i_a] \supset . [\lambda x_a . [Value] g_{22} y_2 x_a] [\iota_{a(0a)} . \lambda x_a . [Value] g_{22} y_2 x_a]$
Sub: .3
- .5 $\vdash \exists i_a [[Value] g_{22} y_2 i_a \wedge \forall j_a . [Value] g_{22} y_2 j_a \supset j_a \stackrel{a}{=} i_a]$
 $\supset . [Value] g_{22} y_2 [\iota x_a . [Value] g_{22} y_2 x_a]$
R: .4, 103; def. of $[\iota x_a A_0]$
- .6 $\vdash \forall j_2 [[Value] g_{22} y_2 j_2 \supset . j_2 \stackrel{2}{=} i_a] \supset . [Value] g_{22} y_2 j_a \supset . j_a \stackrel{a}{=} i_a$
RR: 201, 145
- .7 $\vdash \forall j_2 [[Value] g_{22} y_2 j_2 \supset . j_2 \stackrel{2}{=} i_a] \supset \forall j_a . [Value] g_{22} y_2 j_a$
 $\supset . j_a \stackrel{a}{=} i_a$
143: .6
- .8 $\mathcal{H} \vdash [Assign] g_{22} \supset \exists i_a . [Value] g_{22} y_2 i_a \wedge \forall j_a . [Value] g_{22} y_2 j_a$
 $\supset . j_a \stackrel{a}{=} i_a$
Rule P, 176: .7; Rule S: Premiss; Rule P
- .9 $\mathcal{H} \vdash [Assign] g_{22} \supset . [Value] g_{22} y_2 [\iota x_a . [Value] g_{22} y_2 x_a]$
Rule P: .5, .8
- .10 $\mathcal{H} \vdash \Phi_{022} t_2 z_b$
Sub, Sub_t: 2316; Rule P: Premiss
- .11 $\vdash [[Variable] w_2 t_2 \wedge \Phi_{022} t_2 z_b] \supset . [Assign] h_{22}$
 $\supset [Assign] [[Mod] h_{22} w_2 z_b]$
Sub: 2503
- .12 $\mathcal{H} \vdash [Assign] [[Mod] h_{22} w_2 z_b]$
Rule P: Premiss, .10, .11
- .13 $\mathcal{H} \vdash [Value] [[Mod] h_{22} w_2 z_b] y_2 [\iota x_a . [Value] [[Mod] h_{22} w_2 z_b] y_2 x_a]$
Sub: .9; Rule P: .12

2614.14 $\mathcal{H} \vdash \Phi_{022}u_2[\iota x_a \cdot [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$

Sub: 2603; Rule P: .13, Premiss

.15 $\mathcal{H} \vdash [Value]h_{22}[2^7 * 2^{13} * 2^{w_1} * y_2 * 2^{11}]$

$[\lambda z_b \cdot \iota x_a \cdot [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$

Rule P, Gen: .10, .14, .13, .2.

Theorem 2614 follows from .15 by the Deduction Theorem and Rule P.

Proof of 2615:

The proof is based on 2200. The wff P_0 referred to in the statement of 2200 will be the wff

$$[[Wff]x_2r_2 \wedge \exists a \forall g_{22} \cdot [Assign]g_{22} \supset \exists i_a \cdot [Value]g_{22}x_2i_a \\ \wedge \forall j_2 \cdot [Value]g_{22}x_2j_2 \supset j_2 \stackrel{2}{=} i_a].$$

Note that x_2 and r_2 are the only variables free in P_0 . For convenience we shall write $P(A_2; B_2)$ as an abbreviation for $[S_{A_2B_2}^{x_2r_2}P_0]$.

2615.1 $\vdash [Assign]g_{22} \supset \cdot [Value]g_{22}[2^{17}]_{\iota_{1(01)}} \wedge \forall j_2 \cdot [Value]g_{22}[2^{17}]j_2$

$$\supset j_2 \stackrel{2}{=} \iota_{1(01)} \quad 2604, 2609$$

.2 $\vdash P(2^{17}; 3^5 \times 5^{[3^3 \times 5^5]})$ Rule P, 164, 165, Gen: .1, 2202

.3 $\exists u_a \Phi_{022}y_2u_a, \exists u_b \Phi_{022}y_2u_b, j_2 \stackrel{2}{=} Q_{0bb} \vdash a = b$ 2314

.4 $\exists u_a \Phi_{022}y_2u_a, \exists u_b \Phi_{022}y_2u_b, j_2 \stackrel{2}{=} Q_{0bb} \vdash j_2 = Q_{0aa}$ 300: .3

.5 $\exists u_a \Phi_{022}y_2u_a, \exists b [\exists u_b \Phi_{022}y_2u_b \wedge j_2 \stackrel{2}{=} Q_{0bb}] \vdash j_2 \stackrel{2}{=} Q_{0aa}$ 173: .4

.6 $\vdash [Value]g_{22}[2^{[19^{y_2}]}]j_2 \supset \cdot [TS]y_2 \supset \exists b \cdot \exists u_b \Phi_{022}y_2u_b \wedge j_2 \stackrel{2}{=} Q_{0bb}$ 2610

