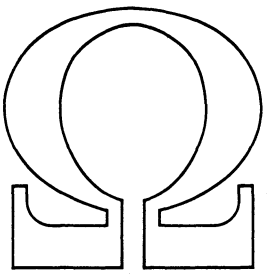
Perspectives in Mathematical Logic

Gerald E. Sacks

Higher Recursion Theory



Springer-Verlag



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Ω-Group:
R. O. Gandy, H. Hermes, A. Levy, G. H. Müller,
G. E. Sacks, D. S. Scott

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Higher Recursion Theory



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In Memory Of My Father

Irwin "Pete" Sacks

Preface to the Series
Perspectives in Mathematic Logic

(Edited by the "Ω-group for Mathematical Logic" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps of guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of* results *and* techniques.

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of values, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R.O. Gandy, A. Levy, G.H. Müller, G. Sacks, D.S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F.K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and*

that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the overall plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the autor was keen to write and which would indeed contribute a perspective.

Oberwolfach, September 1975

Acknowledgements. *In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag und Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970–1973) as an initial help which made our existence as a working group possible.*

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F.K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R.O. Gandy *H. Hermes*
A. Levy *G.H. Müller*
G. Sacks *D.S. Scott*

Author's Preface

Higher recursion theory (HRT) has been one of my two major obsessions for the last twenty years. Nonetheless my interest has not waned. Perhaps because, as Browning claimed:

"The best is yet to be."

I was talked into the subject, skittish all the way, by G. Kreisel. The old devil insisted, in several conversations beginning in 1961, on the existence of golden generalizations of recursion theory in which infinitely long computations converged. I listened for hours, without understanding a word, to his tales of the mother lode of recursion theory hidden far below the peaks of effective descriptive set theory.

My initial reaction to his yarns was naive. One could readily generalize the static (or syntactic) aspects of classical recursion theory (CRT) such as the enumeration theorem, but one could not hope to lift dynamic results such as the Friedberg-Muchnik solution of Post's problem to higher domains. Clearly the dynamic facts of CRT were inseparable from certain combinatoric properties of finite sets, and these properties, being "truly finite" in nature, could not be generalized fruitfully. In 1961 I was trying to prove the density of the recursively enumerable degrees, and that investigation had a no-nonsense flavor that rendered all exotic pursuits unpalatable.

In 1963 I began to understand what Kreisel was talking about. He had unearthed a compactness theorem for ω-logic in which *hyperarithmetic* played the part of *finite*. His result was: if A is a Π^1_1 set of axioms of ω-logic and every hyperarithmetic subset of A has a model, then A has a model. (This eye-opener was the forerunner of Barwise's compactness theorem for $\dot{\Sigma}_1$ admissible sets.) Kreisel's proposal of 1961, as understood by me in 1963, was: replace the natural numbers by a Π^1_1 set I of indices for the hyperarithmetic sets, and recursively enumerable by Π^1_1; most important of all, let "finite" mean hyperarithmetic. The more one scrutinized his initially obscure proposal, the more lucid it became. Right away it was clear that a "finite" union of "finite" sets was "finite". A set A was "recursive" if A and I-A were Π^1_1; in Kreisel's terminology A was hyperarithmetic on I. It was immediate that a "recursive" function restricted to a "finite" set was "finite".

Now it was plausible that the dynamic results of CRT would lift up. The new notions were dubbed metarecursively enumerable, metafinite, and metare-

cursive. For smoothness the set I of indices of hyperarithmetic sets was replaced by a Π^1_1 set of unique notations for recursive ordinals, and finally by the set of recursive ordinals. Thus a metarecursively enumerable set was simply a set of recursive ordinals whose unique notations constituted a Π^1_1 set. The first test of metarecursion theory, Kreisel unerringly pronounced, was to prove the Friedberg-Muchnik theorem. A major technical obstacle stood in the way. It was possible for a metarecursively enumerable subset of a metafinite set not to be metafinite. (For me this obstacle was a source of meta-delight.) It was overcome in 1963 by a partial metarecursive map of ω onto ω_1^{CK}, a foretaste of Jensen's projectum techniques in L.

In 1966 I finished my work on metarecursion theory and conceived a plan for writing a book called *Higher Recursion Theory*. Part A would develop hyperarithmetic theory from scratch and would include connections with forcing and compactness. Part B would expound metarecursion, and Part C would deal with Σ_1 admissible ordinals. The principal flaw in the plan was my ignorance of Kleene's theory of recursion in normal objects of finite type. Platek's lecture on the superjump in Manchester in 1969 gave me my first toehold on the Kleene theory, and long talks with Gandy and Grilliot in 1970 brought me to the top of the subject. Thus Part D was born. In the 1970's and early 1980's considerable progress took place in admissible ordinals and in finite types, and so Parts C and D had to be started over and over again. Even hyperarithmetic theory, fairly stable for some years, saw new developments.

By 1980 I had formed a definite view of the contents of higher recursion theory (HRT). There is only one fork in the road upward. The natural numbers turn into ordinals or into sets. If ordinals, then recursively enumerable becomes Σ_1. If sets, then belonging to a recursively enumerable set means there exists a convergent computation presented as a wellfounded, possibly infinite tree. The ordinal approach was blazed by G. Takeuti, the setcomputational by S. C. Kleene, both in the 1950's. Happily each leads in a different way to a proof of the Friedberg-Muchnik theorem.

One of the early hopes for HRT was the discovery of new theorems in CRT via ideas from above. It is now obvious (as usual, after the case) that CRT is not "low" enough for applications of HRT. A theorem of HRT is proved by overcoming the lack of some combinatoric facts taken for granted in CRT. Moving upward means leaving behind some of the power of CRT. The same loss occurs when moving downward. It is not surprising, then, that HRT has been applied successfully "below" CRT. Consider the splitting theorem of CRT: each non-recursive, recursively enumerable degree is the join of two incomparable such degrees. T. Slaman and H. Woodin have shown that splitting is not provable from a fragment of Peano arithmetic known as Σ_1 bounding. Their argument draws on tricks from HRT. M. Mytilinaios has shown that splitting does follow from Σ_1 induction. His result is inspired by R. Shore's proof of splitting for every Σ_1 admissible ordinal; it uses Σ_2 blocking. J. Shinoda and T. Slaman have shown that the theory of polynomial-time degrees of recursive sets interprets first order arithmetic. They apply an idea from Moschovakis's work (on Kleene's theory of recursion in normal objects of finite type) that views divergence as Σ_1 in character.

The book, on an informal level, is almost self contained. Some of the arguments present, in an easygoing fashion, material that a typical reader has encountered elsewhere in a more formal setting. For example a course on Gödel's L is not assumed. The essential facts about L are given ab initio, but some readers may want more details. No previous acquaintance with forcing is necessary, but it would help clarify the various effective notions of forcing studied here. The more one knows of CRT, the better, but little more than the enumeration theorem is assumed, and at least one well known logician managed to learn priority arguments in the setting of Σ_1 admissible ordinals before applying them to CRT.

The book has four parts:

A. Hyperarithmetic Sets
B. Metarecursion
C. α-Recursion
D. E-Recursion

Part A is perhaps longer than some would wish. It lingers, as if life were not short, on effective transfinite recursion (ETR), the method invented by Church and Kleene in their study of notations for ordinals. My treatment of ETR follows that of H. Rogers, the first to present it intuitively. The classic theorems are proved, most of them in the original spirit, theorems such as recursive ordinals equal constructive ordinals, Kleene's O is Π^1_1 complete, Σ^1_1 bounding, and hyperarithmetic equals Δ^1_1. In addition measure and forcing are developed and applied in a hyperarithmetic context. The set of all real X such that the ordinals recursive in X are the recursive ordinals has measure 1. Forcing with hyperarithmetically encoded perfect sets yields a minimal hyperdegree. Forcing with Σ^1_1 sets (originated by Gandy) leads to Louveau's separation theorem.

Hyperarithmetic theory (HT) is often regarded, and rightly so, as the source of effective descriptive set theory. In this book it is the prologue to higher recursion theory. Many of the major developmets of HRT are foreshadowed in HT. Part B carries out of the program of metarecursion sketched above. It is easy to follow once one becomes accustomed to thinking of hyperarithmetic as "finite". Part B speedily verifies that the priority method of Friedberg and Muchnik can be executed in a higher domain. Simpson's dichotomy applies metarecursion to create new categories of Π^1_1 sets.

Part C tackles α, an arbitrary Σ_1 admissible ordinal. Post's problem is solved by combining fine structure of L with priority. The catch phrase here is "Σ_1 doing the work of Σ_2". The priority method, when applied in CRT, needs Σ_2 replacement. In α-recursion Σ_1 suffices with the assistance of effective approximations, downward projecta and Gödel-Jensen condensation. Shore's density theorem is an example of a Σ_3 (more precisely $\Sigma_{2.5}$) argument of CRT lifted to every Σ_1 admissible α. An early result that points to the flexibility of such α's is the regular sets theorem. A subset of α is said to be regular if its intersection with each ordinal less than α is α-finite (that is, belongs to $L(\alpha)$). An α-recursively enumerable set may fail to be regular, but it always has the same α-degree as some regular, α-recursively enumerable set.

Part D assigns a meaning to $\{e\}(x)$ for every set x via a notion of computation following schemes devised by Normann, and (subsequently) by

Moschovakis. A structure is E-closed if it is closed under application of the partial E-recursive function $\{e\}$ for all e. The biggest twist in E-closed structures is the existence of reflection phenomena and their application to priority and forcing thanks to a crucial connection between reflecting ordinals and Moschovakis divergence witnesses uncovered by Harrington and Kechris. On a more mundane level, the key to progress in E-recursion is usually a new selection theorem, such as those proved by Gandy, Grilliot, Moschovakis and Normann. This is true in priority arguments, and even more so in forcing arguments. Intuitively, a selection theorem provides an effective method of selecting a member of a nonempty, "recursively enumerable" set. van de Wiele's theorem explains why some Σ_1 functions are not E-recursive. The book ends, fittingly I think, with Slaman's density theorem for E-closed $L(\kappa)$'s.

I owe a great deal to those of my students who wrote theses on HRT: C. Bailey, G. Driscoll, S. Friedman, E. Griffor, L. Harrington, S. Homer, F. Lowenthal, M. Machtey, J. Macintyre, D. MacQueen, J. Owings, R. Shore, S. Simpson, T. Slaman, J. Sukonick and S. Thomason.

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Lastly there is the debt to J. B. Rosser (1907–1989), A. Nerode, H. Rogers and B. Dreben. The final four gave me my start and kept me going. If they are pleased with HRT, then so am I.

Gerald E. Sacks
June 1990
Cambridge, Chicago, Pasadena and Princeton

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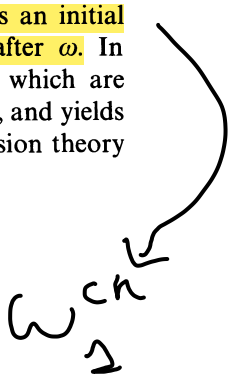
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Part A

Hyperarithmetic Sets

Hyperarithmetic theory is the first step beyond classical recursion theory. It is the primary source of ideas and examples in higher recursion theory. It is also a crossroads for several areas of mathematical logic. In set theory it is an initial segment of Gödel's L. In model theory, the least admissible set after ω. In descriptive set theory, the setting for effective arguments, many of which are developed below. It gives rise directly to metarecursion theory (Part B), and yields the simplest example of both α-recursion theory (Part C) and E-recursion theory (Part D).



Chapter I
Constructive Ordinals and Π^1_1 Sets

It is shown that a universal quantifier ranging over the real numbers is equivalent in certain circumstances to an existential quantifier ranging over the recursive ordinals, a countable set. Along the way notations for ordinals and the method of defining partial recursive functions by effective transfinite recursion are developed.

1. Analytical Predicates

The analytical predicates are obtained by applying function quantifiers to recursive predicates. Chapter I focuses on analytical predicates in which at most one function quantifier occurs, since in that case an analysis based on ordinals goes smoothly.

1.1 Partial Recursive Functions. Some conventions, occasionally violated, in this book are:

ω is $\{0, 1, 2, \ldots\}$, the set of natural numbers.
b, c, e, m, n are constants that denote natural numbers.
$x, y, z, \ldots$ are variables that range over ω.
$f, g, h, \ldots$ are total functions from ω into ω.
$X, Y, Z, \ldots$ are subsets of ω.
ϕ, ψ, θ are partial functions from ω into ω, that is functions whose graphs are subsets of ω^2.
$\phi(b) \simeq c$ is true iff (if and only if) $\phi(b)$ is defined and equal to c. $\phi(b) \simeq \psi(c)$ iff both $\phi(b)$ and $\psi(c)$ are defined and equal, or neither is defined.
$\{e\}^f$ is the e-th item in the standard enumeration of functions partial recursive in f. There exist a recursive predicate T and a recursive function U, both devised by Kleene, such that

$$(1) \qquad \{e\}^f(b) \simeq c \text{ iff } (\mathrm{E}y)[T(\bar{f}(y), e, b, y) \quad \& \quad U(y) = c].$$

$\bar{f}(y)$ encodes $\{\langle i, f(i)\rangle \mid i < y\}$:

$$\bar{f}(y) = \prod_{i<y} p_i^{1+f(i)}.$$

p_i is the i-th smallest prime; $p_0 = 2$. The right side of (1) says there is a computation y derived from the e-th set of equations, and the values of f restricted to $i < y$, whose final outcome is c. All of the above extends to

$$\{e\}^{f_0, \ldots, f_{m-1}}(x_0, \ldots, x_{n-1})$$

for all nonnegative m and n.

1.2 Function Quantifiers. A predicate $R(f, x)$ is recursive if there is an e such that:

(i) $(f)(x)[\{e\}^f(x)$ is defined]; and
(ii) $(f)(x)[R(f, x) \leftrightarrow \{e\}^f(x) = 0]$.

Thus the truth value of $R(f, x)$ is determined by a finite computation. As in subsection 1.1 the definition of recursive predicate extends routinely to predicates of the form $R(f_0, \ldots, f_{m-1}, x_0, \ldots, x_{n-1})$ for all $m, n \geq 0$. For simplicity $R(f, x)$ will be used somewhat ambiguously to denote a recursive predicate with an arbitrary number of function and number variables.

A predicate is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and function quantifiers. Thus

$$(1) \qquad (\mathrm{Ex})(f)(\mathrm{Eg})R(x, y, f, g, h) \quad \text{and} \quad (\mathrm{Ef})(h)S(f, h, z)$$

are analytical if R and S are recursive. A predicate is *arithmetic* if it is analytical but includes no function quantifiers.

There is an aspect of the classification of predicates which will seem picayune now but which will matter a great deal later. A predicate may be classified by virtue of its explicit form, as were the predicates of (1), or by being proved equivalent to another predicate already classified. For example, the predicate, "f is constructible in the sense of Gödel", is seen to be analytical only after it is shown that every constructible number-theoretic function is constructible via a countable ordinal.

1.3 Theorem (Kleene 1955). *If $P(f, x)$ is analytical, then it can be put in one of the following forms*:

$$A(f, x) \qquad \begin{array}{ll} (\mathrm{Eg})(y)R(f, x, g, y), & (\mathrm{Eg})(h)(\mathrm{Ey})R(f, x, g, h, y) \ldots \\ (g)(\mathrm{Ey})R(f, x, g, y), & (g)(\mathrm{Eh})(y)R(f, x, g, h, y) \ldots \end{array}$$

where A is arithmetic and R is recursive.

Proof. First $P(f, x)$ is put in prenex normal form with a recursive matrix by the usual quantifier manipulations associated with first order logic. Then the resulting prefix is put in one of the desired forms by applying the following rules. K is arbitrary.

$$(1) \qquad (x)(\mathrm{Ef})K(f, x) \leftrightarrow (\mathrm{Ef})(x)K((f)_x, x).$$

$(f)_x$ is defined by $(f)_x(y) = f(2^x \cdot 3^y)$.

f is thus interpretable as a code for $\{f_n | n < \omega\}$. Rule (1) is a nontrivial consequence of the axiom of choice. If K is $(\mathrm{Eg})(h)A(f, x, g, h)$ for some arithmetic A, then (1) is provable in Zermelo–Fraenkel set theory (ZF). (See Chapter III, Section 9.) On the other hand there is a K of the form $(g)(\mathrm{Eh})B(f, x, g, h)$ with B arithmetic such that (1) is not provable in ZF.

The dual of (1) is

$$(1^*) \qquad (\mathrm{Ex})(f)K(f, x) \leftrightarrow (f)(\mathrm{Ex})K((f)_x, x).$$

Each of the remaining three rules has a dual.

$$(2) \qquad (\mathrm{Ex})K(x) \leftrightarrow (\mathrm{Ef})K(f(0)).$$

$$(3) \qquad (\mathrm{Ef})(\mathrm{Eg})K(f, g) \leftrightarrow (\mathrm{Ef})K((f)_0, (f)_1).$$

$$(4) \qquad (\mathrm{Ex})(\mathrm{Ey})K(x, y) \leftrightarrow (\mathrm{Ex})K((x)_0, (x)_1).$$

$(x)_i$ is the exponent of p_i, the i-th smallest prime, in the unique factorization of x. Note that the substitution of $(f)_x$ or $(x)_i$ for some of the variables of a recursive predicate leaves it recursive. The following illustrates the normalization of a prefix.

$(\mathrm{Ef})(x)(\mathrm{Ey})(h)(z)$	is given.
$(\mathrm{Ef})(x)(h)(\mathrm{Ey})(z)$	by (1*).
$(\mathrm{Ef})(g)(h)(\mathrm{Ey})(z)$	by (2*).
$(\mathrm{Ef})(g)(\mathrm{Ey})(z)$	by (3*).
$(\mathrm{Ef})(g)(\mathrm{Ey})(h)$	by (2*).
$(\mathrm{Ef})(g)(h)(\mathrm{Ey})$	by (1*).
$(\mathrm{Ef})(g)(\mathrm{Ey})$	by (3*).

Observe that a prefix can be normalized by deleting all number quantifiers, collapsing each block of function quantifiers of the same sort to a single one of that sort, and adding a single number quantifier on the right dual to the rightmost function quantifier. □

Each of the nonarithmetic normal forms of Theorem 1.3 has a Greek name. An analytical predicate in normal form is said to be Σ^1_n (Π^1_n respectively) if its prefix begins with an existential (universal respectively) function quantifier and encompasses $n-1$ alternations of function quantifiers. Thus the forms of Theorem 1.3 are

$$\Sigma^1_1, \Sigma^1_2, \Sigma^1_3, \ldots$$

Arithmetic,

$$\Pi^1_1, \Pi^1_2, \Pi^1_3, \ldots$$

A predicate is said to be Δ^1_n if it is both Σ^1_n and Π^1_n. The Δ^1_1 predicates will eventually be proved to be the same as the hyperarithmetic predicates.

1.4 Set Quantifiers. The effect of applying function quantifiers to predicates can also be realized by applying set quantifiers. X encodes a function if

$$(1) \qquad (x)[E_1 y][2^x \cdot 3^y \in X].$$

If (1) holds, then the function encoded by X is denoted by f_X. Thus

$$f_X(x) = y \leftrightarrow 2^x \cdot 3^y \in X.$$

A predicate $R(X, y)$ is said to be recursive if it is equivalent to some recursive predicate $R(c_X, y)$, where c_X is the characteristic function of X. Let $R_0(X, x, y)$ be a recursive predicate such that (1) is equivalent to $(x)(\mathrm{Ey})R_0(X, x, y)$. The rule for replacing function quantifiers by set quantifiers is:

$$(2) \qquad (\mathrm{Ef})K(f) \leftrightarrow (\mathrm{EX})(x)(\mathrm{Ey})[R_0(X, x, y) \quad \& \quad K(f_X)].$$

Rule (2), and its dual, are all that is needed to transform Theorem 1.3 into Theorem 1.5. It is a fact that the single alternation of number quantifiers occurring in the normal forms of Theorem 1.5 cannot be reduced to a single number quantifier. There exists a recursive predicate $R(X, y, z, x)$ such that $(\mathrm{EX})(y)(\mathrm{Ez})R(X, y, z, x)$ is not equivalent to $(\mathrm{EX})(y)S(X, y, x)$ for any recursive S.

A predicate $P(Z, x)$ is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and set quantifiers; it is *arithmetic* if no set quantifiers are allowed.

1.5 Theorem (Kleene (1955).) *If $P(Z, x)$ is analytical, then it can be put in one of the following forms:*

$$A(Z, x) \qquad \begin{array}{ll} (\mathrm{EX})(y)(\mathrm{Ez})R(X, y, z, Z, x), & (\mathrm{EX})(Y)(\mathrm{Ey})(z)R \ldots \\ (X)(\mathrm{Ey})(z)R(X, y, z, Z, x), & (X)(\mathrm{EY})(y)(\mathrm{Ez})R \ldots \end{array}$$

where A is arithmetic and R is recursive.

There is no harm in mixing set and function variables. Thus a predicate is analytical if it is built up from recursive predicates by application of propositional connectives, number quantifiers, function quantifiers and set quantifiers. It is arithmetic if all quantifiers are number-theoretic. The resulting forms are again denoted by Σ^1_n or Π^1_n $(n \geq 1)$.

The most important of all Π^1_1 predicates is: X encodes a countable wellordering. It turns out to be universal Π^1_1, hence not Σ^1_1. It gives rise to a bounding principle with numerous applications. For example, it is used in Chapter IV to compute the Lebesgue measure of a Π^1_1 set of reals.

If $K \subseteq 2^\omega$, then K is said to be Π^1_n (Σ^1_n respectively) if $X \in K$ is Π^1_n (Σ^1_n respectively). Similar conventions are in force when $K \subseteq \omega$ or $K \subseteq \omega^2$ etc.

1.6 Theorem (Spector 1955). *Suppose* $A(X)$ *is* Σ^1_1.
(*i*) $\cap\{X|A(X)\}$ *is* Π^1_1.
(*ii*) *If* $(E_1X)A(X)$, *then the unique* X *that satisfies* $A(X)$ *is* Δ^1_1.

Proof.
(i) Let B be $\cap\{X|A(X)\}$. Then

$$x\in B\leftrightarrow(X)[A(X)\rightarrow x\in X].$$

(ii) Let C be the unique solution of $A(X)$. Then

$$\begin{aligned} x\in C &\leftrightarrow (\mathrm{E}X)[A(X) \quad \& \quad x\in X] \\ &\leftrightarrow (X)[A(X)\rightarrow x\in X]. \quad \square \end{aligned}$$

In Chapter III, Section 6, it will be shown that every Σ^1_1 set with a non-Δ^1_1 member has a continuum of members. The proof will require more than trivial quantifier manipulations, namely an analysis of Σ^1_1 predicates by means of recursive trees with infinite branching.

Part (i) of Theorem 1.6 is often alluded to as follows: a set (of numbers) is Π^1_1 if it is the closure of a Π^1_1 set under a Σ^1_1 closure condition. A predicate $A(X)$ is a *closure condition* if the intersection of any non-empty collection of solutions of $A(X)$ is a solution of $A(X)$, and if every set (of numbers) is contained in some solution of $A(X)$. Let $A(X)$ be a closure condition. It follows that for each Y there is a least X, call it Y_0, such that $Y\subseteq X$ and $A(X)$:

$$Y_0=\cap\{X|Y\subseteq X \quad \& \quad A(X)\}.$$

Y_0 is called the closure of Y under A. By Theorem 1.6(i), $Y_0\in\Pi^1_1$ if $Y\in\Pi^1_1$ and $A(X)\in\Sigma^1_1$, because then

$$Y\subseteq X \quad \& \quad A(X)$$

is Σ^1_1.

1.7 Proposition. $f\in\Sigma^1_n\leftrightarrow f\in\Pi^1_n\leftrightarrow f\in\Delta^1_n$.

Proof. Since f is a function,

$$f(x)=y\leftrightarrow(z)[y\neq z\rightarrow f(x)\neq z]. \tag{1}$$

If the left side of (1) is Σ^1_n (Π^1_n respectively), then the right side is Π^1_n (Σ^n_1 respectively). $\square$

1.8–1.12 Exercises

1.8. Show there exists a universal Π^1_n predicate, that is a Π^1_n predicate $P(e,f,x)$ such that for each Π^1_n predicate $Q(f,x)$ there is a c for which $P(c,f,x)$ and $Q(f,x)$ are equivalent for all f and x.

1.9. Show $\Pi^1_n \subseteq \Sigma^1_{n+1}$, $\Sigma^1_n \subseteq \Pi^1_{n+1}$, $\Pi^1_n \nsubseteq \Sigma^1_n$ and $\Sigma^1_n \nsubseteq \Pi^1_n$.

1.10. Let ω have the discrete topology and ω^ω the product topology. Basic closed subsets of ω^ω can be coded by finite sequences of natural numbers, hence by natural numbers. A closed subset of ω^ω, regarded as an intersection of basic closed sets, can be coded by a subset of ω. Show "X codes a closed subset of ω^ω" is arithmetic. Show "X codes a countable, closed subset of ω^ω" is Π^1_1.

1.11. Let L be a first order language whose set of primitive symbols is recursive. Let $S(X)$ be "X codes a countable set of sentences of L". Show "$S(X)$ and X is consistent (that is yields no contradiction via first order logic)" is arithmetic. Show "$S(X)$ and X has a model" is Σ^1_1.

1.12. Suppose $<_u$ and $<_v$ are order-isomorphic, recursive wellorderings of ω. Show there exists a $\Delta^1_1 f$ such that $x <_u y \leftrightarrow f(x) <_v f(y)$ for all $x, y \in \omega$.

2. *Notations for Ordinals*

Suppose $<$ is a wellordering of ω. $<$ is said to be recursive if the predicate $x < y$ is recursive. An ordinal is called recursive if it is finite or the ordertype of a recursive wellordering. The recursive ordinals form a countable, initial segment of the countable ordinals with strong closure properties. They constitute an effective analogue of the countable ordinals. For example, the Cantor–Bendixson analysis of a recursively encodable, countable closed set terminates at a recursive ordinal.

The definition of recursive ordinal is, in a manner of speaking, from above. The question of whether or not a recursive linear ordering ℓ is wellordered is complex. A straightforward resolution would require examination of every function f from ω into ω to see if f defines an infinite descending sequence in ℓ. A more constructive approach would be from below. It seems reasonable to expect that the successor of a constructive ordinal be constructive, and that the limit of a recursive sequence of constructive ordinals be constructive. The constructive approach is made precise with the help of notations. Afterwards it is shown the approaches from above and below yield the same result; the recursive and constructive ordinals coincide. The notion of ordinal notation is useful in proof theory as well as in recursion theory. It facilitates delicate recursions and inductions.

2.1 Kleene's *O*. The formula $x <_O y$ is to be read: x and y are notations for constructive ordinals and x is less than y according to the ordering of notations. The ordering $<_O$ is not linear, because the same ordinal may have two different notations.

The predicate $x <_O y$, regarded as a set of ordered pairs, is the closure of a finite set under a Σ^1_1 closure condition as described in the remarks following subsection 1.6.

The closure condition $A(X)$ has three clauses.

(1) $(u)(v)[\langle u, v\rangle \in X \rightarrow \langle v, 2^v\rangle \in X]$.
(2) $(n)\ [\{e\}(n)$ is defined $\&\ \langle \{e\}(n), \{e\}(n+1)\rangle \in X] \rightarrow (n)[\langle \{e\}(n), 3 \cdot 5^e\rangle \in X]$.
(3) $(u)\ (v)\ (w)\ [\langle u, v\rangle, \langle v, w\rangle \in X \rightarrow \langle u, w\rangle \in X]$.

(1) deals with successors, (2) with limits, and (3) with transitivity. All three are positive in nature: if some elements belong to X, then some other elements belong to X. The positiveness of $A(X)$ implies it is a closure condition. Hence there is a least X such that $\langle 1, 2\rangle \in X$ and $A(X)$. $<_O$ is defined to be that least X.

Kleene's O, the set of notations for constructive ordinals, is the field of $<_O$.

A binary relation r is said to be *wellfounded* if there is no f such that $(x)r(f(x+1), f(x))$.

2.2 Theorem

(i) $<_O$ *and* O *are* Π^1_1.
(ii) $<_O$ *is a wellfounded partial ordering*
(iii) If $v \in O$, *then the restriction of* $<_O$ to $\{u | u <_O v\}$ *is linear.*

Proof.

(i) $<_O$ is Π^1_1 by Theorem 1.6(i), since $A(X)$ is arithmetic. Then O is Π^1_1 by rule (1*) of the proof of Theorem 1.3, since $u \in O \leftrightarrow (\mathrm{Ew})[w <_O u \vee u <_O w]$.

(ii) The following *natural enumeration* of $<_O$ is equivalent to a redefinition of $<_O$ by transfinite recursion on the ordinals.

Stage 0: enumerate $1 <_O 2$.

Stage $\delta + 1$: enumerate $v <_O 2^v$ and $u <_O 2^v$ if $u <_O v$ was enumerated at stage δ.

Stage λ (limit): enumerate $\{e\}(n) <_O 3 \cdot 5^e$ and $u <_O 3 \cdot 5^e$, if not enumerated earlier, if for each n, $\{e\}(n) <_O \{e\}(n+1)$ has been enumerated earlier, and if for some n, $u <_O \{e\}(n)$ has been enumerated earlier.

By induction on γ, a pair enumerated at stage γ belongs to $<_O$. On the other hand the set of all pairs enumerated is a solution of $A(X)$ and hence contains $<_O$. Also by induction, if $u <_O v$ and $v <_O w$, then $u <_O v$ is enumerated at an earlier stage than $v <_O w$. It follows that $<_O$ is wellfounded, since otherwise there would be an infinite, descending sequence of ordinals.

The natural enumeration also makes clear there is no x such that

(1) $u <_O x$ & $x <_O 2^u$, or $x <_O 1$, or
(2) $(n)[\{e\}(n) <_O \{e\}(n+1) <_O x]$ & $x < 3 \cdot 5^e$.

Consequently 2^u is said to be the successor of u, and $3 \cdot 5^e$ the limit of $\{e\}(n)\,(n < \omega)$.

(iii) is proved by induction on $<_O$. Assume $u_1, u_2 <_O v$ to check $u_1 <_O u_2$ or $u_1 = u_2$ or $u_2 <_O u_1$. If $v = 2^u$, then (1) implies $u_1, u_2 \leq_O u$ and the desired result follows by induction. If $v = 3 \cdot 5^e$, then apply (2). □

In Part B of this book a generalization of recursive enumerability is offered that allows $<_O$ to be viewed as a higher kind of recursively enumerable relation. The natural enumeration of $<_O$ becomes a proof that $<_O$ is metarecursively enumerable. Elements enter metarecursively enumerable sets by means of metafinite computations, which are infinite but have many of the properties of finite computations.

2.3 Constructive Ordinals. The function $|\ |: O \to$ Ordinals is defined by transfinite recursion on $<_O$.

$$|1| = 0.$$

$$|2^u| = |u| + 1;$$

$$|3 \cdot 5^e| = \lim_{n \to \infty} |\{e\}(n)|.$$

The definition is sound by 2.2(1)–(2). If $u \in O$, then u is said to be a notation for the ordinal $|u|$. An ordinal δ is constructive if $\delta = |u|$ for some $u \in O$. There are no gaps in the constructive ordinals. They form a countable, initial segment of the ordinals. The least non-constructive ordinal is called Church–Kleene omega-one and is written ω_1^{CK}.

The fact that each infinite constructive ordinal has many notations is a consequence of the "approach from below". The choice of a preferred notation for ω is no simple matter. A bad choice for ω might make choices difficult further on (cf. Exercise 2.4). Later it will be seen that there exists a Π^1_1 subset of O, linearly ordered by $<_O$, and of ordertype ω_1^{CK}. Such a subset is called a set of unique notations. It will be defined from above.

2.4–2.6 Exercises

2.4. A path in O is a set $Z \subseteq O$ such that Z is linearly ordered by $<_O$ and $(u)(v)[u <_O v \in Z \to u \in Z]$. A path can be continued if there is a $w \in O$ such that $(u)[u \in Z \to u <_O w]$. Find a path in O of ordertype less than ω_1^{CK} but which cannot be continued.

2.5. Spell out the details omitted from the proof of Theorem 2.2(ii).

2.6. Prove Theorem 2.2(ii) without any reference to ordinals. For example, prove $\sim(\text{Ex})[x <_O 1]$ by showing $A(X) \to A(X - \{\langle x, 1\rangle\})$.

3. *Effective Transfinite Recursion*

Let f map ω into ω, and let $f \upharpoonright n$ denote the restriction of f to $\{m | m < n\}$. To say f is defined by recursion on ω is to say there exists an iterater I such that

$$f(n) = I(f \upharpoonright n) \text{ for all } n. \tag{1}$$

(1) is called a recursion equation. For each I there is a unique f such that (1) holds. If I is computable, then f is computable by virtue of a straightforward, but limiting, intuition. $f(n)$ is computable from I and the set of previous values, $\{f(m) | m < n\}$. More precisely, $f(n)$ is computed by iterating I n times. The record of that n-fold iteration is the computation of $f(n)$. With this intuition in mind, it appears

farfetched to replace the standard wellordering of ω by some arbitrary wellordering of ω and still expect f to be computable when I is. For one thing, n may have infinitely many predecessors and the iteration of I infinitely many times is not a finite computation. For another, it is no longer clear what is meant by the effectiveness of I, since a typical argument of I in (1) may be an infinite object.

Church and Kleene made the remarkable discovery that (1) remains a valid scheme for defining recursive functions when the standard wellordering of ω is replaced by an arbitrary one, so long as I remains effective in an appropriate sense. It is tempting to think that the wellordering should be recursive, but that limitation is unnecessary, and fortunately so, since many of the applications are to $<_O$. Rogers was the first to use the phrase, "effective transfinite recursion", and to provide a general result similar to Theorem 3.2.

First a technical result of classical recursion theory, Kleene's fixed point theorem. The fact that (1) above has a unique solution f for each I is often described in a set theoretic setting by noting that f is a fixed point of I. The next result is an effective counterpart of the essential existence argument employed in the set theoretic treatment of definition by transfinite recursion.

3.1 Theorem (Kleene). *Suppose* $I\colon \omega \to \omega$ *is recursive. Then for some* c, $\{I(c)\} \simeq \{c\}$.

Proof. Let t be a recursive function such that

$$\{t(e)\} \simeq \{\{e\}(e)\}$$

for all e. Choose b so that

$$\{b\}(x) = I(t(x)) \tag{1}$$

for all x. Then

$$\{I(t(b))\} \simeq \{\{b\}(b)\} \simeq \{t(b)\}.$$

Thus $t(b)$ will serve for c. □

Note that a fixed point c of I is computable in a uniform manner from a Gödel number of I, since t is independent of I, and since the composition that occurs in (1) is effective. It follows that if I depends effectively on some parameter p, then the fixed point c can be construed as a recursive function of p.

3.2 Theorem. *Let* $<_R$ *be a wellfounded relation whose field is a subset of* ω, *and* I: $\omega \to \omega$ *a recursive function. Suppose for all* $e < \omega$ *and* x *in the field of* $<_R$, $\{e\}(y)$ *defined for all* $y <_R x$ *implies* $\{I(e)\}(x)$ *defined. Then for some* c, $\{c\}(x)$ *is defined for all* x *in the field of* $<_R$, *and* $\{c\} \simeq \{I(c)\}$.

Proof. By theorem 3.1 there is a c such that $\{c\} \simeq \{I(c)\}$. Suppose x is a minimal (in the sense of $<_R$) element such that $\{c\}(x)$ is not defined. Thus $\{c\}(y)$ is defined for all $y <_R x$. But then $\{I(c)\}(y) \simeq \{c\}(y)$ is defined. □

Warning: definition by effective transfinite recursion (ETR) is more than an effective version of the set theoretic method of definition by transfinite recursion. There is an element of self-reference in ETR with no counterpart in set theory. There is also a use of indices to transform potentially infinite computations into finite ones. Thus $f(n)$ is computed not by iterating I n times, but by having I act on an index for $f \upharpoonright n$. An excellent example of ETR is the definition of $+_O$.

3.3 Addition of Notations. The key property of $+_O$, the addition function for notations in O, is: if $a, b \in O$, then $a +_O b \in O$ and $|a +_O b| = |a| + |b|$. The definition of $+_O$ by effective transfinite recursion is aimed at realizing this key property.

Let h be a recursive function such that

$$\{h(e, a, d)\}(n) \simeq \{e\}(a, \{d\}(n)) \tag{1}$$

for all e, a, d and n. By the use of pleonasms (that is, each partial recursive function has infinitely many Gödel numbers), it is safe to insist h be one-one from ω^3 into ω.

Let I be a recursive function such that

$$\{I(e)\}(a, b) \simeq \begin{cases} a & \text{if } b = 1 \\ 2^{\{e\}(a,m)} & \text{if } b = 2^m \\ 3 \cdot 5^{h(e,a,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases} \tag{2}$$

The first three clauses of (2) mimic the definition of $+$ for ordinals. Thus $\alpha + \beta = \alpha$ if $\beta = 0$, $\alpha + \beta = (\alpha + \gamma) + 1$ if $\beta = \gamma + 1$, and $\alpha + \lambda = \lim_n (\alpha + \gamma_n)$ if $\lambda = \lim_n \gamma_n$. I is recursive, despite the non-recursiveness of $<_O$, because the splitting of O into notations for zero, successors, and limits is effective. Also the instruction coded by $I(e)$ makes sense whether or not a and b belong to O.

By Theorem 3.1, I has a fixed point c. Define $a +_O b$ to be $\{c\}(a, b)$. Since $\{I(c)\} \simeq \{c\}$,

$$a +_O b \simeq \begin{cases} a & \text{if } b = 1 \\ 2^{a +_O m} & \text{if } b = 2^m \\ 3 \cdot 5^{h(c,a,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases}$$

Note that $\{h(c, a, d)\}(n) \simeq a +_O \{d\}(n)$ by (1).

Nothing in Theorem 3.1 requires $+_O$ to be defined anywhere. The proof of Theorem 3.2 shows the domain of $+_O$ contains O. (2) has a quirk that compels $+_O$

to be total. Suppose $a +_O b$ is not defined. Then b must be 2^m, since h is total, and so $a +_O m$ is not defined for some $m < b$. Thus an induction on ω shows $+_O$ is total.

3.4 Theorem (Kleene). *The recursive function $+_O$ has the following properties for all a and b.*

(*i*) $a, b \in O \leftrightarrow a +_O b \in O$.
(*ii*) $a, b \in O \rightarrow |a +_O b| = |a| + |b|$.
(*iii*) $a, b \in O \ \& \ b \neq 1 \rightarrow a <_O a +_O b$.
(*iv*) $a \in O \ \& \ c <_O b \leftrightarrow a +_O c <_O a +_O b$.
(*v*) $a \in O \ \& \ b = c \in O \leftrightarrow a +_O b = a +_O c$.

Proof. First all inferences from left to right are proved simultaneously by induction on b, an induction that proceeds according to $<_O$.
Case 1: $b = 1$. Then $a +_O b = a$, and (i) and (ii) from left to right are immediate. (iv) is vacuous by 2.2(1).
Case 2: $b = 2^m$. Then $m \in O$. Hence by induction $a +_O m \in O$, $|a +_O m| = |a| + |m|$, and $a \leq_O a +_O m$. So $a +_O b \in O$, $|a +_O b| = |a| + |b|$, and $a <_O a +_O b$.

Suppose $c <_O b$. Then $c \leq_O m$ by 2.2(1). Hence by induction $a +_O c \leq_O a +_O m$, and so $a +_O c <_O a +_O b$.
Case 3: $b = 3 \cdot 5^e$. Then by 2.2(2), $\{e\}(n)$ is defined and $\{e\}(n) <_O \{e\}(n+1) <_O 3 \cdot 5^e$ for all n. By induction $a \leq_O a +_O \{e\}(n) <_O a +_O \{e\}(n+1)$ and $|a +_O \{e\}(n)| = |a| + |\{e\}(n)|$ for all n. Hence $a +_O b = 3 \cdot 5^{h(c,a,e)} \in O$, $|a +_O b| = \lim_n |a +_O \{e\}(n)| = |a| + |b|$, and $a <_O a +_O b$.

Suppose $c <_O b$. Then $c <_O \{e\}(n)$ for some n by Theorem 2.2(1) and 2.2(2). By induction $a +_O c <_O a +_O \{e\}(n) <_O a +_O b$.

Now the inferences from right to left are proved by induction on $a +_O b$ with respect to $<_O$.
Case 1: $a +_O b = a$ and $b = 1$. Trivial.
Case 2: $a +_O b = 2^{a +_O m}$ and $b = 2^m$. Then $a +_O m \in O$, and by induction $a, m, b \in O$.

Subcase 2a: $a +_O c = a +_O b$. Then c is of the form 2^n, and $a +_O n = a +_O m$. By induction $n = m$, so $c = b$.

Subcase 2b: $a +_O c < a +_O b$. Then $a +_O c \leq_O a +_O m$, $c \leq_O m$ and $c <_O b$.

Case 3: $a +_O b = 3 \cdot 5^{h(c,a,e)}$ and $b = 3 \cdot 5^e$. Then $a +_O \{e\}(n) <_O a +_O \{e\}(n+1) <_O 3 \cdot 5^{h(c,a,e)}$ for all n. By induction $a \in O$ and $\{e\}(n) <_O \{e\}(n+1)$ for all n, hence $b \in O$.

Subcase 3a: $a +_O c = a +_O b$. Then $c = 3 \cdot 5^d$ for some d such that $h(c, a, d) = h(c, a, e)$. Recall that h is one-one. Thus $d = e$ and $c = b$.

Subcase 3b: $a +_O c < a +_O b$. Then $a +_O c <_O a +_O \{e\}(n)$ for some n. By induction $c <_O \{e\}(n)$, hence $c <_O b$. □

From now on W_e is the e-th recursively enumerable subset of ω, that is the domain of $\{e\}$, the e-th partial recursive function.

3.5 Theorem (Kleene). *There exist recursive functions p and q such that for all $b \in O$:*

(i) $W_{p(b)} = \{a | a <_O b\}$;
(ii) $W_{q(b)} = \{\langle u, v\rangle | u <_O v <_O b\}$.

Proof. The essential properties of p are:

$$\begin{aligned} &W_{p(1)} = \phi, \\ &W_{p(2^a)} = \{a\} \cup W_{p(a)}, \\ &W_{p(3\cdot 5^d)} = \cup \{W_{p(\{d\}(n))} | \{d\}(n) \text{ is defined}\}. \end{aligned} \tag{1}$$

An induction on $<_O$ shows that any p that satisfies (1) also satisfies 3.5(i). The existence of a recursive such p is obtained by an effective transfinite recursion on $<_O$. Let e_o be a Gödel number and j and k recursive functions such that

$$\begin{aligned} &W_{e_o} = \phi, \\ &W_{j(e,a)} = \{a\} \cup W_{\{e\}(a)}, \\ &W_{k(e,d)} = \cup \{W_{\{e\}(\{d\}(n))} | n < \omega\}. \end{aligned} \tag{2}$$

In (2) it is intended that $W_{\{e\}(a)} = \phi$ when $\{e\}(a)$ is undefined; similarly for $W_{\{e\}(\{d\}(n))}$.

There exists a recursive I such that

$$\{I(e)\}(b) \simeq \begin{cases} e_o & \text{if } b = 1 \\ j(e,a) & \text{if } b = 2^a \\ k(e,d) & \text{if } b = 3\cdot 5^d \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 3.1, I has a fixed point c: $\{I(c)\} \simeq \{c\}$. Define $p(b)$ to be $\{c\}(b)$. Then

$$p(b) = \begin{cases} e_o & \text{if } b = 1 \\ j(c,a) & \text{if } b = 2^a \\ k(c,d) & \text{if } b = 3\cdot 5^d \\ 0 & \text{otherwise.} \end{cases}$$

Note that p is total because j and k are. (2) implies that p satisfies (1).

The definition of q is similar to that of p. □

Some applications of Theorem 3.5 need only the fact that $\{a | a <_O b\}$ is recursively enumerable whenever $b \in O$. But it does not seem possible to establish this fact without establishing it uniformly, that is without developing a uniform method p for enumerating the predecessors of an element of O. Not surprising, since the only general approach to showing every element of O has some constructive

property is by effective transfinite recursion, an approach that can only be made by means of an effective recursion step.

3.6–3.7 Exercises

3.6. Prove there exists a recursive function q that satisfies Theorem 3.5(ii).

3.7. Fill in the details of the proof of Theorem 3.4.

4. *Recursive Ordinals*

An ordinal is said to be recursive if it is finite or the ordertype of some recursive wellordering of ω. In a moment it will be shown that the recursive ordinals coincide with the constructive ordinals. This theme will recur in Part D, where it will be seen that the ordinals E-recursive in x are cofinal with the ordinals constructive in x for every set x. It is sometimes said that the notion of recursive ordinal is more intrinsic than that of constructive ordinal. This merely means there is no element of chaos in the notion of recursive predicate, but there is great freedom in the development of notations for ordinals. In Part D it will be the case that every ordinal constructive in x is recursive in x, and every ordinal recursive in x is less than some ordinal constructive in x. In the present section every ordinal less than a constructive ordinal is constructive, but that will not be so in Part D.

There exist very small subclasses of the recursive predicates which give rise to the recursive ordinals. Thus each recursive wellordering is order isomorphic to some primitive recursive wellordering and even to some rudimentary (in the sense of Smullyan) wellordering. In fact O'Neill has observed that predicates computable in polynomial time suffice. In the other direction if $\{\langle x, y\rangle | x <_u y\}$ is Σ^1_1 and $<_u$ is a wellordering, then the ordertype of $<_u$ is a recursive ordinal.

The next lemma states that every recursively enumerable subset of O is bounded in a highly effective manner. Later it will be shown that every Σ^1_1 subset of O is bounded in a somewhat less effective manner.

4.1 Lemma. *There exists a recursive g such that for all e:*

(i) $g(e) \in O \leftrightarrow W_e \subseteq O$;
(ii) $g(e) \in O \rightarrow |a| < |g(e)|$ *for all* $a \in W_e$.

Proof. $g(e)$ is an "infinite sum" of W_e. Let r be a total recursive function such that for all e: $\{r(e)\}$ is total and the range of $\{r(e)\}$ is $W_e \cup \{1\}$. There exists a total recursive function s defined by recursion on n such that for all e:

$$\{s(e)\}(0) = \{r(e)\}(0) = 1,$$
$$\{s(e)\}(n+1) = \{s(e)\}(n) +_O 2^{\{r(e)\}(n+1)}.$$

$+_O$ was defined in subsection 3.3.

Let $g(e)$ be $3 \cdot 5^{s(e)}$.

First assume $g(e) \in O$ to see $W_e \subseteq O$. Clearly $\{s(e)\}(n) \in O$ for all n. Fix $n > O$. By Theorem 3.4(i), $2^{\{r(e)\}(n)} \in O$, and so $\{r(e)\}(n) \in O$.

Now assume $W_e \subseteq O$. Then for each n, $\{r(e)\}(n) \in O$, and by Theorem 3.4(iii) $\{s(e)\}(n) <_O \{s(e)\}(n+1)$. Hence $g(e) \in O$.

Lastly assume $g(e) \in O$ and $1 \neq a \in W_e$. Choose $n > 0$ so that $\{r(e)\}(n) = a$. Then

$$\{s(e)\}(n) = \{s(e)\}(n-1) +_O 2^a.$$

Hence $2^a <_O \{s(e)\}(n)$, and so $|a| < |g(e)|$. □

4.2 Enumeration of Wellfounded Relations. A binary relation $R(s, y)$ is said to be a wellordering if it is:

(i) connected . . . $R(x, y) \vee R(y, x) \vee x = y$;
(ii) transitive . . . $R(x, y)$ & $R(y, z) \to R(x, z)$; and
(iii) wellfounded . . . if S is a nonempty subset of the field of R, then $(\mathrm{E}y)_{y \in S}(x)_{x \in S} \sim R(x, y)$.

Note that (iii) implies R is
(iv) irreflexive . . . $\sim R(x, x)$, and
(v) antisymmetric . . . $R(x, y) \to \sim R(y, x)$.

Certain aspects of the effective study of ordinals make it necessary to consider wellfounded relations rather than wellorderings. In addition a computation is a wellfounded tree rather than a wellordering.

If a binary relation R is wellfounded, then it has a height denoted by $|R|$ and measured by an ordinal. Let β be an ordinal variable. Read "$\mu\beta$" as "the least β such that". Define

$$|x| = \mu\beta[R(y, x) \to |y| < \beta],$$
$$|R| = \mu\beta(x)[x \in \text{field of } R \to |x| < \beta].$$

The notion of height is useful for proving theorems about wellfounded relations by transfinite induction.

Let R_e be the e-th recursively enumerable binary relation, that is

$$R_e(x, y) \leftrightarrow \{e\}(x, y) \text{ is defined.}$$

Thus $\{R_e | e < \omega\}$ is a simultaneous recursive enumeration of all recursively enumerable binary relations.

4.3 Lemma. *There exists a recursive f such that for all e:*

(i) R_e is wellfounded $\leftrightarrow f(e) \in O$; and
(ii) R_e is wellfounded $\to |R_e| \leq |f(e)|$.

Proof. The idea is to define a one-one, order-preserving map from the field of R_e into O by an effective transfinite recursion on R_e. One difficulty is the uncertain nature of the field of R_e. It is recursively enumerable, but may be empty or finite.

Let h be a total recursive function such that

$$R_{h(e,n)}(x, y) \leftrightarrow R_e(x, y) \;\&\; R_e(x, n) \;\&\; R_e(y, n)$$

for all e, n, x and y. $R_{h(e,n)}$ is the initial segment of R_e below n. $R_{h(e,n)}$ is empty if n is not in the field of R_e. There exists a total recursive t such that

$$W_{t(b,e)} = \begin{cases} \phi & \text{if } R_e = \phi, \\ \{\{b\}(h(e, n)) | n < \omega\} & \text{otherwise.} \end{cases} \tag{1}$$

Recall g from Lemma 4.1. Let k be a recursive function such that

$$\{k(b)\}(e) \simeq g(t(b, e));$$

let c_0 be a fixed point of k, that is $\{k(c_0)\} \simeq \{c_0\}$. Define

$$f(e) \simeq \{c_0\}(e), \quad \text{and}$$
$$t(e) = t(c_0, e).$$

Then

$$W_{t(e)} = \begin{cases} \phi & \text{if } R_e = \phi, \\ \{f(h(e, n)) | n < \omega\} & \text{otherwise,} \end{cases}$$

and $f(e) = g(t(e))$.

Suppose R_e is wellfounded to show $f(e) \in O$ and $|R_e| \leq |f(e)|$. If the field of R_e is empty, then $W_{t(e)} = \phi$ and $f(e) \in O$ by Lemma 4.1. Assume $R_e \neq \phi$. Then $|R_{h(e,n)}| < |R_e|$ for all $n < \omega$, and the desired result follows by transfinite induction on $|R_e|$ with the aid of Lemma 4.1.

Now suppose $f(e) \in O$ to show R_e is wellfounded. For all n, $|f(h(e, n))| < |f(e)|$ by Lemma 4.1. So by transfinite induction on $<_O$, $R_{h(e,n)}$ is wellfounded for all n. Hence R_e is wellfounded. □

It can be shown that "$\leq$" in 4.3(ii) cannot be improved to "$=$".

Clause (i) of Theorem 4.3 implies that the predicate "R_e is wellfounded" is many-one reducible to Kleene's O. This is the first indication that O is some kind of complete or universal set. In Section 5 it will be shown that O is a complete Π^1_1 set, that is every Π^1_1 set is many-one reducible to O. With the aid of insight supplied by metarecursion theory in Part B, it will be seen that O is a complete Π^1_1 set in the same sense that

$$K = \{\langle e, n\rangle | \{e\}(n) \text{ is defined}\}$$

is a complete, recursively enumerable set. From the viewpoint of Part B, a number belongs to a Π^1_1 set if and only if some metafinite computation says it does. A

metafinite computation will be a wellfounded relation of recursive ordinal height and of Δ^1_1 complexity.

4.4 Theorem (Kleene, Markwald). *The recursive ordinals equal the constructive ordinals.*

Proof. Suppose β is a recursive ordinal. Let R be a recursively enumerable, wellfounded binary relation of height β. By Theorem 4.3 there is a $b \in O$ such that $|R| \leq |b|$. Since every ordinal less than $|b|$ is constructive, β must be constructive.

Now suppose β is constructive. Then $\beta = |b|$ for some $b \in O$. By Theorem 3.5(ii) there is a recursively enumerable wellordering R such that $|R| = \beta$, namely $W_{q(b)}$. Assume β is infinite. Then there exists a one-one recursive f that maps ω onto the field of R. Define $x < y$ by $\langle f(x), f(y)\rangle \in W_{q(b)}$. $<$ is recursive and $|<| = \beta$. □

4.5–4.6 Exercises

4.5. Show each recursive wellordering is order-isomorphic to some primitive recursive wellordering.

4.6. Recall q from Theorem 3.5. Suppose $a, b \in O$ and $|a| = |b|$. Let f be the unique, one-one, order-preserving map from $W_{q(a)}$ onto $W_{q(b)}$. Show f is Δ^1_1. Show f need not be partial recursive.

5. *Ordinal Analysis of* Π^1_1 *Sets*

In this section Π^1_1 sets are analyzed by means of recursive ordinals represented by recursive wellfounded relations on sequence numbers. The analysis is applied to show Kleene's O is a complete Π^1_1 set, and to obtain Spector's bounding principle for Σ^1_1 subsets of O.

5.1 The Partial Ordering of Sequence Numbers. Define $\bar{f}(x)$ as in subsection 1.1. y is said to be a *sequence number* if $y = \bar{f}(x)$ for some f and x. $\bar{f}(x)$ is thought of a code for the sequence $\langle f(0), f(1), \ldots, f(x-1)\rangle$. $\bar{f}(0) = 1$, and 1 encodes the null sequence. The length of $\bar{f}(x)$ is x, and is denoted by $\ell h(\bar{f}(x))$. If y is a sequence number, then y encodes

$$\langle (y)_0, (y)_1, \ldots, (y)_{\ell h(y)-1}\rangle.$$

If y and z are sequence numbers, then y is properly extended by z (symbolically $y > z$) if $\ell h(y) < \ell h(z)$ and $(y)_i = (z)_i$ for all $i < \ell h(y)$.

Let Seq be the set of all sequence numbers. Seq is a recursive set, and $>$ is a recursive, antisymmetric, transitive binary relation. Seq (with $>$) is useful in the study of Π^1_1 sets because it presents ω^ω effectively as a tree.

5.2 Normalization of Π^1_1 Predicates. Suppose $R_1(f, x, y)$ is recursive. Then there is an e such that for all f, x and y:

$$\{e\}^f(x, y) \text{ is defined, and}$$
$$R_1(f, x, y) \leftrightarrow \{e\}^f(x, y) = 0.$$

As in subsection 1.1, there exist recursive T and U such that

$$\{e\}^f(x, y) = 0$$
$$\leftrightarrow (\mathrm{Ez})[T(\bar{f}(z), e, x, y, z) \quad \& \quad U(z) = 0].$$

It follows that $(f)(\mathrm{Ex})R_1(f, x, y)$ is equivalent to

$$(f)(\mathrm{Ex})R(\bar{f}(x), y) \tag{1}$$

for some recursive R. (1) is the preferred normal form for a Π^1_1 predicate. Since Kleene's T-predicate is "universal", an enumeration of the Π^1_1 predicates is provided by $(f)(\mathrm{Ex})T(\bar{f}(x), e, y, x)$ $(e = 0, 1, 2, \ldots)$.

For each y, let $S_R(y)$ be the restriction of Seq (and $>$) to those sequence numbers $\bar{f}(x)$ such that $(i)_{i<x} \sim R(\bar{f}(i), y)$. Clearly $S_R(y)$ is recursive uniformly in y.

5.3 Proposition. *$(f)(\mathrm{Ex})R(\bar{f}(x), y)$ iff $S_R(y)$ is wellfounded.*

Proof. Fix y. $\sim (f)(\mathrm{Ex})R(\bar{f}(x), y)$ iff there is an f such that $(x) \sim R(\bar{f}(x), y)$ iff there is an f such that $\bar{f}(0) > \bar{f}(1) > \bar{f}(2) \ldots$ is an infinite descending sequence in $S_R(y)$ iff $S_R(y)$ is not wellfounded. □

Proposition 5.3 equates the problem of checking membership in Π^1_1 sets with the problem of checking wellfoundedness of recursive relations. Thus recursive ordinals suffice to analyze Π^1_1 sets.

5.4 Theorem (Kleene). *Each Π^1_1 set is many-one reducible to O.*

Proof. Suppose $B \in \Pi^1_1$. According to subsection 5.2 there is a recursive R such that

$$y \in B \leftrightarrow (f)(\mathrm{Ex})R(\bar{f}(x), y)$$

for all y. By proposition 5.3,

$$y \in B \leftrightarrow S_R(y) \text{ is wellfounded.}$$

Since $S_R(y)$ is recursive uniformly in y, there is a recursive function t such that $S_R(y) = R_{t(y)}$. (R_e is the e-th recursively enumerable, binary relation as defined in subsection 4.2.) Let f be the recursive function of Lemma 4.3. Then

$$y \in B \leftrightarrow f(t(y)) \in O. \tag{1}$$ □

There is a uniformity concealed in the proof of Theorem 5.4 worth making explicit. As in subsection 5.2, let

$$(2) \qquad (f)(\mathrm{Ex})T(\bar{f}(x), e, y, x)$$

be the e-th Π^1_1 predicate. If $Q(y)$ is equivalent to (2) for all y, then e is said to be an index for the predicate $Q(y)$. An index for a Π^1_1 set B is an index for the predicate $y \in B$. The recursive function t of 5.4(1) is a function of two variables, e and y, where e is an index for B. Thus there is a uniform method for passing from an index for B to a recursive function that reduces B to O. In short the Π^1_1 sets are uniformly many-one reducible to O.

5.5 Corollary. $O \notin \Sigma^1_1$.

Proof. Analogous to a proof that a complete, recursively enumerable subset of ω is not recursive. Observe that any set many-one reducible to a Σ^1_1 set is also Σ^1_1. Hence by Theorem 5.3, if O were Σ^1_1, then every Π^1_1 set would be Σ^1_1. So it suffices to find a Π^1_1 set that is not Σ^1_1.

Define $Q(y)$ by $(f)(\mathrm{Ex})T(\bar{f}(x), y, y, x)$. Suppose $\sim Q(y)$ is Π^1_1. Then $\sim Q(y)$ is equivalent to $(f)(\mathrm{Ex})T(\bar{f}(x), e, y, x)$ for some e. But then $\sim Q(e)$ iff $Q(e)$. □

5.6 Corollary (Spector 1955). *Suppose $X \subseteq O$ and $X \in \Sigma^1_1$. Then there exists $b \in O$ such that $|x| \leq |b|$ for all $x \in X$ (Σ^1_1 boundedness).*

Proof. As in the proof of Theorem 5.4, with O in place of B, there is a recursive function t such that for all y,

$$y \in O \leftrightarrow R_{t(y)} \text{ is wellfounded,}$$

where R_e is the e-th recursively enumerable relation. Let $Q(y)$ be

$$(\mathrm{Ez})[z \in X \ \& \ (\mathrm{Ef})$$
$$(u)(v)(R_{t(y)}(u, v) \rightarrow \langle f(u), f(v)\rangle \in W_{q(z)})].$$

q is the recursive function of Theorem 3.5 $Q(y)$ is Σ^1_1. If $Q(y)$ holds, then $R_{t(y)}$ is wellfounded.

Suppose the hoped-for b does not exist. If $R_{t(y)}$ is wellfounded, then by Lemma 4.3 there is a $z \in X \subseteq O$ such that $|R_{t(y)}| \leq |z|$, and so $Q(y)$ holds. But then $y \in O$ is Σ^1_1 despite Corollary 5.5. □

There is a uniformity lacking in the statement of Corollary 5.6, whose existence will be established in Corollary 3.4.II. e is said to be an index for X as a Σ^1_1 set if

$$y \in X \leftrightarrow (\mathrm{Ef})(x) \sim T(\bar{f}(x), e, y, x)$$

for all y. The effective version of Corollary 5.6 is: there exists a recursive function f such that for all e and X, if e is a Σ^1_1 index for X, then

(i) $X \subseteq O \leftrightarrow f(e) \in O$, and
(ii) $X \subseteq O \rightarrow (z)[z \in X \rightarrow |z| \leq f(e)|]$.