.7 $[TS]y_2, \exists u_a \Phi_{022}y_2u_a \vdash [Value]g_{22}[2^{[19^{y_2}]}]j_2 \supset j_2 \stackrel{2}{=} Q_{0aa}$.5, .6

.8 $[TS]y_2, \exists u_a \Phi_{022}y_2u_a, [Assign]g_{22} \vdash [Value]g_{22}[2^{[19^{y_2}]}]Q_{0aa}$

$$\wedge \forall j_2 \cdot [Value]g_{22}[2^{[19^{y_2}]}]j_2 \supset j_2 \stackrel{2}{=} Q_{0aa} \quad .7, 2605$$

$$2615.9 [TS]y_2, \exists u_a \Phi_{022}y_2u_a \vdash P(2^{[19y_2]}; 3^{[3^3 \times 5y_2]} \times 5^{y_2}) \quad .8, 2203$$

$$.10 [TS]y_2 \vdash \exists a \exists u_a \Phi_{022}y_2u_a \supset P(2^{[19y_2]}; 3^{[3^3 \times 5y_2]} \times 5^{y_2})$$

Deduction Theorem, 171: .9

$$.11 \vdash [TS]y_2 \supset \exists a \exists u_a \Phi_{022}y_2u_a \quad 2308$$

$$.12 \vdash \forall y_2 . [TS]y_2 \supset P(2^{[19y_2]}; 3^{[3^3 \times 5y_2]} \times 5^{y_2}) \quad .10, .11.$$

Let $\mathcal{H}^1$ be $[Variable]w_2t_2, \exists x_a \Phi_{022}t_2x_a, [Assign]g_{22}$.

$$.13 \mathcal{H}^1 \vdash [Var]w_2 \quad \text{Premiss, 2107}$$

$$.14 \mathcal{H}^1 \vdash [Value]g_{22}[2^{w_2}] [g_{22}w_2] \quad \text{Premiss, .13, 2606}$$

$$.15 \mathcal{H}^1 \vdash \forall j_2 . [Value]g_{22}[2^{w_2}]j_2 \supset j_2 \stackrel{2}{=} g_{22}w_2 \quad 2611$$

$$.16 \mathcal{H}^1, g_{22}w_2 \stackrel{2}{=} i_a \vdash [Value]g_{22}[2^{w_2}]i_a \wedge \forall j_2 . [Value]g_{22}[2^{w_2}]j_2 \supset j_2 \stackrel{2}{=} i_a \quad \text{Rule P, 200: .14, .15, Premiss}$$

$$.17 \mathcal{H}^1 \vdash \exists i_a [g_{22}w_2 \stackrel{2}{=} i_a] \supset \exists i_a . [Value]g_{22}[2^{w_2}]i_a \wedge \forall j_2 . [Value]g_{22}[2^{w_2}]j_2 \supset j_2 = i_a \quad \text{Deduction Theorem, 176: .16}$$

$$.18 \mathcal{H}^1 \vdash [Variable]w_2t_2 \supset \Phi_{022}t_2[g_{22}w_2] \quad \text{Rule S: Premiss; Def. of [Assign]}$$

$$.19 \mathcal{H}^1 \vdash \exists i_a [g_{22}w_2 \stackrel{2}{=} i_a] \quad 2320, \text{Premiss, .18}$$

$$.20 [Variable]w_2t_2, \exists x_a \Phi_{022}t_2x_a \vdash P(2^{w_2}; t_2) \quad \text{Deduction Theorem, Rule P, Gen, Exist}_1 (167): .17, .19, 2204$$

$$.21 \vdash [Variable]w_2t_2 \supset . \exists a \exists x_a \Phi_{022}t_2x_a \supset P(2^{w_2}; t_2) \quad 173: .20$$

$$.22 \vdash [Variable]w_2t_2 \supset \exists a \exists x_a \Phi_{022}t_2x_a \quad 2104, 2308$$

$$.23 \vdash \forall w_2 \forall t_2 . [Variable]w_2t_2 \supset P(2^{w_2}; t_2) \quad .21, .22.$$

Let $\mathcal{H}^2$ be

$$[Wff]y_2[3^{t_2} \times 5^{u_2}], [Wff]z_2u_2, [TS]t_2, \forall g_{22}[[Assign]g_{22}$$

$$\supset \exists i_c . [Value]g_{22}y_2i_c \wedge \forall j_2 . [Value]g_{22}y_2j_2 \supset j_2 \stackrel{2}{=} i_c],$$

$$\forall g_{22}[[Assign]g_{22} \supset \exists k_d . [Value]g_{22}z_2k_d \wedge \forall j_2 . [Value]g_{22}z_2j_2 \supset j_2 \stackrel{2}{=} k_d].$$

- 2615.24 $\vdash [Value]_{g_{22}y_2i_c} \wedge [Wff]_{y_2}[3^{t_2} \times 5^{u_2}] \supset \Phi_{022}[3^{t_2} \times 5^{u_2}]i_c$ 2603
- .25 $\mathcal{H}^2 \vdash \exists i_c \Phi_{022}[3^{t_2} \times 5^{u_2}]i_c$ Premiss, .24, 2410
- .26 $\mathcal{H}^2 \vdash \exists k_d \Phi_{022}u_2k_d$ Premiss, 2603, 2410
- .27 $\mathcal{H}^2 \vdash [TS]u_2$ Premiss, 2201
- .28 $\vdash [[TS]t_2 \wedge [TS]u_2 \wedge \exists i_c \Phi_{022}[3^{t_2} \times 5^{u_2}]i_c] \supset \exists a \exists b \exists x_{ab} . i_c \stackrel{2}{=} x_{ab}$ 2310, 2316
- .29 $\vdash i_c \stackrel{2}{=} x_{ab} \supset .c = (ab)$ Axiom 10
- .30 $\vdash \exists a \exists b \exists x_{ab} [i_c \stackrel{2}{=} x_{ab}] \supset \exists a \exists b . c = (ab)$.29
- .31 $\mathcal{H}^2 \vdash \exists a \exists b . c = (ab)$ Premiss, .25, .27, .28, .30
- .32 $\mathcal{H}^2, c = (ab) \vdash \Phi_{022}[3^{t_2} \times 5^{u_2}]i_{ab}$ 300: .25; 2316
- .33 $\mathcal{H}^2, c = (ab) \vdash \exists n_b \Phi_{022}u_2n_b$ Premiss, .27, .32, 2309
- .34 $\mathcal{H}^2, c = (ab) \vdash d = b$.26, .33, 2314.

Let G_0 be the wff

- $\forall g_{22} [[Assign]_{g_{22}} \supset \exists i_{ab} . [Value]_{g_{22}y_2i_{ab}} \wedge \forall j_2 . [Value]_{g_{22}y_2j_2} \supset j_2 \stackrel{2}{=} i_{ab}]$
 $\wedge \forall g_{22} [[Assign]_{g_{22}} \supset \exists k_b . [Value]_{g_{22}z_2k_b}$
 $\wedge \forall j_2 . [Value]_{g_{22}z_2j_2} \supset j_2 \stackrel{2}{=} k_b].$
- .35 $\mathcal{H}^2, c = (ab) \vdash G_0$ 300, Premiss, .34
- .36 $\mathcal{H}^2 \vdash \exists a \exists b [c = (ab)] \supset \exists a \exists b G_0$ 177, .35
- .37 $\mathcal{H}^2 \vdash \exists a \exists b G_0$.31, .36
- .38 $[Wff]_{y_2}[3^{t_2} \times 5^{u_2}], [Wff]_{z_2}u_2, [TS]t_2,$
 $\exists c \forall g_{22} [[Assign]_{g_{22}} \supset \exists i_c . [Value]_{g_{22}y_2i_c}$
 $\wedge \forall j_2 . [Value]_{g_{22}y_2j_2} \supset j_2 \stackrel{2}{=} i_c],$
 $\exists d \forall g_{22} [[Assign]_{g_{22}} \supset \exists k_d . [Value]_{g_{22}z_2k_d}$
 $\wedge \forall j_2 . [Value]_{g_{22}z_2j_2} \supset j_2 \stackrel{2}{=} k_d] \vdash \exists a \exists b G_0$ 173: .37

$$\begin{aligned}
 2615.39 \vdash & [P(y_2; 3^{t_2} \times 5^{u_2}) \wedge P(z_2; u_2) \wedge [TS]t_2] \\
 & \supset . [Wff]y_2[3^{t_2} \times 5^{u_2}] \wedge [Wff]z_2u_2 \wedge [TS]t_2 \wedge \exists a \exists b G_0 \quad .38.
 \end{aligned}$$

Let $\mathcal{H}^3$ be $[Wff]y_2[3^{t_2} \times 5^{u_2}]$, $[Wff]z_2u_2$, $[TS]t_2$, $[Value]g_{22}y_2i_{ab}$,
 $\forall j_2[[Value]g_{22}y_2j_2 \supset j_2 \stackrel{2}{=} i_{ab}]$, $[Value]g_{22}z_2k_b$, $\forall j_2[[Value]g_{22}z_2j_2 \supset j_2 \stackrel{2}{=} k_b]$, $[Assign]g_{22}$.

$$.40 \mathcal{H}^3 \vdash \Phi_{022}[3^{t_2} \times 5^{u_2}]i_{ab} \wedge \Phi_{022}u_2k_b \quad 2603, \text{Premiss}$$

$$.41 \mathcal{H}^3 \vdash [Value]g_{22}[2^7 * y_2 * z_2 * 2^{11}] [i_{ab}k_b] \quad 2607, \text{Premiss}, .40.$$