5.7–5.8 Exercises

5.7. Show each Π^1_1 set is one-one reducible to O.

5.8. Suppose A is a Π^1_1 set such that every Π^1_1 set is many-one reducible to A. Show every Π^1_1 set is one-one reducible to A.

Chapter II
The Hyperarithmetic Hierarchy

The hyperarithmetic sets are defined by iterating the Turing jump through the recursive ordinals, and are shown to equal the Δ^1_1 sets. The equality is important for two reasons. First, it reveals that Δ^1_1 is more constructive than it appears to be. Second, it allows properties of Δ^1_1 sets to be proved by induction, since hyperarithmetic sets fall into a hierarchy and can be assigned ordinal ranks less than ω_1^{CK}.

Hyperarithmetic reducibility, hyperdegrees and the hyperjump are defined.

1. Hyperarithmetic Implies Δ^1_1

The H-sets are defined after some properties of the Turing jump are reviewed. A set is defined to be hyperarithmetic if it is recursive in some H-set. Then an effective transfinite recursion produces an effective method for passing from the index of an H-set X to a Δ^1_1 index for X.

1.1 *H*-Sets. Let c_X be the characteristic function of the set X. Y is said to be Turing reducible to (or recursive in) Z if

$$(\text{Ee})[c_Y = \{e\}^{c_Z}]. \tag{1}$$

$\{e\}$ is sketched in Chapter I, subsection 1.1. Formula (1) is often rendered as $X \leq_T Y$.

The Turing jump of X is denoted by X'_1 and is defined by

$$\{e|\{(e)_0\}^X((e)_1) \text{ is defined}\}.$$

X' can be regarded as the effective disjoint union of all sets recursively enumerable in X.

The following elementary facts about Turing reducibility and jump are proved in Rogers 1967.

(2) X' is not recursive in X.

(3) X is recursive in X' uniformly. (There is an e_0 such that for all X, $X = \{e_0\}^{X'}$.)

(4) There is a recursive function t such that

$$X = \{e\}^Y \to X' = \{t(e)\}^{Y'}$$

for all X, Y and e.

(5) There is a recursive function θ such that

$$X = \{d\}^Y \ \& \ Y = \{e\}^Z \to X = \{\theta(d, e)\}^Z$$

for all X, Y, Z, d and e.

The H-sets are defined by recursion on $<_O$:

(6) $$H_1 = \phi,$$

(7) $$H_{2^a} = (H_e)'$$

(8) $$H_{3 \cdot 5^e} = \{\langle x, n\rangle | x \in H_{\{e\}(n)}\} \quad (\langle x, n\rangle = 2^x \cdot 3^n).$$

Let $H: O \to 2^\omega$ be the unique function that satisfies (6)–(8). X is an H-set if $X = H_b$ for some $b \in O$. X is said to be *hyperarithmetic* if X is recursive in some H-set.

Let HYP be the set of all hyperarithmetic sets. HYP theory is largely the creation of Kleene, although pioneering work was done by M. Davis, and by A. Mostowski. Recall that X and Y are said to have the same Turing degree (in symbols $X \equiv_T Y$) if $X \leq_T Y$ and $Y \leq_T X$. It will be seen in Section 2 that the Turing degree of H_b is determined by $|b|$.

1.2 Lemma (Kleene). *There exists a recursive function k such that for all a, b*

$$a <_O b \to H_a = \{k(a, b)\}^{H_b}.$$

Proof. By effective transfinite recursion on b. Let e_0 and θ be as in subsection 1.1, (3) and (5). Fix a and assume $a <_O b$. The definition of k has three cases.

(i): $b = 2^a \cdot k(a, b) = e_0$, since $H_b = H_a'$.

(ii): $b = 2^d \neq 2^a \cdot k(a, b) = \theta(k(a, d), e_0)$.

(iii): $b = 3 \cdot 5^z$. $k(a, b)$ is defined thus. Let p be the recursive function of Theorem 3.5 of Chapter I. Simultaneously enumerate $W_{p(\{z\}(n))}$ $(n \geq 0)$ until an n is found such that $a \in W_{p(\{z\}(n))}$. Such an n must exist if $a <_O 3 \cdot 5^z$. $k(a, b)$ is

$$\theta(k(a, \{z\}(n)), h(n)),$$

where h is a recursive function such that

$$\{h(n)\}^X = \{v | \langle v, n\rangle \in X\}$$

for all n and X. Thus $\{h(n)\}^{H_{3 \cdot 5^z}}$ is $H_{\{z\}(n)}$ when $3 \cdot 5^z \in O$.

Before the definition of k can be made precise, it is necessary to elaborate case (iii). A trick is needed to cover the possibility that $3 \cdot 5^z$ may not belong to O. In

that event there may not be an n such that $\{z\}(n)$ is defined and $a \in W_{p(\{z\}(n))}$. The trick is to define $k(a, 3 \cdot 5^z)$ so as to embody the search for n without worrying about whether such an n exists. Let $\theta_*(e, a, z)$ be a recursive function with the following properties. For all e, a, and z, $\theta_*(e, a, z)$ is the index of a Turing reduction procedure. $\{\theta_*(e, a, z)\}^X$ makes sense for all X but may be partial for some X. The instructions for computing $\{\theta_*(e, a, z)\}^X$ are: simultaneously enumerate $W_{p(\{z\}(n))}$ $(n \geq 0)$ until an n is uncovered such that $\{z\}(n)$ is defined and $a \in W_{p(\{z\}(n))}$. Let n_O be the first such n uncovered. (If n_O does not exist, then $\{\theta_*(e, a, z)\}^X(v)$ is undefined for all v.) Then

$$\{\theta_*(e, a, z)\}^X \simeq \{\theta(\{e\}(a, \{z\}(n_O)), h(n_O))\}^X.$$

The recursive iterater I needed for the definition of k by effective transfinite recursive is given by:

$$\{I(e)\}(a,b) = \begin{cases} e_O & \text{if } b = 2^a, \\ \theta(\{e\}(a,d), e_O) & \text{if } b = 2^d \neq 2^a, \\ \theta_*(e, a, z) & \text{if } b = 3 \cdot 5^z \\ 0 & \text{otherwise.} \end{cases},$$

Let $\{c\} \simeq \{I(c)\}$. k is $\{c\}$. k is total because θ and θ_* are. □

1.3 Theorem. *Each of the following predicates is* Π^1_1.

(*i*) $x \in O \ \& \ y \in H_x$.
(*ii*) $x \in O \ \& \ y \notin H_x$.

Proof. (i) Let $A(X)$ be the conjunction of:

$(X)_1 = \phi$,

(*a*) $[a \in O \to (X)_{2^a} = (X)'_a]$, and

(*e*) $[3 \cdot 5^e \in O \to (X)_{3 \cdot 5^e} = \{\langle x, n\rangle | x \in (X)_{\{e\}(n)}\}$

Recall that $(X)_m = \{n | \langle m, n\rangle \in X\}$ and that $\langle m, n\rangle = 2^m \cdot 3^n$. Define X^* to be the set of all $\langle x, y\rangle$ satisfying predicate 1.3.(i). Then $A(X^*)$ holds by induction on $<_O$.

Suppose $A(X)$ to show $X^* \subseteq X$. If $a \notin O$, then $(X^*)_a = \phi$ and so $(X^*)_a \subseteq (X)_a$. Assume $a \in O$. Then $(X^*)_a = (X)_a$ by induction on $<_O$.

Thus X^* is the intersection of all solutions of $A(X)$. By Theorem 1.6. I, X^* is Π^1_1, since O is Π^1_1.

(ii) Similar to (i). □

1.4 Corollary (Kleene)

(*i*) *If X is hyperarithmetic, then X is* Δ^1_1.
(*ii*) *The predicate, $X \in$ HYP, is* Π^1_1.

Proof. Consider the predicate

$$(1) \qquad \begin{aligned} &(\mathrm{Es})\,(\mathrm{Ez})\,[T(s,e,y,z)\ \&\ \ell h(s)=z\ \&\ U(z)=i \\ &\&\,(j)_{j<z}((s)_j=1\to x\in O\ \&\ j\in H_x) \\ &\&\,(j)_{j<z}((s)_j\neq 1\to x\in O\ \&\ j\notin H_x)]. \end{aligned}$$

By Theorem 1.3, (1) is Π^1_1. Fix x and e, and assume $X=\{e\}^{H_x}$. Then

$$\begin{aligned} &y\in X \leftrightarrow (1) \text{ holds with } i=1, \\ &y\notin X \leftrightarrow (1) \text{ holds with } i=0. \end{aligned}$$

Hence X is Δ^1_1.

The predicate, $X\in \mathrm{HYP}$, is equivalent to

$$(2) \qquad (\mathrm{Ex})\,(\mathrm{Ee})\,[x\in O\ \&\ X=\{e\}^{H_x}].$$

The Π^1_1-ness of (1) implies (2) is Π^1_1. □

More information is to be had concerning Corollary 1.4(i). Suppose X is Δ^1_1. $2^c\cdot 3^d$ is said to be an index for X as a Δ^1_1 set (or simply a Δ^1_1-index for X) if c (d respectively) is a Π^1_1 index for X ($\omega - X$ respectively). Thus

$$\begin{aligned} &y\in X \leftrightarrow (f)\,(\mathrm{Ex})\,[T(\bar{f}(x),c,y,x)\ \&\ U(x)=1], \\ &y\notin X \leftrightarrow (f)\,(\mathrm{Ex})\,[T(\bar{f}(x),d,y,x)\ \&\ U(x)=0]. \end{aligned}$$

1.5 Theorem (Kleene). *There exists a recursive function f such that*

$$f(b) \text{ is a } \Delta^1_1\text{-index for } H_b$$

for all $b\in O$.

Proof. By effective transfinite recursion on $<_O$. First a recursive function j is defined such that

$$m \text{ is a } \Delta^1_1\text{-index for } X \to j(m)$$

is a Δ^1_1-index for X' for all m and X. The definition of X' yields

$$(1) \qquad \begin{aligned} y\in X' \leftrightarrow (\mathrm{Es})[&T(s,(y)_0,(y)_1,\ell h(s)) \\ &\&\ (i)_{i<\ell h(s)}((s)_i=1\to i\in X) \\ &\&\ (i)_{i<\ell h(s)}((s)_i\neq 1\to i\notin X)]. \end{aligned}$$

Let m be a Δ^1_1-index for X:

$$i \in X \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), (m)_0, i, x) \quad \& \quad U(x)=1], \tag{2}$$

$$i \notin X \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), (m)_1, i, x) \quad \& \quad U(x)=0]. \tag{3}$$

Substitute the right side of (2) for $i \in X$, and the right side of (3) for $i \notin X$, in (1). Let c be the index of the resulting Π^1_1 formula after normalization. Thus c is a Π^1_1-index for X'. In a similar fashion a Π^1_1-index d can be found for $\omega - X'$. Let $j(m) = 2^c \cdot 3^d$.

Next a recursive function r is defined so that for all z and all $\{Y_n | n < \omega\}$,

$$\text{if } (n)\,[\{z\}(n) \text{ is a } \Delta^1_1\text{-index for } Y_n],$$
$$\text{then } r(z) \text{ is a } \Delta^1_1\text{-index for } \{\langle y, n\rangle | y \in Y_n\}.$$

Define

$$\langle y, n\rangle \in Z \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), \{z\}(n), y, x) \quad \& \quad U(x) = 1]. \tag{4}$$

(The right side of (4) is false if $\{z\}(n)$ is undefined.) Let u be the Π^1_1-index for the right side of (4) after normalization. Similarly a Π^1_1-index v for $\omega - Z$ can be found. Let $r(z) = 2^u \cdot 3^v$.

Now for the definition of f by transfinite recursion. Let c be a recursive function such that

$$\{c(e, z)\}(n) \simeq \{e\}(\{z\}(n))$$

for all n. There exists a recursive I such that

$$\{I(e)\}(b) \simeq \begin{cases} e_0 & \text{if } b = 1 \\ j(\{e\}(m)) & \text{if } b = 2^m \\ r(c(e, z)) & \text{if } b = 3 \cdot 5^z \\ 0 & \text{otherwise.} \end{cases}$$

Choose d so that $\{d\} \simeq \{I(d)\}$. Then $\{d\}$ is the sought-after f. □

Theorem 1.5 yields another proof of Theorem 1.4(i), and some information concerning *persistent* Δ^1_1 definitions. Fix $X \in \Delta^1_1$. Suppose

$$y \in X \leftrightarrow (\mathrm{Ef})A(f, y) \quad \text{and} \quad y \notin X \leftrightarrow (\mathrm{Ef})B(f, y)$$

for some arithmetic A and B. Clearly

$$(y)(\mathrm{Ef})[A(f, y) \vee B(f, y)].$$

($\vee$ is the exclusive "or".) For each y, choose an f_y that satisfies the matrix above, and

let W be $\{f_y | y < \omega\}$. Then

$$y \in X \leftrightarrow (\mathrm{Ef})_{f \in V} A(f, y) \quad \text{and} \quad y \notin X \leftrightarrow (\mathrm{Ef})_{f \in V} B(f, y)$$

whenever $2^\omega \supseteq V \supseteq W$. The above state of affairs is described by saying: X has an upward persistent Δ_1^1 definition over W. It will be shown in the next section that each H_b has an upward persistent Δ_1^1 definition over $\{H_a | a <_O b\}$.

There is a persistence phenomenon hidden in the proof of Theorem 1.5. Define b^* by: $1^* = 1$, $(3 \cdot 5^e)^* = 3 \cdot 5^e$, and $(2^a)^* = 2^{2^{(a)^*}}$. Let V_b be $\{g | g' \leq_T H_b\}$ for each $b \in O$. If f is the recursive function defined in the proof of Theorem 1.5, then $f(b)$ is the index of a Δ_1^1 definition of H_b upward persistent over V_{b^*} (Exercise 1.9).

An early persistence result is due to Gödel: L has a Σ_1 definition upward persistent over L. To be more precise, there is a Δ_0 formula $P(x, y)$ of set theory such that

$$x \in L \leftrightarrow (\mathrm{Ey})_{y \in V} P(x, y)$$

for all models $V \supseteq L$.

1.6 Lemma. *If X has a Δ_1^1 definition upward persistent over $\{f | f \leq_T B\}$, then X' has a Δ_1^1 definition upward persistent over $\{f | f \leq_T B''\}$.*

Proof. Recall formulas (1)–(3) from the proof of Theorem 1.5. Assume the Δ_1^1 definition of X given by (2) and (3) is upward persistent over $\{f | f \leq_T B\}$. If the right side of (2) is substituted for $i \in X$, and the right side of (3) for $i \notin X$, in (1), then $y \in X'$ becomes

$$(\mathrm{Es})_{s \in \mathrm{Seq}}(f)[f \leq_T B \to (\mathrm{Ex}) R(\bar{f}(x), s, y)] \tag{4}$$

for some recursive R. Since f is restricted, the quantifier manipulations that transform (4) into a Δ_1^1 formula upward persistent over $\{f | f \leq_T B''\}$ have to be considered with care. $y \notin X'$ is

$$(s)_{s \in \mathrm{Seq}}(\mathrm{Ef})[f \leq_T B \quad \& \quad \sim (\mathrm{Ex}) R(\bar{f}(x), s, y)]. \tag{5}$$

In order to move the universal quantifier on s in (5) past the existential quantifier of f, it is necessary to choose, for each s, an f that satisfies the matrix of (5). Since $f \leq_T B$, a choice of f amounts to a choice of n such that $\{n\}^B$ is total. The set of all such n is many-one reducible to B''. It follows that the set of all n such that $\{n\}^B$ is a total function f and satisfies $\sim (\mathrm{Ex}) R(\bar{f}(x), s, y)$ is also many-one reducible to B''. Hence (5) is equivalent to

$$(\mathrm{Ef})[f \leq_T B'' \quad \& \quad (s)_{s \in \mathrm{Seq}} \sim (\mathrm{Ex}) R((\bar{f})_s(x), s, y)]. \tag{6}$$

The f of (6) is such that $(f)_s = \{t(s)\}^{B''}$ for some $t \leq_T B''$. A similar argument begins with $y \notin X'$ and ends with a Σ_1^1 formula for $y \in X'$ upward persistent over $\{f | f \leq_T B''\}$. □

1.7–1.9 Exercises

1.7. Recall $A(X)$ from the proof of Theorem 1.3. Is $A(X)$ a closure condition in the sense of the remarks following the proof of Theorem 1.6.I?

1.8. Obtain Theorem 1.5 as an immediate corollary of Theorem 1.3.

1.9. Let f be the recursive function developed in the proof of Theorem 1.5. Show, for each $b \in O$, $f(b)$ is a Δ^1_1-index for H_b upward persistent over V_{b*}. $1^* = 1$, $(3 \cdot 5^e)^* = 3 \cdot 5^e$, and $(2^a)^* = 2^{2^{(a*)}}$. $V_b = \{f \mid f \leq_T B\}$.

2. Δ^1_1 *Implies Hyperarithmetic*

The main result of this section is that every Δ^1_1 set is hyperarithmetic. Along the way some related boundedness, uniformization and selection principles are proved.

For each $b \in O$, define

$$O_b = \{a \mid a \in O \quad \& \quad |a| < |b|\}.$$

2.1 Lemma. *Each of the following predicates is* Π^1_1.

(i) $x \in O \ \& \ y \in O_x$.
(ii) $x \in O \ \& \ y \notin O_x$.

Proof. Similar to that of Theorem 1.3. This time $A(X)$ is the conjunction of

$$\begin{aligned}
&(1) \qquad (X)_1 = \phi,\\
&(e)\ [3 \cdot 5^e \in O \to (X)_{3 \cdot 5^e} = \bigcup_n (X)_{\{e\}(n)}], \text{ and}\\
&(a)\ [2^a \in O \to (X)_{2^a} = \{1\} \qquad (1)\\
&\qquad \cup \{y \mid (\mathrm{Em})(y = 2^m \quad \& \quad m \in (X)_a)\}\\
&\qquad \cup \{y \mid (\mathrm{Ee})[y = 3 \cdot 5^e \quad \& \quad (n)(\{e\}(n) \in (X)_a)]\}].
\end{aligned}$$

The set of all $\langle x, y \rangle$ satisfying (1) is the intersection of all X satisfying $A(X)$. □

2.2 Uniformization. $P(x, y)$ is said to uniformize $Q(x, y)$ if

$$\begin{aligned}
&(x)(y)[P(x, y) \to Q(x, y)], \text{ and}\\
&(x)[(\mathrm{Ey})Q(x, y) \to (E_1 y)P(x, y)].
\end{aligned}$$

In short Q contains P, and P is the graph of a function whose projection on the x-axis is the same as that of Q. (The term, "uniformization", was coined by descriptive set theorists.) Proving a uniformization principle usually amounts to proving a

selection, or choice, principle. If $Q(x, y)$ is recursively enumerable ($x, y \in \omega$), then it is trivial but instructive to uniformize it by a recursively enumerable P as follows. Let e be such that

$$Q(x, y) \leftrightarrow (\mathrm{Ez})T(e, x, y, z).$$

Define $P(x, y)$ by

$$(\mathrm{Ez})[T(e, x, y, z) \ \& \ (w)_{w<z} \sim (\mathrm{Ey})T(e, x, y, w) \ \& \ (v)_{v<y} \sim T(e, x, v, z)].$$

$P(x, y)$ is recursively enumerable because $T(e, x, y, w)$ implies $y < w$ (a useful convention satisfied by Kleene's T-predicate). Fix x and assume $(\mathrm{Ey})Q(x, y)$. $P(x, y)$ first minimizes the length z of computation needed to uncover some y that satisfies $Q(x, y)$, and then singles out the least y associated with the minimum computation. This selection procedure is adequate for the proof of the next theorem, additional evidence that a Π_1^1 set is some kind of recursively enumerable set.

2.3 Theorem (Kreisel). Π_1^1 *predicates (of numbers) can be uniformized by* Π_1^1 *predicates.*

Proof. By Theorem 5.4.I, there is a recursive function g such that

$$Q(x, y) \leftrightarrow g(x, y) \in O$$

for all $x, y \in \omega$. Let $P(x, y)$ be

$$g(x, y) \in O \ \& \ (z)_{z \neq y}[g(x, z) \notin O_{g(x, y)}] \ \& \ (z)_{z<y}[g(x, z) \notin O_{2^{g(x, y)}}].$$

$P(x, y)$ is Π_1^1 according to Lemma 2.1. Fix x and assume $(\mathrm{Ey})Q(x, y)$. $P(x, y)$ first minimizes $|g(x, y)|$ and then singles out the least y that gives rise to the least ordinal. □

Theorem 2.3 remains true when x and y are set variables, but the proof is more complicated because there is no effective wellordering of 2^ω. Nonetheless the idea of minimizing certain ordinals is still pertinent, as will be seen below in Section 9 of Chapter III.

2.4 Theorem (Spector 1955). *There exists a recursive function g such that*

$$O_b = \{g(b)\}^{H_{2^b}}$$

for all $b \in O$.

Proof. By effective transfinite recursion. The most interesting step occurs in the limit case, since it is that case that requires H_{2^b} rather than H_b in the statement of the theorem.

Limit case: $b = 3 \cdot 5^e$. Let k be the recursive function of Lemma 1.2. Define

$$\theta(e, n) \text{ by } k(2^{\{e\}(n)}, 3 \cdot 5^e).$$

If $3 \cdot 5^e \in O$, then for all n, $\theta(e, n)$ is defined and

$$H_{2\{e\}(n)} = \{\theta(e, n)\}^{H_{3 \cdot 5^e}}.$$

Let $c(u, v)$ be a recursive function such that $\{c(u, v)\}^X \simeq \{u\}^{\{v\}^X}$ for all X. The predicate

$$(1) \qquad (\mathrm{En})[\{e\}(n) \ \& \ \{z\}(n) \text{ are defined} \ \& \ y \in \{c(\{z\}(n), \theta(e, n))\}^X]$$

is recursive in X'. Hence, for some recursive t, (1) is equivalent to $y \in \{t(e, z)\}^{X'}$. t has the power to extend any $\{z\}$ that satisfies the theorem below $3 \cdot 5^e$ to one that satisfies it at $3 \cdot 5^e$. Suppose $3 \cdot 5^e \in O$ and for each n, $\{z\}(n)$ is defined and

$$O_{\{e\}(n)} = \{\{z\}(n)\}^{H_{2\{e\}(n)}}$$

Then $O_{3 \cdot 5^e} = \bigcup_n O_{\{e\}(n)} = \{t(e, z)\}^{H_{2 3 \cdot 5^e}}$.

Successor case: $b = 2^a$. Define $e \in E$ by $(n)[\{e\}(n)$ is defined $\& \ \{e\}(n) \in W_{p(\{e\}(n+1))}]$, where p is the recursive function of Theorem 3.5.I. If $1 \neq a \in O$, then $y \in O_a$ iff

$$(2) \qquad y = 1 \vee (\mathrm{Em})_{m<y}[y = 2^m \ \& \ m \in O_a] \vee (\mathrm{Ee})_{e<y}[y = 3 \cdot 5^e \ \& \ e \in E \ \& \ (n)(\{e\}(n) \in O_a)].$$

If $\{s\}^{X'}$ is substituted for O_a in (2), then the resulting predicate is recursive in X'', since E is recursive in ϕ'', and since

$$(n)(\{e\}(n) \in \{s\}^{X'})$$

can be construed as

$$(n)(z) \sim [\{e\}(n) \text{ is defined} \ \& \ T(\overline{X'}(z), s, \{e\}(n), z) \ \& \ U(z) = 0].$$

Hence the result of substituting $\{s\}^{X'}$ for O_a in (2) is equivalent to $y \in \{j(s)\}^{X''}$ for some recursive function j. If $a \in O$ and

$$O_a = \{s\}^{H_{2a}}, \quad \text{then} \quad O_{2^a} = \{j(s)\}^{H_{2 2^a}}.$$

Choose c_0 so that $\phi = \{c_0\}^X$ for all X. There exists a recursive I such that

$$\{I(z)\}(b) \simeq \begin{cases} c_0 & \text{if } b = 1 \\ j(\{z\}(a)) & \text{if } b = 2^a \\ t(e,z) & \text{if } b = 3 \cdot 5^e \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{d\} \simeq \{I(d)\}$, and g be $\{d\}$. □

2.5 Theorem (Kleene). *If X is Δ^1_1, then X is hyperarithmetic.*

Proof. Assume $X \in \Delta^1_1$. Theorem 5.4.I implies there is a recursive g such that

$$y \in X \leftrightarrow g(y) \in O$$

for all y. Define A by

$$z \in A \leftrightarrow (\mathrm{Ey})(y \in X \quad \& \quad z = g(y)). \tag{1}$$

A is Σ^1_1. Spector's boundedness theorem, 5.6.I, yields a $b \in O$ such that $A \subseteq O_b$. Thus

$$y \in X \leftrightarrow g(y) \in O_b.$$

O_b is hyperarithmetic by Theorem 2.4. □

Theorems 1.4(i) and 2.5 combine to produce $\Delta^1_1 = \mathrm{HYP}$ for sets of numbers. The notion of Δ^1_1 is thought to be less clear, or less constructive, than the notion of HYP. This is a way of drawing attention to the fact that Δ^1_1 sets are defined from above by quantification over ω^ω, while hyperarithmetic sets are defined from below by iterating the Turing jump, or number quantifier, through the recursive ordinals. It is sometimes said that the HYP sets constitute a predicative analysis of the Δ^1_1 sets. A more precise statement of the situation is: the HYP sets provide a hierarchy for the Δ^1_1 sets. It is a general problem of considerable interest to develop a hierarchy for a family of sets defined en masse. The interest is more than philosophical, since a hierarchy makes it possible to prove theorems about the family by transfinite induction. This is the approach taken in the next section to show each Δ^1_1 set is implicitly arithmetically definable, that is the unique solution of an arithmetic predicate.

There is a uniformity lacking in the proof of Theorem 2.5, which will be supplied by Corollary 3.5. It consists of recursive functions f and g such that: if e is a Δ^1_1 index for X, then $f(e) \in O$ and $X = \{g(e)\}^{H_{f(e)}}$. The only information needed to make the proof of 2.5 yield f and g is an effective method of passing from a Σ^1_1 index of A, the set defined by 2.5.(1), to the bound b.

2.6 Lemma (Kreisel). *Suppose $Q(x,y)$ is Π^1_1. Then*

$$(x)(\mathrm{Ey})Q(x,y) \rightarrow (\mathrm{Ef})_{f \in \mathrm{HYP}}(x)Q(s, f(x)).$$

Proof. By Theorem 2.3 there is a $\Pi^1_1 P(x,y)$ that uniformizes $Q(x,y)$. Define $f(x) = y$ by $P(x,y)$. f is Δ^1_1 by Proposition 1.7.I, hence hyperarithmetic by Theorem 2.5. □

It is reasonable, but risky, to view Lemma 2.6 as analogous to the following result of elementary recursion theory: suppose $Q(x,y)$ is recursively enumerable, then

$$(x)(\mathrm{Ey})Q(x,y) \to (\mathrm{Ef})\,[f \text{ is recursive} \quad \& \quad (x)Q(x,f(x))].$$

($f(x)$ is the first y such that $Q(x,y)$ is enumerated.) Reasonable because it adds to the evidence that Π^1_1-ness is akin to recursive enumerability. Risky because it suggests that hyperarithmeticity is akin to recursiveness. In Part B of this book it will be seen that the analogy between Π^1_1 and recursively enumerable entails an analogy between hyperarithmetic and finite. This outcome is signaled by Theorem 2.4. If a recursive enumeration of A is cut short, then the result is a finite set. Analogously, if the natural enumeration of Kleene's O (so termed in the proof of Theorem 2.2.I) is cut short, then the result is some O_b, a hyperarithmetic set according to Theorem 2.4.

2.7–2.11 Exercises

2.7. Show each $\Sigma^0_n Q(x,y)$ can be uniformized by some $\Sigma^0_n P(x,y)$ $(n > 0)$.

2.8. Show the range of a total hyperarithmetic function is hyperarithmetic.

2.9. Formulate precisely and prove: a hyperarithmetic union of hyperarithmetic sets is hyperarithmetic.

2.10. Show each hyperarithmetic set is many-one reducible to some H-set.

2.11. Suppose C is a Π^1_1 set of hyperarithmetic reals, $P(X,Y)$ is Π^1_1, and $(X)\,(\mathrm{EY})\,[Y \in C \quad \& \quad P(X,Y)]$. Show there exists a hyperarithmetic map h from 2^ω into C such that $(X)P(X,h(X))$. (Read Section 5 before solving this problem.)

3. *Selection and Reduction*

It is useful to extend the domain of $\|$ from O to ω by defining $|b|$, when $b \notin O$, to be ∞. ∞ is greater than every constructive ordinal. Thus $|a| < \infty$ means $a \in O$.

It is helpful to think of b as a code for some sort of infinite computation that either terminates at some constructive ordinal or fails to terminate. The main result of this section is a selection principle: there exists a recursive function t such that

$$(1) \qquad a \in O \vee b \in O \to t(a,b) \in O \quad \& \quad \min(|a|,|b|) \leq |t(a,b)|.$$

t selects in the following sense. If either a or b terminates, then $t(a,b)$ terminates at a constructive ordinal large enough to serve as a vantage point from which one can look down and select a terminating element of the pair (a, b). In classical recursion theory an element can be selected from a nonempty recursively enumerable set A by simply enumerating all computations until one is found that puts an element into A.

In the proof of Theorem 2.3 it was seen that an element of a nonempty Π^1_1 set B can be selected in essentially the same manner. Let f be a recursive function such that

$$y \in B \leftrightarrow f(y) \in O.$$

Select the unique y such that

$$(2) \qquad f(y) \in O \;\; \& \;\; (z)_{z \neq y}[f(z) \notin O_{f(y)}] \;\; \& \;\; (z)_{z < y}[f(z) \notin O_{2f(y)}].$$

The notion of selection in (1) is somewhat more subtle than that in (2). In (1) a pair of computations is given with an assurance that at least one terminates. In (2) a set $\{f(y) | y \in B\}$ of computations, all of which terminate is given. The counterpart of (1) in classical recursion theory is proved by alternating between a and b until one of them terminates. This is the so-called computing-in-tandem trick. It is needed to show the disjunction of two recursively enumerable predicates is recursively enumerable. It plays an important part in the proof of Gandy selection in Part D.

As in subsection 4.2.I, let R_e be the e-th binary, recursively enumerable relation, and let $|R|$ be the ordinal height of R when R is wellfounded. Define $|R|$ to be ∞ when R is not wellfounded (WF).

3.1. Lemma. *There exists a recursive function k such that:*

(*i*) $[R_c \in WF \vee R_d \in WF] \leftrightarrow R_{k(c,d)} \in WF$;
(*ii*) $[R_c \in WF \vee R_d \in WF] \rightarrow \min(|R_c|, |R_d|) \leq |R_{k(c,d)}|$.

Proof. $R_{k(c,d)}$ is $R_c \otimes R_d$, a certain kind of product. The field of $R_c \otimes R_d$ consists of all numbers that are codes for ordered pairs $\langle r, s \rangle$, where $r \in$ field R_c and $s \in$ field R_d.

$$\langle r_1, s_1 \rangle R_c \otimes R_d \langle r_2, s_2 \rangle \quad \text{iff} \quad r_1 R_c r_2 \quad \text{and} \quad s_1 R_d s_2.$$

(i) Let Z be a nonempty subset of the field of $R_c \otimes R_d$. Z_{R_c} is $\{r | (\text{Es})(\langle r, s \rangle \in Z)\}$, and $Z_{R_d} = \{s | (\text{Er})(\langle r, s \rangle \in Z)\}$. Z has a minimal element iff Z_{R_c} or Z_{R_d} does.

(ii) Suppose either R_c or R_d is wellfounded. By (i) $R_c \otimes R_d$ is wellfounded. $|R_c \otimes R_d|$ is

$$\mu\beta[\langle r, s \rangle \in \text{field } R_c \otimes R_d \rightarrow |\langle r, s \rangle| < \beta].$$

If $r \in \text{field } R_c$ and the restriction of R_c to $\{r_1 | r_1 R_c r\}$ is wellfounded, then

$$|r| = \mu\beta[r_1 R_c r \to |r_1| < \beta].$$

Otherwise let $|r|$ be ∞. Treat R_d similarly. An induction on $\min(|r|, |s|)$ shows

$$\min(|r|, |s|) \leq |\langle r, s\rangle| \tag{1}$$

for all $\langle r, s\rangle \in \text{field } R_c \otimes R_d$, because

$$\min(|r_2|, |s_2|) = \max\{\min(|r_1|, |s_1|) \Big| \langle r_1, s_1\rangle R_c \otimes R_d \langle r_2, s_2\rangle\}. \tag{2}$$

(1) implies $\min(|R_c|, |R_d|) \leq |R_c \otimes R_d|$. (max is *strict* lub.) □

The proof of Lemma 3.1 centers on the min-max trick of equation (2). It is worth pondering, because it will reappear in Section 1.XII.

3.2 Proposition. *There exists a recursive function h such that*

(*i*) $b \in O \leftrightarrow R_{h(b)} \in WF$;
(*ii*) $b \in O \to |b| \leq |R_{h(b)}|$.

Proof. Since $O \in \Pi^1_1$, there is a recursive R such that

$$b \in O \leftrightarrow (f)(\text{Ex})\, R(\bar{f}(x), b).$$

Define $S_R(b)$ as in subsection 5.2.I. Let q be the recursive function of Theorem 3.5.I.

$$b \in O \leftrightarrow S_R(b) \in WF, \text{ and}$$
$$b \in O \to W_{q(b)} \in WF \quad \& \quad |b| = |W_{q(b)}|.$$

Consequently the effective disjoint union of $S_R(b)$ and $W_{q(b)}$ will serve as $R_{h(b)}$. The effective disjoint union of U and V is W:

$$(2u_1)\, W(2u_2) \leftrightarrow (u_1)\, U(u_2),$$
$$(2v_1 + 1)\, W(2v_2 + 1) \leftrightarrow (v_1)\, V(v_2). \quad \square$$

3.3 Theorem. *There exists a recursive function t with the following properties*:

(*i*) $a \in O \vee b \in O \leftrightarrow t(a,b) \in O$;
(*ii*) $a \in O \vee b \in O \to \min(|a|, |b|) \leq |t(a, b)|$.

Proof. Let f be the recursive function of Lemma 4.3.I. $t(a,b)$ is $f(k(h(a), h(b)))$, where h and k are the recursive functions of Proposition 3.2. and Lemma 3.1. □

Theorem 3.3. makes it possible to effectivize Spector's boundedness theorem (Corollary 5.6.I). (There is a substantially different approach based on the "completeness" and "creativity" of O; it avoids the use of t.)

3.4. Corollary. *There exists a recursive function f such that for all c: if c is a Σ^1_1-index of A, then*

(i) $A \subseteq O \leftrightarrow f(c) \in O$, *and*
(ii) $A \subseteq O \rightarrow A \subseteq O_{f(c)}$.

Proof. According to Theorem 5.4.I, there is a recursive function h such that

$$(1) \qquad (\mathrm{E}f)(x) \sim T(\bar{f}(x), c, y, x) \leftrightarrow h(c, y) \notin O$$

for all c and y. Let t be as in Theorem 3.3. Let k be a recursive function such that

$$W_{k(c)} = \{t(h(c, y), y) | y < \omega\}$$

Assume c is a Σ^1_1 index of A. If $A \subseteq O$, then $W_{k(c)} \subseteq O$. Let $f(c)$ be $2^{g(k(c))}$, where g is the bounding function of Lemma 4.1.I.

Suppose $A \subseteq O$. Then

$$y \in A \rightarrow h(c, y) \notin O \rightarrow |y| \leq |t(h(c, y), y)|,$$

and so $A \subseteq O_{f(c)}$.

Suppose $f(c) \in O$. By Lemma 4.1(i) of Chapter I, $W_{k(c)} \subseteq O$; hence by Theorem 3.3(i),

$$h(c, y) \in O \vee y \in O$$

for all y. Then (1) implies $A \subseteq O$. □

Corollary 3.4 yields an effective form of Theorem 2.5.

3.5 Corollary. *There exist recursive functions f and g such that for all e: if e is a Δ^1_1-index for X, then*

$$f(e) \in O \quad \text{and} \quad X = \{g(e)\}^{H_{f(e)}}.$$

Proof. The only noneffective step in the proof of Theorem 2.5 is the transition from a Σ^1_1 index for A to a bound b for A. But that can now be managed by Corollary 3.4. □

3.6 Reduction and Separation. Let Z be a set and F a family of subsets of Z. Reduction is said to hold for F if for each pair $A, B \in F$, there exists a pair $A_0, B_0 \in F$

such that

(i) $A_0 \subseteq A$ and $B_0 \subseteq B$,
(ii) $A_0 \cap B_0 = \phi$, and
(iii) $A_0 \cup B_0 = = A \cup B$.

The notion of reduction originated in descriptive set theory. In this section it is proved for Π^1_1 sets of numbers, and later for Π^1_1 sets of reals (Exercise 9.13.III). It is easily verified for recursively enumerable sets of numbers. Enumerate A and B simultaneously. If a number comes up in A that has not come up in B at an earlier stage, then put it in A_0. If a number comes up in B that has not come up in A at the same or an earlier stage, then put it in B_0.

Separation is said to hold for F if for each pair $C, D \in F$ such that $C \cap D = \phi$, there exists a pair $C_1, D_1 \in F$ such that

(iv) $C \subseteq C_1$ and $D \subseteq D_1$, and
(v) $C_1 = Z - D_1$.

C_1 is said to separate C and D.

Let $\sim F$ be $\{A | Z - A \in F\}$. Observe that reduction for $(\sim F)$ implies separation for F. Suppose $C, D \in F$ and $C \cap D = \varnothing$. Reduce $Z - C$ to A_0, and $Z - D$ to B_0. Then $C \subseteq Z - A_0$, $D \subseteq Z - B_0$, and $Z - A_0 = B_0$.

3.7 Theorem (Kleene). *Suppose* $A, B \in \Pi^1_1$. *Then there exist* $A_0, B_0 \in \Pi^1_1$ *such that* $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$ *and* $A_0 \cup B_0 = A \cup B$.

Proof. By Theorem 5.4.I, there are recursive functions h and j such that for all n,

$$n \in A \leftrightarrow h(n) \in O \quad \text{and} \quad n \in B \leftrightarrow j(n) \in O.$$

Let $n \in A_0 \leftrightarrow n \in A \;\&\; |h(n)| \leq |j(n)|$, and
$n \in B_0 \leftrightarrow n \in B \;\&\; |j(n)| < |h(n)|$.

Lemma 2.1 implies $A_0, B_0 \in \Pi^1_1$. □

It follows from Theorem 3.7 that any two disjoint Σ^1_1 sets can be separated by a Δ^1_1 set. By Corollary 3.5 the separating set is a hyperarithmetic set whose index can be obtained effectively from the indices of the Σ^1_1 sets.

3.8–3.10 Exercises

3.8. Extend $\|$ from O to ω as in the beginning of Section 3. Show $|a| < |b|$ is Π^1_1.

3.9. Suppose $A \in \Sigma^1_1$, $B \in \Pi^1_1$ and $A \subseteq B$. Find a $\Delta^1_1 C$ such that $A \subseteq C \subseteq B$.

3.10. Find two disjoint Π^1_1 sets which cannot be separated by a Δ^1_1 set.

4. Π^0_2 *Singletons*

Suppose $(E_1 X)P(X)$. Let A be the unique X that satisfies $P(X)$. A is said to be implicitly defined by $P(X)$. If the form of $P(X)$ is F, then A is said to be an F singleton. Thus Theorem I.6.I asserts that each Σ^1_1 singleton is Δ^1_1. It is immediate that every Δ^1_1 set is a Σ^1_1 singleton. It follows from Corollary 1.4(i) that every hyperarithmetic set is a Σ^1_1 singleton. In this section it will be shown that every H-set is a Π^0_2 singleton. (Later it will be shown by forcing that some hyperarithmetic set is not an arithmetic singleton, cf. Exercise 3.18.IV.) It follows that the hierarchy of H-sets can be construed as a hierarchy for the Π^0_2 singletons. In order to make the last assertion more precise, Spector's uniqueness theorem will be proved: $[a, b \in O \ \& \ |a| = |b|] \to H_a \equiv_T H_b$.

The next proposition suggests that the Π^0_2 singletons can be generated by iterating the Turing jump.

4.1 Proposition. *If A is a Π^0_2 singleton, then so are A', and any $B \equiv_T A$.*

Proof. Let A be the unique solution of $(u)(\mathrm{Ev})R(u, v, X)$ for some recursive R. Choose e_0 so that $X = \{e_0\}^{X'}$ for all X, and so that $\{e_0\}^Y$ is total for all Y. Then A' is the unique solution of

$$(1) \qquad (u)(\mathrm{Ev})R(u, v, \{e_0\}^Y) \ \& \ (n)[n \in Y \leftrightarrow (\mathrm{Ez})T(\overline{\{e_0\}^Y}(z), (n)_0, (n)_1, z)].$$

Formula (1) is Π^0_2 because the predicate $z \in \{e_0\}^Y$ is recursive. □

4.2 Theorem.

(*i*) *There exists a Π^0_2 predicate $H(a, X)$ such that*

$$(E_1 X)H(a, X) \quad \& \quad H(a, H_a)$$

for all $a \in O$.

(*ii*) *There exists a recursive predicate $R(a, y)$ such that*

$$(\mathrm{Ef})(x)R(a, \bar{f}(x)) \quad \& \quad (f)[(x)R(a, \bar{f}(x)) \to f \equiv_T H_a]$$

for all $a \in O$.

Proof. By effective transfinite recursion on a according to $<_O$. Let $\Pi(e, a, X)$ denote $(u)(\mathrm{Ev})T(\bar{X}(v), e, a, u, v)$, the e-th Π^0_2 predicate. The proof of Proposition 4.1 yields a recursive function h such that: if A is the unique solution of $\Pi(e, m, X)$, then A' is the unique solution of $\Pi(h(e), 2^m, X)$.

Let j be a recursive function such that for all e and d: $\Pi(j(e,d), 3\cdot 5^d, X)$ is:

$$(n)[\{e\}(\{d\}(n)) \text{ is defined} \to \Pi(\{e\}(\{d\}(n)), \{d\}(n), (X)_n)].$$

Let $\Pi(e_0, 1, X)$ be $(n)(n \notin X)$.

There exists a recursive I such that

$$\{I(e)\}(a) \simeq \begin{cases} e_0 & \text{if } a = 1 \\ h(\{e\}(m)) & \text{if } a = 2^m \\ j(e,d) & \text{if } a = 3\cdot 5^d \\ 0 & \text{otherwise.} \end{cases}$$

Let c be a fixed point of I. Then $\{I(c)\} \simeq \{c\}$, and $H(a, X)$ is $\Pi(\{c\}(a), a, X)$. Note that $\{c\}$ is total as in 3.3.I.

To prove (ii) let $H(a, X)$ be $(x)(\mathrm{Ev})R_1(x, v, a, X)$ for some recursive R_1. Define $Q(a, x, g, X)$ by

$$R_1(s, g(x), a, X) \quad \& \quad g(x) = \mu v R_1(s, v, a, X).$$

Q is recursive and

$$H(a, X) \leftrightarrow (\mathrm{Eg})[g \leq_T X \quad \& \quad (x)Q(a, x, g, X)].$$

The normal form theorem for Σ_1^0 predicates implies there is a recursive R such that

$$(2) \qquad (x)R(a, \bar{f}(x)) \leftrightarrow (x)Q(a, x, (f)_0, (f)_1) \quad \& \quad (i)_{i>1}(x)((f)_i(x) = 0).$$

If $a \in O$, then $(f)_0 \leq_T (f)_1$ and $(f)_1$ is the characteristic function of the unique X that satisfies $H(a, X)$. □

The Π_2^0 predicate $H(a, X)$ is less mysterious if viewed as follows. The Π_1^1 predicate

$$(3) \qquad a \in O \quad \& \quad X = H_a$$

has H_a as a unique solution when $a \in O$. (3) can be arithmetically approximated by dropping the requirement that a represent a wellordering. Instead a represents a recursive linear ordering that bears a superficial, arithmetic resemblance to $\{\langle u, v\rangle | u <_O v <_O a\}$. Thus a might be required to represent an ordering of ω such that each member of b of the field of the ordering has 2^b as its immediate successor save for the last member which is a. In addition the arithmetic approximation of (3) would refer to a hierarchy of sets attached to the elements of the field in much the same way that the H-sets are attached to the elements of O.

It turns out that the content of (3) can be expressed by a Π^0_2 formula save for the clause that requires the recursive linear ordering represented by a to be a well-ordering. In short (3) has a Π^0_2 approximation equivalent to (3) when $a \in O$.

From the vantage point of model theory there is another view of Theorem 4.2. Let M be a nonstandard ω-model of a fragment of mathematics strong enough to develop the theory of hyperarithmetic sets. (An ω-model is one whose integers are the same as those of the real world.) Take nonstandard to mean there exist non-wellfounded, recursive linear orderings which are wellfounded as far as M is concerned; they belong to M, but none of their infinite descending sequences do. It follows that the hyperarithmetic hierarchy of M is a proper end extension of its counterpart in the real world.

Let a be a notation for a constructive ordinal in the sense of M but not in the real world. Thus a represents a recursive linear ordering whose maximal wellfounded initial segment is of height ω_1^{CK}. Let H_a^M be the H-set attached to a by M. a and H_a^M satisfy every reasonable Π^0_2 approximation of (3).

These matters are discussed further in Section 2.III.

4.3 Corollary. *Each H-set is a* Π^0_2 *singleton.*

4.4 Corollary. *For each* $a \in O$, H_a *has a* Δ^1_1 *definition upward persistent over* $\{H_b | b <_O a\}$.

Proof. Let $H(a, X)$ be the Π^0_2 formula of Theorem 4.2. Clearly

$$z \in H_1 \leftrightarrow z \neq z,$$

$$z \in H_{2^b} \leftrightarrow (\mathrm{EX})[H(b, X) \ \& \ z \in X']$$

$$\leftrightarrow (X)[H(b, X) \rightarrow z \in X'], \text{ and}$$

$$z \in H_{3 \cdot 5^e} \leftrightarrow (\mathrm{EX})(\mathrm{Ex})(\mathrm{En})[H(\{e\}(n), X) \ \& \ x \in X \ \& \ z = \langle x, n \rangle]$$

$$\leftrightarrow (X)(\mathrm{Ex})(\mathrm{En})[(H(\{e\}(n), X) \rightarrow x \in X) \ \& \ z = \langle x, n \rangle]. \qquad \square$$

4.5 Theorem (Spector 1955). *There exists a recursive function* $h(a, b)$ *such that*

$$a, b \in O \ \ \& \ \ |a| \leq |b| \rightarrow H_a = \{h(a, b)\}^{H_b}$$

Proof. By effective transfinite recursion on O^2 wellordered as follows. $(a_1, b_1) < (a_2, b_2)$ iff

$$|a_1| < |a_2| \vee (|a_1| = |a_2| \ \ \& \ \ |b_1| < |b_2|).$$

The definition of the recursive iterater $I(e, a, b)$ has four cases. $h(a, b)$ will be $\{c\}(a, b)$, where c is a fixed point of I; that is $I(c, a, b) \simeq \{c\}(a, b)$.

Case 1: $a = 1$. $I(e, a, b) = c_1$ for some c_1 such that $\{c_1\}^X = H_1 = \phi$ for all X.

Case 2: $a = 2^m$ & $b = 2^n$. $I(e, a, b)$ is $j(\{e\}(m, n))$, where j is recursive and $\{j(d)\}^{X'} \simeq (\{d\}^X)'$.

Case 3: $a = 3 \cdot 5^z$. Let r and s be recursive functions such that

$$\{e\}(\{z\}(n), b) \simeq \{s(e, z, b)\}(n), \quad \text{and}$$

$$\{r(z)\}^X = \{\langle x, n\rangle | x \in \{\{z\}(n)\}^X\}.$$

Then

$$\{I(e, a, b)\}^X \simeq \{r(s(e, z, b))\}^X.$$

Case 4: $a = 2^m$ & $b = 3 \cdot 5^d$. Let g be as in Theorem 2.4 and k as in Lemma 1.2. If $m, 3 \cdot 5^d \in O$, and $|2^m| \leq |3 \cdot 5^d|$, then there is an n such that $|2^m| < |\{d\}(n)|$. Such an n can be computed from $H_{3 \cdot 5^d}$ because

$$|2^m| < |\{d\}(n)| \leftrightarrow 2^m \in O_{\{d\}(n)}, \text{ and}$$

$$O_{\{d\}(n)} = \{g(\{d\}(n))\}^{\{k(2^{\{d\}(n)}, 3 \cdot 5^d)\}^{H_{3 \cdot 5^d}}} \tag{1}$$

With the above pair of formulas in mind it is straightforward to find an index c_2 such that: if $m, 3 \cdot 5^d \in O$ and $|2^m| \leq |3 \cdot 5^d|$, then

$$\{c_2\}^{H_{3 \cdot 5^d}}(m, d) \text{ is an } n \text{ such that } |2^m| < |\{d\}(n)|.$$

$\{c_2\}^X(m, d)$ is computed as follows. Let $X^*_{d,n}$ be the result of replacing $H_{3 \cdot 5^d}$ by X in the right side of (1). Compute the truth-value of $2^m \in X^*_{d,n}$ for all n simultaneously. $\{c_2\}^X(m, d)$ is the first n (if there is any) for which the computation of $2^m \in X^*_{d,n}$ terminates affirmatively. Then

$$\{I(e, a, b)\}^X \simeq \{\{e\}(2^m, \{d\}(n)\}^{(X)_n},$$

where $n \simeq \{c_2\}^X(m, d)$.

If (a, b) fails to satisfy any of the above four cases, let $I(e, a, b)$ be 0. □

4.6 Corollary. *If $a, b \in O$, then*

$$|a| = |b| \rightarrow H_a \equiv_T H_b.$$

(Spector's uniqueness theorem)

It is tempting to think that Corollary 4.6 remains true when Turing degrees are replaced by many-one degrees. Moschovakis (1966) has shown that such an improvement is impossible.

For each $\delta < \omega_1^{CK}$, let h_δ be the Turing degree of H_a for some a such that $|a| = \delta$. Spector's uniqueness theorem implies h_δ is well defined. $\{h_\delta | \delta \in \omega_1^{CK}\}$ is a hierarchy for the Turing degrees of the H-sets. Its definition is based on constructive ordinal notations whose principal function was to clarify the notion of effective union

needed to define h_λ when λ is a limit ordinal. Sacks 1971 shows

$$h_\lambda = \inf\{d'' | (\delta)_{\delta<\lambda}(h_\delta < d)\}.$$

It is known that h_λ is definable in the partial ordering of Turing degrees with jump. It is likely that h_λ is definable in the Turing degrees without jump. For successful attempts to define h_γ for $\gamma \geq \omega_1^{CK}$, see Jockusch and Simpson 1976.

4.7 Hierarchy of Π^0_2 Singletons. Let $\Pi(e, X)$ be the e-th Π^0_2 formula, namely $(u)(\mathrm{Ev})T(\bar{X}(v), e, u, v)$. The set of Π^0_2 formulas is closed under recursive unions. If r is a recursive function, then the recursive union of $\Pi(r(e), X)(e \geq 0)$ is defined by

$$(1) \qquad \bigcup_e [\Pi(r(e), X)] \leftrightarrow (e)\Pi(r(e), (X)_e)$$

The right side of (1) is readily put in standard form $\Pi(e^*, X)$ for some e^* computable from a Gödel number for r.

The proof of Proposition 4.1 yields a recursive h such that: if A is the unique solution of $\Pi(e, X)$, then A' is the unique solution of $\Pi(h(e), X)$.

A hierarchy for the Π^0_2 singletons is defined by iteration of the Turing jump through the constructive ordinals. Let p be a recursive function that satisfies the following effective transfinite recursion on O.

$$\Pi(p(1), X) \leftrightarrow X = \phi.$$

$$\Pi(p(2^a), X) \leftrightarrow \Pi(h(p(a)), X)$$

$$\Pi(p(3 \cdot 5^d), X) \leftrightarrow \bigcup_e [\Pi(p(\{d\}(e)), X)].$$

For each $\delta < \omega_1^{CK}$, let Π_δ be the Turing degree of the unique solution of $\Pi(p(a), X)$ for some $a \in O$ such that $|a| = \delta$. Π_δ is well defined thanks to Corollary 4.6 and the proof of Theorem 4.2.

$\{\Pi_\delta | \delta < \omega_1^{CK}\}$ is aptly termed a hierarchy for the Turing degrees of the Π^0_2 singletons, since each Π^0_2 singleton is Δ^1_1, hence hyperarithmetic, and consequently one-one reducible to some Π^0_2 singleton in some Π_δ. (Recall: $X \leq_1 X'$; $X \leq_T Y \to X' \leq_1 Y'$.) Thus a cofinal subset of the Π^0_2 singletons are generated by starting with the null set and closing under Turing jump and recursive union.

4.8–4.10 Exercises

4.8. Let $H(a, X)$ be the Π^0_2 predicate of Theorem 4.2. Find b and Y such that $H(b, Y)$ holds and every H-set is recursive in Y.

4.9. Suppose A is Δ^1_1 definable over Δ^1_1 (that is, the function quantifiers range over the Δ^1_1 elements of ω^ω instead of ω^ω). Show A is Δ^1_1.

4.10. (Martin Davis). Suppose $a, b \in O$ and $|a| = |b| < \omega^\omega$. Show H_a and H_b belong to the same one-one degree.

5. *Hyperarithmetic Reducibility*

Let Y be an arbitrary subset of ω. All the results of Part A up to now relativize easily to Y. The relativization is initiated by replacing the recursive predicates of subsection 1.2.I by predicates recursive in Y. Note that $P(f, x)$ is recursive in Y iff there is a recursive predicate $R(X, f, x)$ such that $P(f, x)$ is $R(Y, f, x)$. In short a typical predicate recursive in Y is obtained by substituting the parameter Y for a free set variable in some recursive predicate. The process of *relativization* is straightforward because the presence of Y tends to have little effect on the proofs given so far in this book. Y simply goes along for the ride.

The consequences of relativization are not trifling. Results about number-theoretic predicates are lifted to predicates of reals by regarding the parameter Y as variable. A surprising aspect of relativization is that some recursive functions do not become recursive in Y. For example, $+_{O^Y}$ is recursive rather than recursive in Y.

5.1 Predicates Analytical in Y. A predicate is Σ^1_n (or Π^1_n or Δ^1_n) in Y if it is the result of substituting Y for a free set variable in some Σ^1_n (or Π^1_n or Δ^1_n) predicate. Thus $p(X, Y)$ is Σ^1_n in Y (with free variable X) if $P(X, Y)$ is Σ^1_n (with free variables X, Y). A predicate is said to be boldface Σ^1_n (or Π^1_n or Δ^1_n) if it is Σ^1_n (or Π^1_n or Δ^1_n) in some parameter Y. The collection of boldface Σ^1_n predicates is denoted by $\mathbf{\Sigma}^1_n$. The predicates of subsection 1.3.I are said to be lightface.

5.2 Proposition (Shoenfield). *For $n \geq 1$, the relation, X is Δ^1_n in Y, is transitive.*

Proof. Suppose A is Δ^1_n in B, $B\Delta^1_n$ in C, and

$$n \in A \leftrightarrow P_i(n, B),\ n \in B \leftrightarrow Q_i(n, C),$$

where $P_0, Q_0 \in \Sigma^1_n$ and $P_1, Q_1 \in \Pi^1_n$. Then

$$\begin{aligned} n \in A \leftrightarrow (\mathrm{E}Y)[&P_0(n, Y) \ \ \& \ \ Y = B] \\ \leftrightarrow (\mathrm{E}Y)[&P_0(n, Y) \ \ \& \ \ (n)(n \in Y \to Q_0(n, C)) \\ &\& \ \ (n)(n \notin Y \to \sim Q_1(n, C))], \end{aligned}$$

and so A is Σ^1_n in C. Similarly A is Π^1_n in C. □

It follow from Proposition 5.2 that "$A\Delta^1_n$ in B and $B\Delta^1_n$ in A" is an equivalence relation. Two sets are said to have the same Δ^1_n degree if each is Δ^1_n in the other. The Δ^1_1 degrees have been studied extensively. In Chapter IV a minimal Δ^1_1 degree will

be constructed, that is a degree greater than 0 with nothing between it and 0. (0 is the degree of the empty set.)

5.3 Ordinals Constructive in Y. The definition of O^Y differs from that of O only in the limit case. $3 \cdot 5^e \in O^Y$ iff $\{e\}^Y$ is total and

$$\{e\}^Y(n) <_{O^Y} \{e\}^Y(n+1)$$

for all n. $|a|^Y$ is the ordinal constructive in Y represented by a when $a \in O^Y$. ω_1^Y is the least ordinal not constructive in Y. O^Y is Π_1^1 in Y uniformly: $x \in O^Y$ is a Π_1^1 predicate whose free variables are x and Y.

The function $u +_{O^Y} v$ is recursive rather than recursive in Y, because the function h of subsection 3.3(1).I can be replaced by a recursive function h^* such that

$$\{h^*(e, a, d)\}^Y(n) \simeq \{e\}^Y(a, \{d\}^Y(n))$$

for all Y.

The functions p and q of Theorem 3.5.I remain recursive, but their meanings change. $W_{p(b)}$ becomes $W^Y_{p(b)}$, the $p(b)$-th set of numbers recursively enumerable in Y.

The function g of Lemma 4.1.I remains recursive.

As in Theorem 4.4.I, the ordinals recursive in Y equal those constructive in Y.

5.4 Π_1^1 Predicates of Reals. Let $P(n)$ be a number-theoretic predicate Π_1^1 in Y. The relativization of Theorem 5.4.I provides a recursive function k such that for all n,

$$P(n) \leftrightarrow k(n) \in O^Y, \tag{1}$$

uniformly in Y. k does not depend on Y. It is determined by the Π_1^1 predicate $Q(n, Z)$ (n and Z are free) such that $P(n)$ is $Q(n, Y)$.

A normal form for $P(n)$ is

$$(f)(\mathrm{Ex})R(Y, \bar{f}(x), n) \tag{2}$$

for some recursive R. If the free variable n of (2) is suppressed by being set equal to 0, and if the parameter Y is regarded as a free variable, then (2) becomes a typical Π_1^1 predicate $N(Y)$ whose only free variable ranges over 2^ω. According to (1)

$$N(Y) \leftrightarrow k(0) \in O^Y \tag{3}$$

for all Y. In this manner relativization to Y lifts the ordinal analysis of Π_1^1 predicates of numbers to Π_1^1 predicates of reals.

Every Π_1^1 predicate $Q(Y)$ of reals can be put in the form

$$\{e\}^Y \text{ is wellfounded,}$$

where e is such that $\{e\}^Y$ is total and a binary relation between numbers for all Y. Suppose $Q(Y)$ is $(f)(\mathrm{Ex})R(Y, \bar{f}(x), 0)$. Then $\{e\}^Y$ is $S_R^Y(0)$, the relativization of $S_R(0)$, from Proposition 5.3.I.

Fix n_0 and suppose $n_0 \in O^Y$ for all Y. Then n_0 can be thought of as defining a function f from 2^ω into ω_1: $f(Y) = |n_0|_Y$. The next result states that f obeys a sharp bounding principle.

5.5 Lemma. *Fix n and suppose $n \in O^Y$ for all Y. Then there exists a recursive ordinal δ such that*

$$|n|_Y < \delta$$

for all Y.

Proof. Suppose not. Then O is Σ_1^1:

$$b \in O \leftrightarrow (\mathrm{Ef})(\mathrm{EY})(x)(y)[x, y \in S_R(b) \ \& \ x > y \rightarrow \langle f(y), f(x)\rangle \in W^Y_{q(n)}].$$

S_R is defined as in Proposition 5.3.I so that for all b, $b \in O$ iff $S_R(b)$ is wellfounded. $|f(x)|_Y$ is the height (or rank) of x in $S_R(b)$. q is the recursive function of Theorem 3.5.I. □

It follows from Lemma 5.5 that if $P(Y)$ is Π_1^1 and $(Y)P(Y)$ holds, then the latter is "seen" to be true by some recursive ordinal.