Let J_0 be the wff

$$\begin{aligned}
 & [\Phi_{022}[3^{t_2} \times 5^{u_2}]\bar{y}_{ab} \wedge \Phi_{022}u_2\bar{z}_b \wedge [Value]g_{22}y_2\bar{y}_{ab} \wedge [Value]g_{22}z_2\bar{z}_b \wedge j_2 \\
 & \stackrel{2}{=} [\bar{y}_{ab}\bar{z}_b]].
 \end{aligned}$$

$$\begin{aligned}
 .42 \vdash & [Value]g_{22}[2^7 * y_2 * z_2 * 2^{11}]j_2 \supset . [[Wff]y_2[3^{t_2} \times 5^{u_2}] \\
 & \wedge [Wff]z_2u_2 \wedge [TS]t_2] \supset \exists \bar{a} \exists b \exists \bar{y}_{ab} \exists \bar{z}_b J_0 \quad 2612
 \end{aligned}$$

$$.43 \mathcal{H}^3 \vdash [Value]g_{22}y_2\bar{y}_{ab} \supset \bar{y}_{ab} \stackrel{2}{=} i_{ab} \quad \text{Rule S: Premiss}$$

$$.44 \mathcal{H}^3 \vdash [Value]g_{22}z_2\bar{z}_b \supset \bar{z}_b \stackrel{2}{=} k_b \quad \text{Rule S: Premiss}$$

$$.45 \vdash \bar{y}_{ab} \stackrel{2}{=} i_{ab} \supset . (\bar{ab}) = (ab) \quad \text{Axiom 10}$$

$$.46 \vdash [(\bar{ab}) = (ab)] = . \bar{a} = a \wedge b = b \quad \text{Axiom 9}$$

$$.47 \mathcal{H}^3, J_0 \vdash \bar{a} = a \quad \text{Premiss, .43, .45, .46}$$

$$.48 \mathcal{H}^3, J_0 \vdash \bar{b} = b \quad \text{Premiss, .43, .45, .46}$$

$$.49 \mathcal{H}^3, J_0 \vdash \bar{y}_{ab} \stackrel{2}{=} i_{ab} \quad \text{Premiss, .43, 300, .47, .48}$$

$$.50 \mathcal{H}^3, J_0 \vdash \bar{z}_b \stackrel{2}{=} k_b \quad \text{Premiss, .44, 300, .48}$$

$$.51 \mathcal{H}^3, J_0 \vdash \bar{y}_{ab} \stackrel{ab}{=} i_{ab} \quad 205: .49$$

$$.52 \mathcal{H}^3, J_0 \vdash \bar{z}_b \stackrel{b}{=} k_b \quad 205: .50$$

$$.53 \mathcal{H}^3, J_0 \vdash j_2 \stackrel{2}{=} [\bar{y}_{ab}\bar{z}_b] \quad 300: \text{Premiss, .47, .48}$$

$$2615.54 \mathcal{H}^3, J_0 \vdash j_2 \stackrel{2}{=} [i_{ab}k_b] \quad \text{R': .51, .52, .53}$$

$$.55 \mathcal{H}^3, \exists \bar{a} \exists \bar{b} \exists \bar{y}_{\bar{a}\bar{b}} \exists \bar{z}_{\bar{b}} J_0 \vdash j_2 \stackrel{2}{=} [i_{ab}k_b] \quad 172, 173: .54$$

$$.56 \mathcal{H}^3 \vdash \forall j_2. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]j_2 \supset j_2 \stackrel{2}{=} [i_{ab}k_b] \\ \text{Premiss, .42, .55}$$

$$.57 \mathcal{H}^3 \vdash \exists i_a. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]i_a \\ \wedge \forall j_2. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]j_2 \supset j_2 \stackrel{2}{=} i_a \\ \text{Rule P, Exist (166): .41, .56}$$

$$.58 [Wff]_{y_2}[3^{t_2} \times 5^{u_2}], [Wff]_{z_2}u_2, [TS]t_2, \exists i_{ao}[[Value]_{g_{22}y_2}i_{ab} \\ \wedge \forall j_2. [Value]_{g_{22}y_2}j_2 \supset j_2 \stackrel{2}{=} i_{ab}], \exists k_b[[Value]_{g_{22}z_2}k_b \\ \wedge \forall j_2. [Value]_{g_{22}z_2}j_2 \supset j_2 \stackrel{2}{=} k_b] \vdash [Assign]_{g_{22}} \\ \supset \exists i_a. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]i_a \\ \wedge \forall j_2. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]j_2 \supset j_2 \stackrel{2}{=} i_a \quad .57$$

$$.59 [Wff]_{y_2}[3^{t_2} \times 5^{u_2}], [Wff]_{z_2}u_2, [TS]t_2, G_0 \vdash [Assign]_{g_{22}} \\ \supset \exists i_a. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]i_a \\ \wedge \forall j_2. [Value]_{g_{22}}[2^7 * y_2 * z_2 * 2^{11}]j_2 \supset j_2 \stackrel{2}{=} i_a \quad .58$$

$$.60 [Wff]_{y_2}[3^{t_2} \times 5^{u_2}], [Wff]_{z_2}u_2, [TS]t_2, G_0 \vdash P(2^7 * y_2 * z_2 * 2^{11}; t_2) \\ .59, 2205$$

$$.61 [Wff]_{y_2}[3^{t_2} \times 5^{u_2}], [Wff]_{z_2}u_2, [TS]t_2, \\ \exists a \exists b G_0 \vdash P(2^7 * y_2 * z_2 * 2^{11}; t_2) \quad 173: .60$$

$$.62 \vdash \forall y_2 \forall z_2 \forall t_2 \forall u_2. [P(y_2; 3^{t_2} \times 5^{u_2}) \wedge P(z_2; u_2) \wedge [TS]t_2] \\ \supset P(2^7 * y_2 * z_2 * 2^{11}; t_2) \quad .39, .61.$$

Let $\mathcal{H}^4$ be $[Variable]_{w_2}t_2, [Wff]_{y_2}u_2,$

$$\forall g_{22}[[Assign]_{g_{22}} \supset \exists i_a. [Value]_{g_{22}y_2}i_a \wedge \forall j_2. [Value]_{g_{22}y_2}j_2 \supset j_2 \stackrel{2}{=} i_a], \\ \exists m_b \Phi_{022}t_2m_b, [Assign]_{h_{22}}.$$

$$.63 \mathcal{H}^4 \vdash [Value]_{h_{22}}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}] \\ [\lambda z_b. \iota x_a. [Value] [[Mod]_{h_{22}w_2}z_b]y_2x_a] \quad 2614$$

2615.64 $\mathcal{H}^4 \vdash [Assign] [[Mod]h_{22}w_2z_b]$

2614.12

(Note that our present $\mathcal{H}^4$ is the same as the $\mathcal{H}$ used in the proof of 2614.)

.65 $\mathcal{H}^4 \vdash [Value] [[Mod]h_{22}w_2z_b]y_2[x_a \cdot [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$
2614.13.

Let H_0 be the wff

$$[j_2 \stackrel{2}{=} f_{cd} \wedge \forall z_d \cdot \Phi_{022}t_2z_d \wedge \Phi_{022}u_2[f_{cd}z_d] \\ \wedge [Value] [[Mod]h_{22}w_2z_a]y_2[f_{cd}z_a]].$$

.66 $\mathcal{H}^4 \vdash [Value]h_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]j_2 \supset \exists c \exists d \exists f_{cd}H_0$

Premiss, 2613

.67 $H_0 \vdash \Phi_{022}t_2v_d \wedge \Phi_{022}u_2[f_{cd}v_d]$

Rule P, Rule S: Premiss

.68 $\mathcal{H}^4 \vdash \Phi_{022}t_2z_b$

2614.10

.69 $\mathcal{H}^4 \vdash \Phi_{022}u_2[x_a \cdot [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$

2614.14

.70 $\mathcal{H}^4, H_0 \vdash d = b$

.67, .68, 2313

.71 $\mathcal{H}^4, H_0 \vdash c = a$

.67, .69, 2313

.72 $\mathcal{H}^4, H_0 \vdash j_2 \stackrel{2}{=} f_{ab} \wedge [Value] [[Mod]h_{22}w_2z_b]y_2[f_{ab}z_b]$

Rule P, Rule S: Premiss; 300: .70, .71

.73 $\vdash \forall j_2 [[Value]g_{22}y_2j_2 \supset j_2 \stackrel{2}{=} i_a] \supset .[[Value]g_{22}y_2m_2 \supset m_2 \stackrel{2}{=} i_a]$

$\wedge .[[Value]g_{22}y_2k_2 \supset k_2 \stackrel{2}{=} i_a]$

Rule P: 145

.74 $\vdash m_2 \stackrel{2}{=} i_a \supset .[k_2 \stackrel{2}{=} i_a] = [m_2 \stackrel{2}{=} k_2]$

204

.75 $\vdash \exists i_a [[Value]g_{22}y_2i_a \wedge \forall j_2 . [Value]g_{22}y_2j_2 \supset j_2 \stackrel{2}{=} i_a]$

$\supset .[[Value]g_{22}y_2m_2 \wedge [Value]g_{22}y_2k_2] \supset m_2 \stackrel{2}{=} k_2$

Rule P, 170: .73, .74

.76 $\mathcal{H}^4 \vdash [Assign]g_{22} \supset .[[Value]g_{22}y_2m_2 \wedge [Value]g_{22}y_2k_2]$