5.6 Hyperarithmetic Predicates of Reals. The H^Y-sets are defined by recursion on O^Y.

$$H_1^Y = Y. \qquad H_{2^m}^Y = (H_m^Y)'.$$

$$H_{3\cdot 5^e}^Y = \{\langle x, n\rangle | x \in H^Y_{\{e\}(n)}\}.$$

X is said to be hyperarithmetic in Y (in symbols $X \leq_h Y$) if X is recursive in some H^Y-set. As in Corollary 1.4(ii), $X \leq_h Y$ is a Π_1^1 predicate (with X and Y as free variables). The relativization of Spector's uniqueness theorem (Corollary 4.6) to Y implies that the Turing degree of H_b^Y depends only on $|b|_Y$. Thus it makes sense to refer to $Y^{(\delta)}$, the δ-th iterate of the Turing jump of Y, when $\delta < \omega_1^Y$.

A predicate $P(Y)$ is said to be *hyperarithmetic* if there exist $b \in O$ and e such that for all Y

$$P(Y) \leftrightarrow \{e\}^{H_b^Y}(0) \text{ is defined.} \tag{1}$$

$\langle b, e\rangle$ is said to be a code for $P(Y)$. The set of all codes for hyperarithmetic predicates is Π_1^1. Inserting a superscript Y in appropriate places in the proof of Theorem 1.3(i) shows

$$b \in O \quad \& \quad n \in H_b^Y$$

is Π_1^1 (with n and Y free). It follows, as in the proof of Corollary 1.4, that $P(Y)$ is Δ_1^1.

Suppose $Q(Y)$ is Δ^1_1 to show it hyperarithmetic. According to formula 5.4(3), there exist k_0 and k_1 such that for all Z,

$$Q(Z) \leftrightarrow k_0 \in O^Z \leftrightarrow k_1 \notin O^Z.$$

The relativization of Theorem 3.3 to Z implies:

$$\min(|k_0|_Z, |k_1|_Z) \leq |t(k_0, k_1)|_Z$$

for all Z. Let n be $2^{t(k_0,k_1)}$.Then

$$Q(Z) \leftrightarrow k_0 \in O^Z_n. \tag{2}$$

It follows from the relativization of Theorem 2.4 to Z that for some e,

$$O^Z_n = \{e\}^{H^Z_{2^n}} \tag{3}$$

for all Z. The function g of 2.4 remains recursive, rather than becoming recursive in Z, when 2.4 is relativized to Z. $n \in O^{\varnothing}$ since $n \in O^Z$ for all Z. ($\varnothing$ is the empty set.) There is very little difference between O and $O^{\varnothing}$, so n can be regarded as a member of O (cf. Lemma 7.5). Thus (2) and (3) imply $Q(Z)$ is hyperarithmetic.

5.7–5.12 Exercises

5.7. Clarify the assertion concerning O and $O^{\varnothing}$ made at the end of subsection 5.6.

5.8 Verify that $X \leq_h Y$ is Π^1_1.

5.9. Let $P(X)$ be Σ^1_1. Suppose

$$(X)[P(X) \rightarrow \{e\}^X \text{ is total and is a wellfounded relation}].$$

Show there is a $\delta < \omega_1^{CK}$ such that

$$(X)[P(X) \rightarrow |\{e\}^X| < \delta].$$

5.10. Suppose $P(X, y)$ is Π^1_1 and

$$(X)(\text{Ey})[y \in O^X \ \& \ P(X, y)].$$

Show there is a $\delta < \omega_1^{CK}$ such that

$$(X)(\text{Ey})[y \in O^X \ \& \ |y|_X < \delta \ \& \ P(X, y)].$$

5.11. (Kleene separation). Let $A, B \subseteq \omega^\omega$ be Σ^1_1 and disjoint. Find a $\Delta^1_1 C$ such that $A \subseteq C$ and $B \cap C = \varnothing$.

5.12. Suppose $A \subseteq B \subseteq \omega^\omega$, $A \in \Sigma^1_1$ and $B \in \Pi^1_1$. Find a $\Delta^1_1 C$ such that $A \subseteq C \subseteq B$.

6. *Incomparable Hyperdegrees Via Measure*

X and Y belong to the same hyperdegree if $X \leq_h Y$ and $Y \leq_h X$. $\leq_h$ is transitive because "Δ^1_1 in" is by Proposition 5.2. Thus the hyperdegrees constitute a partition of 2^ω. The hyperdegree of X is denoted by $\underline{X}$. $\underline{X} \leq \underline{Y}$ iff $X \leq_h Y$. $\varnothing$, the hyperdegree of the empty set, is the least hyperdegree. $\underline{X}$ and $\underline{Y}$ have a least upper bound, the hyperdegree of

$$\{2n | n \in X\} \cup \{2n + 1 | n \in Y\},$$

denoted by $\underline{X} \cup \underline{Y}$.

In this section the existence of two incomparable hyperdegrees is established by a measure-theoretic argument of extraordinary simplicity. In Chapter IV the same result will be obtained by a forcing argument analogous to the Kleene–Post construction of incomparable Turing degrees. The approach via measure is swift once the measurability of Π^1_1 sets is established.

6.1. Measurable Subsets of 2^ω. The subbasic open subsets of 2^ω are obtained by fixing single coordinates.

$$\{X | X \in 2^\omega \ \& \ m \in X\} \quad \text{and} \quad \{X | X \in 2^\omega \ \& \ n \notin X\}$$

are typical subbasic open sets. A basic open set is a finite intersection of subbasic sets. The measure μ of a basic open set b is 2^{-i}, where i is the number of coordinates fixed by b. Thus $\mu(b) = 2^{-i}$ if

$$b = \{X | m_1 \in X \ \& \ \ldots \ \& \ m_j \in X \ \& \ m_{j+1} \notin X \ \& \ \ldots \ \& \ m_i \notin X\}$$

and $m_1, m_2, \ldots, m_i$ are distinct. An open set is a union of basic open sets.

Let J be an arbitrary subset of 2^ω. An open cover of J is a family K of basic open sets such that $J \subseteq \cup K$. Define

$$I(K) = \Sigma\{\mu(b) | b \in K\}.$$

The outer measure of J, denoted by $\mu_0(J)$, is

$$\inf\{I(K) | K \text{ is an open cover of } J\}.$$

$\mu_i(J)$, the inner measure of J, is $1 - \mu_0(2^\omega - J)$. J is said to be *measurable* if $\mu_0(J) = \mu_i(J)$. If J is measurable, then its measure $\mu(J)$ is $\mu_0(J)$.

Every open set is measurable. The family of all measurable sets is a σ-algebra, that is a Boolean algebra closed under countable unions. The operations of meet, joint and complementation are the set-theoretic operations of intersection, union

and complementation. μ is countably additive: if $\{J_i | i < \omega\}$ is a sequence of pairwise disjoint, measurable sets, then

$$\mu\left(\bigcup_i J_i\right) = \sum_i \mu(J_i).$$

The proof of countable additivity makes essential use of the countable axiom of choice. In order to find a suitable open set containing $\bigcup_i J_i$, it is necessary to choose a suitable open set containing J_i for each i.

A subset of 2^ω is *Borel* if it belongs to the least σ-algebra containing all the open sets. It follows that every Borel set is measurable, but not that every measurable set is Borel. A set is measurable iff it differs from a Borel set by a subset of a Borel set of measure 0. Thus J is measurable iff there exist Borel sets B_0 and B_1 such that

$$(B_0 - J) \cup (J - B_0) \subseteq B_1 \quad \text{and} \quad \mu(B_1) = 0.$$

Each Borel set can be fabricated from open sets in countably many steps. Let B_0 be the family of all subsets of 2^ω that are either open or closed. For each countable ordinal δ, let $B_{\delta+1}$ be the result of adding to B_δ all countable intersections of elements of B_δ, and the complements of such intersections. Let B_{ω_1} be $\cup \{B_\delta | \delta < \omega_1\}$. B_{ω_1} is a σ-algebra containing all the open sets, and clearly the least such. Hence a set is Borel iff it belongs to B_δ for some countable δ.

Each hyperarithmetic set is Borel since it can be fabricated in δ steps for some $\delta < \omega_1^{CK}$. According to Exercise 6.4, the Borel sets are the same as the boldface Δ_1^1 sets.

6.2 Lemma (Lusin). Π_1^1 *subsets of* 2^ω *are measurable.*

Proof. Suppose $J \subseteq 2^\omega$ is Π_1^1. As in subsection 5.4 there is an integer k such that

$$X \in J \leftrightarrow k \in O^X$$

for all X. The measurability of J will follow easily from the measurability of Borel sets once a countable bound is found on the ordinals represented by elements of O^X as X ranges over J. First it must be checked that: for each integer j and countable ordinal δ, the set

$$\{X | j \in O^X \ \& \ |j| = \delta\} \tag{1}$$

is Δ_1^1 in any Y such that Y is a wellordering of ω of height δ. Let $S^X(j)$ be a linear ordering of a set of sequence numbers, recursive uniformly in X, such that $j \in O^X$ iff $S^X(j)$ is wellfounded, as in Proposition 5.3.I relativized to X. $X \in (1)$ iff there is an f that maps $S^X(j)$ in a one-one, orderpreserving fashion onto Y. $X \notin (1)$ iff there is a one-one, orderpreserving f that maps one of $S^X(j)$ and Y onto a proper initial segment of the other.

Since (1) is boldface Δ^1_1, it is Borel by Exercise 6.4, hence measurable. Since measure is countably additive, there must be a δ_∞ such that (1) has measure 0 for all δ when $\delta \geq \delta_\infty$. (Corollary 1.6.IV: $\delta_\infty = \omega_1^{CK}$.) Let

$$J_1 = \{X | k \in O^X \ \& \ |k|_X < \delta_\infty\}.$$

Clearly $J_1 \subseteq J$. Since J_1 is Borel, it remains only to show that $J - J_1$ is contained in a Borel set of measure O. Observe that

$$X \in (J - J_1) \to (\mathrm{Ej})[j \in O^X \ \& \ |j|_X = \delta_\infty]. \tag{2}$$

The set of all X that satisfy the right side of (2) is Borel and has measure zero thanks to the choice of δ_∞ and the countable additivity of measure. □

6.3 Theorem (Spector 1958). *There exist X and Y such that $X \not\leq_h Y$ and $Y \not\leq_h X$.*

Proof. As noted in subsection 5.6, $\leq_h$ is Π^1_1, hence measurable by Lemma 6.2. According to Fubini's theorem the measure of $\{(X, Y) | X \leq_h Y\}$ can be computed by integrating the measure of $\{X | X \leq_h Y\}$ along the Y-axis. But $\{X | X \leq_h Y\}$ is countable, hence of measure O. So $\{(X, Y) | X \leq_h Y\}$ has measure 0. In the same manner $\{(X, Y) | Y \leq_h X\}$ has measure 0. Thus almost every pair satisfies the conclusion of the theorem. □

One shortcoming of Spector's proof of Theorem 6.3 is the absence of a concrete pair of incomparables. In Chapter III Kleene's basis theorem will be applied to the conclusion of 6.3 to produce a pair of incomparables recursive in O.

In Chapter IV the measure-theoretic approach will be refined to show $\omega_1^Y = \omega_1^{CK}$ for almost all Y.

6.4–6.7 Exercises

6.4. (Addison). Show a subset of 2^ω is Borel iff it is boldface Δ^1_1.

6.5. (Addison). Call a subset of 2^ω *analytic* if it is the projection of a Borel subset of $2^\omega \times 2^\omega$. Show a subset of 2^ω is analytic iff it is boldface Σ^1_1.

6.6. Show every analytic subset of 2^ω is measurable. (Gödel has shown the consistency of ZFC and the existence of an unmeasurable Δ^1_2 set, if ZF is consistent.)

6.7. Repeat 6.4–6.6 for ω^ω.

7. *The Hyperjump*

The hyperjump of X is O^X. According to subsection 5.3, the graph of the hyperjump function is Π^1_1. There is an imperfect analogy between the Turing- and hyper-

jumps. Each is a completion. For the hyperjump this means: O^X is Π^1_1 in X, and every set Π^1_1 in X is many-one reducible to O^X (the relativization of Theorem 5.4.I). However, the partial ordering of hyperdegrees of Π^1_1 sets differs radically from that of the Turing degrees of recursively enumerable sets.

Iteration of the Turing jump generated the hyperarithmetic sets, which turned out to be a hierarchy for the Turing degrees of the Π^0_2 singletons. Iteration of the hyperjump generates a hierarchy for some, but not all, of the hyperdegrees of the Π^1_1 singletons. To say a set is generated by the hyperjump is to say it is E-recursive in the hyperjump function. E-recursion is defined in part D.

The next proposition implies the hyperjump is well defined for hyperdegrees.

7.1 Proposition. $A \leq_h B \leftrightarrow O^A \leq_m O^B$.

Proof. Assume $O^A \leq_m O^B$. Since A, $\omega - A$ are Π^1_1 in A, they are many-one reducible to O^A by the relativization of Theorem 5.4.I to A. The transitivity of $\leq_m$ implies A, $\omega - A \leq_m O^B$. Thus A, $\omega - A$ are Π^1_1 in B, and so A is Δ^1_1 in B, hence hyperarithmetic in B by the relativization of Theorem 2.5 to B.

Assume $A \leq_h B$. Then A is Δ^1_1 in B. Since O^A is Π^1_1 in A, it follows that O^A is Π^1_1 in B as in the proof of Proposition 5.2. Therefore $O^A \leq_m O^B$ by the completeness of O^B with respect to sets Π^1_1 in B. □

The next lemma implies that every Π^1_1 set has the same hyperdegree as the null set or Kleene's O. It is the first result to suggest that Π^1_1 sets are not analogous to recursively enumerable sets. Friedberg 1955 and Muchik 1955 independently found a pair of incomparable Turing degrees of recursively enumerable sets. Kreisel alone resisted the suggestion, on the grounds that Δ^1_1 sets were not analogous to recursive sets, but bore a relation to Π^1_1 sets much like that of finite sets to recursively enumerable sets. His insight led to the creation of metarecursion theory and a Friedberg–Muchnik-type theorem for the Π^1_1 sets, as detailed in Part B.

7.2 Proposition (Spector 1955). *If* X, $Y \in \Pi^1_1$ *and* $Y \notin \mathrm{HYP}$, *then* $X \leq_h Y$.

Proof. It suffices to show O is Δ^1_1 in Y. By Theorem 5.4.I there is a recursive f such that

$$x \in Y \leftrightarrow f(x) \in O.$$

If there were a $b \in O$ such that $f[Y] \subseteq O_b$, then $Y \leq_m O_b$ and $Y \in \mathrm{HYP}$ by Theorem 2.4. Thus $f[Y]$ is unbounded in O. Hence

$$a \in O \leftrightarrow (\mathrm{Ex})[x \in Y \ \&\ f(x) \in O \ \&\ a \in O_{f(x)}],$$
$$a \notin O \leftrightarrow (x)[x \in Y \rightarrow (f(x) \in O \ \&\ a \notin O_{f(x)})].$$

It follows from Lemma 2.1 that O is Δ^1_1 in Y. □

Try for a moment to think of the proof of Proposition 7.2 in terms of generalized computations. If $a \in O$, then this is seen to be so by a computation of height $|f(x)|$

for some $x \in Y$. If $a \notin O$, then a set of computations makes it so. The supremum of the heights of computations in the set is at most ω_1^{CK}. Thus $a \in O$ is decided by a computation of height at most ω_1^{CK}. It would be more just to allow only one computation of height less than ω_1^{CK} to decide $a \in O$, since the computations that enumerate O all have height less than ω_1^{CK}. Allowing a computation of height ω_1^{CK} in the classification of Π_1^1 sets is no more fair than allowing a computation of height ω in the classification of recursively enumerable sets.

7.3 Theorem (Spector 1955). $A \leq_h B \to \omega_1^A \leq \omega_1^B$.

Proof. Suppose $A \leq_h B$ but $\omega_1^B < \omega_1^A$. Fix $b \in O^A$ such that $|b|_A = \omega_1^B$ with the intent of showing O^B is Σ_1^1 in B, an impossibility according to the relativization of Corollary 5.5.I. It suffices to show O^B is Σ_1^1 in A, since A is Δ_1^1 in B.

When Proposition 3.2 is relativized to B, the function h remains recursive. Thus

$$x \in O^B \leftrightarrow R^B_{h(x)} \text{ is wellfounded,}$$
$$x \in O^B \to |x|_B \leq |R^B_{h(x)}|,$$

where R^B_n is the n-th binary relation recursively enumerable in B. As noted in subsection 5.3, the function q of Theorem 3.5.I remains recursive when 3.5 is relativized. Thus

$$W^A_{q(a)} = \{\langle x, y\rangle | x <_{O^A} y <_{O^A} a\}$$

when $a \in O$. The following formula is Σ_1^1 and equivalent to $x \in O^B$.

(Ef) [f is an orderpreserving map of $R^B_{h(x)}$ into $W^A_{q(b)}$]. Remember that b was chosen so that $|W^A_{q(b)}| = \omega_1^B$. □

7.4 Corollary (Spector 1955). $X \in \mathrm{HYP} \to \omega_1^X = \omega_1^{CK}$.

The converse of Corollary 7.4 is false. In fact the set of all X such that $\omega_1^X = \omega_1^{CK}$ has measure 1, as will be proved in Chapter IV.

The next lemma makes it possible to regard O^A as an initial segment of O^B when A is recursive in B.

7.5 Lemma. *Suppose A is recursive in B. Then there exists a recursive function f such that*

(*i*) $(x)(y)[x <_{O^A} y \leftrightarrow f(x) <_{O^B} f(y)]$, *and*
(*ii*) $(x)[x \in O^A \to |x|_A = |f(x)|_B]$.

Proof. Let $A = \{e_0\}^B$. There is a recursive I such that

$$\{I(c)\}(b) \simeq \begin{cases} 1 & \text{if } b = 1 \\ 2^{\{c\}(m)} & \text{if } b = 2^m \\ 3 \cdot 5^{h(c,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases}$$

h is a recursive function such that

$$\{h(c, d)\}^B(n) \simeq \{c\}(\{d\}^A(n)) \simeq \{c\}(\{d\}^{\{e_0\}^B}(n)).$$

Choose e so that $\{I(e)\} \simeq \{e\}$, and let f be $\{e\}$. f is total by induction on ω. An induction on $<_{O^A}$ establishes (ii) and the left-to-right direction of (i). The other direction of (i) requires an induction on $<_{O^B}$. □

7.6 Theorem (Spector 1955)

(i) $O^A \leq_h B \to \omega_1^A < \omega_1^B$
(ii) $\omega_1^A < \omega_1^B \;\&\; A \leq_h B \to O^A \leq_h B$.

Proof

(i) Suppose $O^A \leq_h B$. By Theorem 7.3, $\omega_1^{O^A} \leq \omega_1^B$, so it need only be shown that $\omega_1^A < \omega_1^{O^A}$. The proof of Lemma 4.3.I is unchanged by relativization to Y. Let Y be O^A. $<_{O^A}$ is a wellfounded relation recursively enumerable in O^A, and so $|<_{O^A}|$ is an ordinal constructive in O^A.

(ii) First consider the special case of $A \leq_T B$. Let f be the recursive function of Lemma 7.5. Suppose $\omega_1^A < \omega_1^B$. Then there is a $c \in O^B$ such that $\omega_1^A = |c|_B$ and

$$x \in O^A \leftrightarrow f(x) \in O_c^B.$$

Hence $O^A \leq_h B$, by the relativization of Theorem 2.4 to B. Now suppose $A \leq_h B$. Then $A \leq_T H_b^B$ for some $b \in O^B$. Theorem 7.3 implies

$$\omega_1^B = \omega_1^{H_b^B}.$$

By the special case, $O^A \leq_h H_b^B$. □

7.7 Corollary. $\omega_1^{CK} < \omega_1^X \leftrightarrow O \leq_h X$.

7.8–7.10 Exercises

7.8. Suppose $P(x, y)$ is Σ_1^1 and $\{\langle x, y\rangle | P(x, y)\}$ is a wellordering. Show its ordinal height is less than ω_1^{CK}.

7.9. Suppose $A <_h O$. Show $O^A \equiv_h O$.

7.10. (Platek). Call an ordinal $\gamma \Pi_1^1$ if there is a Π_1^1 binary relation $P(x, y)$ such that $P(x, y)$ is wellfounded and $\gamma = |P(x, y)|$. Show that ω_1^O is the least non-Π_1^1 ordinal.

Chapter III
Σ^1_1 Predicates of Reals

This chapter concentrates on basis theorems for Σ^1_1 and Π^1_1 predicates. In addition hyperarithmetic quantifiers are classified, and closure points of Π^0_1, and monotonic Π^1_1, inductive definitions are computed.

1. Basis Theorems

Let B be a set of functions and $\mathscr{P}$ a family of predicates of the form $P(f)$. B is said to be a basis for $\mathscr{P}$ if for each $P(f) \in \mathscr{P}$,

$$(\mathrm{E}f)P(f) \to (\mathrm{E}f)_{f \in B} P(f).$$

A basis B for $\mathscr{P}$ is of maximal interest when the complexity of a typical member of B is minimal relative to that of a typical member of $\mathscr{P}$. Basis theorems permeate the literature of higher recursion theory, possibly because they clarify the scope of certain quantifiers. Thus Kleene's basis theorem implies that the scope of an existential function quantifier, when applied to a Σ^1_1 predicate, can be reduced to those functions recursive in Kleene's O. A result of Shoenfield states that the Δ^1_2 reals are a basis for the Σ^1_2 predicates. The Kleene and Shoenfield arguments have in common the analysis of quantifiers by ordinals as in Section 5.I. In general basis theorems are proved by exploiting an appropriate assignment of ordinals to finite partial functions.

Every Σ^1_1 predicate $P(f)$ can be put in the form

$$(\mathrm{E}g)(x)R(\bar{f}(x), \bar{g}(x))$$

for some recursive R. Therefore finding a solution f of $P(f)$ is equivalent to finding a solution $\langle f, g \rangle$ of $(x)R(\bar{f}(x), \bar{g}(x))$. Thus the basis problem for Σ^1_1 predicates is the same as that for Π^0_1 predicates. The recursive predicates have a trivial basis (Exercise 1.5), but the next result indicates the surprising complexity introduced by a single universal number quantifier.

1.1 Theorem (Kleene). *The hyperarithmetic functions are not a basis for the* Π^0_1 *predicates.*

Proof. Let $P(f)$ be $f \notin \mathrm{HYP}$. $P(f)$ is Σ^1_1 by Corollary 1.4(ii).II. Put $P(f)$ in the form $(\mathrm{Eg})(x)R(\bar{f}(x), \bar{g}(x))$. Then $(x)R(\bar{f}(x), \bar{g}(x))$ has no hyperarithmetic solution $\langle f, g\rangle$. □

1.2 Branches of Trees. According to Theorem 1.1 there is a recursive predicate R such that no hyperarithmetic f satisfies $(x)R(\bar{f}(x))$, but some f does. R corresponds to a recursive tree T_R with the property that f is a solution of $(x)R(\bar{f}(x))$ if and only if f is an infinite branch of T_R. A tree, by definition, is a partial ordering with a greatest element and such that every pair of incomparables has no lower bound. T_R is the partial ordering of sequence numbers (as in subsection 5.1.I) consisting of those $\bar{f}(x)$ that satisfy $(i)_{i<x}R(\bar{f}(i))$. (Visualize the tree upside down; the greatest element is on top and the branches go down.) f satisfies $(x)R(\bar{f}(x))$ iff $\bar{f}(0) > \bar{f}(1) > \bar{f}(2) > \ldots$ is an infinite branch.

The next theorem says that a recursive tree with an infinite branch must have one recursive in Kleene's O. If it still seems hard to believe that such a tree need not have a hyperarithmetic branch, pay attention to the difference between finite branching and infinite branching. According to Exercise 1.7, a recursive tree with finite branching and an infinite branch has one recursive in $0'$. Later it will be shown that a recursive tree with less than 2^{ω} branches has only hyperarithmetic branches.

1.3 Theorem (Kleene). *The functions recursive in O are a basis for the Σ^1_1 predicates.*

Proof. Let R be recursive, and assume $(x)R(\bar{f}(x))$ holds for some f. Define $Q(s)$ to be: s is a sequence number in the field of T_R (defined in subsection 1.2) and the part of T_R below s is not wellfounded. Recall: $>$ orders T_R as in 5.1.I.

$Q(s)$ is Σ^1_1, hence recursive in O by Theorem 5.4.I. Define f recursively in Q so that

$$\bar{f}(n+1) = \mu s[Q(s) \quad \& \quad \ell h(s) = n+1 \quad \& \quad \bar{f}(n) > s].$$

f is welldefined because

$$(s)(\mathrm{Et})[Q(s) \to Q(t) \quad \& \quad s > t].$$

In short the step-by-step definition of f avoids all wellfounded subtrees of T_R. □

The recursive ordinals play a significant, if hidden, role in the proof of Kleene's basis theorem. For each s in the field of T_R, let T^s_R be the part of T_R below s. If T^s_R is wellfounded, then it has an ordinal height denoted by $|T^s_R|$ defined in subsection 4.2.I. Let $|s|$ be $|T^s_R|$ when T^s_R is wellfounded. Since R is recursive, $|s|$, when defined, is a recursive ordinal. $Q(s)$ is equivalent to: $|s|$ is not defined. The infinite path f defined in the proof of Theorem 1.2 is the "leftmost" branch that omits all s such that $|s|$ is defined. According to Exercise 1.8, the "leftmost" branch has the same hyperdegree as Kleene's O or is hyperarithmetic. H. Friedman and D. A. Martin

have independently shown: if a recursive tree has a nonhyperarithmetic branch, then it has one in each hyperdegree greater than or equal to that of Kleene's O.

1.4 Theorem (Gandy). $\{f | f <_h O\}$ *is a basis for the* Σ^1_1 *predicates.*

Proof. Suppose $P(f)$ is Σ^1_1 and $(\mathrm{Ef})P(f)$. Then

$$P(f) \quad \& \quad g \not\leq_h f \tag{1}$$

is Σ^1_1 according to subsection 5.6.II.
(1) has a solution since only countably many g's are hyperarithmetic in each f. By Theorem 1.3, (1) has a solution $\langle f, g\rangle \leq_T O$. Thus

$$P(f) \quad \& \quad f \leq_h O \quad \& \quad g \leq_h O \quad \& \quad g \not\leq_h f.$$

Hence $f <_h O$. $\cdot \square$

1.5 Corollary *The* Σ^1_1 *set* $\{f | \omega_1^f = \omega_1^{\mathrm{CK}}\}$ *is a basis for the* Σ^1_1 *predicates.*

Proof. Suppose $P(f)$ is Σ^1_1 and $(\mathrm{Ef})P(f)$. By Theorem 1.4 there is an f such that $P(f)$ and $f <_h O$. Corollary 7.7.II implies $\omega_1^f = \omega_1^{\mathrm{CK}}$. $\square$

References to model theory are in order. The proof of Theorem 1.3 is analogous to a Henkin-style construction of a countable, consistent set of sentences. In the setting of first order logic, a Henkin construction can be regarded as a step-by-step development of an infinite path through a tree with finite branching. The infinite branching allowed in the proof of Theorem 1.2 corresponds to sentences of countably infinite length having the form of a disjunction. In the same vein the finite proofs of first order logic correspond to infinite proofs encoded by hyperarithmetic reals. This vein will be mined further later, but before leaving it, observe that Corollary 1.5 has the appearance of a type-omitting result. The type omitted is: x is a non-recursive ordinal recursive in f.

Another basis result of the type-omitting sort will be proved later in this Chapter. If states: if $A \notin \mathrm{HYP}$, then $\{f | A \not\leq_h f\}$ is a basis for the Σ^1_1 predicates. Still another will come up in Chapter IV. Theorem 2.2.IV states: if a Π^1_1 set has positive measure, then it has a hyperarithmetic member.

1.6–1.9 Exercises

1.6. Let F be the set of all f unequal to 0 on only finitely many coordinates. Show F is a basis for the recursive predicates.

1.7. Suppose T is a recursive tree with finite branching and an infinite branch. Show T has an infinite branch recursive in $0'$. ($0'$ is the complete recursively enumerable subset of ω.)

1.8. Suppose T is a recursive tree with infinite branching. Let f and g be infinite branches. f is said to be to the left of g if for some $x, \bar{f}(x) = \bar{g}(x)$ and $f(x) < g(x)$. Assume T has an infinite branch. Show the leftmost branch of T is either hyperarithmetic or has the same hyperdegree as Kleene's O. Show T has a branch of hyperdegree less than that of O.

1.9. (H. Friedman). Use Corollary 1.4 to show there is a model M of set theory such that the ordinals of M are a proper end extension of the recursive ordinals, and such that $\sup\{\delta | \delta \in M \ \& \ \delta \text{ is standard}\} = \omega_1^{CK}$. (Thus M omits ω_1^{CK}.)

2. *Unique Notations for Ordinals*

A is said to be a set of unique notations for the constructive ordinals if $A \subseteq O$ and the restriction of $<_O$ to A^2 is a linear ordering whose ordertype is ω_1^{CK}. In brief A is a branch that goes all the way through O. Clearly each constructive ordinal has just one notation in A. It will soon be proved that there exists a Π_1^1 set O_1 of unique notations. O_1 will be useful in the analysis of hyperarithmetic quantifiers in the next section, and in the definitions of the basic concepts of metarecursion theory in Part B.

2.1 Lemma (Kleene). *There exists a recursive linear ordering with an infinite descending sequence, but no hyperarithmetic such sequence.*

Proof. As noted in subsection 1.2, there is a recursive R such that no hyperarithmetic f satisfies $(x)R(\bar{f}(x))$ but some f does. T_R, the partial ordering of sequence numbers $\bar{g}(x)$ such that $(i)_{i<x}R(\bar{g}(x))$, is a recursive tree with an infinite branch but no hyperarithmetic such branch. T_R can be linearized by ordering sequence numbers in the Kleene–Brouwer fashion: $\bar{g}(u) <_{KB} \bar{h}(v)$ if

$$(1) \qquad \begin{array}{l} \text{(i) } u > v \ \ \& \ \ (i)_{i<v}(g(i) = h(i)), \ \text{ or} \\ \text{(ii) } (\text{Ek})[k < \min(u, v) \ \ \& \ \ g(k) < h(k) \ \ \& \ \ (i)_{i<k}(g(i) = h(i))]. \end{array}$$

Let $<_R$ be the restriction of $<_{KB}$ to the sequence numbers in the field of T_R. Each solution f of $(x)R(\bar{f}(x))$ defines an infinite descending sequence of $<_R$ according to (1)(i). Suppose $<_R$ has a hyperarithmetic infinite descending sequence $x_0 >_R x_1 >_R x_2 \ldots$ with the intent of showing $(x)\mathrm{R}(\bar{f}(x))$ has a hyperarithmetic solution. Note that any set of sequence numbers of bounded length is wellordered by $<_{KB}$. Consequently the lengths of the x_i's are unbounded. Hence it is safe to assume that $\ell h(x_i)$ is a strictly increasing function of i, since some hyperarithmetic subsequence of the x_i's must have that property. An induction on n shows

$$(n)(\text{Ej})(i)_{i \geq j}[(x_i)_n = (x_j)_n].$$

Define $t(n)$ by $\mu j(i)_{i \geq j}[(x_i)_n = (x_j)_n]$, and $f(n)$ by $(x_{t(n)})_n - 1$. Then f is a hyperarithmetic solution of $(x)R(\bar{f}(x))$. $\square$

Model theory yields another view of Lemma 2.1. Suppose M is a model of a suitable fragment of set theory and the ordinals of M are a proper end extension of the recursion ordinals. Thus the recursive ordinals in the sense of M include some nonstandard ordinals as well as the standard recursive ordinals. Let δ be a nonstandard recursive ordinal of M, and let $<_R \in M$ be a recursive wellordering whose ordertype is δ. Then $<_R$ has an infinite descending sequence outside M but not inside M. If $<_R$ had a hyperarithmetic, infinite descending sequence, then that sequence would belong to M, since all standard recursive ordinals belong to M, hence all standard H-sets belong to M. The next lemma computes the ordertype of a nonstandard recursive ordinal.

2.2 Lemma. *Let $<_R$ be a recursive linear ordering with an infinite descending sequence but no hyperarithmetic such sequence.*

(i) (Gandy) Let Q be the maximum, wellordered initial segment of $<_R$. Then Q is Π^1_1 but not Σ^1_1, and the restriction of $<_R$ to Q has ordertype ω_1^{CK}.

(ii) (Harrison) The ordertype of $<_R$ is $\omega_1^{CK}(1+\eta)+\gamma$, where η is the ordertype of the rationals, and γ is a recursive ordinal.

Proof

(i) Q is Π^1_1 since

$$x \in Q \leftrightarrow x \in <_R \quad \& \quad \sim(\mathrm{E}f)(n)[f(n+1) <_R f(n) <_R x].$$

To see that Q is not Σ^1_1, choose b in the field of $<_R$ but not in Q. Then there is an infinite descending sequence below b and above Q. Suppose Q is Σ^1_1. Then the Π^1_1 relation

$$x <_R b \quad \& \quad y <_R x \quad \& \quad y \notin Q$$

can be uniformized by some Π^1_1 $P(x, y)$ according to Theorem 2.3.II. Let $g(O)$ be some x below b and above Q, and $g(n+1)$ the unique y such that $P(g(n), y)$. Then g is a hyperarithmetic, infinite descending sequence in $<_R$.

Suppose the ordertype of $<_R$ restricted to Q were less than ω_1^{CK}. Then there would be an $a \in O$ such that for all x,

$$(1) \qquad x \in Q \leftrightarrow x \in \text{field of } <_R \ \& \ (\mathrm{E}f)[f \text{ is a one-one orderpreserving map of } <_R^x \text{ into } W_{q(a)}].$$

$<_R^x$ is the initial segment of $<_R$ below x, and $W_{q(a)}$ is the recursively enumerable initial segment of $<_O$ below a, as in Theorem 3.5.I. (1) implies Q is Σ^1_1.

If the ordertype of $<_R$ restricted to Q were greater than ω_1^{CK}, then ω_1^{CK} would be a recursive ordinal.

(ii) Let x, y belong to the field of $<_R$. Call $x \sim y$ if $[x, y]$ or $[y, x]$ is a closed interval of $<_R$ that contains no infinite descending sequences. $\sim$ is a Π^1_1 equivalence relation that partitions $<_R$ into intervals. Each such interval is Π^1_1, since a typical interval is $\{y | y \sim x\}$.

Each interval has an R-least member. Otherwise

$$(u)(\mathrm{Ev})[u \sim x \to v \sim x \quad \& \quad v <_R u].$$

Π^1_1 uniformization (Theorem 2.3.I) makes it possible to construe v as a hyperarithmetic function of u, and so yields a hyperarithmetic, infinite descending sequence in $<_R$.

Since $\{y|y \sim x\}$ has a least member, it follows from the definition of $\sim$ that the restriction of $<_R$ to $\{y|y \sim x\}$ is a wellordering. To determine the ordertype of $\{y|y \sim x\}$, consider $U_x = \{y|x <_R y\}$. If U_x contains no infinite descending sequence, then the ordertype of U_x, hence of $\{y|y \sim x\}$ is a recursive ordinal. If U_x contains an infinite descending sequence, then (i) implies that the ordertype of the maximal, wellfounded initial segment of U_x, hence of $\{y|y \sim x\}$, is ω_1^{CK}. Note there is at most one $\{y|y \sim x\}$ whose ordertype is less than ω_1^{CK}.

If $x <_R y$ and $x \not\sim y$, then there is a z such that $x <_R z <_R y$, $x \not\sim z$ and $z \not\sim y$; otherwise ω_1^{CK} would be a recursive ordinal. □

2.3 Proposition. *There exists a recursive g such that for all e: if $W_e \subseteq O$ and the restriction of $<_O$ to W_e is linear, then*

$$g(e) \in O \quad \& \quad (a)[a \in W_e \to a <_O g(e)].$$

Proof. Let $W_e^n = \{u|(\mathrm{Ey})_{y<n} T(e, u, y)\}$. Then $W_e = \cup\{W_e^n|n < \omega\}$, and each W_e^n is finite. Let q be the recursive function of Theorem 3.5.I. Define

$$\langle u, v\rangle \in W^* \quad \text{by} \quad \langle u, v\rangle \in W_{q(2^v)}.$$

Then W^* is recursively enumerable, and the restriction of W^* to O^2 is $<_O$.

Let $\theta(e, i)$ be a partial recursive function with the following property. If $\theta(e, i)$ is defined for all $i < n$, and if the restriction of W^* to

$$W_e^n \cup \{\theta(e, i)|i < n\} \cup \{1\} \tag{1}$$

is linear, then $\theta(e, n)$ is defined and equal to 2^m, where m is the greatest member of (1) with respect to W^*.

There is a recursive function s such that $\{s(e)\}(i) \simeq \theta(e, i)$ for all e and i. Define $g(e)$ to be $3 \cdot 5^{s(e)}$.

Suppose $W_e \subseteq O$ and the restriction of $<_O$ to W_e is linear. By induction on i, $\theta(e, i)$ is defined and belongs to O, and the restriction of $<_O$ to $W_e \cup \{\theta(e, j)|j < i\} \cup \{1\}$ is linear. In addition if $a \in W_e$, then $a <_O \theta(e, i)$ for some i, and so $a <_O g(e)$. □

2.4 Theorem (Feferman & Spector 1962). *There exists a Π^1_1 set O_1 of unique notations for the constructive ordinals, that is $O_1 \subseteq O$ and the restriction of $<_O$ to O_1 is linear.*

Proof. As in Gandy 1960. Let $<_R$ be the recursive linear ordering of Lemma 2.1. $<_R$ has a maximum wellordered initial segment Q of ordertype ω_1^{CK} according to Lemma 2.2(i). The function g of Proposition 2.3 will be used to imbed Q in O. The range of the imbedding will be cofinal in O_1.

There exist recursive functions j and I such that

$$W_{j(e,x)} = \{\{e\}(a) | a <_R x\}, \text{ and}$$
$$\{I(e)\}(x) = g(j(e, x)).$$

Fix c so that $\{I(c)\} \simeq \{c\}$. Let f be $\{c\}$. Then

$$W_{j(c,x)} = \{f(a) | a <_R x\}, \text{ and}$$
$$f(x) = g(j(c, x)).$$

An induction on $<_R$ shows

$$[x, y \in Q \quad \& \quad x <_R y] \to f(x) <_O f(y).$$

The range of f has gaps easily filled. Define

$$a \in O_1 \leftrightarrow (\text{Ex})[x \in Q \quad \& \quad a \in W_{p(f(x))}],$$

where p is the recursive function of Theorem 3.5.I. □

The proof of Theorem 2.4 given above is quite different from the one presented by Feferman and Spector 1962. Their approach is recursion theoretic in appearance and model theoretic in reality. They study O^*, a nonstandard version of O. The elements of O^* are notations for ordertypes of recursive linear orderings without hyperarithmetic, infinite descending sequences. They show O^* is a proper end extension of O (cf. Exercise 2.6). Then for each $e \in O^* - O$, the set

$$\{a | a <_{O^*} e \ \& \ a \in O\}$$

is a Π_1^1 set of unique notations.

2.5–2.7 Exercises

2.5. Simplify the proof of Proposition 2.3.

2.6. (Feferman & Spector 1962). Let $A(X)$ be the arithmetic closure condition of subsection 2.1.I used to define $<_O$. Define $<_{O^*}$ to be $\cap \{X | \langle 1, 2\rangle \in X \ \& \ A(X) \ \& \ X \in \text{HYP}\}$. Show $<_{O^*}$ is a proper end extension of $<_O$. Let $e \in O^* - O$. Show $\{a | a <_{O^*} e \ \& \ a \in O\}$ is a Π_1^1 set of unique notations.

2.7. Let M be a model of set theory in which the integers are standard and ω_1^{CK} is not. Let δ be a nonstandard recursive ordinal. What is the ordertype of δ in the real world?

3. *Hyperarithmetic Quantifiers*

Sometime ago A. Mostowski conjectured that each Π_1^1 predicate $P(x)$ could be put in the form

$$(1) \qquad (\mathrm{E}f)_{f\in \mathrm{HYP}} A(f, x)$$

for some arithmetic A. Earlier Kleene had shown that (1) is Σ_1^1 for every arithmetic A. Consequently Mostowski's conjecture states that with respect to arithmetic predicates, the effect of applying an universal function quantifier is the same as applying an existential hyperarithmetic function quantifier. The conjecture becomes plausible in the light of model theory. Let $\mathscr{F}$ be a sentence of $\mathscr{L}\omega_1^{CK}, \omega$, an extension of first order logic, in which quantifier prefixes are finite in length but conjunctions and disjunctions can be countably infinite. Assume $\mathscr{F}$ is encoded by a hyperarithmetic real. Thus $\mathscr{F}$ is built up in the same manner as a Borel set of recursive ordinal rank and hyperarithmetic complexity. According to an appropriate completeness theorem, (2) and (3) are equivalent.

(2) $\mathscr{F}$ is true in every countable structure.

(3) $\mathscr{F}$ has a proof in $\mathscr{L}\omega_1^{CK}, \omega$.

On its face (2) is a Π_1^1 predicate of $\mathscr{F}$. (3) is clearly existential. Less clear is that proofs in $\mathscr{L}\omega_1^{CK}, \omega$ are hyperarithmetically encodable. Perhaps it helps to note that finite sentences have finite proofs, and to recall claims made above that hyperarithmetic is analogous to finite.

3.1 Lemma (Kleene). *If $A(x,f)$ is arithmetic, then* $(\mathrm{E}f)_{f\in \mathrm{HYP}} A(x,f)$ *is* Π_1^1.

Proof. An immediate consequence of the proof of Corollary 1.4(ii).II. □

The proof of Mostowski's conjecture, or the Spector–Gandy theorem as it is now called, is based on properties of nonstandard end extensions of the hyperarithmetic hierarchy expressed by the next two lemmas.

3.2 Lemma (Enderton & Putnam). *There exists a recursive function g such that for all e: if $3\cdot 5^e \in O$ and*

$$(n)[H_{\{e\}(n)} \leq_T X],$$

then $H_{3\cdot 5^e} = \{g(e)\}^{X''}$.

Proof. Let $H(a, X)$ be the Π^0_2 predicate of Theorem 4.2.II. Define $P(a, z, X)$ by $\{z\}^X$ is total & $H(a, \{y|\{z\}^X(y) = 1\})$.

If $a \in O$, then $P(a, z, X)$ is equivalent to $H_a = \{z\}^X$. $P(a, z, X)$ is Π^0_2, hence recursive in X'' uniformly in X. In other words, for some e_0,

$$P(a, z, X) \leftrightarrow \{e_0\}^{X''}(a, z)$$

for all a, z and X. Let $\phi(e, y, X)$ be the partial function defined by

$$\{\mu z P(\{e\}((y)_1), z, X)\}^X((y)_0).$$

For some e_1, $\phi(e, y, X) \simeq \{e_1\}^{X''}(e, y)$ for all e, y and X. Let g be a recursive function such that

$$\{g(e)\}^{X''}(y) \simeq \phi(e, y, X).$$

Suppose $3 \cdot 5^e \in O$ and $H_{\{e\}(n)} \leq_T X$ for all n. Then

$$\begin{aligned} y \in H_{3 \cdot 5^e} &\leftrightarrow (y)_0 \in H_{\{e\}((y)_1)} \\ &\leftrightarrow (\mathrm{Ez})[P(\{e\}((y)_1), z, X) \quad \& \quad (y)_0 \in \{z\}^X] \\ &\leftrightarrow \{g(e)\}^{X''}(y) = 1. \quad \square \end{aligned}$$

3.3 Lemma. *Suppose $\{A_n|n < \omega\}$ is a sequence of sets such that $A'_{n+1} \leq_T A_n$ for all n. Then $X \leq_T A_0$ for all hyperarithmetic X.*

Proof. An induction on $<_O$ shows $(n)[H_b \leq_T A_n]$ for all $b \in O$.

If b is 2^m, then $H_m \leq_T A_{n+1}$ by induction, and so $H_{2^m} \leq_T A'_{n+1} \leq_T A_n$.

Suppose b is $3 \cdot 5^e$. Then by Lemma 3.2,

$$H_b \leq_T A''_{n+2} \leq_T A'_{n+1} \leq_T A_n. \quad \square$$

The existence of a sequence $\{A_n|n < \omega\}$ such that $A'_{n+1} \leq_T A_n$ for all n is a consequence of the fact that $O \notin \Sigma^1_1$ (cf. exercise 3.6). Model theory also provides such sequences. Let M be a nonstandard model of set theory in which the integers are standard and ω_1^{CK} is not. If δ is a nonstandard recursive ordinal, then there is an infinite descending sequence (outside M) $\delta = \delta_0 > \delta_1 > \delta_2 > \ldots$ of nonstandard recursive ordinals.

The associated H-sets satisfy the hypothesis of Lemma 3.3. $H'_{\delta_{n+1}} \leq_T H_{\delta_n}$, since H-sets are always separated by at least one jump.

Assume $<$ is a linear ordering of a subset of ω. X is said to be a *Turing jump hierarchy* on $<$ if

$$(1) \qquad (m)\,(n)\,[m < n \rightarrow (X)'_m \leq_T (X)_n].$$

If (1) holds, $<$ is said to *support* X. Note that (1) is an arithmetic statement about $<$ and X. Not much is known about those linear orderings that support Turing jump

hierarchies. Obviously every recursive wellordering does. H. Friedman has found a recursive linear ordering of ordertype $\omega_1^{CK}(1+\eta)$ that does not, although another recursive linear ordering of the same ordertype does.

The Spector–Gandy theorem is a consequence of the next lemma, which characterizes recursive wellorderings by an unexpected appeal to definability notions.

3.4 Lemma (Spector, Gandy). *Let $<$ be a recursive linear ordering. $<$ is a wellordering iff $<$ supports some hyperarithmetic Turing jump hierarchy.*

Proof. If $<$ is a wellordering, then it supports some proper initial segment of the H-sets.

Suppose $<$ supports some hyperarithmetic X and has an infinite descending sequence $m_0 > m_1 > \ldots$. Then $(X)'_{m_{i+1}} \leq_T (X)_{m_i}$ for all i, and Lemma 3.3 implies X is recursive in $(X)_{m_1}$. But then

$$X' \leq_T (X)'_{m_1} \leq_T (X)_{m_0} \leq_T X. \quad \square$$

3.5 Theorem (Spector 1959, Gandy 1960). *Each Π^1_1 predicate $P(x)$ can be put in the form*

$$(\mathrm{EY})_{Y \in \mathrm{HYP}} A(Y, x)$$

for some arithmetic A.

Proof. According to Proposition 5.3 of Chapter I, there is a partial ordering $S(x)$ of sequence numbers, recursive in x uniformly, such that

$$P(x) \leftrightarrow S(x) \text{ is wellfounded.}$$

$S(x)$ can be linearized as in the proof of Lemma 2.1. Let $S_{KB}(x)$ be the Kleene–Brouwer ordering of $S(x)$. Then

$$P(x) \leftrightarrow S_{KB}(x) \text{ is wellordered.} \tag{1}$$

By Lemma 3.4 the right side of (1) is equivalent to: $S_{KB}(x)$ supports some hyperarithmetic, Turing jump hierarchy. $\square$

3.6–3.12 Exercises

3.6. Find a sequence $\{A_n | n < \omega\}$ of sets such that $A'_{n+1} \leq_T A_n$ for all n. (Cf. Exercise 2.6.)

3.7. Find a sequence $\{A_n | n < \omega\}$ of sets such that $A'_{n+1} \equiv_T A_n$ for all n.

3.8. (Enderton & Putnam). For each $n \geq 0$, let $X^{(n+1)}$ be $(X^{(n)})'$, and let $X^{(0)}$ be X. Suppose $X^{(n)} \leq_T Y$ for all n. Let $X^{(\omega)}$ be $\{\langle u, v\rangle | u \in X^{(v)}\}$. Show $X^{(\omega)} \leq_T Y''$.

3.9. Let M be a nonstandard model of set theory in which the integers are standard and ω_1^{CK} is not. Suppose δ is a nonstandard recursive ordinal. Let $\delta = \delta_0 > \delta_1 > \delta_2 > \ldots$ be an infinite descending sequence not in M. Show $H'_{\delta_{i+1}} \leq_T H_{\delta_i}$ for all i.

3.10. Formulate and prove: $<$ is a wellordering iff $<$ supports a Turing jump hierarchy X such that X is hyperarithmetic in $<$.

3.11. Show $(\mathrm{EB})[B \in 2^\omega \ \& \ B \in \mathrm{HYP} \ \& \ R(B, x, Y)]$ is Π^1_1 if R is Π^1_1.

3.12. Suppose $P(X, Y)$ is Δ^1_1 and $(X)[(\mathrm{EY})P(X, Y) \rightarrow (E_1 Y)P(X, Y)]$. Show $\hat{X}(\mathrm{EY})P(X, Y)$ is Δ^1_1.

3.13. Show each Π^1_1 predicate $P(X)$ can be put in the form

$$(\mathrm{EY})_{Y \leq_h X} A(X, Y)$$

for some arithmetic A by relativizing 3.5 to X.

4. *The Ramified Analytic Hierarchy*

The ramified analytic hierarchy was introduced by Kleene, possibly to show the hyperarithmetic sets could be defined by an iterative process couched in the language of analysis. The length of the process is of course ω_1^{CK}, and a typical stage adds all sets (of numbers) that have analytical definitions over the universe of sets already defined. The equivalence of ramified analytic and hyperarithmetic definability is closely related to the existence of upward persistent Δ^1_1 definitions of H-sets, as in Corollary 4.4.II. The equivalence was clarified by Kreisel, who showed HYP is the least ω-model of the Δ^1_1 comprehension axiom schema.

The following account of the Kleene–Kreisel results is influenced by the needs of Chapter IV, which studies various sideways extensions, generic and otherwise, of the ramified analytic hierarchy.

One caution must be observed. Gandy and Putnam have extended the ramified analytic hierarchy to its natural endpoint, β_0, an ordinal far greater than ω_1^{CK}. Any reference to RAH in this book is a reference to RAH through the recursive ordinals only.

4.1. The Language $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$. Feferman 1965 introduced the language $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ to study Cohen-generic extensions of the ramified analytic hierarchy (equivalent to the hyperarithmetic hierarchy), as in Chapter IV below. $\mathscr{T}$ is a set constant that denotes a subset of ω. The other primitive symbols of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ are: $x, y, z, \ldots$ (number variables); $\underline{0}, \underline{1}, \underline{2}, \ldots$ (numerals); $X^\beta, Y^\beta, Z^\beta, \ldots$ (set variables of rank β for each $\beta < \omega_1^{CK}$); $X, Y, Z, \ldots$ (unranked set variables); $+$ (plus), $\cdot$ (times) and $'$ (successor); and $\in$ (membership).

The number theoretic terms are: numerals, number variables. $t + s$, $t \cdot s$, and t', where t and s are number theoretic terms.

The atomic formulas are $t = s$ and $t \in U$, where t and s are number theoretic terms and U is $\mathscr{T}$ or a set variable.

Formulas are built up from atomic formulas, propositional connectives, and quantifiers over all three types of variables. A formula $\mathscr{F}$ is *ranked* if all of its set variables are ranked. The *ordinal rank* of a ranked $\mathscr{F}$ is the least α such that $\beta \leq \alpha$ for all X^β occurring freely in $\mathscr{F}$ and $\beta < \alpha$ for all bound X^β in $\mathscr{F}$.

4.2. Gödel Numbers. An assignment of Gödel numbers to formulas is needed to classify relations between formulas such as truth or forcing. The Gödel numbering is routine save for the complication caused by the use of recursive ordinals as superscripts of ranked variables. Since the correspondence between primitive symbols and their Gödel numbers should be as effective as possible, it follows that the Gödel number of ranked variable X^β should resemble a notation for β. Let O_1 be the Π^1_1 set of unique notations for the recursive ordinals provided by Theorem 2.4. Then the Gödel number of X^β is $\langle 2, a \rangle$, where $a \in O_1$ and $|a| = \beta$. The remaining primitive symbols are numbered routinely with the result that:

(1) The set of Gödel numbers of formulas of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ is Π^1_1.
(2) For each $\alpha < \omega_1^{CK}$ the set of Gödel numbers of ranked formulas of rank less than α is recursively enumerable uniformly in the unique notation for α in O_1.

(2) is a consequence of Theorem 3.5(i).I, since the restriction of $<_O$ to O_1 is linear.

4.3. The Structure $\mathscr{M}(\omega_1^{CK}, T)$. Assume $T \subseteq \omega$. The structure $\mathscr{M}(\omega_1^{CK}, T)$ consists of those sets ramified analytic in T via ordinals less than ω_1^{CK}. It will be of interest chiefly when $\omega_1^{CK} = \omega_1^T$, and will prove useful in the study of such T's.

A simultaneous recursion on $\beta < \omega_1^{CK}$ defines the structure $\mathscr{M}(\beta, T)$ and truth in $\cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}$ for sentences of rank at most β.

(1) Suppose $\mathscr{F}$ is a sentence of rank at most β. Thus if X^α is a bound variable of $\mathscr{F}$, then $\alpha < \beta$. $\mathscr{F}$ is true in $\cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}$ if $\mathscr{F}$ is true when $\mathscr{T}$ is interpreted as T; $X^\alpha, Y^\alpha, \ldots$ are restricted to $\mathscr{M}(\alpha, T)$; $x, y, \ldots$ range over ω; and $+, \cdot, ', =, \in$ have their usual meaning.

(2) $\mathscr{M}(\beta, T)$ is the collection of all sets defined as follows. Let $\mathscr{G}(x)$ be a ranked formula of rank at most β whose only free variable is x. Define $\hat{x}\mathscr{G}(x)$ to be the set of all n such that

$$\mathscr{G}(\bar{n}) \text{ is true in } \cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}.$$

$\hat{x}\mathscr{G}(x)$ is a typical member of $\mathscr{M}(\beta, T)$.

Define $\mathscr{M}(\omega_1^{CK}, T)$ to be $\cup\{\mathscr{M}(\alpha, T) | \alpha < \omega_1^{CK}\}$. Finally define $\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$ ($\mathscr{F}$ is true in $\mathscr{M}(\omega_1^{CK}, T)$) for an arbitrary sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ by allowing the unranked set variables to range over $\mathscr{M}(\omega_1^{CK}, T)$ and interpreting the remaining symbols as in (1).

A sentence of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ is said to be Σ^1_1 (or existential) if it is ranked, or of the form $(EX_1) \ldots (EX_n)\mathscr{G}$ for some formula $\mathscr{G}$ with no unranked bound variables. Lemma 4.5 says that $\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$, restricted to Σ^1_1 $\mathscr{F}$'s, is Π^1_1. This means that the predicate, e is the Gödel number of an existential sentence true in $\mathscr{M}(\omega_1^{CK}, T)$, is Π^1_1 with e and T as its free variables.

4.4 Full Ordinal Rank. The notion of ordinal rank defined in subsection 4.1 is not fine enough for some of the inductions that lie ahead.

Suppose $\mathscr{F}$ is a ranked sentence. The full ordinal rank of $\mathscr{F}$ is a function

$$r\colon\ [-1, \omega_1^{CK}) \to \omega$$

such that for each $\beta \in \omega_1^{CK}$, $r(\beta)$ is the number of occurrences of (EX^β), (X^β), $(EY^\beta), \ldots$ in $\mathscr{F}$, and such that $r(-1)$ is the number of occurrences of $(x), (Ey), (y), \ldots$ in $\mathscr{F}$. Clearly r is 0 for all but finitely many arguments. Suppose r_0 and r_1 are full ordinal ranks. r_0 is said to be less than r_1 (in symbols $r_0 < r_1$) if there is a $\beta \in [-1, \omega_1^{CK})$ such that

$$(1) \qquad r_0(\beta) < r_1(\beta) \quad \& \quad (\alpha)_{\alpha > \beta}(r_0(\alpha) = r_1(\alpha)).$$

If $r_0 < r_1$, then at most one β satisfies (1), namely the greatest β such that $r_0(\beta) < r_1(\beta)$. $<$ is a wellordering of height ω_1^{CK}.

The key property of $<$ is the reduction of full ordinal rank occasioned by the substitution of an appropriate set term for a bound variable. Consider a sentence of the form $(EX^\beta)\mathscr{F}(X^\beta)$. Let $\mathscr{G}(x)$ be a ranked formula of rank at most β whose only free variable is x. The result of substituting $\hat{x}\mathscr{G}(x)$ for X^β in $\mathscr{F}(X^\beta)$ is denoted by $\mathscr{F}(\hat{x}\mathscr{G}(x))$, and is accomplished by substituting $\mathscr{G}(b)$ for each occurrence in $\mathscr{F}(X^\beta)$ of $b \in X^\beta$, where b is a number theoretic term. The full ordinal rank of $\mathscr{F}(\hat{x}\mathscr{G}(x))$ is less than that of $(EX^\beta)\mathscr{G}(X^\beta)$ since all the quantified set variables occurring in $\mathscr{G}(x)$ have superscripts less than β.

It is often convenient to assume an arbitrary ranked sentence $\mathscr{F}$ is in prenex normal form. The standard syntactical manipulations that put a sentence in prenex normal form do not increase full ordinal rank.

4.5 Lemma. *The predicate* $\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$, *restricted to* Σ^1_1 $\mathscr{F}$*'s, is* Π^1_1.

Proof. It is enough to show $\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$, restricted to ranked $\mathscr{F}$'s, is Π^1_1, since

$$\mathscr{M}(\omega_1^{CK}, T) \models (EX)\mathscr{G}(X)$$

is equivalent to

$$(Eb)[b \in O_1 \ \& \ \mathscr{M}(\omega_1^{CK}, T) \models (EX^{|b|})\mathscr{G}(X^{|b|})].$$

The definition of $\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$ for ranked $\mathscr{F}$'s is a recursion on the full ordinal rank of $\mathscr{F}$ and logical complexity of $\mathscr{F}$. To see that the definition is Π^1_1 consider the following Σ^1_1 closure conditions with Theorem 1.6(i). in mind. (0 is true and 1 is false.)

$$(1) \qquad \langle \mathscr{F}, 0 \rangle \in X \ \& \ \langle \mathscr{G}, 0 \rangle \in X \to \langle \mathscr{F} \ \& \ \mathscr{G}, 0 \rangle \in X.$$

(1) is a Σ^1_1 closure condition because the intended meaning of (1) is:

(1*) If c and d are Gödel numbers of ranked sentences, both of which are true, then their conjunction is true.

Since the set of Gödel numbers of ranked sentence is Π^1_1, it follows that (1*) is Σ^1_1. The same applies to (2)–(12).

(2) $\langle \mathscr{F}, 1\rangle \in X \vee \langle \mathscr{G}, 1\rangle \in X \rightarrow \langle \mathscr{F} \ \& \ \mathscr{G}, 1\rangle \in X.$

(3) $\langle \mathscr{F}, 0\rangle \in X \rightarrow \langle \sim \mathscr{F}, 1\rangle \in X.$

(4) $(\mathscr{F}, 1\rangle \in X \rightarrow \langle \sim \mathscr{F}, 0\rangle \in X.$

(5) $\langle \mathscr{F}(\underline{n}), 0\rangle \in X \rightarrow \langle (\mathrm{Ex})\mathscr{F}(x), 0\rangle \in X.$

(6) $(n)[\langle \mathscr{F}(\underline{n}), 0\rangle \in X] \rightarrow \langle (x)\mathscr{F}(x), 0\rangle \in X.$

(7) $b \in O_1 \ \& \ \mathrm{rank}(\mathscr{G}(x)) \leq |b|$
$\& \ \langle \mathscr{F}(\hat{x}\mathscr{G}(x)), 0\rangle \in X \rightarrow \langle (\mathrm{E}X^{|b|})\mathscr{F}(X^{|b|}), 0\rangle \in X.$

(8) $b \in O_1 \ \& \ [\langle \mathscr{F}(\hat{x}\mathscr{G}(x)), 1\rangle \in X$ whenever rank of $\mathscr{G}(x) \leq |b|] \rightarrow \langle (\mathrm{E}X^{|b|})\mathscr{F}(X^{|b|}), 1\rangle \in X.$

In (9)–(12) s and t are closed number theoretic terms, and $\mathrm{val}(s)$ is the numerical value of s.