$\supset m_2 \stackrel{2}{=} k_2$

Rule S: Premiss; Rule P: .75

- 2615.77 $\mathcal{H}^4, H_0 \vdash [f_{ab}z_b] \stackrel{2}{=} [\iota x_a . [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$
 Sub (119): .76; Rule P: .64, .72, .65
- .78 $\mathcal{H}^4, H_0 \vdash \forall z_b . [f_{ab}z_b] \stackrel{a}{=} [\lambda z_b . \iota x_a . [Value] [[Mod]h_{22}w_2z_b]y_2x_a]z_b$
 RR: 201, 103, .77; Gen
- .79 $\mathcal{H}^4, H_0 \vdash f_{ab} \stackrel{ab}{=} [\lambda z_b . \iota x_a . [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$
 .78, Axiom 3
- .80 $\mathcal{H}^4, H_0 \vdash j_2 \stackrel{2}{=} [\lambda z_b . \iota x_a . [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$
 Rule P, EE-Rules (205): .72, .79
- .81 $\mathcal{H}^4 \vdash [Value]h_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]j_2 \supset j_2$
 $\stackrel{2}{=} [\lambda z_b . \iota x_a . [Value] [[Mod]h_{22}w_2z_b]y_2x_a]$.66, .80
- .82 $\mathcal{H}^4 \vdash \exists i_{ab} . [Value]h_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]i_{ab}$
 $\wedge \forall j_2 . [Value]h_{22}[2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}]j_2 \supset j_2 \stackrel{2}{=} i_{ab}$
 Gen, Rule P, Exist: .81, .63
- .83 $[Variable]w_2t_2, [Wff]y_2u_2, \forall g_{22} [[Assign]g_{22}$
 $\supset \exists i_a . [Value]g_{22}y_2i_a \wedge \forall j_2 . [Value]g_{22}y_2j_2 \supset j_2 \stackrel{2}{=} i_a],$
 $\exists m_b \Phi_{022}t_2m_b \vdash P(2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}; 3^{u_2} \times 5^{t_2})$
 .82, 2206
- .84 $\vdash [Variable]w_2t_2 \wedge P(y_2; u_2) \wedge \exists b \exists m_b \Phi_{022}t_2m_b$
 $\supset P(2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}; 3^{u_2} \times 5^{t_2})$.83
- .85 $\vdash [Variable]w_2t_2 \supset \exists b \exists m_b \Phi_{022}t_2m_b$ 2104, 2308
- .86 $\vdash \forall w_2 \forall t_2 \forall y_2 \forall u_2 . [[Variable]w_2t_2 \wedge P(y_2; u_2)]$
 $\supset P(2^7 * 2^{13} * 2^{w_2} * y_2 * 2^{11}; 3^{u_2} \times 5^{t_2})$.84, .85
- .87 $\vdash [Wff]x_2r_2 \supset P_0$ 2200: .2, .12, .23, .62, .86.

Theorem 2615 follows from .87 by Rule P and the definition of P_0 .

Proof of 2616:

Let $\mathcal{H}$ be $[Assign]h_{22}, [Wff]y_2u_2, \exists n_a \Phi_{022}u_2n_a$.

Let G_0 be

$$\forall g_{22} [[Assign]_{g_{22}} \supset \exists i_c . [Value]_{g_{22}y_2i_c} \wedge \forall j_2 . [Value]_{g_{22}y_2j_2} \supset j_2 \stackrel{2}{=} i_c].$$

$$2616.1 \mathcal{H}, G_0 \vdash \exists i_c [Value]_{h_{22}y_2i_c} \quad \text{Premiss}$$

$$.2 \vdash [[Value]_{h_{22}y_2i_c} \wedge [Wff]_{y_2u_2}] \supset \Phi_{022}u_2i_c \quad 2603$$

$$.3 \mathcal{H}, G_0 \vdash \exists i_c \Phi_{022}u_2i_c \quad .1, .2$$

$$.4 \mathcal{H}, G_0 \vdash c = a \quad 2314, \text{Premiss}, .3$$

$$.5 \mathcal{H}, G_0 \vdash \forall g_{22} . [Assign]_{g_{22}} \supset \exists i_a . [Value]_{g_{22}y_2i_a} \\ \wedge \forall j_2 . [Value]_{g_{22}y_2j_2} \supset j_2 \stackrel{2}{=} i_a \quad 300: .4, \text{Premiss}$$

$$.6 \mathcal{H}, \exists c G_0 \vdash \forall g_{22} . [Assign]_{g_{22}} \supset \exists i_a . [Value]_{g_{22}y_2i_a} \\ \wedge \forall j_2 . [Value]_{g_{22}y_2j_2} \supset j_2 \stackrel{2}{=} i_a \quad 173: .5$$

$$.7 \mathcal{H} \vdash \exists c G_0 \quad \text{Premiss, 2615.}$$

Theorem 2616 now follows from .6, .7, and 2614 by the Deduction Theorem and Rule P.

Proof of 2617:

$$2617.1 [Value]_{g_{22}x_2i_a} \wedge \forall j_2 [[Value]_{g_{22}x_2j_2} \supset j_2 \stackrel{2}{=} i_a] \\ \vdash [Value]_{g_{22}x_2i_a} \wedge \forall j_2 . [Value]_{g_{22}x_2j_2} \supset i_a \stackrel{2}{=} j_2 \quad \text{RR: 202, Premiss}$$

$$.2 [Value]_{g_{22}x_2i_a} \wedge \forall j_2 [[Value]_{g_{22}x_2j_2} \supset j_2 \stackrel{2}{=} i_a] \\ \vdash \exists i_2 . [Value]_{g_{22}x_2i_2} \wedge \forall j_2 . [Value]_{g_{22}x_2j_2} \supset i_2 \stackrel{2}{=} j_2 \quad \text{Exist (166): .1}$$

$$.3 \exists a \forall g_{22} [[Assign]_{g_{22}} \supset \exists i_a . [Value]_{g_{22}x_2i_a} \\ \wedge \forall j_2 . [Value]_{g_{22}x_2j_2} \supset j_2 \stackrel{2}{=} i_a] \vdash [Assign]_{g_{22}} \\ \supset \exists i_2 . [Value]_{g_{22}x_2i_2} \wedge \forall j_2 . [Value]_{g_{22}x_2j_2} \supset i_2 \stackrel{2}{=} j_2 \\ 172: .2; \text{Deduction Theorem, Rule P, Rule S, 173.}$$

Theorem 2617 now follows from .3 and 2615 by Rule P.

Proof of 2618:

$$2618.1 \vdash \exists i_2 [p_{02}i_2 \wedge \forall j_2 \cdot p_{02}j_2 \supset i_2 \stackrel{2}{=} j_2] \supset p_{02}[\iota_{2(02)}p_{02}] \quad 216$$

$$\begin{aligned} &.2 \vdash \exists i_2 [[Value]_{g_{22}x_2}i_2 \wedge \forall j_2 \cdot [Value]_{g_{22}x_2}j_2 \supset i_2 \stackrel{2}{=} j_2] \\ &\quad \supset \cdot [Value]_{g_{22}x_2}[\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \quad \text{Sub: .1.} \end{aligned}$$

Theorem 2618 follows from .2 and 2617 by Rule P.

Proof of 2619:

$$\begin{aligned} 2619.1 \vdash \forall j_2 [[Value]_{g_{22}x_2}j_2 \supset i_2 \stackrel{2}{=} j_2] \supset \cdot [[Value]_{g_{22}x_2}y_2 \supset i_2 \stackrel{2}{=} y_2] \\ \wedge \cdot [Value]_{g_{22}x_2}[\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \supset i_2 \\ \stackrel{2}{=} [\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \quad \text{Rule P: 145} \end{aligned}$$

$$\begin{aligned} &.2 \vdash i_2 \stackrel{2}{=} [\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \supset \cdot [i_2 \stackrel{2}{=} y_2] \\ &\quad = [\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \stackrel{2}{=} y_2 \quad 203 \end{aligned}$$

$$\begin{aligned} &.3 [Wff]_{x_2r_2}, [Assign]_{g_{22}}, [Value]_{g_{22}x_2}y_2, \\ &\quad [Value]_{g_{22}x_2}i_2 \wedge \forall j_2 [[Value]_{g_{22}x_2}j_2 \supset i_2 \stackrel{2}{=} j_2] \\ &\quad \vdash [\iota_{2(02)} \cdot [Value]_{g_{22}x_2}] \stackrel{2}{=} y_2 \quad \text{Rule P: Premiss, .1, .2, 2618.} \end{aligned}$$

Now apply 172 and the Deduction Theorem to .3; then use Rule P and 2617 to obtain 2619.

Theorem 2620 follows easily from 2618 and 2619.

Theorems 2700 and 2701 follow easily from 212 and 211, respectively, and the definition of $[Val]$.

Proof of 2702:

$$2702.1 \vdash [Wf]_{n_2} \stackrel{0}{=} [Wff]_{n_2}[\iota_{2(02)}[Wff]_{n_2}] \quad 103, \text{ def. of } [Wf]$$

$$\begin{aligned} &.2 \vdash [[Wff]_{n_2}[\iota_{2(02)}[Wff]_{n_2}] \wedge [Assign]_{g_{22}}] \\ &\quad \supset [Value]_{g_{22}n_2}[\iota_{2(02)} \cdot [Value]_{g_{22}n_2}] \quad 2618 \end{aligned}$$

$$\begin{aligned} &.3 [Assign]_{g_{22}}, [Wf]_{n_2} \vdash [\iota_{2(02)} \cdot [Value]_{g_{22}n_2}] \stackrel{2}{=} [Val]_{g_{22}n_2} \\ &\quad 2701, 205 \end{aligned}$$

2702.4 $[Assign]_{g_{22}}, [Wf]_{n_2} \vdash [Value]_{g_{22}n_2} [[Val]_{g_{22}n_2}]$.1, .2, .3.

Theorem 2702 follows from .4 by the Deduction Theorem and Rule P.

Proof of 2703:

2703.1 $[Assign]_{g_{22}}, [Wf]_{n_2} \vdash [Value]_{g_{22}n_2} y_2 \stackrel{0}{=} [\iota_{2(02)} \cdot [Value]_{g_{22}n_2}]$
 $\stackrel{2}{=} y_2$ Sub: 2620; Rule P: Premiss, 2702.1
 .2 $[Assign]_{g_{22}}, [Wf]_{n_2} \vdash [Value]_{g_{22}n_2} y_2 \stackrel{0}{=} [Val]_{g_{22}n_2} \stackrel{2}{=} y_2$
 R': .1, 2702.3.

Theorem 2703 follows from .2 by the Deduction Theorem and Rule P.

Theorem 2704 follows from 2602 and 2703 by Rule P.

Theorems 2705, 2706 and 2707 follow from 2604, 2605 and 2606, respectively, by Rule Sub, Rule P, and 2704.