(9) $\mathrm{val}(s) \in T \rightarrow \langle t \in \mathscr{T}, 0\rangle \in X.$

(10) $\mathrm{val}(s) \notin T \rightarrow \langle t \in \mathscr{T}, 1\rangle \in X.$

(11) $\mathrm{val}(t_1) = \mathrm{val}(t_2) \rightarrow \langle t_1 = t_2, 0\rangle \in X.$

(12) $\mathrm{val}(t_1) \neq \mathrm{val}(t_2) \rightarrow \langle t_1 = t_2, 1\rangle \in X.$

Let $A(X, T)$ be the conjunction of (1)–(12). $A(X, T)$ is Σ^1_1. An induction on the full ordinal rank and logical complexity of $\mathscr{F}$ shows

$$\begin{aligned} &\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F} \\ &\qquad \leftrightarrow (X)[A(X, T) \rightarrow \langle \mathscr{F}, 0\rangle \in X] \\ &\qquad \leftrightarrow \sim (X)[A(X, T) \rightarrow \langle \mathscr{F}, 1\rangle \in X]. \end{aligned} \tag{13}$$

A sample of the reasoning encountered in the induction is as follows. Suppose $\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle \in X$ for all X such that $A(X, T)$ holds. Suppose further there is an X_0 such that $A(X_0, T)$ and either $\langle \mathscr{F}, 0\rangle$ or $\langle \mathscr{G}, 0\rangle$ fails to belongs to X_0. Clause (1) is the only clause that can require $\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle$ to be in X. Therefore $X_0 - \{\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle\}$ is an X that satisfies $A(X, T)$, a contradiction. Hence $\langle \mathscr{F}, 0\rangle$ and $\langle \mathscr{G}, 0\rangle$ belong to every X satisfying $A(X, T)$. By induction $\mathscr{F} \ \& \ \mathscr{G}$ is true in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$, since both $\mathscr{F}$ and $\mathscr{G}$ are less complex than $\mathscr{F} \ \& \ \mathscr{G}$.

(13) is a Π^1_1 definition of truth in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ for ranked sentences.

4.6 Lemma. *For each $\alpha < \omega_1^{CK}$ the predicate $\mathcal{M}(\omega^{CK}, T) \models \mathcal{F}$, restricted to $\mathcal{F}$'s of rank less than α, is Δ^1_1 uniformly in the unique notation for α.*

Proof. Let g be a recursive function such that for each $b \in O_1$, $W_{g(b)}$ is the recursively enumerable set of Gödel numbers of sentences of rank less than $|b|$. The existence of g was noted in subsection 4.1(2). Fix b. The predicate

$$\mathcal{F} \in W_{g(b)} \quad \& \quad \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{F}$$

and its "negative",

$$\mathcal{F} \in W_{g(b)} \quad \& \quad \mathcal{M}(\omega_1^{CK}, T) \models \sim \mathcal{F},$$

are Π^1_1 by Lemma 4.5. □

The next lemma echoes the persistence phenomenon heard in Lemma 1.6.I.

Lemma 4.7. *For each hyperarithmetic set H there is a ranked formula $\mathcal{H}(x)$ such that for all T and n,*

$$n \in \mathcal{H} \leftrightarrow \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{H}(\underline{n}).$$

Proof. There exist $b \in O_1$ and recursive function f such that

$$n \in H \leftrightarrow (\mathrm{Ex})[f(n) = x \ \& \ x \in H_b].$$

by Exercise 2.10.II. The graph of f is representable by a arithmetic formula of $\mathcal{L}(\omega_1^{CK}, T)$ in which T does not appear.

H_b is represented by a formula developed by recursion on $<_O$. For each $a <_O b$ assume there is a ranked formula $\mathcal{H}_a(x)$ such that for all T and n,

$$n \in H_a \leftrightarrow \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{H}_a(\underline{n}))$$

Assume the ordinal rank of $\mathcal{H}_a$ is $|a| + 1$. Then $\mathcal{M}(|b|, T)$ includes $\{H_a | a <_O b\}$ for all T. According to Corollary 4.4.II there is a Σ^1_1 formula, $(\mathrm{E}Y)A(x, Y)$, that defines H_b in any universe containing $\{H_a | a <_O b\}$. Consequently $(\mathrm{E}Y^{|b|})A(x, Y^{|b|})$ defines H_b in $\mathcal{M}(\omega_1^{CK}, T)$ for all T, and has rank $|b| + 1$.

The *ramified analytic hierarchy* is denoted by $\mathcal{M}(\omega_1^{CK})$ and is defined to be $\mathcal{M}(\omega_1^{CK}, \phi)$.

4.8 Theorem (Kleene). $\mathcal{M}(\omega_1^{CK}) = \mathrm{HYP}$.

Proof. Lemma 4.7 implies $\mathrm{HYP} \subseteq \mathcal{M}(\omega^{CK}, T)$ for all T, hence for $T = \phi$.

Suppose $B \in \mathcal{M}(\omega_1^{CK})$. Then there is a ranked formula $\mathcal{G}(x)$ of $\mathcal{L}(\omega_1^{CK}, T)$ such that for all n,

$$n \in B \leftrightarrow \mathcal{M}(\omega_1^{CK}, \phi) \models \mathcal{G}(\underline{n}). \tag{1}$$

By Lemma 4.6 the right side of (1) is Δ^1_1, hence hyperarithmetic. □

4.9 Δ^1_1 **Comprehension.** The following sentence is a typical instance of G. Kreisel's Δ^1_1 comprehension axiom scheme.

$$(1) \qquad (x)[(\mathrm{E}Y)A(x,Y)\leftrightarrow(Z)B(x,Z)] \rightarrow(\mathrm{E}X)(x)[x\in X\leftrightarrow(\mathrm{E}Y)A(x,Y)].$$

$A(x, Y)$ and $B(x, Z)$ are arithmetic. If they contain set parameters, (1) is said to be boldface, otherwise lightface. An ω-model N is simply any nonempty subset of 2^ω. Any sentence $\mathscr{G}$ of analysis can be interpreted in N by restricting the set variables of $\mathscr{G}$ to N and the number variables to ω. The study of ω-models of fragments of analysis originated in proof theory, but turned out to have applications to recursion theory as in the next lemma, a characterization of a recursion-theoretic hierarchy as the least ω-model of some fragment of analysis.

4.10 Theorem (Kreisel). *The intersection of all ω-models of Δ^1_1 comprehension is a model of Δ^1_1 comprehension, namely* HYP.

Proof. HYP is an ω-model of Δ^1_1 comprehension, since Theorem 3.1 implies any set that has a Σ^1_1 definition over HYP must be Π^1_1.

Suppose N is an ω-model of Δ^1_1 comprehension to see HYP $\subseteq N$. As in the proof of Lemma 4.7 it suffices to show $H_b\in N$ by induction on $<_O$. By Corollary 4.4 of Chapter II there is a Δ^1_1 definition of H_b that defines H_b in any ω-model containing $\{H_a | a <_O b\}$, hence in N by induction. □

4.11 Σ^1_1 **Choice.** The following sentence is a typical instance of Kreisel's Σ^1_1 axiom of choice scheme.

$$(1) \qquad (x)(\mathrm{E}Y)A(x,Y)\rightarrow(\mathrm{E}Y)(x)A(x,(Y)_x).$$

($m\in(Y)_x$ means $2^x\cdot 3^m\in Y$.) $A(x, Y)$ is an arithmetic predicate in which set parameters may occur. If none do, (1) is said to be lightface.

Long ago Kreisel asked if Δ^1_1 choice followed from Δ^1_1 comprehension. (A well known consequence of the Kondo–Addison Theorem is that Σ^1_1 choice follows from Δ^1_2 comprehension (Chapter IV).) J. Steel devised an appropriate notion of forcing to show boldface Δ^1_1 comprehension does not imply boldface Σ^1_1 choice. Later L. Harrington showed boldface Δ^1_1 comprehension does not imply lightface Σ^1_1 choice by means of a self-referential argument.

4.12 Proposition. *Suppose N is an ω-model of Σ^1_1 choice and is downward closed under many-one reducibility. Then N is a model of Δ^1_1 comprehension.*

Proof. Suppose

$$(x)[(\mathrm{E}Y)A(x,Y)\leftrightarrow(Z)B(x,Z)]$$

is true in N for some arithmetic A and B. Then

$$(x)(\mathrm{E}Y)[(x\in(Y)_0 \quad \& \quad A(x,(Y)_1) \\ \vee (x\notin(Y)_0) \quad \& \quad \sim B(x,(Y)_1)]$$

holds in N. Σ^1_1 choice provides a Y in N such that for all x,

$$(x)[(x\in(Y)_{x,0} \quad \& \quad A(x,(Y)_{x,1}) \\ \vee (x\notin(Y)_{x,0} \quad \& \quad \sim B(x,(Y)_{x,1})].$$

Let X be $\{x|x\in(Y)_{x,0}\}$. Then

$$(x)[x\in X\leftrightarrow(\mathrm{E}Y)A(x,Y)]. \qquad \square$$

4.13 Theorem (Kreisel). HYP *is the least ω-model of Σ^1_1 choice downward closed under many-one reducibility.*

Proof. Suppose $(x)(\mathrm{E}Y)_{Y\in\mathrm{HYP}}A(x,Y)$ for some arithmetic $A(x,Y)$ with hyperarithmetic parameters. Then

$$(1) \qquad (x)(\mathrm{E}a)(\mathrm{E}e)[a\in O \quad \& \quad A(x,\{e\}^{H_a})].$$

Let $H(a,X)$ be the Π^0_2 predicate of Theorem 4.2.II. Then (1) is equivalent to

$$(2) \qquad (x)(\mathrm{E}a)(\mathrm{E}e)[a\in O \quad \& \quad (Z)(H(a,Z)\rightarrow A(x,\{e\}^Z))].$$

Since the matrix of (2) is Π^1_1, Lemma 2.6.II provides hyperarithmetic functions $a(x)$ and $e(x)$ such that

$$(x)[a(x)\in O \quad \& \quad (Z)(H(a(x),Z)\rightarrow A(x,\{e(x)\}^Z))].$$

The range of $a(x)$ is bounded by some $b\in O$ by Corollary 5.6.I. To be precise, $(x)(a(x)\in O_b)$. It follows from Lemma 2.1.II and Theorem 1.3.II that the predicate

$$(3) \qquad m\in\{e(x)\}^{H_{a(x)}}$$

is Δ^1_1 hence hyperarithmetic. Thus there is a $Y\in\mathrm{HYP}$ such that $m\in(Y)_x$ is equivalent to (3). Clearly $(x)A(x,(Y)_x)$. $\square$

4.14 Σ^1_1 Dependent Choice. A typical instance of Σ^1_1 dependent choice is

$$(Y)(\mathrm{E}Z)A(Y,Z)\rightarrow(\mathrm{E}Y)(x)A((Y)_x,(Y)_{x+1}).$$

$A(Y,Z)$ is an arithmetic predicate that may contain set parameters. According to Exercise 4.22 HYP is an ω-model of Σ^1_1 dependent choice. Σ^1_1 choice is an

immediate consequence of Σ^1_1 dependent choice; H. Friedman found an ω-model of the former in which the latter fails.

In Chapter IV it will be shown that $\mathscr{M}(\omega_1^{CK}, T)$ is a model of Σ^1_1 dependent choice for almost all T, that is a set of T's of measure 1. At that time the equivalences of Lemma 4.16 will prove useful. The proof of 4.16 requires the following Corollary of Lemma 4.6.

4.15 Lemma. *Suppose $X \in \mathscr{M}(\omega_1^{CK}, T)$. Then there exists a $b \in O$ such that X is recursive in H^T_b. (b depends only on a ramified analytic definition of X from T.)*

Proof. Let $G(x)$ be a ranked formula such that

$$n \in X \leftrightarrow \mathscr{M}(\omega_1^{CK}, T) \models G(\underline{n}) \tag{1}$$

for all n. By Lemma 4.6, the right side of (1) is a Δ^1_1 predicate with n and T as free variables. According to subsection 5.6.II, there is a $b \in O$ and an e such that the right side of (1) is $\{e\}^{H^T_b}$ for all T. □

4.16 Lemma. *The following are equivalent.*

(i) $\omega_1^{CK} = \omega_1^T$.
(ii) $\mathscr{M}(\omega_1^{CK}, T) = \mathrm{HYP}(T)$.
(iii) $\mathscr{M}(\omega_1^{CK}, T)$ *satisfies* Δ^1_1 *comprehension.*

Proof. (i) → (ii). Lemma 4.15 implies $\mathscr{M}(\omega_1^{CK}, T) \subseteq \mathrm{HYP}(T)$. Suppose $B \in \mathrm{HYP}(T)$. Let B be H^T_b. $|b|_T < \omega_1^{CK}$. There is a formula $\mathscr{H}_b$ of rank $|b| + 1$ such that

$$b \in H^T_b \leftrightarrow \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{H}_b(\underline{n}) \tag{1}$$

for all n. (1) is proved as was Lemma 4.7 with the aid of Corollary 4.4.II, now relativized to T.

(ii) → (iii). $\mathrm{HYP}(T)$ is an ω-model of Δ^1_1 comprehension by a relativization to T of the proof of Theorem 4.10.

(iii) → (i). By Theorem 4.10 relativized to T, $\mathrm{HYP}(T) \subseteq M(\omega_1^{CK}, T)$. It follows from Lemma 4.15 that each set hyperarithmetic in T is recursive in some H^T_b, where $|b|_T < \omega_1^{CK}$. But then there cannot be an $a \in O^T$ such that $|a|_T = \omega_1^{CK}$. □

4.17–4.21 Exercises

4.17. Prove 4.2(1) and 4.2(2).

4.18. Show $<$, defined in subsection 4.4, is wellfounded.

4.19. Prove 4.13(3) is Δ^1_1.

4.20. Show Σ^1_1 dependent choice implies Σ^1_1 choice.

4.21. Prove there is an $\mathscr{H}_b$ of rank $|b|+1$ that satisfies (1) of 4.16.

4.22. Show HYP is an ω-model of Σ_1^1 dependent choice.

5. *Kreisel Compactness*

Some of the boundedness properties of the hyperarithmetic hierarchy lead to a compactness theorem for ω-logic discovered by G. Kreisel. His result is: if I is a Π_1^1 set of sentences of ω-logic such that every hyperarithmetic subset of I has a model, then I has a model. Note that "hyperarithmetic" takes the place of "finite" in the compactness theorem for first order logic. Kreisel's theorem paved the way for Barwise's compactness theorem for countable Σ_1 admissible sets. Since this is not a book on model theory, the Kreisel result is presented in set theoretic terms.

Recall what is meant by the compactness of 2^ω. 2 is the two-element space with the discrete topology. 2^ω is the product of countably many copies of 2 and has the product topology. If I is a family of closed subsets of 2^ω such that every finite subfamily of I has a nonempty intersection, then I has a nonempty intersection.

Kreisel compactness is concerned with families of Δ_1^1 subsets of 2^ω. At first this idea seems hopeless, because it is easy to find a family of Δ_1^1 subsets with the finite intersection property whose intersection is empty. Kreisel's insight provides a way out. First ask that every hyperarithmetic subfamily have a nonempty intersection. Second require that the family be Π_1^1. As has been observed earlier, finite bears to recursively enumerable a relation similar to that borne by hyperarithmetic to Π_1^1.

If n is a Δ_1^1-index for a subset of 2^ω, then D_n denotes the subset indexed by n.

5.1 Theorem (Kreisel). *Let I be a Π_1^1 set of indices of Δ_1^1 sets. Suppose*

$$\cap\{D_n | n\in H\} \neq \varnothing$$

for every hyperarithmetic $H\subseteq I$. Then

$$\cap\{D_n | n\in I\} \neq \varnothing.$$

Proof. Suppose not. Then

$$(X)(\mathrm{Ey})[y\in I \ \&\ X\notin D_y]. \tag{1}$$

The matrix of (1), call it $Q(X,y)$, is Π_1^1. The proof of Theorem 2.3.II shows $Q(X,y)$ can be uniformized by some Π_1^1 $P(X,y)$. The virtue of $\mathrm{P}(X,y)$ is its Δ_1^1-ness, since

$$\sim P(X,y)\leftrightarrow(\mathrm{Ez})_{z\neq y}P(X,z).$$

Hence the set J of all n such $(\mathrm{EX})P(X,n)$ is a Σ_1^1 subset of I, and

$$\cap\{D_n | n\in J\} = \varnothing.$$

By the remark following Theorem 3.7.II, there is a hyperarithmetic H separating J and $\omega - I$, that is $J \subseteq H \subseteq I$. But then

$$\cap \{D_n | n \in H\} = \varnothing. \quad \square$$

5.2 Corollary. *Let I be a Π^1_1 set of ranked sentences of $\mathscr{L}(\omega_1^{CK}, T)$. Suppose for each hyperarithmetic $H \subseteq I$, there is a T such that $\mathscr{M}(\omega_1^{CK}, T)$ is a model of H. Then there is a T such that $\mathscr{M}(\omega_1^{CK}, T)$ is a model of I.*

Proof. Let $\mathscr{F}$ be a ranked sentence. The set of all T such that $\mathscr{M}(\omega_1^{CK}, T)$ satisfies $\mathscr{F}$ is Δ^1_1 by Lemma 4.6. $\square$

5.3–5.4 Exercises

5.3. Show the matrix of 5.1(1) is Π^1_1.

5.4. Show that $Q(X, y)$ of the proof of Theorem 5.1 can be uniformized by some Π^1_1 $P(X, y)$.

6. *Perfect Subsets of Σ^1_1 Sets*

The real line has the same cardinality as 2^ω. Thus one version of Cantor's continuum hypothesis is: if K is an uncountable subset of 2^ω, then K has the same cardinality as 2^ω. A classical result of descriptive set theory states that the continuum hypothesis holds for every boldface Σ^1_1 set. As might be expected the classical result says more, namely that K contains a perfect set. By combining the classical splitting argument with some effectiveness considerations, still more information is obtained. For example every member of a countable Σ^1_1 set is hyperarithmetic.

6.1 Perfect Subsets of 2^ω. $P, Q, R, \ldots$ denote perfect subsets of 2^ω. "Perfect" means closed, nonempty and without isolated points, in short homeomorphic to 2^ω. An $x \in P$ is isolated if $\{x\} = P \cap U$ for some open set U. The usual picture of 2^ω is that of a tree with binary branching. The members of 2^ω correspond to the infinite branches of the tree. An arbitrary perfect P should be pictured similarly. To make the picture precise, let P^* be the set of all sequence numbers that represent initial segments of elements of P. Thus $s \in P^*$ iff $\mathrm{Seq}(s)$, $(i)_{i<2}((s)_i \in \{1,2\})$, and

$$(\mathrm{E}X)\,[X \in P \;\&\; (i)_{i<lh(s)}(i \in X \leftrightarrow (s)_i = 1)].$$

The elements of P^* are the branchpoints of a tree whose branches are the members of P. The lack of isolated elements in P corresponds to the existence in P^* of incomparable extensions of each member of P^*.

P^* is the standard encoding of P. Assume $V \subseteq \omega$. V encodes a perfect set if $\varnothing \neq V \subseteq \text{Seq}$,

$$(s)(\text{Et})(\text{Eu})[s \in V \to t, u \in V \ \& \ s \geq t \ \& \ s \geq u \ \& \ t|u],$$

(Recall $s \leq t$ means s is extended by t, and $t|u$ means $[t \not\geq u \ \& \ u \not\geq t]$.)

$$\text{and } (s)\,[s \in V \to (i)_{i < \text{lh}(s)}((s)_i \in \{1,2\})].$$

The perfect set P_V coded by V consists of all X such that arbitrarily long initial segments of the characteristic function of X are coded by sequence numbers in V. Note that the standard coding of P_V is recursive in V.

It is helpful to allow $P, Q, R, \ldots$ to denote both perfect sets and their standard encodings. Thus "P is hyperarithmetic" means "the standard encoding of P is a hyperarithmetic set of sequence numbers".

6.2 Theorem. *Let $K \subseteq 2^\omega$ be Σ^1_1. Then (i)–(iv) are equivalent.*

(i) K contains a perfect set recursive in O.

(ii) K contains a perfect set.

(iii) K has a nonhyperarithmetic member.

(iv) For each hyperarithmetic real H, there is a member of K not recursive in H.

Proof. Clearly (i) → (ii), (ii) → (iii) since perfect sets are uncountable, (iii) → (iv). It suffices to show (iv) → (iii) and (iii) → (i).

Assume (iii) fails with the intent of refuting (iv). The failure of (iii) implies

$$(1) \qquad (X)(\text{Eb})[X \in K \to b \in O \quad \& \quad X \leq_T H_b].$$

The matrix of (1) is Π^1_1 and so can be uniformized by some Π^1_1 $Q(X,b)$ as in the proof of Theorem 2.3.II. Since Q defines the graph of a total function, it follows that $Q(X,b)$ is Σ^1_1:

$$\sim Q(X,b) \leftrightarrow (\text{Ec})_{c \neq b} Q(X,c).$$

Thus there is a Δ^1_1 function $b(X)$ such that

$$(X)[X \in K \to b(X) \in O \quad \& \quad X \leq_T H_{b(X)}].$$

By Spector's boundedness theorem (5.6.I), there is a $c \in O$ such that

$$(X)[|b(X)| < |c|].$$

Consequently every member of K is recursive in H_c by Theorem 4.5.II.

Suppose K has a nonhyperarithmetic member in order to show K has a perfect subset encodable by a real recursive in O. It is safe to assume every element of K is nonhyperarithmetic, since

$$X \in K \quad \& \quad X \notin \text{HYP}$$

is Σ^1_1 by Corollary 1.4.II. The normal form of $X \in K$ is

$$(\mathrm{Eg})(x)R(\bar{X}(x), \bar{g}(x))$$

for some recursive R. $X(x)$ is the characteristic function of X. If $(x)R(\bar{X}(x), \bar{g}(x))$ holds, then g is said to be a witness to the fact that $X \in K$.

The construction of the perfect $P \subseteq K$ includes the construction of a witness for each $X \in P$. $Q(s,t)$ is the Σ^1_1 predicate defined by:

(i) s and t are sequence numbers of the same length.
(ii) there exist X and g such that s encodes an initial segment of X, t encodes an initial segment of g, and g witnesses $X \in K$.

Assume $Q(s,t)$ holds to show there exist $\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ such that:

(iii) $s > s_1$, $s > s_2$, $t > t_1$ and $t > t_2$;
(iv) $Q(s_1,t_1)$ and $Q(s_2,t_2)$;
(v) $s_1 | s_2$.

$\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ are said to be incomparable extensions of $\langle s, t \rangle$. Suppose no such exist. Then there is a *unique* X such that s encodes an initial segment X and t encodes an initial segment of some witness to $X \in K$. That unique X is hyperarithmetic by Theorem 1.6(ii).I, contrary to the assumption no member of K is hyperarithmetic.

The set $\{\langle s^i_j, t^i_j \rangle | i < \omega \ \& \ j < 2^i\}$ is defined by recursion on i. s^o_o and t^o_o are null. Fix i and j. Choose $\langle s^{i+1}_{2j}, t^{i+1}_{2j} \rangle$ and $\langle s^{i+1}_{2j+1}, t^{i+1}_{2j+1} \rangle$ to be incomparable extensions of $\langle s^i_j, t^i_j \rangle$. The choices can be made effectively from O since O is a complete Π^1_1 set (Theorem 5.4.I).

Let P be the perfect set encoded by $\{s^i_j | i < \omega \ \& \ j < 2^i\}$. □

Countable Π^1_1 subsets of 2^ω are much more complicated than their Σ^1_1 counterparts. Their elements tend to be scattered throughout the constructible hierarchy. They will be discussed in Section 9.

The next result is a uniform version of part of Theorem 6.2.

6.3 Theorem. *There exists a recursive function h such that: if c is an index for a Σ^1_1 predicate $C(X)$ with no perfect set of solutions, then $h(c) \in O$ and*

$$(X)[C(X) \to X \leq_T H_{h(c)}].$$

Proof. Let $P(X,b)$, clearly Π^1_1, be

$$b \in O \ \& \ [C(X) \to X \leq_T H_b].$$

By Theorem 6.2, (X) (Eb) $P(X,b)$ if $C(X)$ has no perfect set of solutions. According to Theorem 5.4.I relativized to X, there is a recursive f such that

$$P(X,b) \leftrightarrow f(b) \in O^X.$$

Let $Q(X,b)$ be Π^1_1 and uniformize $P(X,b)$ as in the proof of Theorem 6.2. An index for Q is easily derived from one for P. $Q(X,b)$ is Σ^1_1, because $(X)Q(X,b)$ and Q defines a function. Let B be the range of that function. Corollary 3.4.II states that a bound for B can be computed from an index for B as a Σ^1_1 set. Thus there is a recursive function h such that $h(c)$ bounds B. Then $X \leq_T H_{h(c)}$ if $C(X)$. □

The proof of Theorem 6.2 is similar to a Henkin construction of a perfect set of countable models of a countable theory. The definition of P can be regarded as the simultaneous construction of a continuum of models of a theory that says: "I am a nonhyperarithmetic member of K". The Henkin approach will be applied in Section 9 to the study of Π^1_1 sets.

7. *Kreisel's Basis Theorem*

Let K be a set of axioms in the language of analysis, that is the language of subsection 1.2.I. The predicates of analysis are the analytical predicates. Let Z be a real that occurs in every model of K. What can be said about Z? To make sense of the question, some bound on the complexity of K is needed. From a syntactic vantage point Z might correspond to some term in K compelled to denote Z by K and the rules and axioms of ω-logic. From a model theoretic stance Z cannot be omitted from any model of K. Kreisel's result is that Z must be hyperarithmetic if K is Π^1_1.

7.1 Lemma. *Suppose $A \notin \mathrm{HYP}$, $b \in O$ and K is an uncountable Σ^1_1 set. Then for each e there exists an uncountable Σ^1_1 $K^* \subseteq K$ such that*

$$A \neq \{e\}^{H^X_b}$$

for all $X \in K^*$.

Proof. Since K is uncountable and HYP is Π^1_1, it is safe to assume $K \cap \mathrm{HYP} = \varnothing$.

Case 1: $(EX)[X \in K \ \&\ \{e\}^{H^X_b}$ is not total]. Let K^* be the set of all $Y \in K$ such that $\{e\}^{H^Y_b}$ is not total. K^* is Σ^1_1 by subsection 5.6.II, and uncountable by Theorem 6.2, since $K \cap \mathrm{HYP} = \varnothing$.
Case 2: Case 1 fails and

$$\{e\}^{H^X_b}(m) \neq \{e\}^{H^Y_b}(m)$$

for some m and X, $Y \in K$. Choose such an m. Let K^* be the set of all $Z \in K$ such that $A(m) \neq \{e\}^{H^Z_b}(m)$. K^* is uncountable by Theorem 6.2.
Case 3: Cases 1 and 2 fail. Thus there is a function h such that

$$(m)(X)[X \in K \to \{e\}^{H^X_b}(m) = h(m)].$$

h is Σ^1_1, hence Δ^1_1, because

$$h(m) = n \leftrightarrow (\mathrm{EX})\,(X \in K \;\&\; \{e\}^{H^X_b}(m) = n).$$

Let K^* be K. □

The above lemma will make more sense after the introduction of Gandy forcing in Section 6.IV. That notion of forcing employs uncountable Σ^1_1 sets as forcing conditions. The proof of Lemma 7.1 shows: if K forces $\{e\}^{H^G_b}$ to be total and constant, then K forces $\{e\}^{H^G_b}$ to be hyperarithmetic.

7.2 Theorem (Kreisel). *Suppose $A \notin \mathrm{HYP}$ and K is a nonempty Σ^1_1 set. Then for some $X \in K$, $A \not\leq_h X$.*

Proof. By Theorem 6.2 and Corollary 1.5 it is safe to assume

$$K \cap \mathrm{HYP} = \varnothing,\ K \text{ is uncountable and}$$
$$(X)[X \in K \rightarrow \omega_1^X = \omega_1^{\mathrm{CK}}].$$

A contracting sequence $K^i\,(i < \omega)$ of uncountable Σ^1_1 sets, and a sequence h_i of reals, are defined simultaneously by recursion on i such that:

$K^0 = K$;
if $b = (i)_0$, $e = (i)_1$ and $b \in O$, then K^{i+1} has the properties attributed to K^* by Lemma 7.1;

$\cap \{K^i \mid i < \omega\} = \{X\}$;

h_i is an existential witness to $X \in K^i$.

X, K^i and h_i are defined above as f, P_i and g_i are in the proof of Theorem 6.3.IV. (In fact the present theorem follows directly from Lemma 7.1.III and Theorem 6.3.IV; for each $b \in O$ and e, the set of all K^* provided by 7.1 is dense.) □

7.3 Corollary (Kreisel). *Let K be a Π^1_1 set of axioms in the language of analysis. If X belongs to every ω-model of K, then X is hyperarithmetic.*

Proof. The set of all real Z such that Z encodes a countable ω-model of K is Σ^1_1. □

Corollary 7.3 is a special case of a type omitting theorem for ω-logic. If a Π^1_1 set of axioms of ω-logic has a model, then it has one that omits any given nonhyperarithmetic type. The analogous result for first order logic is: if a countable theory has a model, then it has one that omits any given nonprincipal type.

The recursion theorist winding his way through a Σ^1_1 set is a brother to the model theorist threading his way through a Henkin tree.

8. *Inductive Definitions*

The leading role of inductive definitions in recursion theory has been emphasized by R. Gandy for many years. Two prime examples are the definition of O in Chapter I, and the definition of E-recursion in Part D. Two basic cases dominate the subject, monotonic Π^1_1 and arbitrary Π^0_1.

8.1 Closure Ordinals. Let Γ be a map from 2^ω into 2^ω. Define Γ_α by recursion on α.

$$\Gamma_0 = \varnothing. \quad \Gamma_{\alpha+1} = \Gamma(\Gamma_\alpha) \cup \Gamma_\alpha.$$

$$\Gamma_\lambda = \cup\{\Gamma_\alpha | \alpha < \lambda\} \qquad (\lambda \text{ a limit}).$$

$$\Gamma_\infty = \cup\{\Gamma_\alpha | \alpha \text{ is an ordinal}\}.$$

Γ_∞ is said to be the set inductively defined by Γ. Γ_α is the α-th stage of the inductive definition of Γ_∞. The closure ordinal of Γ, denoted by $|\Gamma|$, is the least α such that $\Gamma_{\alpha+1} = \Gamma_\alpha$. Thus $\Gamma_\infty = \Gamma_{|\Gamma|}$.

Γ is *progressive* if $X \subseteq \Gamma(X)$ for all X. From now on assume all Γ's are progressive, a harmless assumption since $\Gamma^*(X)$, defined by $\Gamma(X) \cup X$, is progressive and differs little from Γ.

Γ is *monotonic* if $X \subseteq Y$ implies $\Gamma(X) \subseteq \Gamma(Y)$. Mathematical definitions tend to be monotonic. If Γ is monotonic, then $\Gamma_\infty = \cap\{X | \Gamma(X) = X\}$.

Γ is classified by classifying the predicate $x \in \Gamma(Y)$.

8.2 Theorem. *Suppose* $A \in \Pi^1_1$, $P(X)$ *is* Π^0_1, *and* $P(A)$. *Then* $P(H)$ *for some hyperarithmetic* $H \subseteq A$. (Π^0_1 *reflection.*)

Proof. Let $P(X)$ be $(z)R(\bar{X}(z))$ for some recursive R. Let f be recursive so that

$$n \in A \leftrightarrow f(n) \in O$$

for all n (cf. Theorem 5.4.I). For each $\alpha < \omega_1^{CK}$ let A_α be $\{n | f(n) \in O_\alpha\}$. Each A_α is hyperarithmetic by Theorem 2.4.II; one of them will be the desired H.

Let $P_0(z,b,c)$ be

$$c \in O \quad \& \quad |b| < |c| \quad \& \quad R(\bar{A}_{|c|}(z)).$$

$P_0(z,b,c)$ is Π^1_1. $P(A)$ implies

$$(z)(b)(\mathrm{Ec})[b \in O \to P_0(z,b,c)].$$

More precisely, $R(\bar{A}(z))$ implies $R(\bar{A}_\alpha(z))$ for all sufficiently large recursive α, because only finitely many membership facts about A are needed to compute the truth-value of $R(\bar{A}(z))$. P_0 can be uniformized by some Π^1_1 Q according to Theorem

2.3 of Chapter II. Thus there is a partial Π^1_1 function $t(z,b)$ such that

$$(z)(b)[b \in O \to P_0(z,b,t(z,b))].$$

A recursive subsequence $\{b_n | n < \omega\}$ of O is defined by recursion. $b_0 = 1$. The set $\{t(z,b_n) | z < \omega\}$ is Σ^1_1 by proposition 1.7.I. Corollary 3.4.II makes it possible to compute b_{n+1}, a strict upper bound for $\{|t(z,b_n)| | z < \omega|\}$ in O. Let β, necessarily a recursive limit ordinal, be the sup of the $|b_n|$'s. For each z, β is the limit of ordinals δ such that $R(\bar{A}_\delta(z))$ holds. It follows that $(z)R(\bar{A}_\beta(z))$ holds. Since β is a limit and computations from A are finite, $\sim R(\bar{A}_\beta(z))$ would imply $\sim R(\bar{A}_\delta(z))$ for all sufficiently large $\delta < \beta$. H is A_β. □

Theorem 8.2 was presented to suggest a connection between reflection and inductive definitions, a link developed by Aczel and Richter 1972. Reflection phenomena are crucial in E-recursion theory as will be seen in Part D. The idea behind 8.2, and similar theorems, is: "enumerate" A until some "finite" subset is found that satisfies P.

8.3 Corollary (Gandy). *If* Γ *is* Π^0_1, *then* $|\Gamma| \leq \omega_1^{CK}$ *and* $\Gamma_\infty \in \Pi^1_1$.

Proof. An effective transfinite recursion produces a recursive function f such that: if $b \in O$, then $f(b)$ is a hyperarithmetic index for $\Gamma_{|b|}$. Consequently $\Gamma_{\omega_1^{CK}}$ is Π^1_1. It follows from the proof of Theorem 8.2 that

$$n \in \Gamma_{\omega_1^{CK}+1} \to n \in \Gamma_\alpha$$

for some recursive α depending on n. □

One way to remember the proof of 8.2, and hence of 8.3, is to think of it as a downward Skolem–Löwenheim argument. From the viewpoint of metarecursion theory (Part B), the β of 8.2 is a fixed point of a continuous metarecursive Skolem function. The downward Skolem idea is more evident when Γ is monotone Π^1_1 (cf. subsection 8.7).

Γ is said to be *positive* if the predicate $x \in \Gamma(Y)$ is positive with respect to Y. This means it can be written in prenex normal form with a matrix $\bigvee_i \bigwedge_j \mathscr{F}_{ij}$ in disjunctive normal form such that no $\mathscr{F}_{ij}$ is of the form $z \notin Y$. Thus $x \in \Gamma(Y)$ is a consequence of positive facts about Y. x gets into $\Gamma(Y)$ because certain z's already belong to Y. Note that positivity implies monotonicity.

8.4 Proposition. *If* Γ *is monotonic, then* Γ *is positive.*

Proof. $x \in \Gamma(Y)$ is equivalent to

$$(Z)[Y \subseteq Z \to x \in \Gamma(Z)],$$
$$(Z)(\mathrm{Eu})[(u \in Y \ \ \& \ \ u \notin Z) \vee x \in \Gamma(Z)]. \quad \square$$

8.5 Theorem (Spector 1955). *Suppose A is Π^1_1 and Γ is monotonic Π^1_1. Then*

$$\Gamma(A) = \cup\{\Gamma(H) | H \subseteq A \ \& \ H \in \mathrm{HYP}\}.$$

Proof. $n \in \Gamma(A)$ iff $S^A_R(n)$ is wellfounded, by the relativization of Proposition 5.3.I to A, as in 5.4.II. $S^A_R(n)$ is recursive uniformly in A, n. Fix n and assume $n \in \Gamma(A)$. Then, as in the proof of Proposition 8.4,

$$(1) \qquad (Z)(\mathrm{Eu})[(u \in A \ \ \& \ \ u \notin Z) \vee S^Z_R(n) \text{ is wellfounded}].$$

Let f be a recursive function such that for all m,

$$m \in A \leftrightarrow f(m) \in O.$$

Define A_α to be $\{m | f(m) \in O_\alpha\}$. Then (1) becomes

$$(2) \qquad (Z)(\mathrm{Eb})[b \in O^Z \ \ \& \ \ (A_{|b|} \notin Z \vee |S^Z_R(n)| = |b|)].$$

The matrix of (2) is Π^1_1, hence it can be uniformized by a Π^1_1 predicate $Q(Z, b)$ by adapting the proof of Theorem 2.3.II. For each Z, let $t(Z)$ be the unique b satisfying $Q(Z, b)$. Then t is Δ^1_1 and

$$(3) \qquad (Z)[t(Z) \in O^Z \ \ \& \ \ (A_{|t(Z)|} \notin Z \vee |S^Z_R(n)| = |t(Z)|)].$$

A slight modification of the proof of Lemma 5.5.II yields a recursive ordinal bound δ on $|t(Z)|$. If there is no such δ, then O is Σ^1_1.

$$b \in O \leftrightarrow (\mathrm{Ef})(\mathrm{EZ})(x)(y)(m)[m = t(Z) \rightarrow (x, y \in S_R(b)$$
$$\& \ \ x > y \rightarrow \langle f(y), f(x)\rangle \in W^Z_{q(m)})].$$

Thus for some $\delta < \omega_1^{\mathrm{CK}}$, $|t(Z)| < \delta$ for all Z. (cf. Exercise 5.10.II). (3) becomes

$$(Z)[A_\delta \subseteq Z \rightarrow |S^Z_R(n)| < \delta],$$

and so $n \in \Gamma(A_\delta)$. □

An intuitive summary of Theorem 8.5 and its proof is: if $n \in \Gamma(A)$, then "enumerate" A until some hyperarithmetic $H \subseteq A$ is developed with the property that $n \in \Gamma(H)$.

8.6 Corollary (Spector 1955). *If Γ is monotonic Π^1_1, then $|\Gamma| \leq \omega_1^{\mathrm{CK}}$ and $\Gamma_\infty \in \Pi^1_1$.*

Proof. If $X \in \Pi^1_1$, then $\Gamma(X)$ is Π^1_1 by Theorem 8.5. An effective transfinite recursion on $<_O$ shows $\Gamma_{|b|}$ is Π^1_1 uniformly in b for all $b \in O$. Hence $\Gamma_{\omega_1^{\mathrm{CK}}}$ is Π^1_1. Another application of 8.5 yields $\Gamma(\Gamma_{\omega_1^{\mathrm{CK}}}) = \Gamma_{\omega_1^{\mathrm{CK}}}$. □

A type-omitting argument in the setting of ω-logic clarifies the roles of monotonicity and Π^1_1-ness in Theorem 8.5. Let A be a finite subset of the axioms of set theory with enough strength to establish the essential properties of the hyperarithmetic hierarchy, and to prove the existence and countability of ω_1^{CK}. The statement, M is a real that encodes an ω-model of A, is arithmetic, hence has a solution N such that $\omega_1^N = \omega_1^{CK}$ by Corollary 1.4.

Let V be the real world of sets, and let V_N be the model encoded by N. The reals of V_N are a subset of the reals of V. The encoding has the property that each real of V_N is recursive in N. Note that any Π^1_1 sentence true in V is also true in V_N, since the transition from V to V_N simply restricts a universal function quantifier to a countable set.

A ordinal z in the sense of V_N is called standard if it belongs to V, that is z is truly wellordered. Suppose γ is a standard ordinal countable inside V_N. Thus γ is the ordertype of some linear ordering X of ω for some $X \in V_N$. X is recursive in N, so $\omega_1^X = \omega_1^{CK}$, and so γ is a recursive ordinal. On the other hand each recursive ordinal (of V) is a recursive ordinal in the sense of V_N. Thus the standard recursive ordinals of V_N are the recursive ordinals.

Let $(\omega_1^{CK})^{V_N}$ denote the least nonrecursive ordinal in the sense of V_N. The above considerations imply $(\omega_1^{CK})^{V_N}$ is not standard. It follows that $(\omega_1^{CK})^{V_N}$ is the limit of nonstandard ordinals.

Suppose $A \subseteq w$ is Π^1_1. Let f be a recursive function such that for all m, $m \in A \leftrightarrow f(m) \in O$, and let A_δ be $\{m | f(m) \in O_\delta\}$ for each $\delta < \omega_1^{CK}$. Then $A_\delta^{V_N} = A_\delta$ for each standard recursive δ, and $A_\beta^{V_N} \supseteq A$ for each nonstandard recursive β.

Let Γ be monotonic and Π^1_1. Suppose $n \in \Gamma(A)$. Then for each nonstandard recursive β,

$$n \in \Gamma(A_\beta^{V_N}) \text{ is true in } V \tag{1}$$

by monotonicity of Γ. Since (1) is a Π^1_1 fact, and V_N is an ω-model, (1) is also true in V_N. In V_N let δ_0 be the least β such that $n \in \Gamma(A_\beta^{V_n})$. δ_0 must be standard, hence recursive, because there is no least nonstandard ordinal in V_N. Therefore $A_{\delta_0}^{V_N} = A_{\delta_0}$ and $n \in \Gamma(A_{\delta_0})$.

The above is called a type-omitting argument because the standard ordinal ω_1^{CK} is omitted from V_N.

8.7 Inducively Defined Sets of Reals. Theorem 8.6 has been lifted by Cenzer 1976 to sets of reals. Let Γ map 2^{2^ω} into itself. Let A be a variable that ranges over subsets of 2^ω. A predicate $R(A, f, X)$ is said to be recursive if there is a Gödel number e such that for all A, f and X,

$$\{e\}^{A,f,X} \text{ is defined, and}$$
$$R(A,f,X) \leftrightarrow \{e\}^{A,f,X} = 0.$$

In the language of Part D: $R(A,f,X)$ is E-recursive in f, X with $r \in A$ as an additional predicate. Only two facts are needed here.

(i) The computation of $\{e\}^{A,f,X}$ uses only countably many membership facts about A.

(ii) Let B be a real that encodes a countable set of reals whose complement relative to 2^ω is denoted by A_B. Then $(f)R(A_B,f,X)$ is a Π^1_1 predicate with free variables B and X.

Both (i) and (ii) should seem plausible. (i) is analogous to the fact that the computation of $\{e\}^X$ uses only finitely many membership facts about X. (ii) is reasonable, since A_B is equivalent to a real, and since a Π^1_1 predicate of sets of reals, restricted to reals, should be Π^1_1 in the usual sense.

8.8 Proposition. *If* $\Gamma\colon 2^{2^\omega}\to 2^{2^\omega}$ *is monotonic* Π^1_1, *then it is positive.*

Proof. Similar to 8.4. Let B be the real variable of 8.7(ii). Then $(f)R(A,f,X)$ implies

$$(1)\qquad (B)[A\subseteq A_B\to(f)R(A_B,X,f)].$$

Assume X satisfies (1) to show $X\in\Gamma(A)$. If not, then by 8.7(i) there is an f such that $\sim R(A_B,f,X)$ holds for some B such that $A\subseteq A_B$. □

8.9 Theorem (Cenzer 1976). *Suppose* $\Gamma\colon 2^{2^\omega}\to 2^{2^\omega}$ *is monotonic* Π^1_1, *and* $X\in\Gamma_\infty$. *Then* $X\in\Gamma_\beta$ *for some* $\beta<\omega_1^X$.

Proof. $\Gamma_0=\varnothing$. By the monotonicity of Γ, $X\in\Gamma_1$ iff $(B)(f)R(A_B,f,X)$. Thus Γ_1 is Π^1_1 by 8.7(ii). Similarly, $X\in\Gamma_2$ iff

$$(B)[\Gamma_1\subseteq A_B\to(f)R(A_B,f,X)],$$

and so Γ_2 is Π^1_1. In this manner an effective transfinite recursion shows

$$(1)\qquad e\in O^X\ \ \&\ \ X\in\Gamma_{|e|_X}$$

is Π^1_1 uniformly in X and e. It follows that $\Gamma_{\omega_1^X}$ is Π^1_1 uniformly in X.

To complete the argument, fix X and assume $X\in\Gamma_{\omega_1^X+1}$ with the intent of showing $X\in\Gamma_{\omega_1^X}$. The desired bound on X is obtained as in the proof of Theorem 8.5. Γ is monotonic, so

$$(B)[\Gamma_{\omega_1^X}\subseteq A_B\to(f)R(A_B,f,X)],$$

since $X\in\Gamma_{\omega_1^X+1}$. Let B_0 be the countable set of reals encoded by B. Then

$$(2)\qquad (B)(\mathrm{Eb})[b\in O^{B,X}\ \ \&\ \ (\Gamma_{|b|}\cap B_0\neq\varnothing\vee|S_R^{B,X}|=|b|)].$$

$S_R^{B,X}$ is a binary relation recursive in B, X (uniformly) that is wellfounded iff $(f)R(A_B,X,f)$. The latter formula is Π^1_1 in B and X according to 8.7(ii). $S_R^{B,X}$ is the relativization of S_R (in subsection 5.2.I) to B, X. The matrix of (2) is Π^1_1, and so can

be uniformized by some Π^1_1 predicate $Q(B, b)$ with parameter X (cf. Theorem 2.3.II). Let $b(B)$ be the unique b satisfying $Q(B, b)$. Then

$$(B)[b(B)\in O^{B,X} \quad \& \quad (\Gamma_{b(B)}\cap B_0 = \varnothing \vee |S^{B,X}| = |b|)].$$

As in the conclusion of the proof of 8.5, there is an ordinal $\delta < \omega_1^X$ such that $|b(B)| < \delta$ for all B. Consequently

$$(B)[\Gamma_\delta \subseteq A_B \to (f)R(A_B, f, X)],$$

hence $x\in\Gamma_{\delta+1}$. □

8.10 Corollary (Cenzer 1976). *If* $\Gamma: 2^{2^\omega} \to 2^{2^\omega}$ *is monotonic* Π^1_1, *then* Γ_∞ *is* Π^1_1 *and* $|\Gamma| = \omega_1$.

8.11 Bounding Arguments. The proofs of Theorems 8.5 and 8.9 made use of a bounding principle first stated as Exercise 5.10.II. Let $P(X, b)$ be Π^1_1. Suppose $(X)(\mathrm{Eb})[P(X, b) \ \& \ b\in O^X]$. Then there exists a $\delta < \omega_1^{\mathrm{CK}}$ such that

$$(X)(\mathrm{Eb})\,[P(X, b) \quad \& \quad |b|_X < \delta].$$

A proof of Exercise 5.10 was incorporated into the last part of the proof of Theorem 8.5. The above bounding principle captures a strong closure property of the recursive ordinals. A weaker form (X replaced by x) will be used extensively in the next chapter.

8.12–8.15 Exercises

8.12. Show each Π^1_1 set of numbers is one-one reducible to one inductively defined by a $\Pi^0_1\Gamma$, by a monotonic $\Pi^1_1\Gamma$.

8.13. Find a Π^1_1 set of numbers not inductively definable by any monotonic $\Pi^1_1\Gamma$.

8.14. If A is a set of formulas, let $|A|$ be the supremum of all ordinals γ such that $\gamma = |\Gamma|$ for some Γ definable by a formula in A. Show $|\Pi^0_1| = \omega_1^{\mathrm{CK}}$. Show $|\Sigma^1_1| \neq |\Pi^1_1|$. (S. Aanderaa has shown $|\Pi^1_1| < |\Sigma^1_1|$.)

8.15. Call $P(X)$ monotonic if $P(X)\to P(Y)$ whenever $X\subseteq Y$. Suppose $P(X)$ is monotonic Π^1_1, A is Π^1_1, and $P(A)$. Show $P(H)$ for some hyperarithmetic $H\subseteq A$.

9. Π^1_1 *Singletons*

The subject of Π^1_1 singletons verges on set theory, but the results of this section have been phrased and proved in a mood of sympathy for those new to constructible sets and forcing. Most of the ideas applied have already been used in the earlier

accounts of Π^0_2 singletons and scattered Σ^1_1 sets. A is a Π^1_1 singleton if A is the unique solution of some Π^1_1 predicate. A set is *scattered* if it contains no perfect subset.
(Review: a set K of reals is *perfect* if K is nonempty, closed and has no isolated points; $p \in K$ is *isolated* if $\{p\} = K \cap V$ for some open set V; a perfect set of reals has the same cardinality as the continuum.)

9.1 Proposition. *If A is a Π^1_1 singleton, then so are O^A and every $B \equiv_h A$.*

Proof. Fix $B \equiv_h A$. Thus

$$B = \{e\}^{H^A_b} \quad \& \quad A = \{c\}^{H^B_a}$$

for some $b \in O^A$ and $a \in O^B$. Then B is the unique solution of

$$P(\{c\}^{H^Y_a}) \quad \& \quad Y = \{e\}^{H^{\{c\}^{H^Y_a}}_b},$$

where $P(Y)$ is a Π^1_1 predicate whose unique solution is A. $<_{O^A}$ is the unique solution of $P(\{e_0\}^Y)$ & [Y satisfies closure condition $A(X)$ of subsection 2.1.I relativized to $\{e_0\}^Y$] & Y is wellfounded. $\{e_0\}^{O^Z} = Z$ for all Z. □

9.2 Constructibility. The formulas of Zermelo–Fraenkel set theory (ZF) are well-formed combinations of the symbols of first order logic and ε, the membership symbol. The first order variables $x, y, z, \ldots$ range over the universe of sets. Suppose b and c are sets. b is said to be a first order definable subset of c if there is a formula $\mathscr{F}(x)$ of ZF such that

$$(1) \qquad (d)[d \in b \leftrightarrow d \in c \quad \& \quad \langle c, \varepsilon \rangle \models \mathscr{F}(\underline{d})].$$

$\mathscr{F}(x)$ may mention finitely many sets from c. The last part of (1) means $\mathscr{F}(\underline{d})$ is true when $\underline{d}$ is interpreted as d and the quantifiers of $\mathscr{F}$ range over c.

Gödel's hierarchy L of constructible sets is defined by iteration of Q. $Q(c)$ is the set of all first order definable subsets of c.

$$L(0) = \phi. \quad L(\beta + 1) = Q(L(\beta)).$$
$$L(\lambda) = \cup\{L(\beta) | \beta < \lambda\} \quad \text{if } \lambda \text{ is a limit.}$$
$$L = \cup\{L(\beta) | \beta \in On\}.$$

Gödel showed L is a model of ZF, the axiom of choice and the generalized continuum hypothesis. The central features of L are its downward Skolem–Löwenheim properties. For example, if $2^\omega \cap (L(\beta + 1) - L(\beta))$ is nonempty, then β is countable. These matters are discussed at greater length in Part C and there utilized in the solution to Post's problem for α-recursion theory.

9.3 Lemma. *Each nonempty Π^1_1 set of reals has a member X constructible via an ordinal recursive in X, that is $X \in L(\beta)$ for some $\beta < \omega_1^X$.*

Proof. According to subsection 5.4.II, a typical Π^1_1 predicate has the form, $\{e\}^Y$ is wellfounded, where $\{e\}$ is such that $\{e\}^Y$ is total and a binary relation between numbers for all Y. The Kleene–Brouwer ordering of sequence numbers, invoked in the proof of Lemma 2.1, makes it possible to say:

(1) $$\{e\}^Y \text{ is wellordered}$$

is a typical Π^1_1 predicate, where $\{e\}^Y$ is total and a linear ordering of numbers for all Y.

The solutions of (1) are associated with the infinite branches of a tree T defined as follows. A node of T is a pair $\langle s, g\rangle$ such that:

(i) s is a sequence number that encodes an initial segment of a subset of ω;
(ii) let $\{e\}^s_\ell$ be the finite linear ordering computed from s via e and computations of length less than s; g is an order-preserving map of $\{e\}^s_\ell$ into ω_1.

Say $\langle s, g\rangle >_T \langle t, h\rangle$ if s is properly extended by t and $g \subseteq h$. Let $\{\langle s_i, g_i\rangle | i < \omega\}$ be an infinite branch of T. The union of the s_i's is the characteristic function of some solution Y of (1). The union of the g_i's is a witness to the wellorderedness of $\{e\}^Y$.

Let $\langle t_i, h_i\rangle$ and $\langle s_i, g_i\rangle$ be infinite branches of T. $\langle t_i, h_i\rangle$ is said to be to the *left* of $\langle s_i, g_i\rangle$ if:

(iii) for some i and n, $t_i(n) < s_i(n)$ & $(m)_{m<n}(t_i(m) = s_i(m))$;
(iv) or (i) $(s_i = t_i)$ and for some j and x, $g(x) < h(x)$ & $g(y) = h(y)$ for all y less than x according to the linear ordering $\{e\}^{s_j}_\ell$.

Assume (1) has a solution. Then T has a leftmost infinite branch Z . Z is readily defined by recursion on i. At state $i+1$, suppose $\langle s_i, g_i\rangle$ has already been chosen, and let $\langle s, g\rangle$ be an arbitrary immediate successor of $\langle s_i, g_i\rangle$ in T. Let $T_{s,g}$ be the subtree of T whose nodes extend $\langle s, g\rangle$. Discard $\langle s, g\rangle$ if $T_{s,g}$ is wellfounded. Among the remaining nodes there is a leftmost one in the same sense of leftness as that defined by (iii) and (iv). It is $\langle s_{i+1}, t_{i+1}\rangle$.

Let Z_0 be the subset of ω whose characteristic function is $\cup\{s_i | s_i \in Z\}$, and let δ be $|\{e\}^{Z_0}|$. Define T^δ by restricting T to δ. The nodes of T^δ are the nodes of T that mention only ordinals less than δ. $T^\delta \in L$, because $\delta \in L$ and T^δ is first order definable over $L(\delta)$.

Suppose T^δ had no infinite branch inside L. Then inside L, T^δ is wellfounded and has ordinal height $|T^\delta|$, and there is a map k from T^δ into $|T^\delta|$ such that $k(u) < k(v)$ whenever $u < v$ in T^δ. But then k serves to show T^δ is truly wellfounded and so has no infinite branch anywhere. This last cannot be, because Z is an infinite branch of T^δ.

Let the leftmost branch of T^δ in L be Z^L. That part of T^δ to the left of Z^L is wellfounded inside L, hence truly wellfounded by the argument of the previous paragraph. Hence Z^L is the leftmost path of T^δ not just in L but truly so. It follows that $Z = Z^L$.

All that remains is the location of Z in L. Suppose $Z = \{\langle s_i, g_i\rangle\}$. Let Z_0 be the set encoded by $\{\langle s_i\rangle\}$, and g^Z the function from $\{e\}^{Z_0}$ onto δ encoded by $\{\langle g_i\rangle\}$.

Call a node *radical* if it belongs to T^δ and lies to the left of Z. Let R^{Z_0} be a wellordering of ω of height δ and recursive in Z_0. Then the nodes of T^δ can be construed as elements of $\omega \times \omega$, and g^Z as a function from $\{e\}^{Z_0}$ onto R^{Z_0}. The uniqueness of g^Z as an orderpreserving function implies it is hyperarithmetic in Z_0. It follows that the set of all radical nodes is hyperarithmetic in Z_0. The subtree T^δ_u that issues from a radical node u is wellfounded. Its ordinal height $|T^\delta_u|$ is less than $\omega_1^{Z_0}$, since T^δ_u can be construed as a tree hyperarithmetic in Z_0 (uniformly in u). The matrix of

$$(u)(\mathrm{E}b)[u \in \mathrm{Radical} \to b \in O^{Z_0} \quad \& \quad |T^\delta_u| < |b|_{Z_0}]$$

is Π^1_1. It follows from Kreisel uniformization (2.3.II) that b can be taken to be a hyperarithmetic function of u. Hence Spector bounding (5.6.I) implies the strict supremum of $|T^\delta_u|$ over all radical u is some $\gamma < \omega_1^{Z_0}$.

The recursion that defined Z from T also defines Z from T^δ. At stage $i+1$ of the recursion, the leftmost $\langle s, g\rangle$ is chosen such that $\langle s, g\rangle$ is an immediate successor of $\langle s_i, g_i\rangle$ and $T^\delta_{s,g}$ is not wellfounded. The last clause is equivalent to

(1) $\qquad |T^\delta_{s,g}|$ is not less than γ.

(1) is false iff there is a map f from $T^\delta_{s,g}$ onto $|T^\delta_{s,g}|$ such that $f(u) = |T^\delta_u|$ for all $u \in T^\delta_{s,g}$. If f exists, then it is unique and defined by recursion on the wellfounded relation $T^\delta_{s,g}$. The recursion can be performed at level $\delta + |T^\delta_{s,g}|$ of L. Thus f, if it exists, is first order definable over $L(\delta + |T^\delta_{s,g}|)$. It follows that the recursion that defines Z can be carried out at level $\delta + \gamma$ of L. □

9.4 Corollary (Addison, Kondo). *Each nonempty* Π^1_1 *set of reals contains a* Π^1_1 *singleton.*

Proof. Suppose K is Π^1_1 and nonempty. By Lemma 9.3 there is an X in K such that $X \in L(\omega_1^X)$. With each such X, associate a pair $\langle \gamma, \delta\rangle_X$ of ordinals: $X \in L(\gamma+1) - L(\gamma)$ and $|\{e\}^X| = \delta$. (K is the set of all X such that $\{e\}^X$ is wellfounded.) The formula, $X \in K$ and $X \in L(\omega_1^X)$ and $\langle \gamma, \delta\rangle_X$ has the least possible value, is Π^1_1, and almost has a unique solution. To clinch the uniqueness, minimize the Gödel number of the formula that defines X as a subset of $L(\gamma)$. The Gödel number has to include parameters needed for the definition of X over $L(\gamma)$. The proof of Lemma 9.3 makes it safe to assume those parameters are ordinals. □

9.5 Theorem (Mansfield, Solovay). *If* K *is* Π^1_1, *then* (*i*), (*ii*) *and* (*iii*) *are equivalent.*

(*i*) K *contains a perfect set.*

(*ii*) K *contains a perfect set with constructible code.*

(*iii*) K *has a member* $X \notin L(\omega_1^X)$.

Proof. (i) → (ii). The set of all Z's that encode perfect subsets of K is Π^1_1, hence at least one such is constructible by Lemma 9.3.

(ii) → (iii). Let $Z \in L$ encode a perfect subset of K denoted by P_Z as in subsection 6.1. The predicate

$$X \in P_Z \quad \& \quad X \not\leq_h Z$$

is satisfiable by all but countably many $X \in P_Z$, is Σ^1_1 in Z, and so has a solution X such that $\omega_1^{X,Z} = \omega_1^Z$ by Corollary 1.5 relativized to Z. If $X \in L(\omega_1^X)$, then $X \in L(\omega_1^Z)$, and consequently $X \leq_h Z$ by Exercise 9.11.

(iii) → (i). The second half of the predicate

$$X \in K \quad \& \quad X \notin L(\omega_1^X) \tag{1}$$

is Σ^1_1 according to Exercise 9.7. Let R be a recursive predicate such that $(\mathrm{Ef})(x)R(\bar{s}(x), \bar{f}(x))$ holds iff s is the characteristic function of some X such that $X \notin L(\omega_1^X)$. The solutions of (1) are associated with a tree U that combines tree T of the proof of Lemma 9.3 with tree T_R needed in the proof of Kleene's basis Theorem (1.3). T is needed for the first half of (1), and T_R for the second half. A node of U has the form $\langle s, g, w \rangle$, where $\langle s, g \rangle$ is a node of T, w is a sequence number that encodes an initial segment of a function, length of w = length of s, and $R(s, w)$ holds. An infinite branch through U delivers an $X \in K$, a witness to the fact that $X \in K$, and a witness to the fact that $X \notin L(\omega_1^X)$.

As in the proof of 9.3, $X \in K$ iff $\{e\}^X$ is wellordered. Choose X_0, a solution of (1), to minimize $|\{e\}^{X_0}| = \delta$. Define U^δ by restricting U to δ. Thus each node of U^δ mentions only ordinals less than δ. The perfect set construction takes place within U^δ. Let $U^\delta_{s,g,w}$ be the subtree of U^δ issuing from node $\langle s, g, w \rangle$. For the moment assume U^δ has the following *splitting* property: If $U^\delta_{s,g,w}$ has an infinite branch, then there exist $\langle s_i, g_i, w_i \rangle$ $(i < 2)$ such that $\langle s, g, w \rangle$ is extended by $\langle s_i, g_i, w_i \rangle$, $U^\delta_{s_i, g_i, w_i}$ has an infinite branch, and s_0 is incomparable with s_1.

The construction proceeds as in the proof of Theorem 6.2. $\{\langle s^j_i, g^j_i, w^j_i \rangle | i < \omega \ \& \ j < 2^i\}$ is defined by recursion on i. s^0_0, g^0_0 and w^0_0 are null. $U^\delta_{s^0_0, g^0_0, w^0_0}$ has an infinite branch because X_0 is a solution of (1). $\langle s^{2j}_{i+1}, g^{2j}_{i+1}, w^{2j}_{i+1} \rangle$ and $\langle s^{2j+1}_{i+1}, g^{2j+1}_{i+1}, w^{2j+1}_{i+1} \rangle$ are extensions of $\langle s^j_i, g^j_i, w^j_i \rangle$ provided by the splitting property. $\{s^j_i | i < \omega \ \& \ j < 2^i\}$ encodes a perfect subset of K.

To check the splitting property, assume that $U^\delta_{s,g,w}$ gives rise to a unique solution X of (1). The concluding arguments of Lemma 9.3 will show $X \in L(\omega_1^X)$ contrary to (1). Call a node $u \in U^\delta_{s,g,w}$ false if $(u)_0$ does not encode an initial segment of X. If u is false, then U^δ_u is wellfounded. Since $\omega_1^{X_0} \leq \omega_1^X$, it follows as in 9.3 that: U^δ_u can be construed as a tree hyperarithmetic in X (uniformly in u); the set of all false nodes is hyperarithmetic in X; the strict supremum of $|U^\delta_u|$ over all false u is some $\gamma < \omega_1^X$.

u is a false node iff there is a map f from U^δ_u onto the ordinal $|U^\delta_u|$ such that $f(x) = |U^\delta_x|$ for all $x \in U^\delta_u$. If f exists, then it is unique and defined by recursion on the wellfounded relation U^δ_u. The recursion can be performed at level $\delta + |U^\delta_u|$ of L. Thus f, if it exists, is first order definable over $L(\delta + |U^\delta_u|)$. It follows that the set of false nodes, hence X, is definable at level $\delta + \gamma$ of L. □

The proof of Theorem 9.5 ((iii) → (i)) was modeled on the proof of Theorem 6.2 ((iii) → (i)). The tree U^δ corresponds to a tree T_R not made explicit in 6.2. The nodes of T_R are of the form $\langle s, t \rangle$, where s and t are sequence numbers of the same length and $R(s, t)$ holds. The Σ^1_1 set K of 6.2 is the set of all X such that $(\mathrm{Eg})(x)R(\bar{X}(x), \bar{g}(x))$. The proof of the splitting property for T_R was based on the fact that (A) if a Σ^1_1 predicate has a unique solution, then that solution is hyperarithmetic. The corresponding fact for 9.5 is: (B) if all the infinite branches of U^δ carry the same subset of ω, namely X, then $X \in L(\omega_1^X)$. According to Exercise 9.12, a subset of ω is hyperarithmetic iff it belongs to $L(\omega_1^{\mathrm{CK}})$. It is possible, although tedious, but certainly instructive, to give a proof of (A) in the style of the proof given of (B).

9.6 A Hierarchy of Π^1_1 Singletons. Proposition 9.1 suggests that the Π^1_1 singletons might be generated by iterating the hyperjump. To be precise, define

$$\begin{aligned} h_0 &= \text{hyperdegree of the empty set,} \\ h_{\delta+1} &= \text{hyperjump of } h_\delta, \text{ and} \\ h_\lambda &= \text{least upper bound of } \{h_\delta \mid \delta < \lambda\}. \end{aligned}$$

It turns out that h_λ is undefined for a certain countable ordinal σ_0. σ_0 is fairly large. It is the least Σ_1 admissible ordinal that is a limit of Σ_1 admissible ordinals (Richter 1967, Sacks 1971).

For all $\delta < \sigma_0$, h_δ is the hyperdegree of a Π^1_1 singleton. But $\{h_\delta \mid \delta < \sigma_0\}$ does not exhaust the hyperdegrees of the Π^1_1 singletons.

Suzuki 1964 proved that the partial ordering of hyperdegrees, restricted to the Π^1_1 singletons, is a wellordering. He observed: if X and Y are Π^1_1 singletons, then

$$X \leq_h Y \leftrightarrow \omega_1^X \leq \omega_1^Y. \tag{1}$$

Assume the right side of (1) holds. Suppose X is the unique solution of

$$\{e\}^Z \text{ is wellordered.} \tag{2}$$

The ordertype of $\{e\}^X$ is less than ω_1^Y, hence (2) (with X in place of Z) is equivalent to $|\{e\}^X| < |b|_Y$ for some fixed $b \in O^Y$. Thus (2) is Δ^1_1 in Y. Hence $X \leq_h Y$ by Theorem 2.5.II relativized to Y.

9.7–9.13 Exercises

9.7. Show $X \in L(\omega_1^X)$ is a Π^1_1 predicate.

9.8. (Gaspari, Kechris, Sacks). Call a Π^1_1 subset of 2^ω scattered if it contains no perfect subset. Show the union of all scattered Π^1_1 sets is a scattered Π^1_1 set.

9.9. (Kondo, Addison). Suppose $P(X, Y)$ is Π^1_1. Show $P(X, Y)$ can be uniformized by some Π^1_1 $Q(X, Y)$.

9.10. (Shoenfield). The Δ^1_2 reals are a basis for the Σ^1_2 predicates of reals.

9.11. Show $X \in L(\omega_1^Z, Z)$ iff $X \leq_h Z$. (The definition of $L(\beta, Z)$ is the same as that of $L(\beta)$ in subsection 9.2 save for the initial step, which is replaced by "$L(0, Z) = \{Z\}$".)

9.12. Show X is hyperarithmetic iff $X \in L(\omega_1^{CK})$.

9.13. Prove reduction, as defined in Section 3.6.II, for Π^1_1 sets of reals.

Chapter IV
Measure and Forcing

The measure-theoretic analysis of Π^1_1 sets begun in Section 6.II is continued. It is shown that every Π^1_1 set of positive measure has a hyperarithmetic member. Forcing over the hyperarithmetic hierarchy is developed in order to construct a minimal hyperdegree and to prove Louveau's separation theorem.

1. Measure-Theoretic Uniformity

The goal of this section is to show $\mathscr{M}(\omega_1^{CK}, T)$ is a model of Δ^1_1 comprehension for almost all T. It follows that $\omega_1^{CK} = \omega_1^T$ for almost all T. Recall the notions of ordinal rank (subsection 4.1.III) and full ordinal rank (subsection 4.4.III) for formulas of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$. Define the number quantifier rank of a formula $\mathscr{F}$ to be the number of occurrences of $(\mathrm{Ex}), (y), \ldots$ in $\mathscr{F}$.

Let $\mathscr{F}$ be a ranked sentence of $\mathscr{L}(\omega_1^{CK}, T)$. By Lemma 4.6.II the set

$$\{T \mid \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}\}$$

is Δ^1_1, hence Borel and measurable according to subsection 6.1.II. Denote its measure by $p(\mathscr{F})$, the probability that $\mathscr{F}$ is true. The next task is to show the graph of $p(\mathscr{F})$, as $\mathscr{F}$ ranges over ranked sentences, is Π^1_1.

1.1 Proposition. *Let $\mathscr{F}(X^\alpha)$ be a ranked formula whose only free variable is X^α. Let $\mathscr{G}_i(x)$ $(i \leq n)$ be formulas of rank at most α whose only free variable is x. Then*

$$\bigvee_{i \leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x))$$

is logically equivalent to a sentence of full ordinal rank less than that of $(\mathrm{EX}^\alpha)\mathscr{F}(X^\alpha)$.

Proof. Put $\bigvee_{i \leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x))$ into prenex normal form without changing the pattern of set quantifiers. Then contract like quantifiers as in the proof of Theorem1.5 of Chapter I. The result will have at least one less occurrence of (EX^α) than does $(\mathrm{EX}^\alpha)\mathscr{F}(X^\alpha)$. □

1.2 Proposition. *Let $\mathscr{F}(x)$ be a ranked formula whose only free variable is x. Then $\bigvee_{i \le n} \mathscr{F}(\underline{i})$ is logically equivalent to a formula of the same full ordinal rank but lower number quantifier rank.*

1.3 Theorem. *The predicate $p(\mathscr{F}) \ge r$, restricted to ranked $\mathscr{F}$ and rational r, is Π^1_1.*

Proof. An application of Theorem 1.6(i).I. similar to that made in the proof of Lemma 4.5.III. There exists a Σ^1_1 formula $A(X)$ such that

$$(1) \qquad p(\mathscr{F}) \ge r \leftrightarrow (X)[A(X) \rightarrow \langle \mathscr{F}, r \rangle \in X].$$

The only conceptual difficulty arises from the impossibility of computing $p(\mathscr{G} \,\&\, \mathscr{H})$ from $p(\mathscr{G})$, $p(\mathscr{H})$. The difficulty is avoided by resort to prenex normal form. This logical trick is equivalent to construing a Borel set as a monotone union (or intersection) of Borel sets of lower rank.

Consider $(\mathrm{EX}^\alpha)F(X^\alpha)$. Let $\mathscr{G}_i(x)$ $(i < \omega)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x.

$$p((\mathrm{EX}^\alpha)\mathscr{F}(X^\alpha)) = \sup_n p(\bigvee_{i \le n} \mathscr{F}(\hat{x}\mathscr{G}_i(x))).$$

Hence $p((\mathrm{EX}^\alpha)\mathscr{F}(X^\alpha)) \ge r$ is equivalent to

$$(\delta)(\mathrm{En})[p(\bigvee_{i \le n} \mathscr{F}(\hat{x}\mathscr{G}_i(x))) \ge r - \delta],$$

where δ ranges over the positive rationals. By Proposition 1.2, the needed reduction in full ordinal rank has occurred.

$A(X)$ in full is the conjunction of (2) through (6).

(2) $\mathscr{F}$ is quantifierless & $p(\mathscr{F}) \ge r \rightarrow \langle \mathscr{F}, r \rangle \in X$.

(3) $$b \in O \;\&\; (\delta)(\mathrm{En})\left[\left\langle \bigvee_{i \le n} \mathscr{F}(\hat{x}\mathscr{G}_i(x)), r - \delta \right\rangle \in X\right] \rightarrow \langle (\mathrm{EX}^{|b|})F(X^{|b|}), r \rangle \in X.$$

(4) $$b \in O \;\&\; (n)\left[\left\langle \bigwedge_{i \le n} \mathscr{F}(\hat{x}\mathscr{G}_i(x)), r \right\rangle \in X\right] \rightarrow \langle (X^{|b|})\mathscr{F}(X^{|b|}), r \rangle \in X.$$

(5) $$(\delta)(\mathrm{En})\left[\left\langle \bigvee_{i \le n} \mathscr{F}(\underline{i}), r - \delta \right\rangle \in X\right] \rightarrow \langle (\mathrm{Ex})\mathscr{F}(x), r \rangle \in X.$$

(6) $$(n)\left[\left\langle \bigwedge_{i \le n} \mathscr{F}(\underline{i}), r \right\rangle \in X\right] \rightarrow \langle (x)\mathscr{F}(x), r \rangle \in X.$$

(1) is proved by induction on the full ordinal rank and number quantifier rank of F. □

1.4 Lemma. *Let $\mathscr{F}(x, Y)$ be a formula of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ whose only unranked set variable is the free variable Y. Assume $p((x)(EY)\mathscr{F}(x, Y)) \geq r$. Then for some $\alpha < \omega_1^{CK}$, $p((x)(EY^{\alpha})\mathscr{F}(x, Y) \geq r$.*

Proof. Recall (1) from the proof of Theorem 1.3.I. The assumption yields

$$(1) \qquad (m)(\delta)(Ea)_{a \in O}\left[p\left((EY^{|a|})\left(\bigwedge_{n \leq m} \mathscr{F}(\underline{n}, (Y^{|a|})_n) \right) \right) \geq r - \delta \right].$$

According to Theorem 1.3 the matrix of (1) is Π_1^1. It follows from Theorem 2.6.II that a can be regarded as a hyperarithmetic function of m and δ. Spector's boundedness theorem (5.6.I) implies

$$(m)(\delta)[|a(m, \delta)| < \alpha]$$

for some $\alpha < \omega_1^{CK}$. Consequently

$$(m)\left[p\left((EY^{\alpha}) \bigwedge_{n \leq m} \mathscr{F}(\underline{n}, (Y^{\alpha})_n) \right) \geq r \right],$$

and so

$$p((x)(EY^{\alpha})\mathscr{F}(x, Y^{\alpha})) \geq r. \qquad \square$$

1.5 Theorem. *If $\mathscr{F}$ is an instance of the Δ_1^1 comprehension theorem, then $p(\mathscr{F}) = 1$.*

Proof. Let $A(x, Y)$ and $B(x, Z)$ be arithmetic, and let K be the set of all T such that

$$(x)[(EY)A(x, Y) \leftrightarrow (Z)B(x, Z)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. Thus for each $T \in K$,

$$(1) \qquad (x)(EY)[A(x, Y) \vee \sim B(x, Y)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. By Lemma 1.4, for almost every T that satisfies (1), there is an $\alpha < \omega_1^{CK}$ such that

$$(x)(EY^{\alpha})[A(x, Y^{\alpha}) V \sim B(x, Y^{\alpha})]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. But then for almost all $T \in K$,

$$(EX)(x)[x \in X \leftrightarrow (EY)A(x, Y)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. The "$X$" is $\hat{x}(EY^{\alpha})A(x, Y^{\alpha})$. $\square$

1.6 Corollary (Sacks 1969, Tanaka 1968). *For almost all T, $M(\omega_1^{CK}, T) = HYP(T)$ and $\omega_1^T = \omega_1^{CK}$.*

Proof. Lemma 4.16.III. □

The next proposition says that $\mathscr{L}(\omega_1^{CK}, T)$ is closed under formation of hyperarithmetic conjunctions.

1.7 Proposition. *Let $\mathscr{F}_i$ $(i < \omega)$ be a hyperarithmetic sequence of ranked sentences. Then there exists a ranked sentence $\mathscr{F}$ such that for all T,*

$$\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F} \quad \textit{iff} \quad (i)[\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}_i].$$

Proof. By Lemma 4.6.III, the set of all T such that

$$(i)[\mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}_i] \tag{1}$$

is Δ_1^1, hence hyperarithmetic according to subsection 5.6.II. It suffices to show: for each $b \in O$ there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{CK}, T)$ such that

$$n \in H_b^T \leftrightarrow \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}(\underline{n})$$

for all n and T. This last is managed by the same recursion on $<_O$ used in the proof of Lemma 4.7.III to show H_b is representable by a formula of $\mathscr{L}(\omega_1^{CK}, T)$. The recursion step relies on a relativization of Corollary 4.4.II to T. □

1.8 Lemma. *Let $\mathscr{F}$ be a ranked sentence and δ a positive rational. Then there exists a ranked sentence $\mathscr{G}$ such that*

(i) $p(\mathscr{F} \ \& \ \sim \mathscr{G}) < \delta$, and

(ii) $\{T \mid \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{G}\}$ *is a closed subset of* $\{T \mid \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}\}$.

Proof. Let $P(\mathscr{F}, \delta, \mathscr{G})$ be the conjunction of (i) and (ii). It follows from Lemma 4.5.III and Theorem 1.3 that $P(\mathscr{F}, \delta, \mathscr{G})$ is Π_1^1. ("Closedness" is Π_1^1.)

$(\mathscr{F})(\delta)(\mathrm{E}\mathscr{G})P(\mathscr{F}, \delta, \mathscr{G})$ is proved by induction on the full ordinal rank and logical complexity of $\mathscr{F}$.

Suppose $\mathscr{F}$ is $(Y^\alpha)\mathscr{F}_1(Y^\alpha)$. Let $\mathscr{G}_i(x)$ $(i < w)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x. By induction

$$(i)(\mathrm{E}\mathscr{H})P\left(\mathscr{F}_1(\hat{x}\mathscr{G}_i(x)), \frac{\delta}{2^{i+1}}, \mathscr{H}\right).$$

By Lemma 2.6.II there is a hyperarithmetic sequence $\mathscr{H}_i$ $(i < \omega)$ such that

$$(i)\ P\left(\mathscr{F}_1(\hat{x}\mathscr{G}_i(x)), \frac{\delta}{2^{i+1}}, \mathscr{H}_i\right).$$

Let $\mathscr{H}$ be equivalent to the conjunction of the $\mathscr{H}_i$'s as provided by Proposition 1.7. Then $P(\mathscr{F}, \delta, \mathscr{H})$.

Suppose $\mathscr{F}$ is $(\mathrm{E}Y^{\alpha})\mathscr{F}_1(Y^{\alpha})$. The countable additivity of p implies there is an n such that

$$p\left(\bigvee_{i\leq n}\mathscr{F}_1(\hat{x}\mathscr{G}_i(x))\right)\geq p(\mathscr{F})-\frac{\delta}{2}. \tag{1}$$

By Proposition 1.1 the finite disjunction occurring in (1) is equivalent to a formula of lower full ordinal rank than $\mathscr{F}$. Hence by induction there is an $\mathscr{H}$ such that

$$P\left(\bigvee_{i\leq n}\mathscr{F}_1(\hat{x}\mathscr{G}_i(x)),\frac{\delta}{2},\mathscr{H}\right).$$

Again, $P(\mathscr{F},\delta,\mathscr{H})$. □

1.9–1.11 Exercises

1.9. Show the measure of a hyperarithmetic set of reals is a hyperarithmetic real.

1.10. Show the measure of a Π^1_1 set A of reals is the supremum of the measures of the hyperarithmetic subsets of A.

1.11. Show the predicate, $\mu(A)>\delta$, is Π^1_1, where A ranges over Π^1_1 sets and δ over rationals.

2. *Measure-Theoretic Basis Theorems*

The main result of this section is: if A is a Π^1_1 set of positive measure, then A has a hyperarithmetic member. In addition the concept of measure-theoretic bounding is formulated and proved for the hyperarithmetic hierarchy.

2.1 Lemma. *Suppose $\mathscr{F}$ is ranked. If $p(\mathscr{F})>0$; then $\mathscr{M}(\omega_1^{\mathrm{CK}},T)\models\mathscr{F}$ for some $T\in\mathrm{HYP}$.*

Proof. By Lemma 1.8 $\mathscr{F}$ can be replaced by a "closed subset" $\mathscr{G}$ such that $p(\mathscr{G})>0$. Define $\mathscr{H}_n(n<\omega)$

$$\underline{n}\in\mathscr{T}\quad\text{if}\quad p\left(\mathscr{G}\ \&\ \bigwedge_{i<n}\mathscr{H}_i\ \&\ \underline{n}\in\mathscr{T}\right)>0,$$

$\mathscr{H}_n$ is

$$\underline{n}\notin\mathscr{T}\ \text{otherwise.}$$

It follows from Theorem 1.3 that $T=\{n\mid\mathscr{H}_n\text{ is }\underline{n}\in\mathscr{T}\}$ is hyperarithmetic. Since $\mathscr{G}$ is "closed", $\mathscr{M}(\omega_1^{\mathrm{CK}},T)\models\mathscr{G}$. □

2.2 Theorem (Sacks 1969, Tanaka 1968). *If $P(X)$ is Π^1_1 and $\hat{X}P(X)$ has positive measure, then $P(X)$ has a hyperarithmetic solution.*

Proof. The relativization of Theorem 3.5.III to X yields an arithmetic predicate $A(X, Y)$ such that

$$P(X) \leftrightarrow (\mathrm{E}Y)_{Y \leq_h X} A(X, Y).$$

By Corollary 1.6

$$P(T) \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y) A(\mathscr{T}, Y)$$

for almost all T. Consequently $P((\mathrm{E}Y)A(\mathscr{T}, Y)) > 0$, and by the countable additivity of p, $p((\mathrm{E}Y^\alpha)A(\mathscr{T}, Y^\alpha)) > 0$ for some $\alpha < \omega_1^{\mathrm{CK}}$. By Lemma 2.1 there is an $H \in \mathrm{HYP}$ such that

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, H) = (\mathrm{E}Y)\mathscr{A}(\mathscr{T}, Y).$$

Since $\mathscr{M}(\omega_1^{\mathrm{CK}}, H) = \mathrm{HYP}$, $P(H)$ holds. □

2.3 Measure-Theoretic Bounding. Assume $P(T, x, y)$ is Π^1_1. It follows from Kreisel's uniformization theorem (2.3.II) relativized to T, that

$$(1) \qquad (x)(\mathrm{E}y)P(T, x, y) \rightarrow (\mathrm{E}f)_{f \leq_h T}(x)(\mathrm{E}y)_{y \leq f(x)} P(T, x, y).$$

For almost all T the bounding function f of (1) can be taken to be hyperarithmetic, that is

$$(2) \qquad (x)(\mathrm{E}y)P(T, x, y) \rightarrow (\mathrm{E}f)_{f \in \mathrm{HYP}}(x)(\mathrm{E}y)_{y \leq f(x)} P(T, x, y).$$

Fix a rational $\delta > 0$. Suppose $f \in \mathrm{HYP}$ is sought so that

$$(x)(\mathrm{E}y)_{y \leq f(x)} P(T, x, y)$$

holds for all T, save those in a set of measure less than δ, for which the left side of (2) holds. Choose a rational m such that

$$m \leq \mu((x)(\mathrm{E}y)P(T, x, y)) < m + \frac{\delta}{2}.$$

Then

$$(3) \qquad (x)(\mathrm{E}z)\left[\mu((\mathrm{E}y)_{y \leq z} P(T, x, y)) > m - \frac{\delta}{2^{x+2}}\right].$$

The matrix of (3) is Π^1_1 by Exercise 1.11. By Kreisel's selection Lemma (2.6.II), z can be taken to be $f(x)$ for some hyperarithmetic f. Hence

$$\mu((x)(\mathrm{E}y)_{y \leq f(x)} P(T, x, y)) > m - \frac{\delta}{2}. \qquad \square$$

For a formulation of measure-theoretic bounding over a countable model of ZF & $V = L$, see Sacks 1969.

2.4–2.6 Exercises

2.4. Suppose $A \in 2^\omega - \mathrm{HYP}$. Show $\{X \mid A \leq_h X\}$ has measure 0.

2.5. Call T Δ^1_1-random if T belongs to every Δ^1_1 set of measure 1. Define Σ^1_1 and Π^1_1 randomness similarly. Show Π^1_1-randomness is equivalent to Δ^1_1-randomness, but not to Σ^1_1-randomness.

2.6. Show Σ^1_1 dependent choice holds in $\mathscr{M}(T_0, T_1, T_2, \ldots)$ for almost all $\langle T_0, T_1, T_2, \ldots \rangle$. The language $\mathscr{L}(\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \ldots)$ and the structure $\mathscr{M}(T_0, T_1, T_2, \ldots)$ are defined in virtually the same way as $\mathscr{L}(\mathscr{T})$ and $\mathscr{M}(\mathscr{T})$ were in Section 4.III. The only difference occurs in the definition of atomic formula, where $\mathscr{T}_i (i < w)$ takes the place of $\mathscr{T}$.

3. *Cohen Forcing*

The method of Cohen forcing over the hyperarithmetic hierarchy was developed by S. Feferman 1965. It has many applications, the simplest of which is the direct construction of two hyperarithmetically incomparable reals, each recursive in Kleene's O.

A Cohen forcing condition is a consistent, finite conjunction of formulas of the form $\underline{m} \in \mathscr{T}$ or $\underline{n} \notin \mathscr{T}$. It helps to think of them as nonempty subbasic open subsets of 2^ω. They are denoted by $p, q, r, \ldots, ;$ $T \in p$ means T satisfies p, or T is a member of $p \subseteq 2^\omega$. $p \geq q$ (p is extended by q) means $(T)[T \in q \to T \in p]$.

3.1 Cohen's Forcing Relation $\Vdash$. Let $\mathscr{F}$ be a sentence of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$, as defined in subsection 4.1.III. Define $p \Vdash \mathscr{F}$ (p forces $\mathscr{F}$) by recursion on the full ordinal rank and logical complexity of $\mathscr{F}$ as follows.

$p \Vdash \underline{n} \in \mathscr{T}$	iff	$(T)[T \in p \to n \in T]$.
$p \Vdash (\mathrm{E}X^\alpha)\mathscr{F}(X^\alpha)$	iff	$p \Vdash \mathscr{F}(\hat{x}\mathscr{G}(x))$
for some $\mathscr{G}(x)$		of rank at most α.
$p \Vdash (\mathrm{E}X)\mathscr{F}(X)$	iff	$(E\alpha)[p \Vdash (\mathrm{E}X^\alpha)\mathscr{F}(X^\alpha)]$.
$p \Vdash \mathscr{F}$ & $\mathscr{G}$	iff	$p \Vdash \mathscr{F}$ and $p \Vdash \mathscr{G}$.
$p \Vdash (\mathrm{Ex})\mathscr{F}(x)$	iff	$(\mathrm{En})[p \Vdash \mathscr{F}(\underline{n})]$.
$p \Vdash \sim \mathscr{F}$	iff	$(q)_{p \geq q} \sim [q \Vdash \mathscr{F}]$.
$p \Vdash t_1 = t_2$	iff	$\mathrm{val}(t_1) = \mathrm{val}(t_2)$.
$p \Vdash t \in \mathscr{T}$	iff	$\mathrm{val}(t) = n$ and $p \Vdash \underline{n} \in \mathscr{T}$.

t_1, t_2 and t are closed number theoretic terms as in the proof of Lemma 4.5.III. Note that at the ground level, forcing and truth coincide. After that, forcing

respects the logical connectives. If one clause of the definition can be said to give the essence of forcing, it is the clause that defines the forcing of a negation.

One way to make sense out of the definition of Cohen forcing for ranked sentence of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ is to consider how to approximate a hyperarithmetic set of reals by an open set. At the ground level a subbasic open set is approximated by itself. At higher levels the approximation is built up from approximations available at lower levels. It turns out that $\cup\{p|p \Vdash \mathscr{F}\}$ is a good approximation of $\{T|\mathscr{M}(\omega_1^{CK}, T) \vDash \mathscr{F}\}$, and that $\cup\{p|p \Vdash \sim\sim\mathscr{F}\}$ is slightly better. Some calculation is needed to show that the latter open set is hyperarithmetic and regular, and that the error of approximation is meager (cf. Exercise 3.10).

3.2 Lemma. *The predicate* $p \Vdash \mathscr{F}$, *restricted is* Σ_1^1 $\mathscr{F}$*'s, is* Π_1^1.

Proof. Same as that of Lemma 4.5.III. The clauses of the definition of $\Vdash$, when $\mathscr{F} \in \Sigma_1^1$, correspond to closure conditions whose conjunction is some Σ_1^1 formula $A(X, T)$. An induction on full ordinal rank and logical complexity shows

$$p \Vdash \mathscr{F} \text{ iff } (X)[A(X,T) \to \langle p, \mathscr{F}, 0\rangle \in X]$$
$$\text{iff } \sim(X)[A(X,T) \to \langle p, \mathscr{F}, 1\rangle \in X].$$

3.3 Genericity. T is generic (in the sense of Cohen) with respect to a sentence $\mathscr{F}$ if there exists a p such that

$$T \in p \quad \text{and} \quad [p \Vdash \mathscr{F} \text{ or } p \Vdash \sim\mathscr{F}].$$

T is *generic* (over $\mathscr{M}(\omega_1^{CK})$) if T is generic with respect to every sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$.

3.4 Proposition. (*i*) $(p)(\mathscr{F}) \sim\{p \Vdash \mathscr{F} \,\&\, p \Vdash \sim\mathscr{F}]$.

(*ii*) $(p)(\mathscr{F})(\mathrm{Eq})_{p \geq q}\, [q \Vdash \mathscr{F} \vee q \Vdash \sim\mathscr{F}]$.

(*iii*) $(p)(q)\,(\mathscr{F})\,[p \Vdash \mathscr{F} \,\&\, p \geq q \to q \Vdash \mathscr{F}]$.

Proof. (i) and (ii) follow from the definition of $p \Vdash \sim\mathscr{F}$. (iii) is proved by induction on the full ordinal rank and logical complexity of $\mathscr{F}$. □

It follows from Proposition 3.4(ii) that generic T's exist. Let $\mathscr{F}_i$ $(i<\omega)$ be a list of all sentences of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$. Choose p_i $(i<\omega)$ so that for all i,

$$p_i \geq p_{i+1} \quad \text{and} \quad [p_i \Vdash \mathscr{F}_i \vee p_i \Vdash \sim\mathscr{F}_i].$$

Since every sentence of the form $\underline{n} \in \mathscr{T}$ appears among the $\mathscr{F}_i$'s, it must be that $\bigcap_{i<\omega} p_i$ has a unique member, call it T. Clearly T is generic. According to Exercise 3.18, there exists a generic T hyperarithmetic in O.

3.5 Lemma. *Let T be generic. Then*

$$\mathcal{M}(\omega_1^{CK}, T) \models \mathcal{F} \quad \textit{iff} \quad (\mathrm{Ep})\,[T \in p \;\&\; p \Vdash \mathcal{F}]$$

for every sentence $\mathcal{F}$ of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$.

Proof. By induction on the full ordinal rank and logical complexity of $\mathcal{F}$.

Suppose $\mathcal{F}$ is $\mathcal{G}_0 \& \mathcal{G}_1$ and $\mathcal{M}(\omega^{CK}, T) \models \mathcal{F}$. By induction there exist $p_i\,(i < z)$ such that $p_i \Vdash \mathcal{G}_i$ and $T \in p_i$. Note that $p_0 \;\&\; p_1$ is a forcing condition, because $T \in p_0 \cap p_1$. By Proposition 3.4(iii) $p_0 \;\&\; p_1 \Vdash \mathcal{G}_0 \;\&\; \mathcal{G}_1$.

Suppose $\mathcal{F}$ is $(\mathrm{E}X^\alpha)\,\mathcal{H}(X^\alpha)$ and $p \Vdash \mathcal{F}$. Then $p \Vdash \mathcal{H}(\hat{x}\mathcal{G}(x))$ for some $\mathcal{G}(x)$ of rank at most α. The full ordinal rank of $\mathcal{H}(\hat{x}\mathcal{G}(x))$ is less than that of $\mathcal{F}$, so $\mathcal{M}(\omega_1^{CK}, T) \models \mathcal{H}(\hat{x}\mathcal{G}(x))$. Hence $\mathcal{F}$ is true in $\mathcal{M}(\omega_1^{CK}, T)$.

Suppose $\mathcal{M}(\omega_1^{CK}, T) \models \sim\mathcal{G}$. $\mathcal{G}$ is less complex than $\sim\mathcal{G}$, so there is no p such that $T \in p$ and $p \Vdash \mathcal{G}$. But T is generic, so there is a p such that $T \in p$ and $p \Vdash \sim\mathcal{G}$.

Suppose $T \in p$ and $p \Vdash \sim\mathcal{G}$. For the sake of a contradiction suppose $\mathcal{M}(\omega_1^{CK}, T) \models \mathcal{G}$. By induction there is a q such that $T \in q$ and $q \Vdash \mathcal{G}$. Then Lemma 3.4(iii) implies $p \;\&\; q \Vdash \mathcal{G}$ and $p \;\&\; q \Vdash \sim\mathcal{G}$, an impossibility according to 3.4(i). □

3.6 Theorem (Feferman 1965). *If T is generic, then Δ_1^1 comprehension holds in $\mathcal{M}(\omega_1^{CK}, T)$.*

Proof. Suppose $\mathcal{M}(\omega_1^{CK}, T)$ satisfies

$$(x)\,[(\mathrm{E}Y)A(x, Y) \leftrightarrow (Z)B(x, Z)]$$

for some arithmetic A and B. Then

$$(1) \qquad (x)(\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)]$$

holds. Since T is generic, there is a p such that $T \in p$ and $p \Vdash (1)$. The definition of $\Vdash$ implies:

$$p \Vdash \sim(\mathrm{Ex})\sim(\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)],$$

$$(q)_{p \geq q} \sim [q \Vdash (\mathrm{Ex}) \sim (\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)]$$

$$(q)_{p \geq q}\,(n) \sim [q \Vdash \sim(\mathrm{E}Y)\,[A(\underline{n}, Y) \vee \sim B(\underline{n}, Y)]],$$

$$(2) \qquad (q)_{p \geq q}\,(n)(\mathrm{Er})_{q \geq r}[r \Vdash (\mathrm{E}Y)\,[A(\underline{n}, y) \vee \sim B(\underline{n}, Y)]].$$

The matrix of (2) is equivalent to

$$(3) \qquad (\mathrm{Ea})_{a \in O}\,[r \Vdash (\mathrm{E}Y^{|a|})\,[A(\underline{n}, Y^{|a|}) \vee B(\underline{n}, Y^{|a|})]].$$

(3) is Π_1^1 by Lemma 3.2. Kreisel's selection Lemma (2.6.II) yields r and a as hyperarithmetic functions of q and n. By Spector's boundedness theorem (5.6.I)

there is a recursive upper bound γ on $|a(q, n)|$. Thus

$$(n)(q)(\mathrm{Er})_{q \geq r}\,[r \Vdash (\mathrm{E}Y^{\gamma})\,[A(\underline{n}, Y^{\gamma}) \vee \sim B(\underline{n}, Y^{\gamma})]],$$

and so p forces

$$(n)(\mathrm{E}Y^{\gamma})[A(\underline{n}, Y^{\gamma}) \vee \sim B(\underline{n}, Y^{\gamma})]]. \tag{4}$$

Since T is generic and a member of p, $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies (4). But then $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies

$$(\mathrm{E}X)(x)[x \in X \leftrightarrow (\mathrm{E}Y)A(x, Y)].$$

(The "X" is $\hat{x}(\mathrm{E}Y^{\gamma})\,A\,(\underline{n}, Y^{\gamma})$). □

3.7 Theorem. *Assume T is generic and $(\mathrm{E}X)\mathscr{F}(X)$ has no unranked variables save X. If*

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}X)\mathscr{F}(X),$$

then $\mathscr{M}(\omega_1^{\mathrm{CK}}) \models (\mathrm{E}X)\mathscr{F}(X)$.

Proof. Suppose $p \Vdash (\mathrm{E}X)\mathscr{F}(X)$. Then $p \Vdash (\mathscr{F}(\hat{x}\mathscr{G}(x))$ for some $\mathscr{G}(x)$ of rank $\alpha < \omega_1^{\mathrm{CK}}$. The relation $p \Vdash \mathscr{K}$, restricted to sentences $\mathscr{K}$ of rank at most α, is Δ_1^1 by Exercise 3.14. Recall the construction of a generic T following Proposition 3.4. Repeat that construction with the $\mathscr{F}_i$'s replaced by a hyperarithmetic enumeration of all formulas of rank at most α. Then the constructed set, call it H, can be taken to be hyperarithmetic, since it can be defined by recursion on ω relative to a hyperarithmetic predicate. H obeys Lemma 3.5 with respect to all sentences of rank at most α. $H \in p$, since the construction of H can start with p. Hence $\mathscr{M}(\omega_1^{\mathrm{CK}}, H) \models \mathscr{F}(\hat{x}\mathscr{G}(x))$. $\mathscr{M}(\omega_1^{\mathrm{CK}}, H)$ is $\mathscr{M}(\omega_1^{\mathrm{CK}})$ by Lemma 4.16(ii).III. □

3.8 Corollary. *If T is generic, then O is Σ_1^1 definable over $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$.*

Proof. By Lemma 3.5.III,

$$x \in O \leftrightarrow (\mathrm{E}Y)_{Y \in \mathrm{HYP}}\, A(x, Y)$$

for some arithmetic A. Theorem 3.7 implies

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y)A(\underline{n}, Y).$$ □

3.9 Category Versus Measure. There is an analogy between the results of Sections 3 and 2 based on a standard analogy between category and measure. Let A be a subset of 2^{ω}. (Recall the topology assigned to 2^{ω} in subsection 6.1.II.) A is said to be *nowhere dense* if it has an empty interior. A is said to be *meager* if it is contained in a countable union of nowhere dense closed sets. Baire's category theorem states: the

complement of a meager set is dense. Consequently meager sets are thought to be small and analogous to sets of measure 0.

It follows from Proposition 3.4(ii) that the set of T's such that T is generic with respect to a given sentence $\mathscr{F}$ is dense open, and the set of all generic T is co-meager. Thus Theorem 3.6 is analogous to Theorem 1.5, and Exercise 3.11 to Theorem 2.2.

3.10–3.18 Exercises

3.10. An open subset of 2^ω is said to be regular if it equals the interior of its closure. Let A be a Δ^1_1 subset of 2^ω. Find a Δ^1_1 B such that B is regular open and $(A-B) \vee (B-A)$ is meager.

3.11. (Hinman, Thomason). Let A be a non-meager, Π^1_1 subset of 2^ω. Show A has a hyperarithmetic element.

3.12. If T is generic, then $T \notin \mathrm{HYP}$.

3.13. If T is generic, then $O^T \leq_h T, O$. (O^T is the hyperjump of T, defined in Section 7.II.)

3.14. Use 4.1(2).III and the proof of 3.2 to show $p \Vdash \mathscr{F}$, restricted to $\mathscr{F}$ of rank at most α, is Δ^1_1 when $\alpha < \omega_1^{\mathrm{CK}}$.

3.15. Assume T is generic. Let $T_i (i<2)$ be $\{n | 2n+i \in T\}$. Show T_i is generic. Show $T_0 \not\leq_h T_1$ and $T_1 \not\leq_h T_0$.

3.16. (H. Friedman). Find a T which is generic with respect to all ranked sentences (of $\mathscr{L}(\omega_1^{\mathrm{CK}}, T)$), but which is not generic.

3.17. (S. Feferman). Cohen forcing for $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \ldots)$ is defined by taking as forcing conditions consistent finite conjunctions of formulas of the form $m \in \mathscr{T}_i$, $n \notin \mathscr{T}_j$ $(i, j < \omega)$. Show $\mathscr{M}(T_0, T_1, T_2, \ldots)$ satisfies Σ^1_1 dependent choice when $\langle T_0, T_1, T_2, \ldots \rangle$ is generic (cf. Exercise 2.6).

3.18. Show there exists a generic $T \leq_h O$.

3.19. Find a hyperarithmetic set that is not an arithmetic singleton.

4. *Perfect Forcing*

Let $P, Q, R, \ldots$ denote perfect subsets of 2^ω as in subsection 6.1.III. In this section Cohen's forcing method is extended from finite conditions to perfect ones. The resulting generic reals differ vastly from those of Section 3. In particular a real generic in the sense of the present section has minimal hyperdegree. According to Exercise 3.15, a real generic in the sense of subsection 3.3 lies above two hyperarithmetically incomparable reals.

Let $p, q, r, \ldots$ be sequence numbers that encode finite initial segments (of characteristic functions) of subsets of ω. As in subsection 6.1.III, a perfect subset of 2^ω can be encoded by the set of codes for the initial segments of its members. The natural homeomorphism between 2^ω and a perfect set P gives rise to a standard encoding of P denoted by $\lambda_{ij}|p_{ij}\,(i<\omega\ \&\ j<2^i)$. $p_{i+1,2j}$ and $p_{i+1,2j+1}$ are incomparable extensions of $p_{i,j}$.

$$P=\hat{X}((i)\,(\mathrm{Ej})_{j<2^i}\,(X\in p_{ij})).$$

P is *hyperarithmetically encodable* if $\lambda_{ij}|p_{ij}$ is a hyperarithmetic function. From now on P ambiguously denotes a hyperarithmetically encodable perfect set and its standard code. Thus the set of all P's is Π^1_1.

4.1 The Perfect Forcing Relation. Let $\mathscr{F}$ be a sentence of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ as defined in subsection 4.1.III. The perfect forcing relation, $P \Vdash \mathscr{F}$, is defined by six closure conditions.

(1) If $\mathscr{F}$ is ranked and $(X)\,[X\in P\to \mathscr{M}(\omega_1^{\mathrm{CK}}, X)\vDash \mathscr{F}]$, then $P \Vdash \mathscr{F}$.

(2) If $\mathscr{F}(Y^\alpha)$ is unranked and $P \Vdash \mathscr{F}(\hat{x}\mathscr{G}(x))$ for some $\mathscr{G}(x)$ of rank at most α, then $P \Vdash (\mathrm{EY}^\alpha)\mathscr{H}(Y^\alpha)$.

(3) If $P \Vdash (\mathrm{EY}^\alpha)\mathscr{F}(Y^\alpha)$ for some $\alpha<\omega_1^{\mathrm{CK}}$, then $P \Vdash (\mathrm{EY})\mathscr{F}(Y)$.

(4) If $\mathscr{F}(x)$ is unranked and $P \Vdash \mathscr{F}(\underline{n})$ for some n, then $P \Vdash (\mathrm{Ex})\mathscr{F}(x)$.

(5) If $\mathscr{F}\ \&\ \mathscr{G}$ is unranked, $P \Vdash \mathscr{F}$ and $P \Vdash \mathscr{G}$, then $P \Vdash \mathscr{F}\ \&\ \mathscr{G}$.

(6) If $\mathscr{F}$ is unranked and $(Q)_{P\supseteq Q}\sim[Q \Vdash \mathscr{F}]$, then $P \Vdash \sim\mathscr{F}$.

Clause (1) may seem unorthodox in that it treats all ranked sentences as if they were at the ground level. (1) makes it easy to show that the perfect forcing relation, restricted to ranked sentences, is Π^1_1, but obscures the fact that

$$(\mathscr{F})(P)\,(\mathrm{EQ})_{P\supseteq Q}\,[Q \Vdash \mathscr{F} \text{ or } Q \Vdash \sim\mathscr{F}].$$

4.2 Lemma. *The relation* $P \Vdash \mathscr{F}$, restricted to Σ^1_1 $\mathscr{F}$'s, is Π^1_1.

Proof. Clause (3) of the definition of H makes it safe to assume $\mathscr{F}$ is ranked. Hence

(1) $$P \Vdash \mathscr{F} \leftrightarrow (X)\,[X\in P\to \mathscr{M}(\omega_1^{\mathrm{CK}}, X)\vDash \mathscr{F}]$$

The right side of (1) is Π^1_1 by Lemma 4.5.II. □

The next lemma establishes the so-called fusion (or splitting) property of perfect forcing in the hyperarithmetic case.

4.3 Lemma. *Let* $\mathscr{F}_i (i<\omega)$ *be a hyperarithmetic sequence of* Σ^1_1 *sentences. Suppose*

$$(i)\ (Q)_{P\supseteq Q}(\mathrm{ER})_{Q\supseteq R}[R \Vdash \mathscr{F}_i].$$

Then $(\mathrm{E}Q)_{P\supseteq Q}(i)[Q \Vdash \mathscr{F}_i]$.

Proof. By lemma 4.2 the predicate

$$Q \supseteq R \ \&\ R \Vdash \mathscr{F}_i$$

is Π^1_1. Kreisel's uniformization theorem (2.6.II) implies R can be construed as a partial Π^1_1 function of Q and i. $R(Q, i)$ is iterated below to produce $\lambda ij/Q_{i,j}$, a hyperarithmetic function as in Exercise 4.16.

A hyperarithmetic family $\{Q_{i,j} | i < \omega \ \&\ j < 2^i\}$ is defined by recursion on i with the aid of $\lambda iQ|R(Q, i)$ and effective splitting.

$$Q_{0,0} = P.$$

(1) $$Q_{i+1,2j} \subseteq Q_{i,j} \quad \text{and} \quad Q_{i+1,2j+1} \subseteq Q_{i,j}.$$

(2) $$Q_{i+1,2j} \cap Q_{i+1,2j+1} = \varnothing.$$
$$Q_{i+1,2j} \Vdash \mathscr{F}_i \quad \text{and} \quad Q_{i+1,2j+1} \Vdash \mathscr{F}_i.$$

Let Q be $\bigcap_i \bigcup_{j<2^i} Q_{i,j}$. Then $Q \in \mathrm{HYP}$ and $Q \subseteq P$. Q is perfect because of the splitting in (1) and (2). Note that if $X \in Q$, then for each i, there is a unique j such that $X \in Q_{i,j}$.

Fix i to check $Q \Vdash \mathscr{F}_i$. Let $\mathscr{F}_i$ be $(\mathrm{EX})\mathscr{G}_i(x)$. Q is contained in $\cup\{Q_{i,j} | j < 2^i\}$, and for each $j < 2^i$,

$$Q_{i+1,j} \Vdash (\mathrm{EX}^{\alpha_j})\mathscr{G}_i(X^{\alpha_j})$$

for some $\alpha_j < \omega_1^{\mathrm{CK}}$. Let α be $\sup_{j<2^i} \alpha_j$. Then $Q \Vdash (\mathrm{EX}^{\alpha})_i(X^{\alpha})$. □

Lemma 4.3 is the central fact of perfect forcing over the hyperarithmetic hierarchy. Its counterpart in the set theoretic case (Sacks 1967) is applied to show aleph-one is preserved in generic extensions of L.

4.4 Lemma. $(\mathscr{F})(P)(\mathrm{E}Q)_{P\supseteq Q}[Q \Vdash \mathscr{F}$ or $Q \Vdash \sim \mathscr{F}]$.

Proof. Suppose $\mathscr{F}$ is unranked. Then the lemma follows from clause (6) of the definition of H.

Suppose $\mathscr{F}$ is ranked. Proceed by induction on the full ordinal rank and logical complexity of $\mathscr{F}$ as defined in subsection 4.4.III. For example, suppose $\mathscr{F}$ is $(\mathrm{EX}^{\alpha})\mathscr{H}(X^{\alpha})$. Let $\mathscr{G}_i(x)$ $(i < \omega)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x. First assume

(1) $$(\mathrm{Ei})(\mathrm{E}Q)_{P\supseteq Q}[Q \Vdash \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

Then $Q \Vdash \mathscr{F}$. Now assume (1) fails. By induction

$$(i)(Q)_{P \supseteq Q}(\mathrm{E}R)_{Q \supseteq R}(R \Vdash \sim \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

Lemma 4.3 yields

$$(\mathrm{E}Q)_{P \supseteq Q}(i)[Q \Vdash \sim \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

But then $Q \Vdash \sim \mathscr{F}$. □

Lemma 4.4 has roughly the same content as: each Δ^1_1 set or its complement contains a perfect Δ^1_1 set. Note that an uncountable Δ^1_1 set need not contain a perfect Δ^1_1 set.

4.5 Perfect Genericity. Suppose $T \subseteq \omega$. T is said to be *generic* (in the sense of perfect forcing) if for each sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$, there is a P such that $T \in P$ and either $P \Vdash \mathscr{F}$ or $P \Vdash \sim \mathscr{F}$. It follows from Lemma 4.4 that generic T's exist inside every P. Let $\mathscr{F}_i$ $(i < \omega)$ be an enumeration of all sentences of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$. Define $P_i (i < \omega)$ so that $P_0 = P$, $P_i \supseteq P_{i+1}$, and

$$P_{i+1} \Vdash \mathscr{F}_i \quad \text{or} \quad P_{i+1} \Vdash \sim \mathscr{F}_i.$$

Then $\cap\{P_i | i < \omega\} = \{T\}$ and T is generic.

The next proposition is a technicality needed for the proof of the truth lemma (4.8).

4.6. Proposition. *Suppose T is generic with respect to all ranked sentences, and $T \in P \cap Q$. Then there exists an R such that $T \in R \subseteq P \cap Q$.*

Proof. The statement $T \in P$ is equivalent to

$$(i)(\mathrm{E}j)[j < 2^i \ \ \& \ \ T \in p_{ij}],$$

where $\lambda_{ij} | p_{ij}$ is the standard hyperarithmetic code for P. It follows from Lemma 4.7.III. that there exists a ranked sentence $\mathscr{G}$ such that

$$(T)[T \in P \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G}].$$

Let $\mathscr{H}$ be a ranked sentence that bears the same relation to Q that $\mathscr{G}$ does to P.

By supposition $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G} \ \& \ \mathscr{H}$. Since T is generic, there is an R such that $T \in R$, and

$$R \Vdash \mathscr{G} \ \& \ \mathscr{H} \quad \text{or} \quad R \Vdash \sim(\mathscr{G} \ \& \ \mathscr{H}).$$

If $R \Vdash \sim(\mathscr{G} \ \& \ \mathscr{H})$, then clause (1) of the definition of $\Vdash$ implies $T \notin P \cap Q$. Hence $R \Vdash \mathscr{G} \ \& \ \mathscr{H}$. And so $R \subseteq P \cap Q$. □

4.7 Proposition. *If* $P \Vdash \mathscr{F}$ *and* $P \supseteq Q$, *then* $Q \Vdash \mathscr{F}$.

Proof. Clause (1) of the definition of H disposes of the matter if $\mathscr{F}$ is ranked. For unranked $\mathscr{F}$ the proof proceeds by induction on the full ordinal rank and logical complexity of $\mathscr{F}$. □

4.8 Lemma. *Suppose* T *is generic. Then*

$$\mathscr{M}(\omega_1^{CK}, T) \vDash \mathscr{F} \quad \textit{iff} \quad (\mathrm{E}P)_{T \in P}[P \Vdash \mathscr{F}]$$

for every sentence $\mathscr{F}$ *of* $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$.

Proof. Clause (1) of the definition of $\Vdash$ takes care of ranked $\mathscr{F}$'s. For unranked $\mathscr{F}$'s the proof is by induction on logical complexity as in Lemma 3.5. The only difficulty occurs when $\mathscr{F}$ is $\mathscr{G}$ & $\mathscr{H}$. Suppose

$$\mathscr{M}(\omega_1^{CK}, T) \vDash \mathscr{G} \ \& \ \mathscr{H}.$$

By induction there exist P and Q such that $T \in P \cap Q$, $P \Vdash \mathscr{G}$ and $Q \Vdash \mathscr{H}$. Lemma 4.6 supplies an $R \subseteq P \cap Q$ such that $T \in R$. By Proposition 4.7 $R \Vdash \mathscr{G}$ & $\mathscr{H}$. □

4.9–4.15 Exercises

4.9. Let A be a Δ_1^1 subset of 2^ω. Find a perfect Δ_1^1 set P such that $P \subseteq A$ or $P \subseteq 2^\omega - A$.

4.10. Show there exists a generic $T \leq_h O$.

4.11. Assume $\mathscr{F}$ is Σ_1^1. Show

$$P \Vdash \mathscr{F} \quad \text{iff} \quad (X)[X \in P \to \mathscr{M}(\omega_1^{CK}, X) \vDash \mathscr{F}].$$

4.12. If T is generic (in the sense of perfect forcing) and $S \equiv_h T$, then S is generic. (Refute this assertion for Cohen genericity.)

4.13. If T is generic, then $T \notin \mathrm{HYP}$.

4.14. If T is generic, then Kleene's O is Σ_1^1 definable over $\mathscr{M}(\omega_1^{CK}, T)$. (First show an arithmetic predicate with a solution in $\mathscr{M}(\omega_1^{CK}, T)$ has a hyperarithmetic solution.)

4.15. Define $\Vdash_1$ by using the perfect conditions of subsection 4.1 and the recursion of subsection 3.1. For example $P \Vdash_1 \sim \mathscr{F}$ iff $(Q)_{P \supseteq Q} \sim Q \Vdash_1 \mathscr{F}$ whether or not $\mathscr{F}$ is ranked. Call T generic in the sense of $\Vdash_1$ if for each sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$, there is a P such that $T \in P$ and either $P \Vdash_1 \mathscr{F}$ or $P \Vdash_1 \sim \mathscr{F}$. Show T is generic in the sense of $\Vdash_1$ iff T is generic as defined in subsection 4.4.

4.16. Let $g(0) = 0$, and $g(n+1) \simeq f(g(n))$ for all n, where f is partial Π_1^1. If g is total, then g is Δ_1^1 by 1.7.I.

5. *Minimal Hyperdegrees*

T is said to be of minimal hyperdegree of $T \notin \mathrm{HYP}$ and $(S)[S \leq_h T \to S \in \mathrm{HYP} \vee T \leq_h S]$. The existence of minimal Turing degrees was proved by Spector 1956. The hyperdegree case combines Spector's ideas with perfect forcing over the hyperarithmetic hierarchy. The principal result of this section is that every perfectly generic T is of minimal hyperdegree.

5.1. $\mathscr{M}$-Definability. Suppose $I \subseteq \omega$. I is said to be *$\mathscr{M}$-definable* if there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$, in which $\mathscr{T}$ does not occur, such that

$$(n)[n \in I \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}) \models \mathscr{F}(\underline{n})].$$

I is said to be *generically definable* if there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ such that

$$(n)[n \in I \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, \mathscr{T}) \models \mathscr{F}(\underline{n})]$$

for all generic T (generic in the sense of perfect forcing).

5.2 Lemma. *The following are equivalent*
(*i*) *I is arithmetic in O.*
(*ii*) *I is $\mathscr{M}$-definable.*
(*iii*) *I is generically definable.*

Proof. By Theorems 3.5.III and 4.8.III, there is an arithmetic $A(x, Y)$ such that for all n,

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}) \models (\mathrm{E}\,Y) A(\underline{n}, Y).$$

It follows from the parenthetical remark in Exercise 4.14 that

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y) A(\underline{n}, Y).$$

for all generic T. Thus O is $\mathscr{M}$-definable and generically definable. Consequently every set arithmetic in O is.

Suppose I is $\mathscr{M}$-definable. Then Lemma 4.5.III implies I is arithmetic in O.

Finally suppose I is generically definable via the formula $\mathscr{F}(x)$. It suffices, by Lemma 4.2, to see

$$n \in I \leftrightarrow (P)(\mathrm{E}Q)_{P \supseteq Q}[Q \Vdash \mathscr{F}(\underline{n})]$$

in order to conclude I is arithmetic in O. First let $n \in I$, T be generic, and $T \in P$. Then

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}(\underline{n}).$$

By Lemma 4.8 there is an R such that $T \in R$ and $R \Vdash \mathscr{F}(\underline{n})$. Proposition 4.6 provides a $Q \subseteq P \cap R$ such that $T \in Q$. But then $Q \Vdash \mathscr{F}(\underline{n})$ by Proposition 4.7.

Now let $n \notin I$. Choose a generic T and a P such that

$$\mathcal{M}(\omega_1^{CK}, T) \models \sim \mathcal{F}(\underline{n}),$$

$T \in P$, and $P \Vdash \sim \mathcal{F}(\underline{n})$. Then

$$\sim(\mathrm{E}Q)_{P \supseteq Q}[Q \Vdash \mathcal{F}(\underline{n})]. \quad \square$$

Let K be a set of P's and Q's, that is a set of hyperarithmetically encodable perfect conditions. K^i is the set of indices of members of K. $2^b \cdot 3^e$ is an index for P if $b \in O$, $\{e\}^{H_b}$ is total, and $\{e\}^{H_b}$ $(= \lambda ij | p_{ij})$ is the standard code for P.

K is *dense* if $(P)(\mathrm{E}Q)[P \supseteq Q \in K]$.

5.3 Theorem. *T is generic iff $T \in \cup\{P | P \in K\}$ for every dense set K arithmetic in O.*

Proof. Suppose T satisfies the density hypothesis. Fix $\mathcal{F}$. The set

$$\{P | P \Vdash \mathcal{F} \quad \text{or} \quad P \Vdash \sim \mathcal{F}\}$$

is dense by Lemma 4.4, and arithmetic in O by Lemma 4.2. Hence T is generic with respect to $\mathcal{F}$.

Now suppose K is dense and arithmetic in O. By Lemma 5.2 there is an $\mathcal{F}(x)$ such that

$$n \in K^i \leftrightarrow \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{F}(\underline{n})$$

for all generic T. Let H be the arithmetic formula of Theorem 4.2.II. Define $\mathcal{G}$ by

$$(\mathrm{Ex})(\mathrm{E}Y)[\mathcal{F}(x) \ \& \ H((x)_0, Y)$$
$$\& \ (i)(\mathrm{Ej})_{j<2^i}(\mathrm{Ez})(\{(x_1)\}^Y(i,j) = z = (\ell k(z)))].$$

$\mathcal{G}$ says: $(\mathrm{E}P)[P \in K \ \& \ T \in P]$. ($z$ is a sequence number that encodes an initial segment of the characteristic function of $\mathcal{T}$.) In fact

$$T \in \cup\{P | P \in K\} \leftrightarrow \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{G} \tag{1}$$

holds for all generic T. It need only be shown that the right side of (1) holds for all generic T. If not, there is a P and a generic $T \in P$ such that $P \Vdash \sim \mathcal{G}$. Since K is dense, $P \supseteq Q$ for some $Q \in K$. Choose a generic $T^* \in Q$. It follows from (1) that

$$\mathcal{M}(\omega_1^{CK}, T^*) \models \mathcal{G},$$

and so $T^* \in R$ for some R that forces $\mathcal{G}$. By Proposition 4.6 there is an $S \subseteq Q \cap R$ such that $T^* \in S$. $S \Vdash \mathcal{G}$ since $R \Vdash \mathcal{G}$ and $S \Vdash \sim \mathcal{G}$ since $S \subseteq P$. $\square$

5.4 Theorem. *If T is generic, then $\mathcal{M}(\omega_1^{CK}, T)$ satisfies Δ_1^1 comprehension.*

Proof. Similar to the proof of Theorem 3.6. It is enough to show: if $\mathcal{M}(\omega_1^{CK}, T) \models (x)(EY)A(x, Y)$ for some arithmetic A, then $\mathcal{M}(\omega_1^{CK}, T) \models (x)(EY^{\alpha})A(x, Y)$ for some $\alpha < \omega_1^{CK}$. Let K be the set of all P such that

$$P \Vdash \sim(x)(EY)A(x, Y)$$
$$\text{or } (Eb)_{b\in O}[P \Vdash (x)(EY^{|b|})A(x, Y^{|b|})].$$

By Lemma 4.2 K is arithmetic in O. To see K is dense, fix P. If P contracts to some Q that forces $\sim(x)(EY)A(x, Y)$, then all is well. If not, then P contracts to some Q that forces $(x)(EY)A(x, Y)$. It follows, as in the derivation of (2) from (1) in the proof of Theorem 3.6, that

$$(n)(R)_{Q\supseteq R}(ES)_{R\supseteq S}[S \Vdash (EY)A(\underline{n}, Y)].$$

The fusion lemma (4.3) yields $S \subseteq Q$ such that

$$(n)[S \Vdash (EY)A(\underline{n}, Y)],$$

or in more detail,

$$(n)(Ec)_{c\in O}[S \Vdash (EY^{|c|})A(\underline{n}, Y^{|c|})].$$

Kreisel's selection Lemma (2.5.II) implies c can be constructed as a hyperarithmetic function of n. Then Spector's boundedness theorem (5.6.I) provides a recursive bound α on c: $|c(n)| < \alpha$ for all n. Thus $P \supseteq S \in K$.

By Theorem 5.3 T belongs to some member of K. □

Let $\mathcal{G}(x)$ be a ranked formula of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$ whose sole free variable is x. For each $T \subseteq \omega$, define

$$\mathcal{G}^T \quad \text{by} \quad \{n | \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{G}(\underline{n})\}.$$

Thus for each T the $\mathcal{G}^T$'s are the members of $\mathcal{M}(\omega_1^{CK}, T)$.

The essential content of the next lemma is: let $f: 2^{\omega} \to 2^{\omega}$ be Δ_1^1; then there exists a perfect Δ_1^1 $A \subseteq 2^{\omega}$ such that f, restricted to A, is either constant or 1-1.

5.5 Lemma. *For each* P and $\mathcal{G}(x)$ *there are* $Q \subseteq P$ *and* $H \in \mathrm{HYP}$ *such that either* (*i*) *or* (*ii*) *holds.*

(*i*) $(T)[T \in Q \to \mathcal{G}^T = H]$.
(*ii*) $(T)[T \in Q \to T$ *recursive in* $\mathcal{G}^T, H]$.

Proof. Case I: there is an $R \subseteq P$ such that

$$(Q_0)_{Q_0\subseteq R}(Q_1)_{Q_1\subseteq R}(n) \sim [Q_0 \Vdash \mathcal{G}(\underline{n}) \quad \text{and} \quad Q_1 \Vdash \sim \mathcal{G}(\underline{n})].$$

Define H by

$$n \in H \leftrightarrow (EQ)_{Q\subseteq R}[Q \Vdash \mathcal{G}(\underline{n})].$$

Observe that $n \notin H \leftrightarrow (\mathrm{E}Q)_{Q \subseteq R}[Q \Vdash \sim \mathscr{G}(\underline{n})]$. It follows from Lemma 4.2 that $H \in \mathrm{HYP}$. The fusion Lemma (4.3) supplies a $Q \subseteq R$ such that for all n,

$$n \in H \to Q \Vdash \mathscr{G}(\underline{n})$$
$$\text{and} \quad n \notin H \to Q \Vdash \sim \mathscr{G}(\underline{n}).$$

Then $\mathscr{G}^T = H$ for all $T \in Q$.