Proof of 2708:

Let $\mathcal{H}$ be $[Assign]_{g_{22}}, [Wff]_{j_2} [3^{t_2} \times 5^{u_2}], [Wff]_{m_2} u_2, [TS]_{t_2},$

$[Val]_{g_{22}j_2} \stackrel{2}{=} y_{ab}, [Val]_{g_{22}m_2} \stackrel{2}{=} z_b.$

2708.1 $\mathcal{H} \vdash [Wf]_{j_2} \wedge [Wf]_{m_2}$ Premiss, 2209
 .2 $\mathcal{H} \vdash [Value]_{g_{22}j_2} y_{ab} \wedge [Value]_{g_{22}m_2} z_b$ Premiss, .1, 2703
 .3 $\mathcal{H} \vdash \Phi_{022} [3^{t_2} \times 5^{u_2}]_{y_{ab}} \wedge \Phi_{022} u_2 z_b$ Premiss, .2, 2603
 .4 $\mathcal{H} \vdash [Val]_{g_{22}} [2^7 * j_2 * m_2 * 2^{l_1}] \stackrel{2}{=} [y_{ab} z_b]$
 Premiss, .2, .3, 2607, 2704.

Theorem 2708 follows from .4 by the Deduction Theorem and Rule P.

Proof of 2709:

Let $\mathcal{H}$ be $[Assign]_{g_{22}}, [Variable]_{w_2} t_2, [Wff]_{y_2} u_2, \exists m_b \Phi_{022} t_2 m_b,$

$\exists n_a \Phi_{022} u_2 n_a.$

2709.1 $\mathcal{H} \vdash [Val]_{g_{22}} [2^7 * 2^{l_3} * 2^{w_2} * y_2 * 2^{l_1}]$
 $\stackrel{2}{=} [\lambda z_b . \iota x_a . [Value] [[Mod]_{g_{22}w_2} z_b] y_2 x_a]$ Premiss, 2616, 2704

- .2 $\mathcal{H} \vdash [Wf]y_2$ Premiss, 2209
 .3 $\mathcal{H} \vdash \Phi_{022}t_2z_b$ Premiss, 2316
 .4 $\mathcal{H} \vdash [Assign] [[Mod]g_{22}w_2z_b]$ Premiss, .3, 2503
 .5 $\mathcal{H} \vdash [Value] [[Mod]g_{22}w_2z_b]y_2x_a \stackrel{0}{=} .[Val] [[Mod]g_{22}w_2z_b]y_2 \stackrel{2}{=} x_a$
.4, .2, 2703
 .6 $\mathcal{H} \vdash [Val]g_{22}[2' * 2^{13} * 2^{w_2} * y_2 * 2^{11}]$
 $\stackrel{2}{=} [\lambda z_b . \iota x_a . [Val] [[Mod]g_{22}w_2z_b]y_2 \stackrel{2}{=} x_a]$ R': .1, .5.

(Note that the restrictions on R' are satisfied here.)

Theorem 2709 follows from .6 by the Deduction Theorem and Rule P.

IV.4. Q strengthens Q_0

By making appropriate changes of variables, we can regard each wff of Q_0 as a wff of Q. It is trivial to see that every theorem of Q_0 is in this sense a theorem of Q. It should be noted, however, that certain wffs of Q_0 are theorems of Q but not theorems of Q_0 .

The proof of this fact follows familiar lines, so we content ourselves with a brief sketch of it. Let [*Infin*] be an Axiom of Infinity for type 1; i.e., [*Infin*] asserts that $\mathfrak{D}_1$ is infinite. If we add [*Infin*] to Q as an axiom, we obtain a system Q^∞ in which we can define the finite cardinals in the usual way as entities of finite type (i.e. as wffs with a type in K^1), and prove Peano's postulates. The numerals thus obtained can be used in place of the numerals of Chapter III for Gödel numbers of wffs of Q_0 , and the formalization of the semantics of Q_0 in Q^∞ using these numerals proceeds as before, with only minor changes in type symbols.

It is then possible to prove in Q^∞ a statement [*Consis*] which expresses the fact that the system Q_0^∞ obtained from Q_0 by adding [*Infin*] as an axiom is consistent. This is done in the usual way, by showing that the axioms of Q_0^∞ are all true and that the rule of inference preserves truth. It is of course trivial to show in Q^∞ that any particular axiom of Q_0^∞ is true by using (the analogues of) Theorems 2710 and 2711; however this is not

sufficient since Q_0 has infinitely many axioms. Nevertheless by using Theorems 2705–2711 one can show in Q^∞ that all axioms of Q_0^∞ are true, and that the rule of inference preserves truth.

Thus one obtains a proof of $[Consis]$ in Q^∞ , and hence a proof of $[[Infin] \supset [Consis]]$ in Q . Now $[[Infin] \supset [Consis]]$ is a wff of Q_0 , but not a theorem of Q_0 . For if it were a theorem of Q_0 then $[Consis]$ would be a theorem of Q_0^∞ , contradicting Gödel's Second Incompleteness Theorem¹³.

(¹³) See [7], p. 196, or [8] p. 70.

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