Case II: Case I fails. Hence

$$(R)_{P \supseteq R}(\mathrm{E}Q_0)_{Q_0 \subseteq R}(\mathrm{E}Q_1)_{Q_1 \subseteq R}(\mathrm{En})[Q_0 \Vdash \mathscr{G}(\underline{n}) \quad \text{and} \quad Q_1 \Vdash \sim \mathscr{G}(\underline{n})].$$

The present situation is similar to that found in the proof of Lemma 4.3. Q_0, Q_1 and n can be construed as partial Π^1_1 functions of R. They are applied in a recursion of length ω to obtain hyperarithmetic functions $\lambda ij | Q_{ij}$ and $\lambda ij | f(i,j)$ such that:

$$Q_{00} = P, \quad Q_{i+1,2j} \cap Q_{i+1,2j+1} = \varnothing,$$
$$Q_{i+1,2j} \subseteq Q_{ij} \quad \text{and} \quad Q_{i+1,2j+1} \subseteq Q_{ij},$$
$$Q_{i+1,2j} \Vdash \mathscr{G}(\underline{f(i,j)}), \quad \text{and}$$
$$Q_{i+1,2j+1} \Vdash \sim \mathscr{G}(\underline{f(i,j)}).$$

Let Q be $\bigcap_i \bigcup_{j<2^i} Q_{ij}$. Q is hyperarithmetic and perfect.

Fix $T \in Q$ to see why T is recursive in $\mathscr{G}^T$, H, where H is $\langle Q, f \rangle$. The idea is: $\mathscr{G}^T$ is 1-1 on Q, so T can be recovered from $\mathscr{G}^T$ with the help of Q and f. Let $t(i)$ be the unique j such that $T \in Q_{i,j}$. The construction of Q implies

$$t(i+1) = 2t(i) \quad \text{if} \quad f(i, t(i)) \in \mathscr{G}^T$$
$$2t(i)+1 \quad \text{if} \quad f(i, t(i)) \notin \mathscr{G}^T.$$

Thus t, hence T, is recursive in $\mathscr{G}^T$, Q, f. □

5.6 Theorem (Gandy & Sacks 1967). *If T is generic in the sense of perfect hyperarithmetic forcing, then T is of minimal hyperdegree.*

Proof. By Exercise 4.13 $T \notin \mathrm{HYP}$. Suppose $X \leq_h T$. By Lemma 5.4 $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies Δ^1_1 comprehension, and so by Lemma 4.16.III, $X \in \mathscr{M}(\omega_1^{\mathrm{CK}}, T)$. Thus $X = \mathscr{G}^T$ for some ranked $\mathscr{G}(x)$. Let K be the set of all Q that satisfy (1) or (2) of Lemma 5.5 for some hyperarithmetic H. Then K is dense, and K is arithmetic in O; in fact K is Π^1_1. Theorem 5.3 implies $T \in Q$ for some $Q \in K$. Hence

$$X \in \mathrm{HYP} \quad \text{or} \quad T \leq_h X. \quad \square$$

5.7–5.9 Exercises

5.7. Suppose $f: 2^\omega \to 2^\omega$ is Δ^1_1. Find a perfect Δ^1_1 set A such that the restriction of f to A is either constant or 1-1.

5.8 (Local Cohen Forcing). Let Q be a hyperarithmetically encodable, perfect subset of 2^ω. The Cohen forcing method of Section 3 is localized to Q as follows. In essence Q replaces 2^ω. $p \Vdash_Q \mathscr{F}$ is defined as in subsection 3.1 save that $p, q, r, \ldots$ now encode finite initial segments of members of Q. Generic is defined as in subsection 3.3, and it follows that all generic T's belong to Q. Recall the trick of the proof of Theorem 3.7. Use local Cohen forcing to give proofs of Lemmas 4.4 and 5.5.

5.9. Try exercises 4.12 and 4.15 again.

6. *Louveau Separation*

The Kleene separation theorem for subsets of ω states: if $A, B \subseteq \omega$ are Σ^1_1 and disjoint, then there exists a Δ^1_1 $C \subseteq \omega$ such that $A \subseteq C$ and $C \cap B = \varnothing$. It was obtained as a corollary to a reduction theorem (3.7.II) for Π^1_1 subsets of ω. Kleene separation for subsets of ω^ω (Exercise 5.11.II) is obtained similarly by relativizing Theorem 3.7.II in the manner of Section 5.II.

Let A and B be disjoint subsets of ω^ω. As in subsection 5.4.II, there are n_A and n_B such that

$$(f)[f \in A \leftrightarrow n_A \in O^f], \quad \text{and}$$
$$(f)[f \in B \leftrightarrow n_B \in O^f].$$

Let $f \in A_0 \leftrightarrow f \in A \ \& \ |n_A|_{O^f} \leq |n_B|_{O^f}$, and

$$f \in B_0 \leftrightarrow f \in B \ \& \ |n_B|_{O^f} < n_A|_{O^f},$$

Lemma 2.1.II, relativized to f, implies A_0 and B_0 are Π^1_1. By construction $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$.

As in subsection 3.6.II, reduction for Π^1_1 implies separation for Σ^1_1. Kleene separation for subsets of ω^ω is: if $A, B \subseteq \omega^\omega$ are Σ^1_1 and disjoint, then there exists a Δ^1_1 C such that $A \subseteq C$ and $B \cap C = \varnothing$. Louveau separation is concerned with the complexity of the separating set C. In order to make the notion of complexity precise, a hierarchy of subsets of ω^ω is defined. Let $D \subseteq \omega^\omega$. D is *subbasic* if D is of the form $\{f \mid f(m) = n\}$. D is *clopen* if D is a finite Boolean combination of subbasic sets.

$D \in \mathbf{\Pi}^o_0$ if D is clopen.

$D \in \mathbf{\Pi}^o_\gamma$ if D is a countable intersection of complements of sets in $\mathbf{\Pi}^o_{<\gamma} = \cup\{\mathbf{\Pi}^o_\delta \mid \delta < \gamma\}$.

D is *open* if it is a union of clopen sets. D is *Borel* if it belongs to the least σ-algebra containing the open sets (cf. subsection 6.1.II). According to Exercise 6.10, D is Borel iff D belongs to $\mathbf{\Pi}^0_{\omega_1}$. If $D \in \mathbf{\Pi}^0_\gamma - \mathbf{\Pi}^0_{<\gamma}$, then D is said to have rank γ. One measure of the complexity of D is its rank. Another is the way D is put together from sets of rank less than D. The "putting together" is described by a code for D. The code for a clopen set is obtained from a fixed, effective one-one correspondence between $\mathbf{\Pi}^0_0$ and $\{5^{n+1} | n < \omega\}$. Each clopen set has a code of the form $\{5^{n+1}\}$. Suppose

$$D = \cap\{\omega^\omega - D_m | m < \omega\},$$
$$D \in \mathbf{\Pi}^0_\gamma, \quad D_m \in \mathbf{\Pi}^0_{\gamma_n} \quad (\gamma_n < \gamma),$$

and c_m is a code for $D_m (m < \omega)$. Then $\{2^m \cdot 3^n | n \in c_m\}$ is a code for D. According to Exercises 6.7 and 6.8: if D has a Δ^1_1 code, then $D \in \mathbf{\Pi}^0_{\omega_1^{CK}}$; D is Δ^1_1 iff D has a Δ^1_1 code. D is said to be a $\mathbf{\Pi}^0_\gamma(\Delta^1_1)$ set if $D \in \mathbf{\Pi}^0_\gamma$ and D has a Δ^1_1 code.

6.1 Theorem (Louveau 1980). *If $A, B \subseteq \omega^\omega$ are Σ^1_1, and A is separable from B by a $\mathbf{\Pi}^0_\gamma$ set, then A is separable from B by a $\mathbf{\Pi}^0_\gamma(\Delta^1_1)$ set.*

6.2 Corollary (Louveau 1980). *If $A \subseteq \omega^\omega$ is Δ^1_1 and $\mathbf{\Pi}^0_\gamma$, then A is $\mathbf{\Pi}^0_\gamma(\Delta^1_1)$.*

Corollary 6.2 is remarkable. It implies, for example, if A is Δ^1_1 and an intersection of open sets, then A is the intersection of a Δ^1_1 sequence of Δ^1_1 open sets. The proof of Theorem 6.1 is a combination of bounding arguments and forcing. The role of forcing is to approximate certain sets by unions of forcing conditions in the manner discussed at the end of subsection 3.1. The conditions are nonempty Σ^1_1 subsets of ω^ω. Σ^1_1 forcing was invented by Gandy circa 1964. One difficulty of Σ^1_1 forcing is that Σ^1_1 conditions are not closed. A contracting sequence of Σ^1_1 conditions can shrink down to nothing. Consequently it is necessary to pay close attention to the existential witnesses that establish membership in Σ^1_1 sets.

A set is open in the *Gandy topology* for ω^ω if it is a union of Σ^1_1 sets. The next result says that the Baire category theorem holds for the Gandy topology. It is needed for the "approximation" aspect of the proof of Louveau's separation theorem.

6.3 Theorem. *Let ω^ω have the Gandy topology. If O_i $(1 \leq i < \omega)$ is a sequence of dense open sets, then $\bigcap_i O_i$ is dense.*

Proof. Let P, P_1, P_2, etc. denote nonempty Σ^1_1 subsets of ω^ω. Fix P with the intent of finding an $f \in P \cap \bigcap_i O_i$. Recall Seq and $>$ from subsection 5.1.I. The construction below defines P_i, f and g_{ij} $(1 \leq i, j < \omega)$ such that:

(a) $P \supseteq P_i \supseteq P_{i+1}$ and $P_i \subseteq O_i$.
(b) $f_i \in \mathrm{Seq}$, $\ell h(f_i) = i$, and $f_i > f_{i+1}$.

(c) $g_{ij} \in \mathrm{Seq}$, $\ell h(g_{ij}) = j$, and $g_{ij} > g_{i,j+1}$.

(d) Let $\bar{f}(i) = f_i$ and $\bar{g}_i(j) = g_{ij}$. g_i is an existential witness to the membership of f in P_i.

It is immediate from (a)–(d) that $f \in P \cap \bigcap_i P_i$.

Suppose Q is $\hat{f}(\mathrm{E}g)(x)R(\bar{f}(x), \bar{g}(x))$ from some recursive R. $Q(s, t)$, a notation needed for the construction, is defined by:

$$\hat{f}[\bar{f}(\ell h(s)) = s \ \&\ (\mathrm{E}g)(\bar{g}(\ell h(t)) = t \ \&\ (x)R(\bar{f}(x), \bar{g}(x)))].$$

The construction proceeds as follows.

(1) Choose $P_1 \subseteq O_1 \cap P$. P_1 exists by density of O_1.
(2) Choose f_1 and g_{11} so that $P_1(f_1, g_{11}) \neq \varnothing$.
(3) Choose $P_2 \subseteq P_1(f_1, g_{11}) \cap O_2$.
(4) Choose f_2 and g_{21} so that $f_1 > f_2$ and $P_2(f_2, g_{21}) \neq \varnothing$.
(5a) Note that $P_1(f_2, g_{11}) \cap P_2(f_2, g_{21}) \neq \varnothing$, because $P_1(f_1, g_{11}) \supseteq P_2(f_2, g_{21})$.
(5b) $P_1(f_2, g_{11}) = \cup\{P_1(f_2, x) | g_{11} > x\}$.
(6) It follows from (5a)–(5b) that g_{12} can be chosen so that $P_1(f_2, g_{12}) \cap P_2(f_2, g_{21}) \neq \varnothing$.
(7) Choose $P_3 \subseteq P_1(f_2, g_{12}) \cap P_2(f_2, g_{21}) \cap O_3$.

Steps (3)–(6) define stage 2 of the construction.

Stage i is similar and begins with the choice of P_i inside

$$P_1(f_{i-1}, g_{1,i-1}) \cap P_2(f_{i-1}, g_{2,i-2}) \cap \ldots \cap P_{i-1}(f_{i-1}, g_{i-1,1}) \cap O_i.$$

Then f_1 and g_{i1} are chosen so that $f_{i-1} > f_i$ and $P_i(f_i, g_{i1}) \neq \varnothing$. Next, as in step (6), g_{1i} is chosen so that

$$P_1(f_i, g_{1i}) \cap P_2(f_i, g_{2,i-2}) \cap \ldots \cap P_{i-1}(f_i, g_{i-1,1}) \cap P_i(f_i, g_{i1}) \neq \varnothing.$$

Then in the same manner $g_{2,i-1}, g_{3,i-2}, \ldots, g_{i-1,2}$ are chosen in succession. □

Louveau separation (Theorem 6.1) is proved by induction on γ, referred to below as the *main induction*. It is safe to assume $\gamma < \omega_1^{\mathrm{CK}}$ by Exercise 5.11.II and Exercise 6.7 below. The proof of 6.1 is broken into three lemmas (6.4–6.6), each of which makes use of the main induction.

Suppose $H \subseteq \omega^\omega$. The γ-closure of H, denoted by $\bar{H}^\gamma$, is defined by:

$$x \notin \bar{H}^\gamma \leftrightarrow (\mathrm{E}Y)[x \in Y \in \mathbf{\Pi}^0_{<\gamma} \cap \mathbf{\Sigma}^1_1 \ \&\ Y \cap H = \varnothing].$$

Note that γ-closure is the usual notion of closure for subsets of ω^ω with respect to the topology generated by $\mathbf{\Pi}^0_{<\gamma} \cap \mathbf{\Sigma}^1_1$.

6.4 Lemma. *If $A, B \subseteq \omega^\omega$ are $\mathbf{\Sigma}^1_1$, and A is separable from B by $\bar{A}^\gamma$, then A is separable from B by some $\mathbf{\Pi}^0_\gamma(\Delta^1_1)$ set.*

Proof. The hypothesis of the lemma yields

$$(1) \qquad (X)_{X\in B}(\mathrm{E}Y)[X\in Y\in \mathbf{\Pi}^o_{<\gamma}\cap\Sigma^1_1 \ \ \&\ \ Y\cap A=\varnothing].$$

The Y of (1) is a $\mathbf{\Pi}^o_{<\gamma}$ set that separates the Σ^1_1 sets Y and A. The main induction implies that Y and A can be separated by a $\mathbf{\Pi}^o_{<\gamma}(\Delta^1_1)$ set. Thus the matrix of (1) can be replaced by

$$(2) \qquad X\in Y\in \mathbf{\Pi}^o_{<\gamma}(\Delta^1_1) \quad \&\quad Y\cap A=\varnothing.$$

Let C be the set of all hyperarithmetic codes for $\mathbf{\Pi}^o_{<\gamma}(\Delta^1_1)$ sets. C is Π^1_1. Let Y_z be the set coded by $z\in C$. (1) becomes

$$(3) \qquad (X)_{X\in B}(\mathrm{E}Z)_{Z\in C}[x\in Y_Z \quad \&\quad Y_Z\cap A=\varnothing].$$

The predicates, $Z\in C \ \&\ X\in Y_Z$ and $Z\in C \ \&\ X\notin Y_C$, are Π^1_1 (as in Lemma 2.1.II), hence the matrix of (3) is Π^1_1. It follows from Exercise 2.11.II, a variant of Kreisel selection (Lemma 2.6.II), that the Z of (3) can be taken to be a hyperarithmetic function of X, call it $m(X)$. Let

$$B_0 \text{ be } \{m(X)\mid X\in B\}, \text{ and}$$
$$B_1 \text{ be } \{Z\mid Z\in C \ \&\ Y_Z\cap A=\varnothing\}.$$

Then $B_0\in\Sigma^1_1$, $B_0\subseteq B_1$ and $B_1\in\Pi^1_1$. By Exercise 5.12.II, a variant of Exercise 3.9.II, there is a Δ^1_1 D such that $B_0\subseteq D\subseteq B_1$. (Note that Kleene separation for Σ^1_1 sets was used to obtain D.)

Let E be $\cup\{Y_Z\mid Z\in D\}$. Then $B\subseteq E$, $E\cap A=\varnothing$ and $(\omega^\omega-E)\in\mathbf{\Pi}^o_\gamma(\Delta^1_1)$. □

The proof of Lemma 6.4 was a bounding argument typical of hyperarithmetic theory. The next lemma is an approximation result inspired by Gandy forcing. Let ω^ω have the Gandy topology as in Theorem 6.3. Suppose $H\subseteq\omega^\omega$. As usual H is said to be *nowhere* dense if the closure of H has an empty interior, and *meager* if H is contained in a countable union of nowhere dense sets. Theorem 6.3 is equivalent to: if H is meager and open, then H is empty.

6.5 Lemma. *If $\gamma<\omega_1^{\mathrm{CK}}$ and $H\in\mathbf{\Pi}^o_\gamma$, then there exists an L such that $L=\bar{L}^\gamma$ and $(H-L)\cup(L-H)$ is meager (in the sense of the Gandy topology).*

Proof. By induction on γ. Not to be confused with the main induction. Let H be $\omega^\omega-\bigcup_n\{H_n\mid n<\omega\}$, where for each n, $H_n\in\mathbf{\Pi}^o_{\gamma_n}$ for some $\gamma_n<\gamma$. By induction there is an L_n such that

$$\bar{L}_n^{\gamma_n}=L_n \quad\text{and}\quad (H_n-L_n)\cup(L_n-H_n) \text{ is meager.}$$

Define $L = -\cup\{Y \mid Y \in \mathbf{\Pi}^0_{<\gamma} \cap \Sigma^1_1 \ \&\ Y \cap H \text{ is meager}\}$.

$H - L$ is meager, since a countable union of meager sets is meager.

$L - H = \cup_n[(L \cap H_n \cap L_n) \cup (L \cap (H_n - L_n))]$. So the meagerness of $L - H$ will follow from that of $L \cap L_n$. $L \cap L_n$ is closed. Suppose C is Σ^1_1 and $C \subseteq L \cap L_n$ with the intent of showing $C = \varnothing$. Since $C \in \Sigma^1_1$, the main induction implies (as in the passage from (1) to (2) in the proof of Lemma 6.4):

$$\bar{C}^{\gamma_n} = -\cup\{Y \mid Y \in \mathbf{\Pi}^0_{<\gamma_n}(\Delta^1_1) \ \&\ Y \cap C = \varnothing\}.$$

The set of codes for members of $\mathbf{\Pi}^0_{<\gamma_n}(\Delta^1_1)$ is a Π^1_1 set of hyperarithmetic reals. Consequently $\bar{C}^{\gamma_n}$ is Σ^1_1.

$$\bar{C}^{\gamma_n} \subseteq \bar{L}^{\gamma_n}_n = L_n; \quad \text{so} \quad \bar{C}^{\gamma_n} \cap H \subseteq L_n \cap H \subseteq L_n - H_n.$$

Since $L_n - H_n$ is meager, $\bar{C}^{\gamma_n} \cap H$ must also be meager. Clearly $\bar{C}^{\gamma_n} \in \mathbf{\Pi}^0_{<\gamma}$. It follows from the definition of L that $\bar{C}^{\gamma_n} \cap L = \varnothing$. Hence $C \cap L = \varnothing$ and so $C = \varnothing$.

If $Y \in \mathbf{\Pi}^0_{<\gamma} \cap \Sigma^1_1$ and $Y \cap H$ is meager, then $Y \cap L = \varnothing$. So $L \supseteq \bar{L}^\gamma$. □

6.6 Lemma. *If $A, B \subseteq \omega^\omega$ are Σ^1_1, and A is separable from B by a $\mathbf{\Pi}^0_{<\gamma}$ set, then $\bar{A}^\gamma \cap B = \varnothing$.*

Proof. Suppose $A \subseteq H$ and $H \cap B = \varnothing$ for some $H \in \mathbf{\Pi}^0_\gamma$. As was noted just before Lemma 6.4, it is safe to assume $\gamma < \omega_1^{CK}$. Lemma 6.5 provides an L such that $L = \bar{L}^\gamma$ and $(H - L) \cup (L - H)$ is meager. $A \subseteq H$, so $A - L$ is meager. L is closed and A is open, so $A - L$ is open. By Theorem 6.3 $A - L = \varnothing$. Hence $\bar{A}^\gamma \subseteq \bar{L}^\gamma = L$.

$H \cap B = \varnothing$, so $L \cap B$ is meager, and $\bar{A}^\gamma \cap B$ is meager. The main induction implies $\bar{A}^\gamma$ is open, as $\bar{C}^{\gamma_n}$ was open in the proof of Lemma 6.5. Thus $\bar{A}^\gamma \cap B$ is a meager, open set, hence empty. □

Louveau's separation theorem (6.1) is an immediate consequence of Lemmas 6.4 and 6.6. The proof of 6.1 in outline is as follows. Suppose A and B are separated by some $H \in \mathbf{\Pi}^0_\gamma$ for some $\gamma < \omega_1^{CK}$. H can be well approximated by some L with the property that $\bar{L}^\gamma = L$. In essence the complement of L is the union of all Σ^1_1 forcing conditions of boldface rank less than γ that force generic reals out of H. A comparison of $\bar{A}^\gamma$ with L shows $\bar{A}^\gamma$ separates A and B. Baire's category theorem for the Gandy topology shows the error introduced by substituting L for H amounts to nothing. A bounding argument, together with the Kleene separation result, shows $\bar{A}^\gamma$ can be replaced by a $\mathbf{\Pi}^0_\gamma(\Delta^1_1)$ set. Induction on γ is used throughout to replace "$Y \in \mathbf{\Pi}^0_{<\gamma} \cap \Sigma^1_1$" by "$Y \in \mathbf{\Pi}^0_{<\gamma}(\Delta^1_1)$". The effect is, in 6.4, to make the matrix of (1) Π^1_1, and, in 6.5–6.6, to show the δ-closure of a Σ^1_1 set is Σ^1_1 ($\delta \leq \gamma$). Less precisely, the effect of the induction on γ is to show: with respect to the forcing of rank γ sentences with Σ^1_1 conditions of rank less than γ, there is no difference between lightface and boldface conditions.

6.7–6.11 Exercises

6.7. If $D \in \mathbf{\Pi}^0_{\omega_1}$ and D has a Δ^1_1 code, then $D \in \mathbf{\Pi}^0_{\omega_1^{CK}}$.

6.8. Suppose $D \in \mathbf{\Pi}^0_{\omega_1}$. Show $D \in \Delta^1_1$ iff D has a Δ^1_1 code.

6.9. Find a Σ^1_1 set $D \subseteq \omega^\omega$ such that $D \in \mathbf{\Pi}^0_{\omega_1^{CK}} - \mathbf{\Pi}^0_{<\omega_1^{CK}}$.

6.10. D is Borel iff $D \in \mathbf{\Pi}^0_{\omega_1}$.

6.11. Use Gandy forcing, as Harrington did, to prove Silver's theorem: Let E be a Π^1_1 equivalence relation whose field is ω^ω. Suppose there are uncountably many E-equivalence classes; show there are continuum many.

Part B

Metarecursion

Metarecursion theory lifts classical recursion theory (CRT) from the natural numbers to the recursive ordinals via definitions in hyperarithmetic terms. It makes precise the vague idea that Π^1_1-ness is analogous to recursive enumerability. As a generalization of classical recursion theory, it is strong enough to carry out the solution of Post's problem and the construction of a maximal set. Thus priority arguments make sense in the context of Π^1_1 sets, and supply results not obtainable by more direct means. As an outgrowth of hyperarithmetic theory, it provides a concrete introduction to the fundamentals of α-recursion beyond ω.

Chapter V
Metarecursive Enumerability

With the aid of a Π^1_1 set of unique notations for recursive ordinals, the fundamental notions of classical recursion are lifted to metarecursion. "Finite" is raised to "metafinite", "recursively enumerable" to "metarecursively enumerable", and "Turing reducible to" "metarecursive in". A set is said to be regular if its intersection with every metafinite set is metafinite. It is shown that each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set.

1. Fundamentals of Metarecursion

Metarecursion theory originated in Kreisel's observation that hyperarithmetic subsets of Π^1_1 sets of natural numbers behave much like finite subsets of recursively enumerable sets. (A similar observation was made independently by Hartley Rogers with respect to Rice's theorem on indices.) The natural enumeration of a Π^1_1 set in ω_1^{CK} steps yields a hyperarithmetic set at each step, just as the enumeration of a recursively enumerable set yields a finite set at each step. If A is Π^1_1, then there is a recursive function f such that

$$(x)[x \in A \leftrightarrow f(x) \in O]$$

(Theorem 5.4.I). Let

$$A_\delta = \{x \mid |f(x)| \leq \delta\}.$$

Then $A = \cup\{A_\delta | \delta < \omega_1^{CK}\}$, and each A_δ is hyperarithmetic (Lemma 2.4.II). Takeuti (1960) was the first to generalize recursion theory from natural numbers to ordinals; he replaced recursive enumerability by a schematic approach equivalent to Σ_1 definability. Kreisel was the first to point out that a generalization of recursion theory ought to pay special attention to the concept of finiteness in addition to that of recursive enumerability. His insight went beyond the idea that finite computations should be replaced by certain infinite ones. For inspiration he drew on model theory. He proved (Kreisel 1961, 1965): a Π^1_1 set A of axioms of ω-logic has a model if every hyperarithmetic subset of A has a model. His result

suggested that generalizations of the compactness theorem of first order logic ought to take generalizations of finiteness into account, and that one such generalization, in the context of Π^1_1 sets, was hyperarithmeticity.

Kreisel also maintained that any generalization of recursion theory worthy of investigation should be able to support the priority method of Friedberg and Muchnik. In particular it should yield a positive solution to Post's problem, that is incomparable degrees of recursively enumerable sets. Spector proved what some thought was a negative solution of Post's problem for Π^1_1 sets of numbers, when he showed that every non-hyperarithmetic Π^1_1 set has the same hyperdegree as Kleene's O (Proposition 7.2.II). Kreisel took an opposing view of Spector's result. The set of natural numbers was not the correct domain when recursively enumerable was replaced by Π^1_1. Since finite was akin to hyperarithmetic, it followed that r.e. sets of numbers were akin to Π^1_1 sets of hyperarithmetic sets. With ω replaced by HYP, Spector's negative argument no longer worked.

Another reason for changing the domain came from consideration of Post's theorem: a set is recursive iff both it and its complement are r.e. With domain ω the Post theorem became: a set is hyperarithmetic iff both it and its complement are Π^1_1. Thus with domain ω, recursive became akin to hyperarithmetic, a connection ruled out by the equating of hyperarithmetic with finite.

A further argument against Spector's theorem as a negative solution to Post's problem lay in the difference between Turing and hyperarithmetic reducibility. A Turing reduction procedure P consists of a recursively enumerable sequence of finite computations. A set B is reduced by applying to B those elements of P that fit B, those computations based on membership statements true of B. A hyperarithmetic reduction procedure Q fails to resemble a Π^1_1 sequence of hyperarithmetic computations. Q does not specify computations in advance. If Q is applied to a non-hyperarithmetic B, then the resulting computations are hyperarithmetic in B rather than hyperarithmetic.

Kreisel's idea that Π^1_1 sets of hyperarithmetic sets of numbers correspond to recursively enumerable sets of numbers with respect to priority arguments was sound and marked the beginning of metarecursion theory. Suppose for a moment that a metarecursively enumerable set is defined to be a Π^1_1 set of hyperarithmetic sets of numbers. Then a metarecursive set is a metarecursively enumerable set whose complement with respect to the domain of hyperarithmetic sets is also metarecursively enumerable. And a metafinite set is a hyperarithmetic collection of hyperarithmetic sets. In short meta r.e. means Π^1_1 *on* HYP, metarec. means Δ^1_1 *on* HYP, and metafinite means hyperarithmetic. Kreisel conjectured, rightly as it turned out, that metarecursively enumerable sets would be amenable to priority arguments. The use of hyperarithmetic sets as individuals was much too awkward, and soon gave way to indices, then to notations for recursive ordinals, and finally to recursive ordinals as in the next subsection.

1.1 Definition of Metarecursive. Let Q be a Π^1_1 set of unique notations of recursive ordinals (Theorem 2.4.III). Let $n: \omega_1^{CK} \to Q$ take each recursive ordinal to its unique notation. Thus $|n(\beta)| = \beta$.

Assume $A \subseteq \omega_1^{CK}$. A is called *metarecursively enumerable* (Kreisel and Sacks 1963) if $n[A]$ is Π_1^1. ($n[A] = \{n(a) | a \in A\}$.) A is called *metarecursive* if A and $\omega_1^{CK} - A$ are metarecursively enumerable. A is said to be *metafinite* if $n[A]$ is hyperarithmetic.

The choice of Q makes no difference. Suppose Q_1 and Q_2 are Π_1^1 sets of unique notations. Let $\theta: Q_1 \to Q_2$ be the unique map such that $|\theta(n)| = |n|$. Then $\theta(x) = y$ iff

$$(1) \quad x \in Q_1 \ \& \ y \in Q_2 \ \& \ (\mathrm{Ef})\,[f \text{ is an order-preserving map of } Q_1^{<x} \text{ onto } Q_2^{<y}].$$

($Q_i^{<x}$ is $\{z | z \in Q_i \ \& \ z <_O x\}$.) The f of (1), if it exists, is unique, hence hyperarithmetic. Thus the graph of θ is Π_1^1. Let $n_i: \omega_1^{CK} \to Q_i$ take each recursive ordinal to its unique notation in Q_i. Then

$$n_1[A] \text{ is } \Pi_1^1 \quad \text{iff} \quad n_2[A] \text{ is } \Pi_1^1,$$

since $n_2 = \theta n_1$.

Let $\Phi \subseteq \omega_1^{CK} \times \omega_1^{CK}$ be a partial function. ϕ is said to be *partial metarecursive* if

$$\{\langle n(\alpha), n(\beta)\rangle | \phi(\alpha) = \beta\}$$

is Π_1^1.

1.2 Proposition. *Let K, $A \subseteq \omega_1^{CK}$*

(*i*) *K is metafinite $\leftrightarrow$ K is metarecursive and bounded above by some $\beta < \omega_1^{CK}$.*

(*ii*) *A is meta r.e. $\leftrightarrow$ A is the range of a partial metarecursive function.*

(*iii*) *If K is metafinite, and ϕ is partial metarecursive and defined on K, then $\phi[K]$ is metafinite.*

(*iv*) *Assume $A \subseteq \omega$. A is metarecursively enumerable $\leftrightarrow$ A is Π_1^1. A is metafinite $\leftrightarrow$ A is hyperarithmetic.*

Proof

(i) Suppose K is metafinite. Then $n[K]$ is hyperarithmetic, $Q - n[K]$ is Π_1^1, so K is metarecursive. By Spector's bounding theorem (5.6.I), there is a $b \in O$ such that K is bounded above by $|b|$.

Suppose K is metarecursive and bounded above by $|b|$. Then $n[K] \subseteq O_b \cap Q = Q_b$ and $\omega - n[K] = (\omega - Q_b) \cup (Q_b - n[K])$. Since $n[K]$ and $Q - n[K]$ are Π_1^1, and Q_b is r.e. (3.5.I), $n[K]$ must be Δ_1^1.

(iii) Let f be the function whose graph is

$$\{\langle n(x), n(y)\rangle | \phi(x) = y\}.$$

Then $n[\phi[K]] = f[n[K]]$. The latter is Δ_1^1 (as in the proof of Proposition 1.7.I), since $n[K]$ is hyperarithmetic and f is partial Π_1^1.

(iv) Assume $A \subseteq \omega$. $n \restriction \omega$ is hyperarithmetic. Hence A is $\Pi_1^1 \leftrightarrow n[A]$ is Π_1^1. □

Proposition 1.2 (i) suggests that *finite*, in the context of recursion theory, is simply *recursive and bounded*. It seems hard to believe that no argument in classical recursion theory needs the fact that a finite set has a greatest element, but this

appears to be the case. Proposition 1.2(iii) is the most important principle of metarecursion theory. In Part C it will be seen to be equivalent to the fact that $L(\omega_1^{CK})$ satisfies Σ_1 replacement, the key principle of α-recursion theory. Proposition 1.2(iv) makes possible the application of metarecursion theory to Π_1^1 sets, and in particular the construction of a maximal Π_1^1 set via a Friedbergian priority argument.

1.3 Bounded Meta r.e. Sets. Lifting arguments from classical recursion theory to metarecursion theory will not in general be routine. It will not as a rule suffice to prefix all key words with "meta". The reader may already have noticed the following significant difference between metarecursion and classical recursion. There exists a bounded, meta r.e. set that is not metarecursive. The simplest example is Kleene's O according to Proposition 1.2(iv). In the setting of classical recursion theory every bounded set is finite, hence recursive. Bounded, non-metarecursive, meta r.e. sets do not belong to the realm of pathology. They are a source of technical problems whose solutions illuminate the workings of recursion theory.

1.4 Theorem (Enumeration). *For each $n \geq 1$ there is a partial metarecursive function $\phi_n(z, x_1, \ldots, x_n)$ such that: for each partial metarecursive $\psi(x_1, \ldots, x_n)$ there is an $e < \omega$ such that*

$$\psi(x_1, \ldots, x_n) \simeq \phi_n(e, x_1, \ldots, x_n).$$

Proof. Let $n = 1$. As in subsection 5.2.I, each Π_1^1 predicate $P_e(x, y)$ can be put in the form

$$(f)\,(\mathrm{Eu})\,T(\bar{f}(u), e, x, y)$$

for some e determined by P. The proof of Kreisel's uniformization Theorem (2.3.II) yields a recursive function g such that $g(e)$ is an index for the Π_1^1 predicate that uniformizes

$$P_e(x, y) \;\&\; x \in Q \;\&\; y \in Q.$$

Define $\phi_1(e, x) \simeq y$ by

$$(f)\,(\mathrm{Eu})\,T(\bar{f}(u), g(e), n(x), n(y)).$$

(n is the notation function of subsection 1.1.) □

1.5 Lemma. *There exist metarecursive functions j and k such that: for each metafinite set K, there exists a unique $\delta < \omega_1^{CK}$ such that*

$$K = \{x \mid x < j(\delta) \quad \& \quad k(\delta, x) = 0\}.$$

Proof. Let Q be a Π_1^1 set of unique notations as in subsection 1.1, and let I be the set of all indices of hyperarithmetic subsets of Q. To be precise, $\langle e, b \rangle \in I$ if $b \in Q$ and

$\{e\}^{H_b}$ is a subset of Q. I is Π^1_1. $<$, a well-ordering of I, is defined by:

$$\langle e_1, b_1\rangle < \langle e_2, b_2\rangle \leftrightarrow (|b_1| < |b_2|) \vee (|b_1| = |b_2| \quad \text{and} \quad e_1 < e_2).$$

For each $b \in I$, let G_b be the subset of Q indexed by b. I_1, the Π^1_1 set of unique indices for hyperarithmetic subsets of Q, is defined by:

$$b \in I_1 \leftrightarrow b \in I \quad \text{and} \quad (y)[y < b \rightarrow G_y \neq G_b].$$

There exists a partial recursive function t whose domain includes Q, and which maps Q 1-1 onto I_1. t can be thought of as enumerating I_1 without repetitions. t is defined by effective transfinite recursion on Q. For each b, $t(b)$ is the result of selecting a member of $I_1 - t\restriction b$. The selection is made via Kreisel's uniformization Theorem (2.3.II). Let $P(e,x)$ be the e-th Π^1_1 set. Then 2.3.II supplies a partial Π^1_1 function h such that for all e:

$$(\mathrm{Ex})P(e,x) \rightarrow h(e) \text{ is defined} \quad \& \quad P(e,h(e)).$$

The recursion equation for t is

$$t(b) = h(e(b)),$$

where e is a recursive function such that

$$\hat{x}P(e(b),x) = I_1 - t\restriction b.$$

Define

$$k(\gamma, x) = 0 \leftrightarrow x \in G_{t(n(\gamma))}$$

and 1 otherwise.

The predicate

$$b, c \in Q \quad \text{and} \quad (x)[x \in G_{t(b)} \rightarrow |x| < |c|]$$

is Π^1_1, so another Kreisel uniformization allows c to be construed as a partial Π^1_1 function g of b. Define

$$j(\gamma) = g(t(n(\gamma))). \qquad \square$$

1.6 Indices for Metafinite Sets. Let j and k be the metarecursive functions of 1.5. For each β define

$$K_\beta = \{x | x < j(\beta) \quad \text{and} \quad k(\beta, x) = 0\}.$$

According to Lemma 1.5 there is for each metafinite K just one β such that $K = K_\beta$. β is said to be the index of K regarded as a metafinite set. β is intended to be a strong index. Post (1944) distinguished between weak and strong indices for finite sets. A strong index for a finite set is an instruction for recursively enumerating the set together with an instruction for stopping. A weak index consists simply of the first instruction. Strong indices are needed in recursion theory so that finite sets will live on the same level of complexity as their elements.

Let f be a partial function from ω_1^{CK} into the set of metafinite sets. f is said to be partial metarecursive if there exists a partial metarecursive function g such that $f(\delta) \simeq K_{g(\delta)}$ for all δ.

1.7 Proposition. *If H is metafinite, then $\cup\{K_\delta | \delta \in H\}$ is metafinite.*

Proof. Let K be the union in question. Then

$$x \in K \leftrightarrow (E\delta)[\delta \in H \quad \& \quad x < j(\delta) \quad \& \quad k(\delta, x) = 0].$$

The definitions of Subsection 1.1. make it easy to verify that K and $\omega_1^{CK} - K$ are metarecursively enumerable. Thus K is metarecursive. By Proposition 1.2(iii), $j[H]$ is metafinite, hence bounded, so K is bounded. Then K is metafinite by Proposition 1.2(i). □

Note that the proof of Proposition 1.7 also shows K is a metarecursive function of H in the sense of subsection 1.6.

1.8 Proposition (Transfinite Recursion). *If I is metarecursive, then the unique solution of*

$$f(\delta) = I(f \restriction \delta) \quad (\delta < \omega_1^{CK})$$

is metarecursive.

Proof. As in set theory, let F be the set of all g such that:

(a) g is a metafinite function;
(b) domain of g is a recursive ordinal;
(c) $g(\delta) = I(g \restriction \delta)$ for all $\delta \in \text{dom } g$.

F is metarecursive because I is. By induction on γ, F is consistent: if $g_1, g_2 \in G$ and $\gamma \in \text{dom } g_1 \cap \text{dom } g_2$, then $g_1(\gamma) = g_2(\gamma)$. Define $f = \cup F$. f is partial metarecursive with domain $d \subseteq \omega_1^{CK}$. Suppose $d < \omega_1^{CK}$. Then g, defined by

$$g \restriction d = f \quad \& \quad g(d) = I(f),$$

belongs to F. □

1.9 Proposition. *If A is metarecursively enumerable but not metafinite, then A is the range of a one-one metarecursive function.*

Proof. $n[A]$ is Π^1_1. By Theorem 5.4.I there is a recursive t such that $x \in n[A]$ iff $t(x) \in O$. Let

$$A^\gamma = \{\delta \mid |t(n(\delta))| < \gamma\}.$$

A^γ is a metarecursive function of γ. Define

$$f(\omega \cdot \gamma + m) = n^{-1}(m) \quad \text{if} \quad m \in n[A^\gamma - \cup_{<\gamma} A^\gamma].$$

Let $f(\omega \cdot \gamma + m) = -1$ otherwise. f is metarecursive and enumerates A without repetitions save for the spurious value -1. A transfinite recursion is needed to eliminate -1. Define g by

$$g(\delta) = \mu x[x \geq \sup^+ g[\delta] \quad \& \quad f(x) \neq -1].$$

μx is read "the least ordinal x such that". $\sup^+ B$ is the least ordinal greater than every ordinal in B. Since $n[A]$ is not hyperarithmetic, there are arbitrarily large x's such that $f(x) \neq -1$. It follows from Proposition 1.8 that g is metarecursive. Let $h(\delta) = f(g(\delta))$. Then h enumerates A without repetitions. □

1.10–1.16 Exercises

1.10. Prove Proposition 1.2(ii).

1.11. Verify that k and j, defined in the proof of Lemma 1.5 are metarecursive.

1.12. Verify that F, defined in the proof of Proposition 1.8, is metarecursive.

1.13. Show that the set of partial metarecursive functions is closed under composition.

1.14. Suppose f is metarecursive and K is metafinite. Show $f \restriction K$ is metafinite.

1.15. Formulate and prove: the class of metarecursive sets is closed under complementation, metafinite unions, and metafinite intersections.

1.16. Formulate and prove: the class of metarecursively enumerable sets is closed under meta r.e. unions and metafinite intersections.

2. *Metafinite Computations*

This section is a sketch of an equation calculus for computing partial metarecursive functions, a calculus that extends Kleene's by allowing certain infinite computations. A sketch is sufficient, because the calculus has little practical use. It

is included only to clarify the concepts of metarecursive function and metarecursive reducibility.

Implementation of the calculus gives rise to metafinite computations. This is seen to be necessarily so, if the definition of metarecursive function is viewed in the light of the remarks made concerning the natural enumeration of O in the proof of Theorem 2.2.I. Suppose f is metarecursive. Then

$$f(x) = y \leftrightarrow \langle n(x), n(y) \rangle \in A$$

for some $\Pi_1^1 A \subseteq O$. Thus the computation of a value of f corresponds to showing some number belongs to O. The verification of $m \in O$ is accomplished by enumerating O up to stage $|m|$. The resulting object is hyperarithmetic, hence metafinite. The object has the tree-like appearance of a wellfounded computation if it is developed backwards, as in the proof of Theorem 3.5.I, where the predecessors of m in O are generated by starting with m and proceeding downward along branches that end with 1.

2.1 An Infinitary Calculus. A calculus of the sort presented here was first devised independently by Tugué (1964), and by Levy (1963) and Machover (1961). Variations were later studied by Kripke (1964) and Platek (1966). All are inspired by the Kleene calculus for classical recursion theory. The sketch below follows Kripke (1964). The primitive symbols are:

numbers $\underline{\gamma}$ for each $\gamma < \omega_1^{CK}$;
variables $x, y, z, \ldots$ ranging over ω_1^{CK};
n-place function symbols $f, g, h, \ldots$;
and $\exists$ (there exists), $\forall$(for all), $<$ (less than), $=$ (equals), and $'$(successor).

Terms are defined recursively. Numerals and variables are terms. If t is a term, then t' is a term. If $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, then $f(t_1, \ldots, t_n)$ is a term. If t_0 and t_1 are terms with x not occurring freely in t_0, then $(\mathrm{E}x < t_0)t_1$ and $(\forall x < t_0)t_1$ are terms.

If t_0 and t_1 are terms, then $t_0 = t_1$ is an equation.

As in Kleene (1952) there are finitely many substitution rules for deriving an equation e from a set S of equations. For example $\underline{\gamma + 1}$ is substituted for $(\underline{\gamma})'$, or $\underline{\gamma}$ for t if $t = \underline{\gamma}$ has already been derived. Kleene's rules are finitary in that each operates on a finite set of premises. It follows that his computatons are finite, because a wellfounded tree with finite branching is necessarily finite. The calculus for metarecursion has only one infinitary rule, W. W operates on

$$\{e\} \cup \{t(\underline{\delta}) = \underline{0} \mid \delta < \gamma\}$$

and yields as conclusion the result of substituting $\underline{0}$ for all occurrences of $(\forall x < \gamma)t(x)$ in e. W is seen to be a kind of ω-rule if 0 is taken to mean true.

Let E be a finite set of equations. The set $C(E)$ of all equations computable from E is defined by transfinite recursion.

$$C(0, E) = E.$$

$e \in C(\beta+1, E)$ if e is an immediate consequence of equations in $C(\beta, E)$ via a rule whose application mentions no ordinals greater than $\beta+1$.

$$C(\lambda, E) = \cup\{C(\beta, E) \mid \beta < \lambda\} \quad \text{if } \lambda \text{ is a limit.}$$
$$C(E) = C(\omega_1^{CK}, E).$$

$C(E)$ has a natural enumeration similar to the one given for O in the proof of Theorem 2.2.I. It follows that $C(E)$ is metarecursively enumerable, and that $C(\beta, E)$ is metafinite for each $\beta < \omega_1^{CK}$. There is no difficulty in metarecursively assigning recursive ordinals to equations as Gödel numbers.

Each equation in $C(E)$ owes its existence to a metafinite computation with roots in E. Abstractly, each computation is simply a metafinite, wellfounded tree. The set Z of all computations can be defined by a recursion that parallels the one that defines C(E) and so is metarecursively enumerable. It is worth noting that Z is not metacursive (Exercise 2.3), since classical recursion theory suggests otherwise.

2.2. Metacomputable Functions. Let $\phi \subseteq \omega_1^{CK} \times \omega_1^{CK}$ be a partial function. ϕ is said to be metacomputable if there exists a finite set E of equations with principal function letter f such that for all $\alpha, \beta < \omega_1^{CK}$:

$$\phi(\alpha) = \beta \leftrightarrow (f(\underline{\alpha}) = \underline{\beta}) \in C(E). \tag{1}$$

The metacomputable partial functions are the same as the metarecursive partial functions. Checking this assertion involves many tedious details but the ideas needed are straightforward and are familiar to readers of Kleene (1952).

Suppose ϕ is metacomputable. By (1) the graph of ϕ is metarecursively enumerable, since $C(E)$ is. Hence ϕ is partial metarecursive according to the definitions of subsection 1.1.

The other direction is more difficult. Suppose ϕ is partial metarecursive. Thus

$$\{\langle n(\alpha), n(\beta)\rangle \mid \phi(\alpha) = \beta\} \tag{2}$$

is Π_1^1, as in subsection 1.1, and so is many-one reducible to O vi a some ordinary recursive function g. To see that O is the range of a metacomputable function, recall the equations that yield the natural enumeration of O. Since g is metacomputable via Kleene's equation calculus, it follows that (2), is the range of a metacomputable function defined by some finite set E of equations. A slight modification of E defines ϕ.

The most important point suppressed above concerns the consistency of E. In general it is simple to find an E that defines a given partial metarecursive ϕ. "Simple" means that

$$\phi(\alpha) = \beta \rightarrow (f(\underline{\alpha}) = \underline{\beta}) \in C(E) \tag{3}$$

is immediate. The converse of (3), the consistency of E, can be troublesome. Kleene (1952) handles the problem in a proof theoretic fashion. He allows substitutions

only under certain narrow restrictions, so narrow no inconsistent computations can slip by. For example, consider the usual recursion equations for $+$. The base equation is

$$f(x, \underline{0}) = x. \tag{4}$$

It allows the computation of $f(\underline{m}, \underline{0}) = m$ for any natural number m. What is to prevent the computation of $f(\underline{m}, \underline{0}) = \underline{n}$ for some $m \neq n$? If such a computation existed, it might have the form

$$f(\underline{m}, t) = \underline{n}, \quad t = \underline{0} \vdash f(\underline{m}, \underline{0}) = \underline{n}. \tag{5}$$

But (5) is not allowed by the Kleene rules. The principal function letter f occurs only on the left sides of equations, and the only substitution permitted on the left is that of a numeral for a variable. Hence $f(\underline{m}, \underline{0})$ can be obtained on the left only by starting with (4) or with an equation whose left side is $f(\underline{m}, y)$. But the recursion equations for $+$ whose left sides have variable second arguments have right sides that cannot give rise to $\underline{0}$, for example $(g(x, y))'$. In short the restrictions on Kleene's rules make it possible to trace an inconsistency back to its source.

2.3–2.4 Exercises

2.3. Formulate and prove: the set of all metafinite computations is not metarecursive.

2.4 (Kripke 1964). Show $C(E, \omega_1^{CK} + 1) = C(E, \omega_1^{CK})$.

3. *Relative Metarecursiveness*

Classical recursion theory centers on the notion of Turing reducibility (or relative recursiveness). The corresponding notion for metarecursion theory is at first obscure, because it is not clear which formulation of Turing reducibility should be lifted. As it turns out, the choice matters a great deal. One formulation, when lifted, fails to be transitive, and so does not lead to a suitable concept of degree. In general two sets are said to have the same degree if each is reducible to the other. If the given notion of reducibility is transitive (and reflexive), then the derived concept of degree is an equivalence relation. The concept of degree is essential to the study of recursively enumerable sets, and of Post's problem, several generalizations of which are solved in this book. The first is the metarecursive version of Chapter VI.

3.1 Weakly Metarecursive In. One formulation of Turing reducibility for sets $A, B \subseteq \omega$ is as follows. A Turing reduction procedure ϕ is a recursively enumerable set of quadruples $\langle h, j, n, i \rangle$. h and j are finite subsets of ω; and $n, i \in \omega$. cB, the

complement of B, is $\omega - B$. A is Turing reducible to B via ϕ if

(1a) $$n \in A \leftrightarrow (\mathrm{Eh})(\mathrm{Ej})[\langle h, j, n, 0\rangle \in \phi \ \& \ h \subseteq B \ \& \ j \subseteq cB]$$

and

(1b) $$n \notin A \leftrightarrow (\mathrm{Eh})(\mathrm{Ej})[\langle h, j, n, 1\rangle \in \phi \ \& \ h \subseteq B \ \& \ j \subseteq cB]$$

hold for all n. To decide $n \in A$ (from B) enumerate the quadruples in ϕ until one is found that satisfies B. Such a quadruple has the form $\langle h, k, n, i\rangle$ and the property that $h \subseteq B$ and $K \subseteq cB$. Then $N \in A$ iff $i = 0$. The associated computation is finite, because only finitely much of ϕ is enumerated, and only finitely much of B is called for. Let $\leq_T$ denote Tuning reducibility.

To see that $\leq_T$ is transitive, suppose $A \leq_T B$ via ϕ, and $B \leq_T C$ via ψ. To decide $n \in A$ (from C) enumerate ϕ and ψ simultaneously. If $n \in A$, then the associated computation, in essence, is:

(2a) $\langle h, j, n, 0\rangle \in \phi$

(2b) $(x)_{x \in h}[\langle h_x, j_x, x, 0\rangle \in \psi]$

(2c) $(y)_{y \in j}[\langle h_y, j_y, y, 1\rangle \in \psi]$

(2d) $U\{h_v | v \in h \cup j\} \subseteq C, \quad U\{j_v | v \in h \cup j\} \subseteq cC.$

By (2a) $n \in A$ if $h \subseteq B$ and $j \subseteq cB$. By (2b) $h \subseteq B$ if $U\{h_x | x \in h\} \subseteq C$ and $U\{j_y | y \in j\} \subseteq cC$.

If $n \notin A$, then the associated computation begins with $\langle h, k, n, 1\rangle \in \phi$ and otherwise resembles (2).

The only combinatoric principle used above is: a finite union of finite sets is finite. A glance at Proposition 1.7 suggests that the above demonstration lifts to metarecursion theory, but that is not the case. To see why not, consider $A, B \subseteq \omega_1^{\mathrm{CK}}$. cB, the complement of B, is now $\omega_1^{\mathrm{CK}} - B$. Define A as *weakly metarecursive* in B (in symbols $A \leq_w B$) by:

(3a) $$\delta \in A \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, 0\rangle \in \phi \ \& \ H \subseteq B \ \& \ J \subseteq cB]$$

and

(3b) $$\delta \notin A \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, 1\rangle \in \phi \ \& \ H \subseteq B \ \& \ J \subseteq cB],$$

where ϕ is metarecursively enumerable, H and K are metafinite, and $\delta < \omega_1^{\mathrm{CK}}$. (3) is the most obvious lifting of (1). The attempt to lift (2) breaks down with (2d). In general

(4) $$\cup\{H_v | v \in H \cup J\}$$

is not metafinite despite the fact that $H \cup J$ and H_v ($v \in H \cup J$) are metafinite. (4) is not a metafinite union of metafinite sets in the sense required by Proposition 1.7. For each v let $\delta(v)$ be the strong index of H_v. Thus $H_v = K_{\delta(v)}$ as in subsection 1.6. (4) becomes

$$\cup\{K_{\delta(v)} | v \in H \cup J\}. \tag{5}$$

To apply Proposition 1.7 to (5) it is necessary to know

$$\{\delta(v) | v \in H \cup J\} \text{ is metafinite.} \tag{6}$$

The truth of (6) depends on C. $\delta(v)$ is defined by enumerating ϕ and ψ simultaneously and occasionally referring to C. If the definition of $\delta(v)$ for all $v \in H \cup J$ draws only on a metafinite set of membership facts about C, then (6) is true. Nothing in (3) guarantees such limited use of C.

Driscoll (1968) showed that $\leq_w$ fails to be transitive even on the metarecursively enumerable sets. His theorem will be proved in Chapter VI.

3.2 Metarecursive In. The remedy for Driscoll's troublesome counterexample was found by Kreisel. Another formulation of Turing reducibility is:

$$j \subseteq A \leftrightarrow (\text{Eh})(\text{Ek})[\langle h,k,j,0\rangle \in \phi \ \& \ h \subseteq B \ \& \ k \subseteq cB] \tag{1a}$$

and

$$j \subseteq cA \leftrightarrow (\text{Eh})(\text{Ek})[\langle h,k,j,1\rangle \in \phi \ \& \ h \subseteq B \ \& \ k \subseteq cB] \tag{1b}$$

for all finite $j \subseteq \omega$. A, $B \subseteq \omega$, h and k are finite, and ϕ is recursively enumerable. The symmetry of (1) with respect to finite neighborhood conditions makes the transitivity of $\leq_T$ immediate. A finite set of membership questions about A is reduced to a finite set about B. Suppose $A \leq_T B$ via ϕ, and $B \subseteq_T C$ via ψ. To show $A \leq_T C$ enumerate ϕ and ψ simultaneously. If $j \subseteq A$, then the associated computation is:

(2) $\langle h,k,j,0\rangle \in \phi$;

$\langle h_0,k_0,h,0\rangle, \langle h_1,k_1,k,1\rangle \in \psi$;

$h_0 \cup h_1 \subseteq C$; $k_0 \cup k_1 \subseteq cC$.

If $j \subseteq cA$, then the computation begins with $\langle h, k, j, 1\rangle \in \phi$ and otherwise resembles (2).

The only combinatoric principle used above is: the union of two finite sets is finite. Consequently the above demonstration lifts instantly to metarecursion theory.

Let A, B be sets of recursive ordinals, J, H be metafinite and ϕ be metarecursively enumerable. Say A *is metarecursive* in B (in symbols $A \leq_M B$) via ϕ if

(3a) $\quad J \subseteq A \leftrightarrow (\mathrm{EH})(\mathrm{EK})[\langle H, K, J, 0\rangle \in \phi \quad \& \quad H \subseteq B \quad \& \quad K \subseteq cB]$

and

(3b) $J \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EK})[\langle H, K, J, 1\rangle \in \phi \ \& \ H \subseteq B \ \& \ K \subseteq cB]$ hold for all metafinite J. (2) lifts to show $\leq_M$ is transitive.

Two sets of recursive ordinals have the same metadegree if each is *metarecursive* in the other (in symbols $A \equiv_M B$).

3.3 Post's Method. Post in a celebrated paper (1944) liberated classical recursion theory from formal arguments by presenting recursive enumerability as a natural mathematical notion safely handled by informal mathematical procedures. He also stressed what may be called a dynamic view of recursion theory. For example, he proves the existence of a simple set S by giving instructions in ordinary language for the enumeration of S and then verifying that the instructions do in fact produce a simple set. A formal approach to S would refer to formulas or equations from some formal system. A static approach would attempt to define S by some explicit formula. The advantages of Post's informal, dynamic method are considerable. Without it arguments in classical recursion theory would be lengthy and hard to devise.

His method, and its advantages, lift to metarecursion theory. Consider the proposition:

(1) If A is metarecursively enumerable, then the collection of all metafinite subsets of A is metarecursively enumerable. Formal static proof of (1): Recall I_1 defined in the proof of Lemma 1.5. I_1 is a Π^1_1 set of unique indices for the hyperarithmetic subsets of Q. $n[A]$ is Π^1_1, so

$$\langle e, b\rangle \in I_1 \quad \& \quad \{e\}^{H_b} \subseteq n[A]$$

is also Π^1_1, and (1) is proved.

Informal dynamic proof of (1): Metarecursively enumerate A. A^γ (defined in the proof of Proposition 1.9) is that part of A enumerated prior to stage γ. If $K_\delta \subseteq A$, then $K_\delta \subseteq A^{\gamma_0}$ for some $\gamma_0 < \omega_1^{\mathrm{CK}}$. If $K_\delta \subseteq A^{\gamma_0}$, then enumerate δ. Thus the enumeration of A gives rise to a simultaneous enumeration (of the indices) of the metafinite subsets of A. The existence of γ_0 follows from Proposition 1.2(iii). Let

$$h(x) \quad \text{be} \quad \mu\gamma[x \in A^\gamma].$$

h is partial metarecursive and defined for all $x \in K_\delta$. *So* $h[K_\delta]$ is metafinite, hence bounded by some δ_0.

From now on proofs in metarecursion theory will follow the informal dynamic style originated by Post.

3.4 Proposition. *Suppose A and ϕ are metarecursively enumerable, and for all metafinite K,*

$$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, K\rangle \in \phi \ \&\ H \subseteq B \ \&\ J \subseteq cB].$$

Then $A \leq_M B$.

Proof. According to (1) of subsection 3.3, the set of all metafinite subsets of A is metarecursively enumerable. Thus there exists a metarecursively enumerable ψ such that

$$K \subseteq A \leftrightarrow K \in \psi$$

for all metafinite K. □

The next proposition is a technical fact needed in Section 4.

3.5 Proposition. *Let A be metarecursively enumerable. Then there exists a metarecursively enumerable A^* such that $A \equiv_M A^*$ and*

$$(X)[A^* \leq_w X \leftrightarrow A^* \leq_M X].$$

Proof. Let A^* be $\{\delta | K_\delta \cap A \neq \phi\}$. A^* is metarecursively enumerable since any enumeration of A induces a simultaneous enumeration of A^*. Proposition 3.4 implies $A \leq_M A^*$, since

$$K_\delta \subseteq cA \leftrightarrow \delta \notin A^*. \tag{1}$$

There exists a metarecursive function g such that

$$K_{g(\delta)} = \cup\{K_\gamma | \gamma \in K_\delta\}$$

Hence

$$K_\delta \subseteq cA^* \leftrightarrow K_{g(\delta)} \subseteq cA, \tag{2}$$

and so $A^* \leq_M A$ by Proposition 3.4.

It follows from (1) and (2) that

$$K_\delta \subseteq cA^* \leftrightarrow g(\delta) \notin A^*.$$

Consequently $A^* \leq_w X$ implies $A^* \leq_M X$. □

3.6 Reducibility for Functions. Suppose $\leq_r$ is a notion of reducibility defined for sets, for example $\leq_w$ or $\leq_M$. $\leq_r$ is extended to functions by identifying functions with their graphs. Thus

$$f \leq_r g \leftrightarrow \mathrm{graph}(f) \leq_r \mathrm{graph}(g).$$

(The graph of f is $\{\langle x, y\rangle \mid f(x) = y\}$.) Similarly

$$f \leq_r g \leftrightarrow \text{graph}\,(f) \leq_r A.$$

Let c_A be the characteristic function of A. As far as reducibility is concerned it is safe to identify A with the graph of c_A. Thus

$$A \leq_r B \leftrightarrow c_A \leq_r c_B.$$

Of special interest is $f \leq_w A$. It is equivalent to: There exists a metarecursively enumerable ϕ such that

$$f(x) = y \leftrightarrow (\text{EH})(\text{EJ})[\langle H, J, x, y\rangle \in \phi \quad \& \quad H \subseteq A \quad \& \quad J \subseteq cA]$$

for all x, $y < \omega_1^{\text{CK}}$.

3.7–3.8 Exercises

3.7. Define g by $K_{g(\delta)} = \cup\{K \mid \gamma \in K_\delta\}$. Show g is metarecursive.

3.8. The equation calculus of Subsection 2.1 is relativized as follows. Let g be an auxillary function letter. For any $B \subseteq \omega_1^{\text{CK}}$ define $\Delta(B)$ to be

$$\{g(\underline{\delta}) = \underline{0} \mid \delta \in B\} \cup \{g(\underline{\delta}) = \underline{1} \mid \delta \notin B\}.$$

Define $C(E, B)$ as $C(E)$ was in subsection 2.1 save that $C(0, E, B)$ is $E \cup \Delta(B)$. Each equation in $C(E, B)$ owes its existence to a computation of recursive ordinal height, abstractly a wellfounded tree with roots in $C(0, E, B)$. (If B is not metarecursive, then some of these computations may not be metafinite objects.) Define $C_m(E, B)$ to be the subset of $C(E, B)$ based on metafinite computations.

Call A *metafinitely computable from* B (in symbols $A \leq_{mc} B$) if for some E with principal function letter f,

$$\delta \in A \leftrightarrow (f(\underline{\delta}) = \underline{0}) \in C_m(E, B)$$

and

$$\delta \notin A \leftrightarrow (f(\underline{\delta}) = \underline{1}) \in C_m(E, B)$$

for all $\delta < \omega_1^{\text{CK}}$.

Show $A \leq_{mc} B \leftrightarrow A \leq_w B$.

4. *Regularity*

A set $A \subseteq \omega_1^{\text{CK}}$ is said to be *regular if* $A \cap K$ is metafinite for every metafinite K. The main result of this section is: each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set. This theorem greatly

facilitates the study of metarecursively enumerable degrees. The definition of regularity is inspired by its frequent, if hidden, use in classical recursion theory, and by Gödel's definition of constructible class, namely a class whose intersection with each constructible set is a constructible set.

A typical use of regularity in classical recursion theory occurs in the proof of the enumeration theorem (Kleene 1952). Finite neighborhood conditions are replaced by finite initial segment conditions in order to obtain an enumeration theorem for partial recursive functions relative to A (uniformly), a result that does not lift to metarecursion theory.

To elaborate let $A \subseteq \omega$, and $\{e\}^A$ be the e-th function partial recursive in A. Initially $\{e\}^A$ is defined as in (1) of subsection 3.1. There is a recursively enumerable ϕ such that $\{e\}^A(m)$ is defined and has the value n iff

$$(1) \qquad \langle h,k,m,n\rangle \in \phi \ \& \ h \subseteq A \ \& \ k \subseteq cA$$

for some finite h and k. Thus an enumeration of the ϕ's yields an enumeration of the $\{e\}^A$'s. It can happen that ϕ is inconsistent, that $\{e\}^A(m)$ has more than one value. Kleene eliminates inconsistencies by giving preference to the "least" neighborhood condition that A satisfies. Let $\phi(i)$ be the i-th 4-tuple in the recursive enumeration of ϕ. Now define $\{e\}^A(m)$ to have the value n if $\phi(i)$ satisfies (1) but $\phi(j)$ does not for all $j < i$. In order to compute the preferred value of $\{e\}^A(m)$ it is necessary to know the relation A bears to all the finite sets mentioned in $\phi(j)$ $(j \leq i)$. Since only finitely many finite sets are involved, the preferred value is determined by a bounded initial segment of the characteristic function of A. The regularity of A implies that a bounded initial segment of A is equivalent to a single finite neighborhood condition.

The initial segment trick is not applicable to metarecursion theory because a typical $A \subseteq \omega_1^{CK}$ is not regular. Machtey (1970) and Ohashi (1970) have shown that metarecursion theory lacks an enumeration theorem of the functions partial metarecursive in A (uniformly). Thus there is no substitute for the initial segment trick. But there does remain an enumeration of the many-valued functions partial metarecursive in A (uniformly).

The next proposition expresses the notion of regularity in dynamic form.

4.1 Proposition. *Let A be the range of some metarecursive f. Then A is regular iff for each δ, f eventually enumerates no new member of A less than δ.*

Proof. Suppose A is regular. Then $A \cap \delta$ is metafinite. For each $\gamma \in (A \cap \delta)$, let $g(\gamma)$ be the least σ such that $f(\sigma) = \gamma$. Since g is metarecursive, $g[A \cap \delta]$ is bounded. □

The study of recursively enumerable sets in classical recursion theory makes frequent use of the fact that every recursively enumerable set is regular in the sense of Proposition 4.1. The regular sets theorem (4.3) makes it safe to assume metarecursively enumerable sets are regular modulo their metadegrees.

The next lemma is inspired by an early result of Dekker (1954), and is the source of regularity in the proof of the regular sets theorem. Suppose A is metarecursively enumerable but not metafinite. By Proposition 1.9 there is a metarecursive f that enumerates A without repetitions. Define the deficiency set of f to be

$$D_f = \{\alpha | (E\beta)_{\beta > \alpha}(f(\beta) < f(\alpha))\}.$$

4.2 Lemma

(*i*) *D_f is metarecursively enumerable and regular.*
(*ii*) *cD_f is unbounded.*
(*iii*) *If $A(=f[\omega_1^{CK}])$ is regular, then $f[cD_f]$ is unbounded.*
(*iv*) *If A is regular, then $A \leq_M D_f$.*
(*v*) *$D_f \leq_M A$.*

Proof.

(i) Fix γ to see $D_f \cap \gamma$ is metafinite. Enumerate $D_f \cap \gamma$ without repetitions as follows. A new member α of $D_f \cap \gamma$ is enumerated when there is a σ such that

$$(1) \qquad \sigma > \alpha \ \& \ f(\sigma) < f(\alpha) \quad \& \quad (\tau)_{\alpha < \tau < \sigma}[f(\tau) \geq f(\alpha)].$$

If there is a bound on the σ's satisfying (1), then $D_f \cap \gamma$ is metafinite. Suppose there is no such bound. If $\gamma < \sigma_1 < \sigma_2$ both satisfy (1), then $f(\sigma_1) > f(\sigma_2)$. Hence there is an infinite, descending sequence of ordinals.

(ii) Fix γ. Let δ be the infimum of $\{f(\tau) | \tau > \gamma\}$. Then $f^{-1}(\delta) \in cD_f$.

(iii) follows from (ii) and Proposition 4.1.

(iv) By (iii) $K \subseteq cA$ iff

$$(E\delta)\,(f(\delta) > \sup K \ \& \ \delta \notin D_f \ \& \ K \subseteq cf[\delta])$$

for all metafinite K. Now apply Proposition 3.4.

(v) For all metafinite K,

$$K \subseteq cD_f \leftrightarrow \bigcup_{\alpha \in K} (f(\alpha) - f[\alpha]) \subseteq cA. \qquad \square$$

4.3 Theorem (Sacks 1966). *Each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set.*

Proof. Suppose A is metarecursively enumerable and $A \cap \gamma$ is not metafinite. Let $n: \omega_1^{CK} \to \omega$ be metarecursive and one-one. (n could be the notation function of subsection 1.1.) Then $n[A \cap \gamma]$ is metarecursively enumerable but not metafinite. Let f be a one-one metarecursive function whose range is $n[A \cap \gamma]$, and let B be $f[A]$. B is not metarecursive, because A is not and

$$x \in A \leftrightarrow f(x) \in B.$$

Let g be a metarecursive function that enumerates B without repetitions. D_g, the deficiency set of g (defined just before Lemma 4.2), is the desired regular set.

By Lemma 4.2 D_g is metarecursively enumerable and regular. The concept of "*finitely metarecursive in*" (in symbols $\leq_f$) is useful in showing $Dg \equiv_M A$. Say $U \leq_f V$ if $U \leq_w V$ via some ϕ that mentions only finite neighborhood conditions. Note that $U \leq_f V$ and $V \leq_w W$ imply $U \leq_w W$.

Observe that

$$x \notin D_g \leftrightarrow (g(x) - g[x]) \subseteq cB$$

and that $g(x)$ is finite. It follows from Proposition 3.4 that

(1) $$D_g \leq_f B.$$

(2) $$B \leq_w A$$

is a consequence of:

$$x \notin B \leftrightarrow (x \notin \text{range } f) \vee (x \in \text{range } f \ \& \ f^{-1}(x) \notin A);$$

$$(x \notin \text{range } f) \leftrightarrow x \notin n[\gamma] \vee (x \in n[\gamma] \ \& \ n^{-1}(x) \notin A).$$

From (1) and (2), $D_g \leq_w A$. Hence $D_g \leq_M A$, because every metafinite subset of cD_g is finite. To prove the latter note that the restriction of g to cD_g is strictly increasing. Hence the ordertype of cD_g is the same as that of $g[cD_g]$, namely ω. In addition, cD_g is an unbounded subset of ω_1^{CK} by Lemma 4.2.

To check $A \leq_M D_g$, it suffices to show $A \leq_w D_g$, since Proposition 3.5 makes it safe to assume

$$(X)[A \leq_w X \leftrightarrow A \leq_M X].$$

$A \leq_f B$ and $\leq_f$ is transitive, so it remains only to show $B \leq_f D_g$. But that is the case, because

$$m \notin B \leftrightarrow (E\alpha)[\alpha \notin D_g \ \ \& \ \ g(\alpha) > m \ \ \& \ \ m \notin g[\alpha]]$$

for all $m < \omega$. □

The above proof, much cleaner than the original, is due to S. Simpson. The only non-effective step is the choice of γ. W. Maass has shown that choice to be eliminable. To be precise he has proved: there exists a metarecursive function t such that for each δ, $R_\delta \equiv_M R_{t(\delta)}$ and $R_{t(\delta)}$ is regular. (R_δ is the δ-th metarecursively enumerable set in a standard enumeration.)

Theorem 4.3 is very much a result about metarecursively enumerable degrees. It fails for some metadegrees below that of Kleene's O. In fact there is a $T \subseteq \omega$ such that T is Turing reducible to O and no set of the same metadegree as T is regular (Macintyre 1973).

It will be seen in Part C that Theorem 4.3 generalizes readily from ω_1^{CK} to an arbitrary Σ_1 admissible ordinal α.

4.4 ω-sets. A subset of ω_1^{CK} is said to be an ω-set (Owings 1969) if its complement is unbounded and of ordertype ω. Note that the complement of an ω-set cannot be metafinite. A typical ω-set is D_g in the proof of Theorem 4.3. ω-sets will figure significantly in the proof of Simpson's dichotomy, Section 3.VI. If C is a meta-recursively enumerable ω-set, then

$$(X)[C \leq_w X \leftrightarrow C \leq_M X],$$

since every metafinite subset of cC is finite.

A is said to be *metacomplete* if A is metarecursively enumerable and

$$(B)[B \text{ meta r.e.} \rightarrow B \leq_M A].$$

4.5 Proposition. *A is metacomplete iff $O \leq_w A$.*

Proof. Suppose $O \leq_w A$ and B is meta r.e. By Proposition 3.5, it is safe to assume

$$B \leq_w A \leftrightarrow B \leq_M A.$$

Let n: $\omega_1^{CK} \rightarrow \omega$ be one-one and metarecursive. Then $n[B]$ is Π_1^1, hence $n[B] \leq_f O$ by Theorem 5.4.I. But $B \leq_f n[B]$, so $B \leq_w A$. □

4.6 Theorem (Owings 1969). *If D is a metarecursively enumerable ω-set, then there exists a metacomplete Π_1^1 set C such that $C \leq_f D$.*

Proof. Let n be a one-one, metarecursive map of ω_1^{CK} onto O. Define

$$t(x) = \mu\delta[\delta \notin D \ \& \ n(\delta) > x] \quad (x < \omega),$$
$$B = \{\delta \mid \delta < tn(\delta)\}, \quad \text{and} \quad C = n[B].$$

t is not metarecursive, so an approximation argument is needed to show C is Π_1^1. Let D be the range of some metarecursive function f. Define

$$D^\sigma = \{f(\tau) \mid \tau < \sigma\}, \quad \text{and}$$
$$t(\sigma, x) = \mu\delta[\delta \notin D^\sigma \ \& \ n(\delta) > x].$$

$t(\sigma, x)$ is metarecursive. More importantly, for each x, $t(\sigma, x)$ is a non-decreasing function of σ whose limit is $t(x)$. Consequently

$$B = \{\delta \mid (E\sigma)[\delta < t(\sigma, n(\delta))]\},$$

hence B is metarecursively enumerable and C is Π_1^1.

Every metafinite subset of cD is finite, so $t \leq_f D$ by Exercise 4.7. It follows that $C \leq_f D$, since for all $x < \omega$,

$$x \notin C \leftrightarrow x \notin n[t(x)].$$

For the metacompleteness of C, assume for the moment that B is an ω-set. By Proposition 4.5 it is enough to show $O \leq_w C$. Recall $x \in O$ iff $2^x \in O$. If $x \in O$, then

$$n^{-1}[\{x, 2^x, 2^{2^x}, \ldots\}].$$

is metafinite, hence touches B because B is an ω-set. Consequently C touches $\{x, 2^x, \ldots\}$. Conversely, if C touches $\{x, 2^x, \ldots\}$, then $x \in O$ because $C \subseteq O$.

To see that B is an ω-set, suppose there is a $\gamma < \omega_1^{CK}$ such that $\gamma - B$ is infinite. Then

$$\{\delta | \delta < \gamma \ \& \ \delta \geq tn(\delta)\}$$

is infinite, and so $\{x | x < \omega \ \& \ t(x) < \gamma\}$ is infinite. Fix x so that $t(x) < \gamma$ and $x \geq \max(n[\gamma - D])$. Such an x exists because D is an ω-set. $t(x) \in \gamma - D$, so $x \geq nt(x)$. But the definition of t implies $nt(x) > x$. □

4.7–4.8 Exercises

4.7. Suppose C is metarecursively enumerable and every metafinite subset of cC is finite. Show $X \leq_w C$ implies $X \leq_f C$. ($\leq_f$ is defined in the proof of Theorem 4.3.)

4.8. Call A *simple* if A is metarecursively enumerable, cA is unbounded, and every metarecursively enumerable subset of cA is bounded. Show each non-metarecursive, meta r.e. set has the same metadegree as some simple set.

Chapter VI
Hyperregularity and Priority

The notion of hyperregularity is introduced and studied with the help of priority arguments. Two solutions of Post's problem are obtained, as well as Simpson's dichotomy, for the metarecursively enumerable degrees.

1. *Hyperregular Sets*

Let $A \subseteq \omega_1^{CK}$. A is said to be hyperregular if for every $f \leq_w A$ and γ, $f[\gamma]$ is bounded. The concept of hyperregularity, suitably generalized, figures prominently in the solutions of Post's problem offered in Part C. Its relation to Σ_1 admissibility is made clear in Chapter VII. (For those who cannot wait: Assume A is regular. Then A is hyperregular iff $\langle L(\omega_1^{CK}), A\rangle$ is Σ_1 admissible.)

$$f(m) = \begin{cases} |m| & m \in O \\ 1 & m \notin O, \end{cases}$$

is weakly metarecursive in O and maps ω onto ω_1^{CK}. The main result of this section is the existence of a metarecursively enumerable set that is hyperregular but not metarecursive.

In general a hyperregular set need not be regular, but the next lemma says otherwise for the case of maximum interest.

1.1 Lemma. *If A is metarecursively enumerable and hyperregular, then A is regular.*

Proof. Suppose $A \cap \gamma$ is not metafinite. Let g metarecursively enumerate $A \cap \gamma$ without repetitions. Define

$$f(\delta) = \begin{cases} 0 & \text{if } \delta \geq \gamma \text{ or } \delta \notin A, \\ \sigma & \text{if } \delta < \gamma \text{ and } g(\sigma) = \delta. \end{cases}$$

Then $f \leq_w A$ and $f[\gamma] = \omega_1^{CK}$. □

If a set is both regular and hyperregular, then it behaves as tamely as possible with respect to reducibility, as is expressed by the next lemma, and by Exercise 1.7.

1.2 Lemma. *Suppose A is regular and hyperregular.*

(i) If $f \leq_w A$ and $\gamma < \omega_1^{CK}$, then $f \restriction \gamma$ is metafinite.
(ii) If $f \leq_w A$, then $f \leq_M A$.
(iii) If $B \leq_w A$, then B is regular and hyperregular.

Proof

(i) Let ϕ be metarecursive and such that

$$f(\beta) = \delta \leftrightarrow (E\rho)(\mathrm{EH})(\mathrm{EJ})[\phi(\rho,\beta) = \langle H,J,\delta\rangle \ \&\ H \subseteq A \ \&\ J \subseteq cA].$$

Let $g(\beta)$ be

$$(1) \qquad \mu\rho(\mathrm{EH})(\mathrm{EJ})(E\rho)[\phi(\rho,\beta) = \langle H,J,\delta\rangle \ \&\ H \subseteq A \ \&\ J \subseteq cA].$$

$g(\beta)$ is the "least computation" of $f(\beta)$ from A via ϕ. To see that $g \leq_w A$, let $\phi(\rho,\beta)$ be $\langle H^\rho_\beta,\ J^\rho_\beta,\ \delta^\rho_\beta\rangle$ and define

$$t(\beta) = \sup{}^+\{H^\rho_\beta \cup J^\rho_\beta \mid \rho \leq g(\beta)\}.$$

($\sup^+$ is the strict least upper bound.) Since A is regular, $A \cap t(\beta)$ is metafinite. The value of $g(\beta)$ can be obtained by substituting $A \cap t(\beta)$ for A in (1). Thus the value of $g(\beta)$ is determined by a metafinite set of facts about A, and so $g \leq_w A$.

It follows from the hyperregularity of A that $g[\gamma]$ is bounded, and consequently that $t[\gamma]$ is bounded by some $s(\gamma)$. The values of $f(\beta)$ $(\beta < \gamma)$ are computable from the metafinite set $A \cap s(\gamma)$ via ϕ, hence $f \restriction \gamma$ is metafinite.

(ii) The proof of (i) showed that $f \restriction \gamma$, regarded as a function of γ, is weakly metarecursive in A. It follows that $f \leq_M A$.

(iii) If $h \leq_w B$ and $B \leq_w A$, then $h \leq_w A$ by (ii). So B is hyperregular because A is. Fix γ and define

$$k(\delta) = \begin{cases} \delta + 1 & \text{if } \ \delta \in B \cap \gamma \\ 0 & \text{otherwise.} \end{cases}$$

By (i) $k[\gamma]$ is metafinite, hence $B \cap \gamma$ is. □

The next result is needed for Simpson's dichotomy (Section 3), and is proved by a *recursive approximation* technique.

1.3 Lemma (Simpson 1971). *If A is metarecursively enumerable but not hyperregular, then some non-hyperarithmetic Π^1_1 C is weakly metarecursive in A.*

Proof. By Owings' ω-sets theorem (4.6) it suffices to find a metarecursively enumerable ω-set $D \leq_w A$. Since A is not hyperregular, there exists an $f\colon \omega \to \omega_1^{CK}$ such that $f \leq_w A$, f is strictly increasing, and $f[\omega]$ is unbounded. Let D be the complement of the range of f. The fact that f is strictly increasing on ω implies

$D \leq_w A$. A metarecursive approximation of f is needed to show D is metarecursively enumerable.

Let ϕ be metarecursive and such that

$$f(n) = \delta \leftrightarrow (\mathrm{EH})(\mathrm{EJ})(E\sigma)[\phi(n,\sigma) = \langle H, J, \delta \rangle \quad \& \quad H \subseteq A \quad \& \quad J \subseteq cA]$$

for all $n < \omega$. f is approximated by guessing at A. Suppose g is metarecursive and $g[\omega_1^{\mathrm{CK}}] = A$. Let

$$A^\delta = \{g(\beta) | \beta < \delta\}.$$

Thus A is the union of a nondecreasing, metarecursive sequence of metafinite sets. Define

$$f(n,\tau) = \delta \leftrightarrow (\mathrm{EH})\,(\mathrm{EJ})(E\sigma)_{\sigma < \tau}[\phi(n,\sigma) = \langle H, J, \delta \rangle \ \& \ H \subseteq A^\tau \quad \& \quad J \subseteq cA^\tau].$$

As defined above, $\lambda n \tau | f(n,\tau)$ is a many-valued, partial metarecursive function. Make it total on $\omega \times \omega_1^{\mathrm{CK}}$ by setting $f(n,\tau) = 0$ when no $\sigma < \tau$ satisfies the defining condition. Make it single-valued by insisting on the least such σ when there is one. By the regular sets theorem (4.3) A may be taken to be regular. It follows that

$$(\gamma)(E\sigma)(\tau)_{\tau \geq \sigma}[A^\tau \cap \gamma = A \cap \gamma].$$

Consequently

$$(1) \qquad (n)\,(E\sigma)(\tau)_{\tau \geq \sigma}[f(n,\tau) = f(n)].$$

In short $\lim_\tau f(n,\tau) = f(n)$. The function $f(n,\tau)$ is said to metarecursively approximate the function $f(n)$.

For the moment assume (2):

(2a) for each n, $\lambda\tau | f(n,\tau)$ is nondecreasing;

$$(2b) \qquad (n)\,(\sigma)(\tau)[\sigma > \tau \quad \text{and} \quad f(n,\sigma) > f(n,\tau) \rightarrow f(n,\sigma) \geq \sup_{m < \omega} f(m,\tau)].$$

If (3) holds, then D is metarecursively enumerable.

$$(3) \qquad \delta \in D \leftrightarrow (E\tau)(\mathrm{Em})[f(m,\tau) > \delta \quad \text{and} \quad (n)_{n < m}(\delta \neq f(n,\tau))]$$

To verify (3) first assume $\delta \in D$. Thus $\delta \neq f(n)$ for any n. Hence there exist m and τ such that

$$f(m) > \delta \quad \text{and} \quad (n)_{n \leq m}(f(n) = f(n,\tau)).$$

It follows from (2a) that m and τ satisfy the right side of (3). Now fix n and suppose $f(n)$ substituted for δ satisfies the right side of (3). Thus there exist τ and m such that

$$f(m,\tau) > f(n) \quad \text{and} \quad (p)_{p < m}(f(n) \neq f(p,\tau)).$$

Since f is strictly increasing, it follows from (2a) that $m > n$. Hence

$$f(m,\tau) > f(n) \neq f(n,\tau).$$

But then (2) implies $f(n,\sigma) \geq f(m,\tau)$ for some $\sigma > \gamma$; hence $f(n,\sigma) > f(n)$, an impossibility according to (1) and (2a).

For the sake of (2) define

$$h(n,\tau) = \begin{cases} \sup\limits_{\delta<\tau} h(n,\delta) & \text{if } f(n,\tau) \leq \sup\limits_{\delta<\tau} h(n,\delta), \\ f(n,\tau) \cup \sup\limits_{\substack{\delta<\tau \\ m<\omega}} h(m,\delta) & \text{otherwise.} \end{cases}$$

Let $h(n) = \lim\limits_{\tau} h(n,\tau)$. $h(n,\tau)$ obeys (2). $h \leqslant_w A$ and $h([\omega]$ is unbounded. □

1.4–1.10 Exercises

1.4. Suppose every set weakly metarecursive in A is regular. Show A is hyperregular.

1.5. Find a set that is hyperregular but not regular. Find such a set metarecursive in O.

1.6. (Macintyre 1969). Find a set that does not have the same metadegree as any regular set. Find such a set metarecursive in O.

1.7. Recall the definitions of $C(E,B)$ and $C_m(E,B)$ given in Exercise 3.8 of Chapter V. B is said to be *subgeneric* if

$$C(\omega_1^{CK}+1, E, B) = C(\omega_1^{CK}, E, B).$$

The computations from B via E of nonrecursive ordinal height add nothing to $C(E,B)$. (In Chapter VII it will be seen that B is subgeneric iff $\langle L[B, \omega_1^{CK}], B\rangle$ is Σ_1 admissible.) Show every subgeneric set is hyperregular.

1.8. Find a subgeneric set that is not regular.

1.9. Assume B is regular and hyperregular. Show B is subgeneric. Show $C(E,B) = C_m(E,B)$.

1.10. Find a hyperregular set that is not subgeneric.

2. *Two Priority Arguments*

The main result of this section is the existence of a hyperregular, metarecursively enumerable set A that is not metarecursive. It follows from Lemma 2.1 that A is not metacomplete and hence constitutes a positive solution to Post's problem for the

metarecursively enumerable degrees. The construction of A is a priority argument in the sense originated by Friedberg 1957b and Muchnik 1956. It suggests that the generalization of finite to metafinite, and r.e. to meta r.e., is solid enough to support the combinatorics associated with nontrivial dynamic arguments of classical recursion theory.

A secondary result of this section is the construction by means of a priority argument of a non-hyperarithmetic Π^1_1 set B of lower metadegree than Kleene's O. The construction of B is intended to suggest that metarecursion theory can be applied to prove theorems about Π^1_1 sets analogous to classical results about recursively enumerable sets. One such is the existence of a maximal Π^1_1 set (Exercise 2.9). It has the virtue of being expressible without any reference to notions of metarecursion theory.

2.1 Theorem (Sacks 1966). *There exists a hyperregular, metarecursively enumerable set that is not metarecursive.*

Proof. Recall the partial metarecursive function $\phi_1(e,x)$ of Theorem 1.4.V. Let R_e be the range of $\lambda x|\phi_1(e,x)$. Then $\{R_e|e<\omega\}$ is an enumeration of the metarecursively enumerable sets. Recall that

$$\phi_1(e,x) \simeq y \leftrightarrow (f)(\mathrm{Eu}) \sim T(\bar{f}(u), g(e), n(x), n(y)). \tag{1}$$

Since the right side of (1) is Π^1_1, it is equivalent to

$$h(g(e), n(x), n(y)) \in O$$

for some recursive function h. Define

$$y \in R^\sigma_e \quad \text{by} \quad (\mathrm{Ex})_{x<\sigma}[|h(g(e), n(x), n(y))| < \sigma].$$

Then R^σ_e is metafinite, $\lambda\sigma e|R^\sigma_e$ is metarecursive, and $R_e = \bigcup_\sigma R^\sigma_e$. $\lambda\sigma e|R^\sigma_e$ provides a simultaneous metarecursive enumeration of the metarecursively enumerable sets.

Say $\{e\}^A(\gamma)$ is defined and equal to δ iff

$$(\mathrm{EH})(\mathrm{EJ})[\langle H, J, \gamma, \delta\rangle \in R_e \ \&\ H \subseteq A \ \&\ J \subseteq cA.] \tag{2}$$

$\{e\}^A$ is a many-valued, partial function. If $\{\mathrm{e}\}^A$ is single-valued and total, then $\{e\}^A \leq_w A$. Conversely, if $f \leq_w A$, then $f = \{e\}^A$ for some $e < \omega$.

The theorem is proved by metarecursively enumerating a set A with three objectives in mind.

(3) cA is unbounded.

(4) If R_e is unbounded, then $R_e \cap A \neq \varnothing$

(5) If $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$, then $\lambda n|\{e\}^{\mathrm{A}}(n)$ is metafinite.

(3) and (4) combine to show cA is not metarecursively enumerable. Hence A is not metarecursive, as in Post's simple set construction in classical recursion theory. (5) implies A is hyperregular, since any infinite recursive ordinal can be put into a matafinite, one-to-one correspondence with ω.

A will be the union of a non-decreasing, metarecursive sequence A^σ ($\sigma < \omega_1^{CK}$) of metafinite sets. Let $A^{<\sigma}$ denote $\cup\{A^\delta \mid \delta < \sigma\}$. Say $\{e\}_\sigma^{A^{<\sigma}}(\gamma)$ is defined and equal to δ iff

$$(6) \qquad (EH)(EJ)[\langle H, J, \gamma, \delta\rangle \in R_e^\sigma \quad \& \quad H \subseteq A^\sigma \quad \& \quad J \subseteq cA^{<\sigma}].$$

$\{e\}^{A^{<\sigma}}(\gamma)$ is a many-valued, partial metarecursive function of e, σ and γ.

If $H \subseteq A$, then $H \subseteq A^\sigma$ for all sufficiently large σ. Hence any $\langle H, J, \gamma, \delta\rangle$ that satisfies the matrix of (2) also eventually satisfies the matrix of (6). Thus $\{e\}^A(\gamma) = \delta$ implies $\{e\}_\sigma^{A^{<\sigma}}(\gamma) = \delta$ for all sufficiently large σ.

For the sake of objectives (4) and (5) attempts are made during the enumeration of A to satisfy the following *requirements*.

Req $2e$: If R_e is unbounded, then there is a σ such that $R_e \cap A^\sigma \neq \varnothing$.

Req $2e + 1$: If $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$, then

$$(7) \qquad \lambda n[\{e\}^A(n) = \lambda n[\{e\}^{A^{<\sigma}}(n)$$

for all sufficiently large σ.

An even-numbered requirement is said to be positive, because it is met by adding an element to A. An odd-numbered requirement is said to be negative, because it is met by keeping every member of some metafinite set out of A. To elaborate, suppose a σ can be found such that the right side of (7) is defined for all $n < \omega$. Then req $2e + 1$ can be met by not adding any element of $cA^{<\sigma}$ *to* A at stage τ for any $\tau \geq \sigma$. This strategy, if successful, preserves forever after computations established at the beginning of stage σ. Consequently an odd-numbered requirement is sometimes called a preservation requirement.

Positive and negative requirements have a potential for conflict. Suppose δ is added to A at stage τ to meet requirement $2e_0$. Suppose further that $\delta \notin A^{<\sigma}$ is a negative fact needed for some computation developed at stage $\sigma < \tau$ and being preserved in the hope of meeting req $2e + 1$. Such an event is termed an *injury* to req $2e + 1$ for the sake of req $2e_0$. The frequency of injury is minimized by Friedberg–Muchnik priorities. Req $2e_0$ is said to be of higher *priority* than req $2e + 1$ if $e_0 \leq e$. An injury to one requirement for the sake of another is allowed to happen only if the injured requirement has lower priority than the other. Thus adding an element to A for the sake of req $2e_0$ never injures req $2e + 1$ when $e < e_0$.

A recursion on σ simultaneously defines metarecursive functions $\lambda\sigma|A^\sigma$, $\lambda\sigma e|m(\sigma, e)$ and $\lambda\sigma e|r(\sigma, e)$. m guides the preservation requirements, and r insures that cA is unbounded.

Stage σ. If $\{e\}_\sigma^{A^{<\sigma}}(n)$ is defined, let $\langle H_{e,n}^\sigma, J_{e,n}^\sigma\rangle$ be the least $\langle H, J\rangle$ that satisfies

$$(E\delta)[\langle H, J, n, \delta\rangle \in R_e^\sigma \quad \& \quad H \subseteq A^{<\sigma} \quad \& \quad J \subseteq cA^{<\sigma}].$$

("least $\langle H, J\rangle$" is given a precise meaning by referring to some metarecursive one-to-one correspondence between ω_1^{CK} and the set of all metafinite $\langle H, J\rangle$'s.) Define

$$n_e = \mu m_{m \le \omega}(n)_{n<m}[\{e\}_\sigma^{A^{<\sigma}}(n) \text{ is defined}],$$

$$m(\sigma, e) = \mu\beta[(\delta)_{\delta<\sigma}(\beta \ge m(\delta, e) \quad \& \quad \beta \ge \sup\{J^\sigma_{e,n} | n < n_e\})]$$

(When possible, $\delta \le m(\sigma, e)$ is kept out of $A^\tau(\tau \ge \sigma)$ in order to preserve $\{e\}_\sigma^{A^{<\sigma}}(n)$ for all $n < n_e$.)

Let p: $\omega_1^{CK} \to \omega$ be a one-one, into, metarecursive function. Define

$$r(\sigma, e) = \mu\beta[\beta \notin A^{<\sigma} \ \& \ \beta \ge \sup_{\delta<\sigma} r(\delta, e) \ \& \ (e \in p[\sigma] \to \beta \ge p^{-1}(e))].$$

Stage σ is completed by attending to req $2e_0$, where $e_0 = h(\sigma)$ and h is a metarecursive function that enumerates every finite ordinal unboundedly often. If

$$R^\sigma_{e_0} \cap A^{<\sigma} = \varnothing, \tag{8}$$

and there is a β such that

$$\beta \in R^\sigma_{e_0} \ \& \ (e)_{e<e_0}[m(\sigma, e) < \beta \ \& \ r(\sigma, e) < \beta], \tag{9}$$

then add the least such β to A. Otherwise $A^\sigma = A^{<\sigma}$.

Condition (8) is all important. It implies:

(10) For each e_0, there is at most one stage σ at which an ordinal is added to A for the sake of req $2e_0$.

If β is added to A at stage σ and $\beta \le m(\sigma, e)$, then that act is said to *injure* req $2e + 1$. (9) and (10) imply:

(11) For each e the set of stages at which req $2e + 1$ is injured has cardinality at most $e + 1$.

To prove (5) fix e and assume $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$. It follows that for each $m < \omega$, and all sufficiently large σ,

$$(n)_{n<m}[\{e\}_\sigma^{A^{<\sigma}}(n) \text{ is defined}]. \tag{12}$$

By (11) there is a stage σ_e after which req $2e + 1$ is never injured. For each $m < \omega$, let $\sigma(m)$ be the least $\sigma > \sigma_e$ that satisfies (12). The "least" computations that make (12) true when $\sigma = \sigma(m)$ are preserved forever after. $\lambda m | \sigma(m)$ is metarecursive, hence $\{\sigma(m) | m < \omega\}$ is bounded above by some τ. Consequently,

$$(n)_{n<\omega}[\{e\}_\tau^{A^{<\tau}}(n) = \{e\}^A(n)],$$

and (5) follows.

To prove (3), fix e and consider the behavior of $r(\sigma,e)$ as σ increases. $r(\sigma,e)$ is nondecreasing. It increases at at most $e+2$ stages: once if $e \in p[\omega_1^{CK}]$, and at most once for the sake of req $2e_0$ for each $e_0 \leq e$. Let

$$r(e) = \lim_{\sigma} r(\sigma,e). \tag{13}$$

The set $\{r(e) | e < \omega\}$ is unbounded, since $r(p(\gamma)) \geq \gamma$ for all $\gamma < \omega_1^{CK}$. For all σ and e, $r(\sigma,e) \notin A^{<\sigma}$. Hence for all e, $r(e) \notin A$.

To prove (4), fix e_0 and assume R_{e_0} is unbounded. To find a β that satisfies (9), it suffices by (13) to show

$$\lim_{\sigma} m(\sigma,e) \text{ exists} \tag{14}$$

for each $e < e_0$. Fix e. Let d_e be the strict upper bound of all m such that $\sigma(m)$, as defined above in the proof of (5), exists. Thus $d_e \leq \omega$. The set $\{\sigma(m) | m < d_e\}$ is either finite, or metafinite as in the proof of (5), hence bounded above by some τ. It follows that $m(\sigma,e) = m(\tau,e)$ for all $\sigma \geq \tau$. □

2.2 The Projectum. Recall the function p: $\omega_1^{CK} \to \omega$ from the proof of Theorem 2.1. It is metarecursive, one-one and into. It injects ω_1^{CK} into ω in a manner effective enough to allow requirements to be indexed by finite ordinals rather than recursive ordinals. It is conventional to sum up the situation by calling ω the projectum of ω_1^{CK}. The role of the projectum in the proof of Theorem 2.2 is seen by considering I_e, the set of stages at which req $2e+1$ is injured. The construction of A is such that I_e is metarecursively enumerable (uniformly in e). To make the proof of 2.2 work it was necessary to bound I_e. The bound followed from the finiteness of I_e, which in turn followed from the finiteness of e. It would have been enough for I_e to have been metafinite. If requirements had been indexed by recursive ordinals, then I_δ, when $\delta \geq \omega$, would have had little chance of being finite or even metafinite, although it would have still been metarecursively enumerable.

The notion of projectum is central in the priority arguments of Part C, where ω_1^{CK} is replaced by an arbitrary Σ_1 admissible ordinal.

2.3 Corollary. *There exists a metarecursively enumerable set that is neither metarecursive nor metacomplete.*

Proof. Let A be an instance of Theorem 2.1. A is not metacomplete by Proposition 4.5.V and Lemma 1.2(iii). □

The proof of Corollary 2.3 provides a hyperregular solution to Post's problem for the metarecursively enumerable degrees. The next theorem supplies a Π_1^1 solution B. The construction of B draws on priorities as in the proof of Theorem 2.2, but there is an additional difficulty. Since $B \subseteq \omega$, a metafinite computation of B can involve unboundedly much of B. As a result the positive and negative requirements can crowd each other more severely than they did above.

Theorem 2.4 should be compared with Spector's Theorem (7.2.II): Kleene's O is hyperarithmetic in every non-hyperarithmetic Π^1_1 set.

2.4 Theorem (Sacks 1966). *There exists a non-hyperarithmetic Π^1_1 set B such that Kleene's O is not weakly metarecursive in B.*

Proof. Two Π^1_1 sets, B and C, are simultaneously metarecursively enumerated with the following objectives in mind.

(1) $\omega - B$ is infinite.

(2) If Q is Π^1_1 and infinite, then $Q \cap B \neq \varnothing$.

(3) $C \not\leq_w B$.

(1) and (2) imply B is not hyperarithmetic. It follows from (3) that $O \not\leq_w B$, since C is many-one reducible to O via a recursive function.

Recall R_e and R_e^σ from the beginning of the proof of Theorem 2.1. Let $Q_e = R_e \cap \omega$ and $Q_e^\sigma = R_e^\sigma \cap \omega$. Then $\{Q_e | e < \omega\}$ is an enumeration of the Π^1_1 sets; $\lambda\sigma e | Q_e^\delta$ is metarecursive; and $Q_e = \bigcup_\sigma Q_e^\sigma$. Define $\{e\}^B(n)$ as in Theorem 2.1, formula (2), with A replaced by B and γ by n. B will be the union of a non-decreasing, metarecursive sequence $B^\sigma (\sigma < \omega_1^{CK})$ of metafinite subsets of ω. Define $\{e\}_\sigma^{B^{<\sigma}}(n)$ as in Theorem 2.1, formula (6), with A replaced by B and γ by n.

Objectives (2) and (3) lead to the following requirements.

Req $2e$: If Q_e is infinite, then $Q_e^\sigma \cap B^\sigma \neq \phi$ for some σ.

Req $2e+1$: There exists an n such that if $\{e\}^B(n)$ is defined and singlevalued, then $C(n) \neq \{e\}^B(n)$. ($C(n)$ is the characteristic function of C.) The construction is a recursion on σ that simultaneously defines metarecursive functions $\lambda\sigma | B^\sigma$, $\lambda\sigma | C^\sigma$, $\lambda\sigma e | r(\sigma, e)$, $\lambda\sigma e | n(\sigma, e)$ and $\lambda\sigma e | J(\sigma, e)$. For each e, $r(\sigma, e)$ converges to the e-th member of $\omega - B$, and $n(\sigma, e)$ converges to an n that satisfies req $2e+1$. $J(\sigma, e)$ is a finite subset of $cB^{<\sigma}$ associated with the preservation of $\{e\}^{B^{<\sigma}}(n(\sigma, e))$.

Let $\lambda mn | z(m, n)$ be a one-one recursive function that maps ω^2 onto ω.

Stage $\sigma = 0$. Set $B^0 = C^0 = \varnothing$, $J(0, e) = \varnothing$, $r(0, e) = e$ and $n(0, e) = z(0, e)$.

Stage $\sigma > 0$. (For any $\lambda\sigma x | f(\sigma, x)$, define

$$f(<\sigma, x) = \sup\{f(\delta, x) | \delta < \sigma\}.)$$

Assume

(4) $$\omega - B^{<\sigma} \text{ is infinite } \&\ r(<\sigma, e) < \omega \quad \&\quad n(<\sigma, e) < \omega$$

for all e. (Assumption (4) will be proved by induction on $\delta \leq \sigma$ in a moment.) Define by induction on e

(5) $$r(\sigma, e) = \mu m[m \in (\omega - B^{<\sigma}) \ \&\ m \geq r(<\sigma, e) \ \&\ (i)_{i<e}(m > r(\sigma, i))].$$

Let h be a metarecursive function that enumerates all finite ordinals unboundedly often. Stage σ splits into two cases corresponding to even and odd numbered requirements.

Case 0: $h(\sigma) = 2e$. If $Q_e^\sigma \cap B^{<\sigma} = \varnothing$ and there is a p such that

$$p \in Q_e^\sigma \quad \& \quad (i)_{i<e}[p \notin J(<\sigma,i) \quad \& \quad p > r(\sigma,i)], \tag{6}$$

then add the least such p to $B^{<\sigma}$ to obtain B^σ, and define

$$n(\sigma,i) = n(<\sigma,i) \text{ for all } i < e; \tag{7}$$

$$n(\sigma,i) = ux[x > n(<\sigma,i) \quad \& \quad (\mathrm{Em})(x = z(i,m))]$$

for all $i \geq e$.

Otherwise $B^\sigma = B^{<\sigma}$ and $n(\sigma,i) = n(<\sigma,i)$ for all i.

Case 1: $h(\sigma) = 2e + 1$. Let $B^\sigma = B^{<\sigma}$ and $n(\sigma,i) = n(<\sigma,i)$ for all i. Define

$$\{e\}^{B<\sigma}_{\sigma,J}(n) \tag{8}$$

as in the proof of Theorem 2.1, formula (6), with A replaced by B, γ by n, and $cA^{<\sigma}$ by $\omega - B^{<\sigma}$, and with the restriction that J be finite. The restriction on J is important because a typical metafinite computation from $B^{<\sigma}$ might mention all of ω and prevent any p from satisfying (6). Of course there is now the fear that (8) is not an adequate approximation of $\{e\}^B(n)$. The fear will become groundless, after it is shown that B satisfies the even-numbered requirements. If

$$n(\sigma,e) \notin C^{<\sigma} \quad \& \quad \{e\}^{B<\sigma}_{\sigma,J}(n(\sigma,e)) = 0, \tag{9}$$

then add $n(\sigma,e)$ to $C^{<\sigma}$ to obtain C^σ, and let $J(\sigma,e)$ be the "least" finite subset of $\omega - B^{<\sigma}$ that includes $J(<\sigma,e)$ and is needed for the computation of the second half of (9).

If (9) is false, then $C^\sigma = C^{<\sigma}$ and $J(\sigma,e) = J(<\sigma,e)$. End of construction. In both cases it is understood that no function changes unless otherwise indicated. For example, in case (1), $r(\sigma,e) = r(<\sigma,e)$.

The first order of business is to prove (4). Assume $\omega - B^{<\delta}$ is infinite & $r(<\delta,e) < \omega$ & $n(<\delta,e) < \omega$ for all $\delta < \sigma$. Fix e and trace the behavior of $r(\delta,e)$ as δ increases from 0 to σ. Each change corresponds to the addition of an element to B for the sake of req. $2e_0$ for some $e_0 \leq e$. At most one such addition occurs on behalf of req. $2e_0$ thanks to the clause, "$Q^\delta_{e_0} \cap B^{<\delta} = \phi$", which precedes any such addition at stage δ. Thus $r(\delta,e)$ charges at most $e+1$ times, and so $r(<\sigma,e) < \omega$. Then $\omega - B^{<\sigma}$ is infinite, since the range of $\lambda e | r(<\sigma,e)$ is infinite and is contained in $\omega - B^{<\sigma}$. The behavior of $n(\delta,e)$ as δ increases from 0 to σ is similar to that of $r(\delta,e)$, so $n(<\sigma,e) < \omega$.

The above reasoning also shows $r(\sigma,e)$ changes only finitely often as σ approaches ω_1^{CK}. Let

$$r(e) = \lim_\sigma r(\sigma,e)$$

Then $\omega - B$ is infinite, since the range of $\lambda e|r(e)$ is infinite and is contained in $\omega - B$.

$J(\sigma, e)$ changes only finitely often as σ approaches ω_1^{CK}, since each such change is occasioned by an attempt to satisfy req $2e_0$ for some $e_0 \leq e$. Since each $J(\sigma, e)$ is finite, $\lim_{\sigma} J(\sigma, e)$ must be finite, and so there is no substantial obstacle to satisfying an even-numbered requirement. Consequently $\omega - B$ contains no infinite meta-finite set, and so (8) is an excellent approximation of $\{e\}^B(n)$.

Fix e to show req $2e + 1$ is met. Let n be $\lim_{\sigma} n(\sigma, e)$, and let σ_0 be the stage at which the final value of $n(\sigma, e)$ is attained. If (9) holds for some $\sigma \geq \sigma_0$ such that $h(\sigma) = 2e + 1$, then $C(n) = 1$. Also

$$J(\sigma, e) \subseteq \omega - B,$$

because otherwise (7) would force an increase in $\lambda\sigma|n(\sigma, e)$ at some stage after stage σ. Hence

$$\{e\}^{B<\sigma}_{\sigma,f}(n) = \{e\}^B(n) = 0.$$

Suppose (9) fails for all $\sigma \geq \sigma_0$. Then $n \notin C$, because $n(\sigma_0, e) \notin C^{<\sigma_0}$, and because the one-one-ness of $\lambda jm|z(j, m)$ prevents n from being put in C for the sake of req $2j + 1$ when $j \neq e$. It follows that $\{e\}^B(n)$ cannot equal 0, since otherwise (9) would hold for all sufficiently large σ. □

The next result, promised in subsection 3.1.V., is the failure of $\leq_w$ to be transitive for Π^1_1 sets. Note that Lemmas 1.1 and 1.2 imply $\leq_w$ is transitive on hyperregular, metarecursively enumerable sets.

2.5 Corollary (Driscoll 1968). *There exist Π^1_1 sets A, B and C such that $A \leq_M B$ and $B \leq_w C$, but $A \not\leq_w C$.*

Proof. By Theorem 2.4 there are Π^1_1 sets A and C such that $A \not\leq_w C$ and C is not hyperarithmetic. Let g metarecursively enumerate C without repetitions, and let Dg be the deficiency set of g. Dg is an ω-set, as defined in subsection 4.4.V. By Theorem 4.6.V there is a metacomplete Π^1_1 set B such that $B \leq_f D_g$. Hence $A \leq_M B$. According to Lemma 4.2.(v).V, $D_g \leq_M C$, so $B \leq_w C$. □

2.6 Further Results. The priority arguments of Theorems 2.3 and 2.4 can be easily modified to obtain a familiar form of the solution to Post's problem, the existence of two metarecursively enumerable sets (and two Π^1_1 sets) such that neither is weakly metarecursive in the other. The e-th negative requirement is injured at most 2^e times, rather than $e + 1$ times as in 2.3.

Call a Π^1_1 set B *maximal* if $\omega - B$ is infinite and every Π^1_1 superset of B differs finitely from either B or ω. The Friedberg maximal set construction of classical recursion theory lifts to the Π^1_1 case via metarecursion theory. As in the proof of Theorem 2.4 requirements are indexed by finite ordinals. Unlike the classical case B

will be infinite at intermediate stages of the construction, and so some care is needed to keep $\omega - B$ infinite.

Driscoll 1968 showed that the metarecursively enumerable degrees are dense. His argument is modeled on the classical case, but he makes several departures. Suppose B and C are metarecursively enumerable, and $B <_M C$. By Theorem 4.3.V it is safe to assume B and C are regular. Without regularity it seems difficult to begin looking for D such that $B <_M D <_M C$. If B is hyperregular, then Driscoll's argument is close to the classical case. If B is not hyperregular, then he observes there is an $f \leq_w B$ such that f is a one-one map of ω onto ω_1^{CK}. Then f is used to make ω, rather than ω_1^{CK}, the domain of preservation requirements associated with D. The regularity of B is used to develop a suitable metarecursive approximation of f. Driscoll's theorem is a special case of Shore's density theorem proved at the end of Part C.

2.7–2.12 Exercises

2.7. Find two hyperregular, metarecursively enumerable sets such that neither is weakly metarecursive in the other.

2.8. Find two Π_1^1 sets such that neither is weakly metarecursive in the other.

2.9. Find a maximal Π_1^1 set. (A Π_1^1 set A is *maximal* iff $\omega - A$ is infinite and for all Π_1^1 B, if $A \subseteq B \subseteq \omega$, then $B - A$ or $\omega - B$ is finite.)

2.10. For each $B \subseteq \omega_1^{CK}$ define $C^+(E, B)$ as $C(E, B)$ was in Exercise 3.8.V. save that computations of arbitrary height are allowed. Kreisel calls A *computable* from B (in symbols $A \leq_c B$) if there is an E with principal function letter f such that

$$\delta \in A \leftrightarrow (f(\underline{\delta}) = \underline{0}) \in C^+(E, B)$$

$$\delta \notin A \leftrightarrow (f(\underline{\delta}) = \underline{1}) \in C^+(E, B)$$

for all $\delta < \omega_1^{CK}$. Show $\leq_c$ is transitive. Show there are only two c-degrees of Π_1^1 sets.

2.11. Suppose B is regular and hyperregular, and $A \leq_c B$. Show $A \leq_w B$.

2.12. Find two metarecursively enumerable sets such that neither is computable from the other.

3. *Simpson's Dichotomy*

Let $\mathscr{RE}$ be the set of all metarecursively enumerable degrees and $\mathscr{H}$ the set of all members of $\mathscr{RE}$ with hyperregular representatives. Lemmas 1.1 and 1.2 imply $\mathscr{H}$ is an initial segment of $\mathscr{RE}$. $\mathscr{RE} - \mathscr{H}$ is non-empty, since it contains the meta-degrees

of all non-hyperarithmetic Π^1_1 sets. Simpson's dichotomy says it contains nothing else.

3.1 Proposition. *Suppose B is a non-hyperarithmetic Π^1_1 set. Then there exists a simple Π^1_1 set C such that $B \equiv_f C$.*

Proof. f is a one-one, metarecursive function whose range is B, and $\{H_\sigma | \sigma < \omega_1^{CK}\}$ is a metarecursive enumeration of all infinite, hyperarithmetic sets. C will be the union of a non-decreasing, metarecursive sequence $\{C^\sigma | \sigma < \omega_1^{CK}\}$ of hyperarithmetic sets.

$C^0 = \varnothing$ and $C^\lambda = \cup\{C^\sigma | \sigma < \lambda\}$ for λ a limit. To define $C^{\sigma+1}$, assume $\omega - C^\sigma$ is infinite, and let

$$\{a_0^\sigma < a_1^\sigma < a_2^\sigma < \dots\} = \omega - C^\sigma,$$

$$b = \mu a[a \in H_\sigma \ \& \ a \geq a^\sigma_{f(\sigma)}],$$

$$C^{\sigma+1} = C^\sigma \cup \{b, a^\sigma_{f(\sigma)}\}.$$

The assumption that $\omega - C^\sigma$ is infinite is proved by induction on σ. Suppose it is true for all $\sigma < \lambda$. Fix m to show $\omega - C^\lambda$ has at least m members. Let τ be such that

$$(\sigma)_{\tau \leq \sigma < \lambda}[f(\sigma) \geq m].$$

Then $(i)_{i<m}(\sigma)_{\tau \leq \sigma < \lambda}[a_i^\tau = a_i^\sigma = a_i^\lambda]$. The argument is valid when $\lambda = \omega_1^{CK}$, hence $\omega - C$ is infinite.

Let $a_i = \lim_\sigma a_i^\sigma$. If C were not simple, then there would be an infinite hyperarithmetic $H \subseteq \omega - C$, and a σ such that $H = H_\sigma$ and $H_\sigma \cap C^{\sigma+1} \neq \varnothing$. Finally

$C \equiv_f B$ because

$x \notin C \leftrightarrow (E\sigma)[x \notin C^\sigma \ \& \ f[\sigma] \supseteq B \cap (x+1)]$,

$x \notin B \leftrightarrow (E\sigma)[x \notin f[\sigma] \ \& \ a_x^\sigma = a_x]$. □

3.2 Theorem (Simpson 1971). *If A is a non-hyperregular, metarecursively enumerable set, then there exists a Π^1_1 set B of the same metadegree as A.*

Proof. By 1.3 and 3.1 there is a simple Π^1_1 set $C \leq_w A$. Let f metarecursively enumerate C without repetitions. Define $B = f[A]$.

Clearly $A \leq_M B$. To see $B \leq_M A$ consider a typical metafinite $K \subseteq cB$. $f^{-1}[K]$ is a meta r.e. subset of cA. By Exercise 4.8.V, A can be assumed to be simple. Hence $f^{-1}[K]$ is bounded above strictly by some δ, and $K \cap C = K \cap f[\delta]$. Thus $K \cap C$ and $K - C$ are metafinite. Since C is simple $K - C$ is finite. In short, for metafinite K,

$$K \subseteq cB \rightarrow K - C \text{ is finite.}$$

Let F range over all finite subsets of ω_1^{CK}. Then

$$K \subseteq c\,B \leftrightarrow (\mathrm{EF})(\mathrm{EH})[K = F \cup H \;\&\; F \subseteq c\,C \;\&\; H \subseteq C \;\&\; f^{-1}[H] \subseteq c\,A].$$

(H ranges over the metafinite sets.) It follows that $B \leq_M A$, since $C \leq_w A$. □

Part C

α-Recursion

α-recursion theory lifts classical recursion theory from ω to an arbitrary Σ_1 admissible ordinal α. Many of the classical results lift to every α by means of recursive approximations and fine structure techniques.

Chapter VII
Admissibility and Regularity

The fundamental notions of α-recursion theory are introduced. Metarecursion is shown equivalent to ω_1^{CK}-recursion. The essential properties of the Σ_1 projectum are established and applied in a priority argument to obtain a non-α-recursive, hyperregular, α-recursively enumerable set. A regular sets theorem is proved.

1. Σ_1 Admissibility

The notion of admissible set was invented by R. Platek (1966) in the course of his investigations of the foundations of recursion theory. Here, for reasons soon to be evident, "admissible" is called "Σ_1 admissible".

1.1 Σ_1 Admissible Sets. Let A be a nonempty set. A is said to be transitive if $x \in A$ whenever $x \in y$ and $y \in A$. A is closed under pairing if $\{x, y\} \in A$ whenever $x \in A$ and $y \in A$. A is closed under union if $x \in A$ implies $(\cup x) \in A$. ($\cup x = \{y | (\mathrm{Ez})(y \in z \in x)\}$.)

The formulas of Zermelo–Fraenkel set theory (ZF) are built up from set variables $x, y, z, \ldots$, atomic formulas ($s \in t$, where s and t are set variables), propositional connectives, and set quantifiers ((Ex) (there exists x), (y) (for all y)). A formula is *bounded* if all its quantifiers are bounded, that is of the form $(\mathrm{Ex})_{x \in y}$ or $(\mathrm{y})_{y \in z}$. The Levy hierarchy of formulas is defined by induction on n. A formula F is $\Delta_0 = \Pi_0 (= \Sigma_0)$ if F is bounded. $(\mathrm{Ex}_1) \ldots (\mathrm{Ex}_m)G$ is Σ_n if G is Π_{n-1} ($m \geq 0$). $(y_1) \ldots (y_m)G$ is Π_n if G is Σ_{n-1} ($m \geq 0$). $\Delta_n = \Pi_n \cap \Sigma_n$.

Typical Δ_0 formulas are: $x \subseteq y$; x is an ordinal. An example of a formula that is Π_1 but not Σ_1 is: x is a cardinal.

A formula G is said to have *parameters* in a set A if G is of the form $H(x_1, \ldots, x_m, a_1, \ldots, a_n)$, where $H(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a formula of ZF and $a_i \in A$ $(1 \leq i \leq n)$.

A satisfies Δ_0 *separation* if every sentence of the form

$$(\mathrm{Ex})(y)[y \in x \leftrightarrow y \in a \quad \& \quad F(y)]$$

is true in $\langle A, \varepsilon \rangle$, where $a \in A$ and $F(y)$ is Δ_0 with parameters in A.

A satisfies Δ_0 *bounding* (or *collection*) if every sentence of the form

$$(x)_{x\in\underline{a}}(\mathrm{Ey})F(x,y)\to(\mathrm{Ez})(x)_{x\in\underline{a}}(\mathrm{Ey})_{y\in z}F(x,y)$$

is true in $\langle A,\varepsilon\rangle$. Again $a\in A$ and $F(x,y)$ is Δ_0 with parameters in A.

A set A is said to be Σ_1 *admissible* if A is transitive, closed under pairing and union, and satisfies Δ_0 separation and Δ_0 bounding. A pairing trick shows a Σ_1 admissible set satisfies Δ_1 separation and Σ_1 bounding (Exercise 1.15). Suppose $B\subseteq C$. The structure $\langle C,B\rangle$ is said to be Σ_1 admissible if C is Σ_1 admissible when $x\in B$ is added to the list of Δ_0 formulas.

The least Σ_1 admissible set is HF, the set of all hereditarily finite sets. (x is hereditarily finite iff x is finite and every member of x is hereditarily finite.) HF can serve as the setting for classical recursion theory. It turns out that a partial function $f\colon\omega\to\omega$ is partial recursive in the classical sense iff the graph of f is Σ_1 definable over HF.

A set $B\subseteq A$ is Σ_n definable over A if there is a Σ_n formula $G(x)$ with parameters in A such that

$$a\in B\leftrightarrow\langle A,\in\rangle\models G(\underline{a})$$

for all $a\in A$. Similarly, Π_n and Δ_n definability.

Call a set *A-recursive* if it is Δ_1 definable over A, and *A-finite* if it belongs to A. Then Δ_1 separation becomes: if B is A-recursive and c is A-finite, then $B\cap c$ is A-finite. Call a function f partial A-recursive if its graph is Σ_1 definable over A. Then Σ_1 bounding and Δ_1 separation imply: if f is partial A-recursive, c is A-finite, and the domain of f includes c, then $f[c]$ is A-finite. In this fashion the beginnings of classical recursion theory can be generalized to every Σ_1 admissible A. A great deal can be lifted if A admits an A-recursive, one-to-one correspondence between A and $\mathrm{ord}(A)$, the least ordinal not in A. Such a correspondence is available when A is an initial segment of L, Gödel's universe of constructible sets, the setting for α-recursion theory.

Metarecursion theory was developed in Part B in terms of Π^1_1 sets and unique notations for recursive ordinals. Another approach, perhaps smoother, is via Σ_1 admissibility. There exists a Σ_1 admissible set called HYP such that the metarecursively enumerable subsets of ω_1^{CK} are just those Σ_1 definable over HYP.

The definition of HYP needs the idea of coding hereditarily countable sets (HC) by reals, that is, subsets of ω. $x\in\mathrm{HC}$ iff x is countable and every member of x belongs to HC. The set K of all codes for members of HC is defined by an arithmetic closure condition C, hence is Π^1_1 by Theorem 1.6.I. C has two clauses:

$\{0\}$ is a code;
if $(i)_{i<\omega}((r)_i$ is a code), then r is a code.

Recall that $(r)_i=\{m\mid 2^i\cdot 3^m\in r\}$. Define $u>_K v$ to be the transitive closure of

$$u,v\in K\quad\&\quad(\mathrm{Ei})_{i<\omega}(v=(u)_i).$$

$>_K$ is wellfounded by the natural enumeration approach that showed $<_O$ is well-

founded (Theorem 2.2.I). The set $x(r)$ encoded by $r \in K$ is defined by transfinite recursion on $>_K$:

$$x(r) = \{x((r)_i) | i < \omega\}.$$

Thus each member of K encodes a member of HC, and each member of HC has at least one code in K. (The latter is a consequence of AC_ω, the countable axiom of choice.)

Let HC(HYP) be the set of all x such that x has a hyperarithmetic code in K.

1.2 Proposition. HC(HYP) *is* Σ_1 *admissible.*

Proof. To verify Δ_0 bounding, suppose

$$\text{HC(HYP)} \models (x)_{x \in \underline{a}} (\text{Ey}) F(x, y) \tag{1}$$

for some $a \in \text{HC(HYP)}$ and $F(x, y) \in \Delta_0$ with parameters in HC(HYP). Since each member of HC(HYP) is countable in HC(HYP), a can be taken to be ω. (1) becomes

$$(n)(\text{Er})[r \in \text{HYP} \cap K \quad \& \quad \text{HC(HYP)} \models F(\underline{n}, \underline{x}(\underline{r}))]. \tag{2}$$

The notion of truth in HC(HYP) for Δ_0 sentences is Π^1_1, because it can be defined inductively by a Σ^1_1 closure condition (cf. proof of Lemma 4.5.III). K is Π^1_1, so the matrix of (2) is Π^1_1. In addition the real variable r of (2) can be regarded as a number variable ranging over indices of hyperarithmetic reals. Then Lemma 2.6.II applies, and r can be construed as a hyperarithmetic function of n. That function is in itself a hyperarithmetic code for a set suitable for bounding y in (1). □

1.3 Proposition. *Let* $A \subseteq \omega_1^{CK}$. *Then* A *is metarecursively enumerable iff* A *is* Σ_1 *definable over* HC(HYP).

Proof. First suppose A is metarecursively enumerable. Thus

$$(\delta)[\delta \in A \leftrightarrow |\delta| \in A_1]$$

for some $\Pi^1_1 A_1$. By Theorem 3.5.III, there is an arithmetic $A(X, b)$ such that

$$\delta \in A \leftrightarrow (\text{Eb})(\text{EX})[X \in \text{HYP} \quad \& \quad A(X, b) \quad \& \quad |b| = \delta]. \tag{1}$$

To say $|b| = \delta$ is the same as saying $W_{q(b)}$ (from Theorem 3.5.I) is isomorphic to $\{\langle u, v \rangle | u \in v \in \delta\}$. It follows from 1.2 that the right side of (1) is Σ_1.

Now suppose A is Σ_1 is definable over HC(HYP). Then

$$\delta \in A \leftrightarrow (\text{Er})[r \in \text{HYP} \cap K \quad \& \quad F(x(r), \delta)]$$

for some Δ_0 F. Let A_1 be the set of all b such that

$$b \in O \quad \& \quad (\mathrm{Er})[r \in \mathrm{HYP} \cap K \quad \& \quad F(x(r), |b|)].$$

Then A_1 is Π^1_1. □

1.4 Let X and Y be sets. X is said to be a *first order definable* subset of Y if there is a formula $F(x)$ of ZF with parameters in Y such that

$$X = \{a | a \in Y \quad \& \quad \langle Y, \in \rangle \models F(\underline{a})\}.$$

The set of all first order definable subsets of Y is denoted by $\mathrm{Fod}(Y)$. Gödel's universe of constructible sets is defined by iterating Fod through the ordinals.

$$L(0) = \phi.$$
$$L(\delta + 1) = \mathrm{Fod}(L(\delta)).$$
$$L(\lambda) = \cup \{L(\delta) | \delta < \lambda\} \ (\lambda \text{ a limit}).$$
$$L = \cup \{L(\delta) | \delta \text{ is an ordinal}\}.$$

Some excellent sources of information about L are Jech (1978) and Devlin (1984). Gödel showed that L is a model of ZF, the axiom of choice and the generalized continuum hypothesis. He obtained these extraordinary results by means of ideas which can be refined to yield detailed information about initial segments of L. Some of these refinements will be discussed in future sections. They combine smoothly with ideas from classical recursion theory to produce methods for classifying the Σ_1 definable subsets of $L(\alpha)$ when $L(\alpha)$ is Σ_1 admissible. The ordinal α is said to be Σ_1 admissible if $L(\alpha)$ is a Σ_1 admissible set. α-recursion theory is the study of α-recursively enumerable (Σ_1 over $L(\alpha)$) subsets of α.

Recursion theory on initial segments of ordinals was invented by Takeuti (1960). Σ_1 admissible initial segments of ordinals were first studied by Kripke (1964) and Platek (1966).

1.5 Proposition. *α is Σ_1 admissible $\leftrightarrow$ $L(\alpha)$ satisfies Δ_0 bounding.*

Proof. A straightforward induction on δ shows that $L(\delta)$ is transitive and that the ordinals in $L(\delta)$ are just those less than δ. If α is a limit, then $L(\alpha)$ is closed under pairing and union, and satisfies Δ_0 separation. Suppose $\alpha = \beta + 1$. Consider the Δ_0 function f defined by $f(x) = \beta$ for all $x \in L(\alpha)$. Δ_0 bounding implies $\beta \in z$ for some $z \in L(\alpha)$. But then $z \subseteq L(\beta)$ and so $\beta \in L(\beta)$, an impossibility. □

From now on α is a Σ_1 admissible ordinal. The fundamental notions of α-recursion theory are defined as in subsection 1.1 with $L(\alpha)$ in place of A. A set $B \subseteq \alpha$ is said to be *α-recursively enumerable* if B is Σ_1 definable over $L(\alpha)$. Remember that parameters from $L(\alpha)$ are permitted (in the Σ_1 definition of B). A function $f \subseteq \alpha^2$ is

said to be partial α-recursive if its graph is α-recursively enumerable; if the domain of f is α, then f is said to be α-recursive. The *α-finite* subsets of α are those that belong to $L(\alpha)$. There is a natural enumeration of $L(\alpha)$ that maps α onto $L(\alpha)$ (subsection 1.7). It is only slightly more complicated than the natural enumeration of all hereditarily finite sets.

The first two Σ_1 admissible ordinals are ω and ω_1^{CK} (Exercise 1.10). The classical partial recursive functions are the same as the partial ω-recursive functions. Metarecursion theory is equivalent to ω_1^{CK}-recursion theory by Proposition 1.3, once it is seen that $L(\omega_1^{CK}) = \mathrm{HC(HYP)}$ (Exercise 1.11). ω_1^T is Σ_1 admissible for every $T \subseteq \omega$. Conversely, every countable Σ_1 admissible ordinal greater than ω is of the form ω_1^T for some $T \subseteq \omega$ (Sacks (1976)).

1.6 Recursion on α. The Σ_1 admissibility of $L(\alpha)$ makes it possible to define α-recursive functions by Σ_1-on-$L(\alpha)$ transfinite recursions on α. Let I: $L(\alpha) \to \alpha$ be Σ_1 definable on $L(\alpha)$. There is at most one f such that

$$(\delta)_{\delta<\alpha}[f(\delta) = I(f \restriction \delta)]. \tag{1}$$

Let Z be the set of all α-finite g such that $g(\gamma) = I(g \restriction \gamma)$ for all $\gamma \in \mathrm{dom}\, g$. Z is α-recursive, and any two functions in Z agree on all arguments in common. Define

$$f(\delta) = y \quad \text{by} \quad (\mathrm{Eg})[g \in Z \ \ \& \ \ \delta \in \mathrm{dom}\, g \ \ \& \ \ f(\delta) = y].$$

f is Σ_1 on $L(\alpha)$. Furthermore, $\mathrm{dom}\, f = \alpha$, otherwise f belongs to Z and extends to some other member of Z.

The definition of f can be thought of as a process. At stage δ it is assumed that all activity at previous stages is encapsulated in an α-finite object, $s \restriction \delta$. Instruction I is given in advance. It tells how to compute $f(\delta)$ from $s \restriction \delta$. In general it will be necessary to search through $L(\alpha)$ for some existential witness w that bears a specified Δ_0 relationship to $s \restriction \delta$. $s(\delta)$ encodes not only $f(\delta)$ but also w and the location of w. The Σ_1 admissibility of $L(\alpha)$, in particular Σ_1 bounding, is needed to bound the search for w below α so long as δ is bounded below α.

1.7 The Natural Enumeration of $L(\alpha)$. The elements of $L(\alpha)$ lend themselves to an enumeration in α stages. At stage δ the elements of $L(\delta)$ are enumerated one at a time. If δ is a limit ordinal, then the enumerations of $L(\gamma)$ $(\gamma < \delta)$ are stacked one on top of another and repeated. Suppose $\delta = \gamma + 1$. A typical element x of $L(\delta)$ is a first order definable subset of $L(\gamma)$. A code for x is a sequence $\langle e, b_1, \ldots, b_m \rangle$. e is the Gödel number of a formula $F(y, z_1, \ldots, z_m)$, $b_i \in L(\gamma)(1 \leq i \leq m)$, and

$$x = \{a \mid a \in L(\gamma) \ \ \&^{\cdot} \ \ L(\gamma) \models F(\underline{a}, \underline{b}_1, \ldots, \underline{b}_m)\}.$$

The enumeration of $L(\delta)$ parallels the enumeration of all codes of elements of $L(\delta)$. The latter is derived from the given enumeration of $L(\delta)$.

The main question avoided above is how to pass from a code for x to x. The answer has to do with the definability of truth in $L(\gamma)$. If γ is a limit, then the

relation $L(\gamma) \models F$ is definable over $L(\gamma+1)$. In fact, the usual Tarski-style definition of truth works.

1.8 Proposition. *There exists a one-one, α-recursive f that maps α onto $L(\alpha)$.*

Proof. By appeal to the picture of Σ_1 recursion sketched in subsection 1.6 and to the natural enumeration of $L(\alpha)$ given in Subsection 1.7. Let $h(\delta)$ be the enumeration of $L(\delta)$ alluded to in 1.7. The α-finite object $h(\delta)$ is derived from the α-finite object $h \restriction \delta$ by an instruction I easily (but tediously) seen to be Σ_1 over $L(\alpha)$. Let g be the α-recursive function that enumerates $L(\alpha)$, repetitions permitted, by stacking the $h(\delta)$'s one on top of another. The desired f is g with the repetitions omitted. □

1.9 Proposition. *There exists a partial α-recursive function $\phi(\varepsilon, \delta)$ such that for each partial α-recursive $\psi(\delta)$, there exists an ε such that*

$$\phi(\varepsilon, \delta) \simeq \psi(\delta)$$

for all δ. (Enumeration Theorem)

Proof. By appeal to subsections 1.6 and 1.7, as in 1.8. Let ψ be partial α-recursive. There must be a Δ_0 formula $F(u, v, z)$ with parameters in $L(\alpha)$ such that $\psi(\delta) = \gamma$ iff for some $w \in L(\alpha)$,

$$(1) \qquad F(\delta, \gamma, w) \quad \& \quad (y)(z)_{\langle y, z\rangle < \langle \gamma, w\rangle} \sim F(\delta, y, z).$$

In (1) w is a witness to the fact that $\psi(\delta) = \gamma$, and $<$ is the α-recursive wellordering of $L(\alpha)$ provided by Proposition 1.8, and derived from the natural enumeration of $L(\alpha)$. The truth or falsity of (1) can be determined by examining any limit structure $L(\beta)$ whose members include δ, γ, w, the parameters of $F(u, v, z)$, and $<$ below $\langle \gamma, w\rangle$. Thus the natural enumeration of $L(\alpha)$ gives rise to an α-recursive enumeration of T, the set of all true sentences of type (1). A suitable code for (1) is $\langle \varepsilon, \delta, \gamma, w\rangle$, where ε is a code for $F(u, v, z)$. Let T_1 be the set of codes of elements of T. Define

$$\psi(\varepsilon, \delta) \simeq \gamma \quad \text{by} \quad (\mathrm{E}w)[\langle \varepsilon, \delta, \gamma, w\rangle \in T_1].$$

ψ is partial α-recursive since T, hence T_1, is α-recursively enumerable. In fact, T is α-recursive, because all false sentences of type (1) can be enumerated simultaneously with T. □

The immense power of Σ_1 admissibility is not needed to prove Propositions 1.8 and 1.9. They hold for any $L(\beta)$ such that β is a limit. In that case the proof of 1.8 relies on a condensation argument of the sort given in the next section rather than the natural enumeration of subsection 1.7. If $L(\beta)$ is closed with respect to some primitive recursions, then the natural enumeration of 1.7 is available.

1.10–1.15 Exercises

1.10. Show ω_1^{CK} is the first Σ_1 admissible ordinal after ω.

1.11. Show $L(\omega_1^{CK}) = \mathrm{HC(HYP)}$.

1.12. Suppose $A \subseteq \omega_1^{CK}$. Show A is metarecursively enumerable iff A is ω_1^{CK}-recursively enumerable.

1.13. Show ω_1^T is Σ_1 admissible for all $T \subseteq \omega$.

1.14. Suppose $A \subseteq \alpha$ is α-recursively enumerable. Find a partial, one-one, α-recursive g whose range is A and whose domain is an initial segment of α.

1.15. Show every Σ_1 admissible set satisfies Δ_1 separation and Σ_1 bounding.

1.16. Let f be partial and α-recursive. Find a one-one, partial α-recursive g with the same range as f.

2. *The Σ_1 Projectum*

In part B considerable use was made of a metarecursive, one-one map of ω_1^{CK} into ω. The priority arguments of Section 2.VI relied strongly on the fact that each requirement was injured only for the sake of finitely many requirements of higher priority. Thus some of the combinatoric tricks of classical recursion theory were lifted to ω_1^{CK} by mapping ω_1^{CK} into ω. In general there is no hope of mapping α down to ω, or even down to something less than α, but the idea of mapping α downward makes sense for every α. The Σ_1 *projectum of* α, denoted by $\sigma 1 p\alpha$ or α^*, is defined to be

$$\mu\beta(\mathrm{Ef})\left[f \text{ is } \alpha\text{-recursive: } \alpha \xrightarrow[\text{into}]{\text{one-one}} \beta \right].$$

Thus $(\omega_1^{CK})^* = \omega$. Mapping ω_1^{CK} into ω proved useful because every metarecursively enumerable subset of a finite set is finite, hence metafinite. In this manner some troublesome, bounded, metarecursively enumerable, but not metafinite sets were avoided. The Σ_1 projectum of α has a similar virtue for every α according to the next proposition.

2.1 Proposition. *Assume $A \subseteq \delta < \sigma 1 p\alpha$. If A is α-recursively enumerable, then A is α-finite.*

Proof. Suppose not. Let g be a partial, one-one, α-recursive function whose range is A, and whose domain is an initial segment of α (Exercise 1.14). Since $\delta < \sigma 1 p\alpha$, the domain of g must be less than α. But then the range of g is α-finite. □

2.2 Proposition. *$\sigma 1 p\alpha$ is the least β such that some α-recursively enumerable subset of β is not α-finite.*

Proof. Let β_0 be the least β. By Proposition 2.1, $\sigma 1 p\alpha \leq \beta_0$. To see that $\sigma 1 p\alpha \geq \beta_0$, let f be a one-one, α-recursive map of α into $\sigma 1 p\alpha$. If $f[\alpha]$ were α-finite, then $f^{-1}f[\alpha]$ would be α-finite, hence bounded below α. □

If $\sigma 1 p\alpha = \omega$, then there is little difference between metarecursion theory, that is ω_1^{CK}-recursion theory, and α-recursion theory. If $\sigma 1 p\alpha > \omega$, then $L(\alpha)$ merits a finer analysis in terms of the notion of α-cardinal. Suppose $\beta < \alpha$; β is said to be an *α-cardinal* if there is no α-finite, one-to-one correspondence between β and some $\delta < \beta$. In other words,

$$L(\alpha) \models [\beta \text{ is a cardinal}].$$

Note that $\sigma 1 p\alpha < \alpha$ implies $\sigma 1 p\alpha$ is the greatest α-cardinal. Each α-finite set has an α-cardinality, namely the least ordinal with which it can be put into one-to-one, α-finite correspondence. If an α-finite set H has α-cardinality less than $\sigma 1 p\alpha$, then every α-recursively enumerable subset of H is α-finite by Proposition 2.1.

Each α-cardinal is either regular or singular. β is a *regular* α-cardinal if

$$L(\alpha) \models [\beta \text{ is a regular cardinal}].$$

Equivalently, there is no α-finite function with domain $p < \beta$ and range an unbounded subset of β.

A collection $\{A_\delta | \delta < \gamma\}$ of subsets of α is said to be *simultaneously α-recursively enumerable* if there exists a partial α-recursive function $f(\delta, \sigma)$ such that

$$A_\delta = \{f(\delta, \sigma) | \sigma < \alpha\}$$

for all $\delta < \alpha$.

2.3 Lemma (Sacks & Simpson 1972). *Suppose $\gamma < \beta$ and β is a regular α-cardinal. Let $\{A_\delta | \delta < \gamma\}$ be a simultaneously α-recursively enumerable collection of subsets of α. If each A_δ is α-finite and has α-cardinality less than β, then $\cup \{A_\delta | \delta < \gamma\}$ is α-finite and has α-cardinality less than β.*

Proof. It is safe to assume the A_δ's are pairwise disjoint, since otherwise each can be replaced by $A_\delta \times \{\delta\}$. The given simultaneous enumeration of the A_δ's gives rise to an overall enumeration of $\cup \{A_\delta | \delta < \gamma\}$ without repetitions. At each stage of the overall enumeration, just one element of just one A_δ is enumerated. If the number of stages needed is less than β, then the lemma is proved. Otherwise, let A_δ^β be that part of A_δ enumerated during the first β stages of the overall enumeration. Then $\{A_\delta^\beta | \delta < \gamma\}$ is an α-finite collection of α-finite sets whose union has α-cardinality equal to β. But $\gamma < \beta$ and each A_δ^β has α-cardinality less than β, so β is not regular. □

Lemma 2.3 says that a regular α-cardinal is more regular than it appears; it is in fact $\mu - \Pi_1^\alpha$ regular (Exercise 2.12). Note the strong use of Σ_1 admissibility in the proof of 2.3. In the next Chapter, Lemma 2.3 will be needed to show that the

α-recursively enumerable sets of "injuries" that arise in the solution to Post's problem are α-finite.

2.4 α-Stability. Let M and N be sets. M is said to be a Σ_1 *substructure* of N (in symbols $M \prec_1 N$) if $M \subseteq N$ and

$$[\langle N, \in \rangle \models F] \to [\langle M, \in \rangle \models F] \tag{1}$$

for every Σ_1 sentence of ZF with parameters in M. If $M \prec_1 N$, then (1) holds for every Π_2 sentence of ZF with parameters in M. β is said to be α-*stable* if $\beta < \alpha$ and $L(\beta) \prec_1 L(\alpha)$. α-stable ordinals provide bounds for partial α-recursive functions. Suppose ϕ is partial α-recursive and the parameters occurring in the Σ_1 definition of ϕ are ordinals less than β. If β is α-stable, then

$$\gamma < \beta \ \& \ \phi(\gamma) \text{ defined} \to \phi(\gamma) < \beta. \tag{2}$$

It will be seen very shortly that every α-cardinal beyond ω is α-stable by arguments similar to those invented by Gödel to show the generalized continuum hypothesis. The first result of this sort was obtained by Takeuti (1960). He showed (2) holds if β is a cardinal of L and ϕ is a partial recursive function of ordinals.

2.5 Lemma (H. Putnam). *There exists a Π_2 sentence, $[V = L]$, such that for every transitive set M,*

$$M \models [V = L] \quad \textit{iff} \quad M = L(\lambda)$$

for some limit ordinal λ.

Proof. The sentence $[V = L]$ is

$$(x)(E\delta)(\mathrm{E}y)[y = L(\delta) \ \& \ x \in y]. \tag{1}$$

The function $\lambda\delta | L(\delta)$ is defined by a Σ_1 transfinite recursion. (Cf. Subsection 1.6.) Hence (1) is Π_2.

If $x \in L(\lambda)$ and λ is a limit, then $x \in L(\delta)$ for some $\delta < \lambda$. Also $L(\delta) \in L(\lambda)$. Thus $[V = L]$ is true in $L(\lambda)$.

Suppose M is transitive and $M \models [V = L]$. Let λ be the least ordinal not in M. Then $\lambda = M \cap \lambda$. For each $\delta < \lambda$, let $L(\delta)^M$ be $L(\delta)$ in the sense of M, that is, the result of executing within M the Σ_1 recursion that defines $\lambda\gamma | L(\gamma)$. By induction on δ, $L(\delta)^M = L(\delta)$ for all $\delta \in M$. So $L(\lambda) \subseteq M$. On the other hand, if $x \in M$, then $x \in L(\delta)^M$ for some $\delta < \lambda$. Hence $x \in L(\delta) \subseteq L(\lambda)$. □

2.6 Lemma. *Suppose $\omega \leq \delta < \alpha^*$. Then there exists an α-stable ordinal $\delta_0 \geq \delta$ of the same α-cardinality as δ.*

Proof. The natural enumeration of $L(\alpha)$ outlined in subsection 1.7 yields a partial α-recursive function $h(e, \langle x_1, \ldots, x_n \rangle)$ $(n \geq 0)$ with properties similar to those of

the enumeration function ϕ of Proposition 1.9. Suppose $P(x_1, \ldots, x_n, y)$ is the Σ_1 formula of ZF whose Gödel number is e. If

$$L(\alpha) \models (\mathrm{Ey})\, P(\underline{a}_1, \ldots, \underline{a}_n, y)$$

for some $a_1, \ldots, a_n \in L(\alpha)$, then $h(e, \langle a_1, \ldots, a_n \rangle)$ is defined and

$$L(\alpha) \models P(\underline{a}_1, \ldots, \underline{a}_n, \underline{h(e, \langle a_1, \ldots, a_n \rangle)});$$

otherwise h is undefined. $h(e, \langle a_1, \ldots, a_n \rangle)$ is the "first" y that is seen to satisfy $P(a_1, \ldots, a_n, y)$ as the natural enumeration of $L(\alpha)$ unfolds.

h is *lightface* Σ_1; its Σ_1-on-$L(\alpha)$ definition does not require any parameters from $L(\alpha)$. For any $x \subseteq L(\alpha)$, let

$$h[x] \text{ be } \{h(e, \langle a_1, \ldots, a_n \rangle) \mid e < \omega \ \& \ a_i \in x\}.$$

$h[x]$ is called the Σ_1 hull of x. Note that $h^2[x] = h[x] \subseteq x$. By design h is a universal, partial Σ_1 Skolem function for $L(\alpha)$.

Let $\omega \leq \delta < \alpha^*$. Then

$$h[\delta] \prec_1 L(\alpha), \quad \text{and so} \quad h[\delta] \models [V = L].$$

Suppose for the moment that $h[\delta]$ is transitive. By Lemma 2.5, $h[\delta] = L(\delta_0)$ for some limit δ_0. Clearly $\delta \leq \delta_0$ and δ_0 is α-stable. The domain of $h \restriction \delta$ is α-finite by Proposition 2.1, so $h[\delta]$ is α-finite and δ_0 has the same α-cardinality as δ.

To check that $h[\delta]$ is transitive, define an α-recursive map

$$t: h[\delta] \to L(\alpha)$$

by transfinite recursion:

$$t(x) = \{t(y) \mid y \in x \quad \& \quad y \in h[\delta]\}.$$

An induction on rank shows: for all x, $y \in h[\delta]$,

$$y \in x \leftrightarrow t(y) \in t(x).$$

The induction succeeds because $h[\delta]$ is *extensional*:

$$u, v \in h[\delta] \ \& \ u \neq v \to (\mathrm{Ew})[w \in h[\delta] \ \& \ w \in (u - v) \cup (v - u)].$$

Thus t maps $\langle h[\delta], \in \rangle$ isomorphically to $\langle M, \in \rangle$, where M is $t[h[\delta]]$. The definition of t implies M is transitive. t is often called (Mostowski's) collapsing map, since it isomorphically collapses any set satisfying extensionality to a transitive set.

$t \restriction \delta$ is the identity, because $\delta \subseteq h[\delta]$. Fix $b \in h[\delta]$. Then $b = h(\gamma)$ for some $\gamma < \delta$. The sentence "$\underline{b} = \underline{h(\gamma)}$" is Σ_1 and true in $L(\alpha)$, hence true in $h[\delta]$, and con-

sequently true in M when b and γ are replaced by their t-images. $t(\gamma) = \gamma$ so "$\underline{t(b)} = \underline{h(\gamma)}$" is true in M. The transitivity of M implies $t(b) = h(\gamma)$ is outright true, and so $t(b) = b$. Thus $t \restriction h[\delta]$ is the identity. □

Corollary 2.7. *Every α-cardinal greater than ω is α-stable.*

Proof. Every α-cardinal is less than or equal to α^*, hence by Lemma 2.6 the limit of stable ordinals. □

The method of proof of Lemma 2.6 was originated by Gödel. It consists of forming a Skolem hull in L and identifying its transitive collapse with an initial segment of L. The method possesses great power, but must be applied with caution. The proof of 2.6 shows that the Σ_1 hull of a transitive set is transitive. This principle can fail for Σ_2 hulls of transitive sets, and for Σ_1 hulls of intransitive sets. It is worth noting that the collapse of a Σ_1 substructure of $L(\alpha)$ need not be a Σ_1 substructure of $L(\alpha)$.

2.8–2.12 Exercises

2.8. Assume σ is α-stable and show σ is Σ_1 admissible.

2.9. Give an example of a Σ_1 admissible α such that $\omega < \alpha^* < \alpha$ and α^* is not the greatest α-stable ordinal.

2.10. (R. Jensen). Prove Proposition 2.2 without assuming α is Σ_1 admissible. Assume only that α is a limit.

2.11. (S. Friedman). Show Lemma 2.3 is false for some limit α that is not Σ_1 admissible.

2.12. A function $f\colon \alpha \to \alpha$ is said to be $\mu - \Pi_1^\alpha$ if

$$f(x) = \mu y P(x, y)$$

for some $P(x, y) \in \Pi_1^\alpha$. Show: $g \in \Sigma_1^\alpha \to g \in \mu - \Pi_1^\alpha \to g \in \Sigma_2^\alpha$. Suppose β is a regular α-cardinal, $\gamma < \beta$, and $f \in \mu - \Pi_1^\alpha$.Show $f[\gamma]$ is bounded below β if $f[\gamma] \subseteq \beta$. In short: a regular α-cardinal is $\mu - \Pi_1^\alpha$ regular.

3. Relative α-Recursiveness

In this section the notion of α-degree is defined, and the first steps are taken towards the proof of the regular sets theorem of Section 4.

3.1 Indices for α-Finite Sets. The proof of Proposition 1.8 is easily modified to yield a one-one, α-recursive function f_0 that maps α onto the collection of all α-finite sets. $f_0(\delta)$ is simply the δ-th α-finite set to occur in the natural enumeration of $L(\alpha)$.

Define

$$K_\delta \text{ to be } f_0(\delta).$$

If K is α-finite, then the unique δ such that $K_\delta = K$ is said to be the index of K as an α-finite set. The 2-place predicate, $\gamma \in K_\delta$, is α-recursive. A collection of α-finite sets,

$$\{K_\delta \mid \delta \in J\}, \tag{1}$$

is said to be α-finite if J is. In that event the union of (1) is α-finite.

3.2 α-Recursive In. R is called a *reduction procedure* if R is α-recursively enumerable and every member of R is of the form $\langle H, J, \gamma, \delta \rangle$, where H and J are disjoint α-finite sets. f: $\alpha \to \alpha$ is *weakly α-recursive in* $B \subseteq \alpha$ if there exists a reduction procedure R such that

$$f(\gamma) = \delta \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, \gamma \rangle \in R \ \& \ H \subseteq B \ \& \ J \subseteq cB] \tag{1}$$

for all γ, $\delta < \alpha$. (cB is $\alpha - B$). (1) is rendered in symbols as $f \leq_{w\alpha} B$. Let ϕ be the enumerating partial function of Theorem 1.9. For each $\varepsilon < \alpha$, define

$$R_\varepsilon = \{\delta \mid \phi(\varepsilon, \delta) \text{ is defined}\}.$$

R_ε is called the ε-th α-recursively enumerable set. If R is α-recursively enumerable, then $R = R_\varepsilon$ for some ε.

For any $B \subseteq \alpha$ the expression $\{\varepsilon\}^B(\delta)$ is defined and equal to γ iff

$$(\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, \gamma \rangle \in R_\varepsilon \ \& \ H \subseteq B \ \& \ J \subseteq cB]. \tag{2}$$

Note that $\{\varepsilon\}^B(\delta)$ may have more than one value. If f is a function, then

$$f \leq_{w\alpha} B \leftrightarrow f = \{\varepsilon\}^B \text{ for some } \varepsilon.$$

The many-valued character of $\{\varepsilon\}$ is a fact of life that can be avoided for some, but not all, α. This oddity was discussed in subsection 4.V.

A is said to be *weakly α-recursive in B* (symbolically $A \leq_{w\alpha} B$) if $c_A \leq_{w\alpha} B$, where c_A is the characteristic function of A. (It is customary to write $A(n)$ in place of $c_A(n)$, and $A = \{\varepsilon\}^B$ in place of $c_A = \{e\}^B$.) It is a result of Driscoll (1968) (Corollary 2.3.VI) that $\leq_{w\alpha}$ fails to be transitive on the α-recursively enumerable sets when $\alpha = \omega_1^{\mathrm{CK}}$. Shore (1975) characterizes those α's for which $\leq_{w\alpha}$ does not have the above failing.

The following notion of reducibility, suggested by Kreisel, is transitive by virtue of its symmetry: an α-finite neighborhood condition on A is established by one on B. A is *α-recursive in B* (symbolically $A \leq_\alpha B$) if there exist reduction procedures R_0 and R_1 such that

$$K \subseteq A \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, K \rangle \in R_0 \ \& \ H \subseteq B \ \& \ J \subseteq cB] \tag{3}$$

(4) $$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H,J,K\rangle \in R_1 \ \& \ H \subseteq B \ \& \ J \subseteq cB]$$

for all α-finite K. Note that $A \leq_\alpha B$ implies $A \leq_{w\alpha} B$. Define A and B to be of the same α-degree (symbolically $A \equiv_\alpha B$) if $A \leq_\alpha B$ and $B \leq_\alpha A$.

3.3 Proposition. *Assume A is α-recursively enumerable. Then for all B: $A \leq_\alpha B$ iff there exists a reduction procedure R such that for all α-finite K,*

$$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H,J,K\rangle \in R \ \& \ H \subseteq B \ \& \ J \subseteq cB].$$

Proof. Since A is α-recursively enumerable, so is

$$\{K \mid K \subseteq A \quad \& \quad K \text{ is } \alpha\text{-finite}\}.$$

Thus clause (3) of the definition of $\leq_\alpha$ can be expressed without reference to B. □

3.4 Proposition. *Let $A^* = \{\delta \mid K_\delta \cap A \neq \varnothing\}$. If A is α-recursively enumerable, then A^* is α-recursively enumerable, $A^* \equiv_\alpha A$, and*

$$(B)[A^* \leq_{w\alpha} B \leftrightarrow A \leq_\alpha B].$$

Proof. The α-recursive enumerability of A^* follows from that of A and the α-recursiveness of the predicate $\gamma \in K_\delta$. Proposition 3.3 implies

(1) $$A^* \leq_{w\alpha} B \leftrightarrow A \leq_\alpha B$$

for all B. (1), with A^* in place of B, yields $A \leq_\alpha A^*$. (1), with A in place of B, yields $A^* \leq_{w\alpha} A$. Let t be an α-recursive function such that

$$K_{t(\delta)} = \cup\{K_\gamma \mid \gamma \in K_\delta\}$$

for all δ. Then

$$K_\delta \subseteq c(A^*) \leftrightarrow K_{t(\delta)} \subseteq cA \leftrightarrow t(\delta) \in c(A^*).$$

Hence $A^* \leq_{w\alpha} A$ implies $A^* \leq_\alpha A$ with the aid of Proposition 3.3. □

3.5 Regularity. Let $A \subseteq \alpha$. A is said to be *regular* if $A \cap \delta$ is α-finite for every $\delta < \alpha$. The notion of regular set was inspired by Gödel's notion of constructible class, that is, a class whose intersection with each constructible set is constructible. In Jensen's terminology A is amenable. In the next section it will be shown that each α-recursively enumerable set has the same α-degree as some regular, α-recursively enumerable set.

It is immediate from Proposition 2.2 that there exists a non-regular, α-recursively enumerable set iff $\sigma 1 p(\alpha) < \alpha$.

For a non-α-finite, α-recursively enumerable A, regularity is equivalent to a dynamic property important in priority arguments. Assume $A = f[\alpha]$ for some one-one, α-recursive f. Then A is regular iff

$$(1) \qquad (\delta)(E\sigma)(\tau)_{\tau \geq \sigma}[f(\tau) \geq \delta].$$

(1) says that for each α-finite K, there is a stage in the enumeration (without repetitions) of A after which no member of K is added to A.

For all $A \subseteq \alpha$, regularity can be expressed in static, set theoretic terms. Introduce a predicate $x \in \underline{A}$. (Of course $\underline{\delta} \in \underline{A}$ is true iff $\delta \in A$.) The formulas of ZF^A are defined as were the formulas of ZF in subsection 1.1 save that $x \in \underline{A}$ is added to the list of atomic formulas. Fod^A is defined as in subsection 1.4 with ZF^A in place of ZF. The definition of $L[A]$ parallels that of L.

$$L[A, 0] = \phi.$$
$$L[A, \delta + 1] = \mathrm{Fod}^A(L[A, \delta]).$$
$$L[A, \lambda] = \cup\{L[A, \delta] \mid \delta < \lambda\} \quad (\lambda \text{ a limit}).$$
$$L[A] = \cup\{L[A, \delta] \mid \delta \text{ an ordinal}\}.$$

$L[A]$ is said to be *constructible from $\underline{A}$ as an additional predicate*, and is not to be confused with $L(A)$, said to be constructible from A as a set, and to be defined later. The structure $\langle L[A,\alpha], \in, A \rangle$ has $L[A,\alpha]$ as universe, and $x \in y$ and $x \in A \cap \alpha$ as atomic predicates.

3.6 Proposition. *Assume $A \subseteq \alpha$. A is regular iff $L[A,\alpha] = L(\alpha)$.*

Proof. By induction on δ the ordinals in $L[A, \delta]$ are just those less than δ. A further induction shows

$$L[A,\delta] = L[A \cap \delta, \delta].$$

Hence A is regular iff $L[A, \delta]$ is α-finite for all $\delta < \alpha$. □

The formulas of ZFC^A fall into quantifier complexity classes ($\Delta_0^A, \Sigma_1^A, \Pi_1^A$, etc.) in the same manner as the formulas of ZF in subsection 1.1. If F is a Σ_n^A formula with parameters in $L[A,\alpha]$, then F is said to be $\Sigma_n^{\alpha, A}$. Suppose $B \subseteq L[A,\alpha]$. If B is first order definable over $\langle L[A,\alpha], \in, A \rangle$ by means of a formula in $\Sigma_n^{\alpha, A}$, then B is said to be $\Sigma_n^{\alpha, A}$. Similarly for Π_n and Δ_n. If $A = \phi$, then $\Sigma_n^{\alpha, A}$ is written Σ_n^{α}. Thus B is α-recursively enumerable iff $B \subseteq \alpha$ and $B \in \Sigma_1^{\alpha}$.

3.7–3.8 Exercises

3.7. Suppose $B \leq_{w\alpha} A$. Show $B \in \Delta_1^{\alpha, A}$.

3.8. Suppose A is regular, $B \subseteq \alpha$, and $B \in \Delta_1^{\alpha, A}$. Show $B \leq_{w\alpha} A$.

4. *Existence of Regular Sets*

Let f be a one-one, α-recursive map of α into α. The deficiency set of f, in symbols D_f, is defined by:

$$\gamma \in D_f \leftrightarrow (\mathrm{E}\delta)[\gamma < \delta \ \& \ f(\delta) < f(\gamma)].$$

Deficiency sets are the source of regularity in α-recursion theory. They were invented by Dekker (1954) in order to prove theorems about simple sets in the setting of classical recursion theory.

4.1 Lemma
(1) *D_f is a regular, α-recursively enumerable set.*
(2) *$D_f \leq_\alpha f[\alpha]$.*
(3) *If $f[\alpha]$ is regular, then $f[\alpha] \leq_\alpha D_f$.*

Proof. Let $g(\tau)$ be the unique z such that

$$f(z) = \mu x[x \in f[\alpha - \tau]].$$

Then $g \leq_{w\alpha} f[\alpha]$ and range $g = cD_f$.
D_f is regular, because

$$\gamma \in D_f \cap \tau \leftrightarrow \gamma < \tau \ \& \ (\mathrm{Ey})[\gamma < y \leq g(\tau) \ \& \ f(y) < f(\gamma)].$$

Proposition 3.3 implies $D_f \leq_\alpha f[\alpha]$, since

$$K \subseteq cD_f \leftrightarrow (\gamma)_{\gamma \in K}(y)[\gamma < y \leq g(\sup K) \rightarrow f(y) > f(\gamma)].$$

cD_f is unbounded; otherwise there would be an infinite descending sequence of ordinals. $f \restriction (cD_f)$ is strictly increasing. If $f[\alpha]$ is regular, then $f \restriction (cD_f)$ is unbounded, and consequently

$$K \subseteq cf[\alpha] \leftrightarrow (\mathrm{Ev})[v \notin D_f \ \& \ \sup K < f(v) \quad \& \quad K \subseteq cf[v]].$$

4.2 Theorem (Sacks 1966). *Let A be α-recursively enumerable. Then there exists a regular, α-recursively enumerable B of the same α-degree as A.*

Proof. Following Maass (1978a). Let $A^* = \{\delta \mid K_\delta \cap A \neq \phi\}$. It is safe to assume $0 \in A$. Then A^* is unbounded, and there exists a one-one, α-recursive map f of α onto A^*. Let p: $\alpha \rightarrow \alpha^*$ be α-recursive, one-one and into. α^* is the Σ_1 projectum of α as in section 2. Define

$$\langle w, x \rangle \in B \text{ by}$$
$$(\mathrm{Ey})[x < y \ \& \ pf(y) < pf(x) \ \& \ f(y) < w].$$

Note that $\{x|(\text{Ew})(\langle w,x\rangle \in B\}$ is $D_{p\circ f}$, the deficiency set of $p\circ f$. A typical member of B consists of a deficiency point x with an upper bound w on a witness $f(y)$ to the deficiency of x. The regularity of B will be a consequence of its connection with $D_{p\circ f}$. The use of $D_{p\circ f}$, rather than D_f, makes it possible to show $A \leq_\alpha B$ without the assumption, as in Lemma 4.1, that A is regular. The bound w is needed to show $B \leq_\alpha A$.

To see that B is regular, fix τ and define

$$\langle w,x\rangle \in B_y^\tau \text{ by}$$
$$\langle w,x\rangle \in \tau^2 \;\&\; x<y \;\&\; pf(y)<pf(x) \;\&\; f(y)<w.$$

Observe that

$$(1)\qquad \tau \leq y_0 < y_1 \;\&\; B_{y_0}^\tau \not\supseteq B_{y_1}^\tau \to pf(y_1)<pf(y_0) \vee f(y_1)<f(y_0).$$

Suppose $B\cap\tau^2$ is not α-finite. Then there exists an infinite sequence $y_0<y_1<y_2<\ldots$ such that for all $i<j$,

$$\tau \leq y_i < y_j \quad \& \quad B_{y_i}^\tau \not\supseteq B_{y_j}^\tau.$$

But then (1) yields an infinite descending sequence of ordinals.

To show $A \leq_\alpha B$ it suffices by Proposition 3.4 to show $A^* \leq_{w\alpha} B$. Fix z. The set

$$K=\{v|v\leq p(z) \quad \& \quad v\in p[A^*]\}$$

is α-finite by Proposition 2.1, and so $p^{-1}[K]$ is α-finite. It follows there is a y_0 such that

$$(x)[x>y_0 \to pf(x)>p(z)].$$

Let $x_0>y_0$ be such that $pf(x_0)=\min\{pf(x)|x>y_0\}$. Then

$$pf(x_0)>p(z) \;\&\; \langle z+1,x_0\rangle \notin B.$$

Observe that for all x,

$$pf(x)>p(z) \;\&\; \langle z+1,x\rangle\notin B \to z\notin A^*-f[x].$$

Thus

$$z\notin A^* \leftrightarrow (\text{Ex})\,[pf(x)>p(z) \;\&\; \langle z+1,x\rangle\notin B \quad \& \quad z\notin f[x]].$$

To prove $B\leq_\alpha A$ let J be an α-finite subset of α^2 and define

$$J_p=\bigcup_{\langle w,x\rangle\in J}(\{v|v<w \;\&\; pv<pf(x)\}-f[x]).$$

J_p is α-finite thanks to the bound on v. Then

$$J \subseteq cB \leftrightarrow J_p \subseteq cA^*,$$

and so $B \leq_\alpha A$ by Propositions 3.3 and 3.4. □

The above proof of the regular sets theorem provides a uniformity absent from earlier versions: there exists an α-recursive function t such that for all ε, $R_{t(\varepsilon)}$ is regular and of the same α-degree as R_ε. Maass has obtained a further uniformity by finding a definition of t independent of α. The above proof has a troublesome parameter in the Σ_1^α definition of p: $\alpha \to \alpha^*$.

5. *Hyperregularity*

The notion of hyperregularity is useful in the study of relative α-recursiveness. The main result of this section is the existence of a non-α-recursive, hyperregular, α-recursively enumerable set via a mild priority argument. Here "mild" means that the assignment of priorities is based on properties of α^*, mild in contrast to Section 2.VIII, where a tame Σ_2 projection of α is invoked to solve Post's problem.

Suppose $A \subseteq \alpha$. A is said to be *hyperregular* if

$$(1) \qquad (\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \to (E\gamma)_{\gamma<\alpha}(f[\delta] \subseteq \gamma].$$

If $B \leq_\alpha A$ and A is hyperregular, then B is hyperregular. The hyperregularity of A can be viewed as a weak form of Σ_1 admissibility relative to A. A strong form would be:

$$(2) \qquad (\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \to f \restriction \delta \text{ is } \alpha\text{-finite}].$$

In general (1) and (2) are distinct, but they do coincide, as will be seen shortly, when A is α-recursively enumerable.

5.1 Proposition. *If A is α-recursively enumerable and hyperregular, then A is regular.*

Proof. Fix δ to see $A \cap \delta$ is α-finite. Assume A is non-empty, hence the range of some α-recursive function g. Let

$$f(x) = \begin{cases} \mu\sigma(g(\sigma) = x) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $f \leq_{w\alpha} A$, there is a $\gamma \supseteq f[\delta]$. But then $A \cap \delta = g[\gamma] \cap \delta$. □

Sets that are both regular and hyperregular figure prominently in the study of α-degrees. They make it possible to lift certain constructions of classical recursion

theory to α. Friedberg's jump theorem (1957a) is a good example: each Turing degree above $0'$ is the Turing jump of some lesser degree. Simpson's jump theorem (1974a) for α says: if an α-degree above $0'$, the α jump of the empty set, has a regular representative, then it is the α-jump of some lesser α-degree with a regular, hyperregular representative. Simpson's result is proved in Chapter VIII.

Lemma 5.2 captures the property of regular, hyperregular sets that makes them so welcome in recursion-theoretic constructions.

5.2 Lemma. *A is regular and hyperregular iff*

$$(\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \to f \restriction \delta \text{ is } \alpha\text{-finite}].$$

Proof. Suppose A has the latter property. Then A is hyperregular since each α-finite function is bounded below α. Let c be the characteristic function of A. Then $c \restriction \delta$ is α-finite, hence $A \cap \delta$ is α-finite.

Now suppose A has the former property. Fix $\delta < \alpha$ and $f \leq_{w\alpha} A$. Let R be α-recursively enumerable and such that

$$(1) \quad f(x) = y \leftrightarrow (\mathrm{EH})(\mathrm{EK})[\langle H, K, x, y\rangle \in R \quad \& \quad H \subseteq A \quad \& \quad K \subseteq cA].$$

R is the range of some α-recursive t. For simplicity write

$$t(\gamma) = \langle H^\gamma, K^\gamma, x^\gamma, y^\gamma\rangle.$$

Define: $g(x) = \mu\gamma[x^\gamma = x \ \& \ H^\gamma \subseteq A \ \& \ K^\gamma \subseteq cA]$;

$$h(x) = \sup\{H^\rho \cup K^\rho \mid \rho \leq g(x)\}.$$

$h(x) < \alpha$ since t is α-recursive. The regularity of A implies $h(x) \cap A$ and $h(x) - A$ are α-finite. Hence $g \leq_{w\alpha} A$, since the value of $g(x)$ is determined by α-finitely much of A.

Since A is hyperregular there is a β such that

$$\cup\{H^{g(x)} \cup K^{g(x)} \mid x < \delta\} \subseteq \beta.$$

$A \cap \beta$ is α-finite. Consequently an α-finite definition of $f \restriction \delta$ is obtained from the right side of (1) by replacing R by $t\left[\sup_{x<\delta} g(x)+1\right]$, A by $\beta \cap A$, and cA by $\beta - A$. $\square$

It is time for a mild priority argument in the setting of α-recursion theory. The construction of a non-α-recursive, hyperregular set is an excellent choice, since the associated injury sets are simpler than those developed in the solution to Post's problem in the next Chapter. It first appears that the proof of Theorem 5.5 follows the same lines as that of Theorem 2.1.VI with ω_1^{CK} replaced by α, and ω by α^*. But

some unexpected difficulties arise when $\alpha^* > \omega$. They are overcome by Lemma 2.1 of the present chapter.

5.3 Theorem (Sacks 1966). *There exists a non-α-recursive, hyperregular, α-recursively enumerable set.*

Proof. Let p be a one-one, α-recursive map of α into α^*. Recall R_ε and $\{\varepsilon\}^B$ from subsection 3.2. Let

$$x \in R_{p^{-1}\varepsilon} \text{ mean } p^{-1}\varepsilon \text{ is defined and equal to } \gamma \quad \& \quad x \in R_\gamma.$$

Then "$x \in R_{p^{-1}\varepsilon}$" is α-recursively enumerable, and so $\{R_{p^{-1}\varepsilon} | \varepsilon < \alpha^*\}$ is a simultaneous α-recursive enumeration of all α-recursively enumerable sets. Define $\{p^{-1}\varepsilon\}^B$ as in subsection 3.2, formula (2), with R_ε replaced by $R_{p^{-1}\varepsilon}$. Then $f \leq_{w\alpha} B$ iff $f = \{p^{-1}\varepsilon\}^B$ for some $\varepsilon < \alpha^*$.

To prove the theorem a set B will be α-recursively enumerated with three objectives in mind.

(1) cB is unbounded.
(2) If $R_{p^{-1}\varepsilon}$ is unbounded, then $R_{p^{-1}\varepsilon} \cap B \neq \varnothing$.
(3) Let t_ε be α^* if α^* is a regular α-cardinal, and ε otherwise. If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and single-valued for all $\gamma < t_\varepsilon$, then $\{p^{-1}\varepsilon\}^B \restriction t_\varepsilon$ is α-finite.

(1) and (2) combine to make B non-α-recursive. (3) will imply B is hyperregular. B will be the union of an increasing, α-recursive sequence $\{B^\sigma | \sigma < \alpha\}$ of α-finite sets. B^σ is that part of B enumerated by the end of stage σ. Let

$$B^{<\sigma} \text{ be } \cup \{B^\delta | \delta < \sigma\}.$$

E enumerates the even ordinals: $E(0) = 0$; $E(\varepsilon + 1) = E(\varepsilon) + 2$; and $E(\lambda) = \lambda$ for limit λ. Od enumerates the odd ordinals: $\mathrm{Od}(\varepsilon) = E(\varepsilon) + 1$.

For the sake of (2) and (3) attempts are made during the enumeration of B to satisfy the following requirements.

Req $E(\varepsilon)$: If $R_{p^{-1}\varepsilon}$ is unbounded, then $R_{p^{-1}\varepsilon} \cap B^\sigma \neq \varnothing$ for some σ.

Req $\mathrm{Od}(\varepsilon)$: If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and single-valued for all $\gamma < t_\varepsilon$, then $(E\sigma)(\tau)_{\tau \geq \sigma}(\gamma)_{\gamma < t_\varepsilon}$ such that

$$\{p^{-1}\varepsilon\}_\tau^{B^{<\tau}}(\gamma) = \{p^{-1}\varepsilon\}_\sigma^{B^{<\sigma}}(\gamma). \tag{4}$$

The right side of (4) is an α-recursive approximation of $\{p^{-1}\varepsilon\}^B(\gamma)$ at the beginning of stage σ of the enumeration of B, and is defined as follows.

Suppose R is α-recursively enumerable. Then there exists a Δ_0^α formula $F(x, y)$ such that for all $\delta < \alpha$,

$$\delta \in R \leftrightarrow L(\alpha) \models (\mathrm{Ey})\, F(\underline{\delta}, y).$$

Define R^σ by

$$\delta \in R^\sigma \leftrightarrow L(\sigma) \models (\mathrm{Ey})\, F(\underline{\delta}, y).$$

Recall that $F(x, y)$ may contain some parameter $q \in L(\alpha)$. Take R^σ to be empty if $q \notin L(\sigma)$. Each R^σ is α-finite, the function $\lambda\sigma|R^\sigma$ is α-recursive, and $R = \cup\{R^\sigma|\sigma < \alpha\}$. The *right side of* (4) is defined and equal to δ if

$$(5) \qquad (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \gamma, \delta\rangle \in R^\sigma_{p^{-1}\varepsilon} \quad \& \quad H \subseteq B^{<\sigma} \quad \& \quad J \subseteq cB^{<\sigma}].$$

If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and equal to δ, then for all sufficiently large σ,

$$\langle H, J, \gamma, \delta\rangle \in R^\sigma_{p^{-1}\varepsilon} \quad \& \quad H \subseteq B^{<\sigma} \quad \& \quad J \subseteq cB^{<\sigma}.$$

Requirement $E(\varepsilon)$ is said to be *positive* because it is met by adding an element to B. Requirement $\mathrm{Od}(\varepsilon)$ is said to be *negative* because it is met by excluding α-finitely many elements from B. If a stage σ can be found such that the right side of (4) is defined for all $\gamma < t_\varepsilon$, then requirement $\mathrm{Od}(\varepsilon)$ can be met by keeping out of B any element that would falsify any of the facts about $cB^{<\sigma}$ used in the computation of the right side of (4) for any $\gamma < t_\varepsilon$. The preservation of $\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma \restriction t_\varepsilon$ during all stages $\tau \geq \sigma$ implies that (3) holds.

Positive and negative requirements tend to conflict. If an ordinal is added to B at stage τ for the sake of requirement $E(\varepsilon_0)$, then that addition may falsify some computation developed at stage $\sigma < \tau$ and being preserved for the sake of requirement $\mathrm{Od}(\varepsilon)$. Such an event is termed an *injury* to req $\mathrm{Od}(\varepsilon)$ for the sake of req $E(\varepsilon_0)$. The adverse effect of injuries is limited by a system of Friedberg-Muchnik priorities. Req $E(\varepsilon_0)$ is said to be of *higher priority than* req $\mathrm{Od}(\varepsilon)$ if $\varepsilon_0 \leq \varepsilon$. An injury occurs only for the sake of a requirement of higher priority. It will follow from the fact that $\varepsilon < \alpha^*$ that req $\mathrm{Od}(\varepsilon)$ is injured only α-finitely often.

The construction of B consists of simultaneous α-recursive definitions of $\lambda\sigma|B^\sigma$, $\lambda\sigma|n(\sigma, \varepsilon)$, $\lambda\sigma\varepsilon|r(\sigma, \varepsilon)$ and $\lambda\sigma\varepsilon|m(\sigma, \varepsilon)$. r insures the unboundedness of B. m defines the preservation requirements on B.

Stage σ. $r(\sigma, \varepsilon)$ is the least β such that

$$\beta \geq \sup_{\delta < \sigma} r(\delta, \varepsilon) \quad \& \quad \beta \notin B^{<\sigma} \quad \& \quad [\varepsilon \in p[\sigma] \to \beta \geq p^{-1}\varepsilon].$$

$n(\sigma, \varepsilon)$ is the greatest $m \leq t_\varepsilon$ such that $\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma(\gamma)$ is defined for all $\gamma < m$. $J^\sigma_{\varepsilon,\gamma}$ is the least J (least according to the ordering of $L(\alpha)$ supplied by Proposition 1.8) such that: $\langle H, J, \gamma, \delta\rangle$ satisfies the matrix of (5) and $\sup J$ is as small as possible. $m(\sigma, \varepsilon)$ is the least β such that

$$\beta \geq \sup_{\delta < \sigma} m(\delta, \varepsilon) \quad \& \quad \beta \geq \sup_{\gamma < n(\sigma, \varepsilon)} J^\sigma_{\varepsilon,\gamma}.$$

When possible, ordinals less than or equal to $m(\sigma, \varepsilon)$ will be kept out of $B^\tau (\tau \geq \sigma)$ in order to preserve

$$\{\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma(\gamma) | \gamma < n(\sigma, \varepsilon)\}.$$

Stage σ is completed by attacking req $E(\varepsilon_0)$, where $\varepsilon_0 = h(r)$ and h is an α-recursive function that enumerates every ordinal less than α^* unboundedly often. If $R^{\sigma}_{p^{-1}\varepsilon_0} \cap B^{<\sigma} = \varnothing$ and there exists a β such that

$$\beta \in R^{\sigma}_{p^{-1}\varepsilon_0} \quad \& \quad (\varepsilon)_{\varepsilon<\varepsilon_0}[m(\sigma,\varepsilon)<\beta \quad \& \quad r(\sigma,\varepsilon)<\beta], \tag{6}$$

then put the least such β in B. Otherwise $B^{\sigma} = B^{<\sigma}$. End of stage σ.

The clause, $R^{\sigma}_{p^{-1}\varepsilon_0} \cap B = \varnothing$, is important. It implies:

(7) For each ε_0 there is at most one stage σ at which an ordinal is added to B^{σ} for the sake of req $E(\varepsilon_0)$.

If β is added to B at stage σ and $\beta \leq m(\sigma,\varepsilon)$, then $\mathrm{Od}(\varepsilon)$ is said to be injured. Define

$$I_{\varepsilon} = \{\sigma \mid \text{req } \mathrm{Od}(\varepsilon) \text{ is injured at stage } \sigma\}.$$

I_{ε} is α-recursively enumerable. The priorities were designed to insure

$$I_{\varepsilon} \text{ is } \alpha\text{-finite}. \tag{8}$$

To prove (8) fix $\varepsilon < \alpha^*$. (6) implies $h(\sigma) \leq \varepsilon$ for all $\sigma \in I_{\varepsilon}$. $\varepsilon < \alpha^*$, so $h[I_{\varepsilon}]$ is α-finite by Proposition 2.1. $h \restriction I_{\varepsilon}$ is one-one by (7). Hence

$$I_{\varepsilon} = h^{-1}[h[I_{\varepsilon}]]$$

is α-finite (and has α-cardinality less than α^*).

$\lim_{\sigma} r(\sigma,\varepsilon)$ is said to exist if there are τ and ℓ such that $r(\sigma,\varepsilon) = \ell$ for all $\sigma \geq \tau$.

$\lambda\sigma \mid r(\sigma,\varepsilon)$ is non-decreasing, so $\lim_{\sigma} r(\sigma,\varepsilon)$ exists if $\lambda\sigma \mid r(\sigma,\varepsilon)$ increases only α-finitely often. Suppose $r(\sigma-1,\varepsilon) \neq r(\sigma,\varepsilon)$. This could be because $\varepsilon = p(\sigma) > r(\sigma-1,\varepsilon)$. Otherwise $r(\sigma-1,\varepsilon)$ was put in $B^{\sigma-1}$ for the sake of req $E(\varepsilon_0)$ for some $\varepsilon_0 \leq \varepsilon$. The set of all such σ's is α-finite by the same reasoning used to prove (8). Thus

$$r(\varepsilon) = \lim_{\sigma} r(\sigma,\varepsilon)$$

exists for all $\varepsilon < \alpha^*$.

To prove (1) observe that $r(\varepsilon) \notin B$, and that $r(p\gamma) \geq \gamma$ for all γ.

To prove (3) assume $\{p^{-1}\varepsilon\}^{B}(\gamma)$ is defined and single-valued for all $\gamma < t_{\varepsilon}$. (8) implies there is a stage σ_{ε} after which req $\mathrm{Od}(\varepsilon)$ is not injured. It follows that

$$\sigma > \sigma_{\varepsilon} \quad \& \quad \gamma < n(\sigma,\varepsilon) \rightarrow \{p^{-1}\varepsilon\}^{B^{<\sigma}}_{\sigma}(\gamma) = \{p^{-1}\varepsilon\}^{B}(\gamma). \tag{9}$$

The function $\lambda\sigma \mid n(\sigma,\varepsilon)$, restricted to $\sigma > \sigma_{\varepsilon}$, is non-decreasing. The set S_{ε} of stages after stage σ_{ε} at which $n(\sigma,\varepsilon)$ increases is α-recursive. Since $n(\sigma,\varepsilon)$ is bounded above by $t_{\varepsilon} < \alpha$, it follows that the ordertype of S_{ε} is less than α. Hence $n(\sigma,\varepsilon)$ increases

only α-finitely often and

$$n(\varepsilon)=\lim_{\sigma} n(\sigma, \varepsilon)$$

exists. Suppose $n(\varepsilon)<t_\varepsilon$. Then $\{p^{-1}\varepsilon\}^B(n(\varepsilon))$ is defined and single-valued. By the remark following (5),

$$\{p^{-1}\varepsilon\}^B(n(\varepsilon))=\{p^{-1}\varepsilon\}^{B<\sigma}_{\sigma}(n(\varepsilon))$$

for all sufficiently large σ. But then $n(\varepsilon)<n(\sigma, \varepsilon)$ for all sufficiently large σ. So $n(\varepsilon)$ must equal t_ε, and (3) follows from (9).

The remainder of the proof of Theorem 5.5 splits into two lemmas.

5.4 Lemma. *B is hyperregular.*

Proof. Assume $\{p^{-1}\varepsilon\}^B\restriction\delta$ is defined and single-valued for some $\delta<\alpha$. The proof that $\{p^{-1}\varepsilon\}^B\restriction\delta$ is α-finite breaks into three cases.

Case 1: $\alpha^*=\alpha$. Then p can be the identity map. Choose $\eta>\delta$ so that $\{\varepsilon\}^X\simeq\{\eta\}^X$ for all $X\subseteq\alpha$. η exists because of the endless repetitions that inevitably occur in any standard enumeration of the α-recursively enumerable sets. η codes up the same instructions as ε. Thus $\{\varepsilon\}^B\restriction\delta=\{\eta\}^B\restriction\delta$. $t_\eta=\eta>\delta$, so by (3), proved above, $\{\eta\}^B\restriction\delta$ is α-finite.

Case 2: $\alpha^*<\alpha$ and α^* is a regular α-cardinal. Then α^* is the greatest α-cardinal and there is an α-finite map v of α^* onto δ. Choose $\varepsilon_0<\alpha^*$ so that

$$\{p^{-1}\varepsilon_0\}^B(x)\simeq\{p^{-1}\varepsilon\}^B(vx)$$

for all $x<\alpha^*$. $t_{\varepsilon_0}=\alpha^*$, so by (3), $\{p^{-1}\varepsilon_0\}^B\restriction\alpha^*(\equiv\{p^{-1}\varepsilon\}^B\restriction\delta)$ is α-finite.

Case 3: $\alpha^*<\alpha$ and α^* is a singular α-cardinal. As in case 2, it suffices to show $\{p^{-1}\varepsilon_0\}^B\restriction\alpha^*$ is α-finite. Let λ_0 be the cofinality of α^* in $L(\alpha)$. Then for some $\lambda_0<\alpha^*$, there is a sequence $\{\gamma_i\mid i<\lambda_0\}$ of α-cardinals whose limit is α^*. For each $i<\lambda_0$, there is an $n_i>\gamma_i$ such that $\{p^{-1}n_i\}^X\simeq\{p^{-1}\gamma_i\}^X$ for all $X\subseteq\alpha$ (as in case 1). By (3) $\{p^{-1}n_i\}^B\restriction\gamma_i$ is α-finite.

Choose ε_1 so that

$$\{p^{-1}\varepsilon_1\}^B(i)=\{p^{-1}\varepsilon_0\}^B\restriction\gamma_i$$

for all $i<\lambda_0$. Then $\{p^{-1}\varepsilon_1\}^B\restriction\lambda_0$ is α-finite since $\lambda_0<\alpha^*$. □

So far no reason has been given for setting t_ε equal to ε when α^* is a singular α-cardinal. Thus case 3 of the proof of Lemma 5.6 may appear to be an unnecessary complication that could have been eliminated by setting t_ε equal to α^* whenever $\alpha^*<\alpha$. The reason for distinguishing between regular and singular α^*'s is made evident in the next lemma.

5.5 Lemma. *If R_{ε_0} is unbounded, then $R_{\varepsilon_0}\cap B\neq\varnothing$.*

Proof. It suffices to show

$$\{m(\sigma, \varepsilon) \mid \varepsilon < \varepsilon_0 \quad \& \quad \sigma < \alpha\} \quad \text{and} \quad \{r(\sigma, \varepsilon) \mid \varepsilon < \varepsilon_0 \quad \& \quad \sigma < \alpha\}$$

are bounded sets. Recall from Theorem 5.5 the proof that $\lim_{\sigma} r(\sigma, \varepsilon)$ exists. $\lambda\varepsilon \mid r(\varepsilon)$ is Σ_2^α and unbounded; nonetheless the restriction of $\lambda\varepsilon \mid r(\varepsilon)$ to ε_0 is bounded. It follows from Proposition 2.1 and (8) of Theorem 5.5 that there is a σ_0 such that:

$$\{\varepsilon \mid \varepsilon < \varepsilon_0 \quad \& \quad \varepsilon \in p[\alpha]\} \subseteq p[\sigma_0]; \tag{10}$$

and no element of R_ε is added to B at or after stage σ_0 for the sake of req $E(\varepsilon)$ for any $\varepsilon < \varepsilon_0$.

The bounds on m are less apparent than those on r. Two cases have to be considered. The second contains the principal twist in the proof of Theorem 5.5.
Case 1: $\alpha^* = \alpha$ or α is a singular α-cardinal. Fix $\varepsilon < \varepsilon_0$. Let σ_0 be as in (10) above. A change in the value of $m(\sigma, \varepsilon)$ occurs at stage $\sigma > \sigma_0$ only if $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}$ is defined on a longer initial segment than it was at any previous stage after σ_0. All such initial segments are bounded by $\varepsilon + 1$. Thus the set of stages after σ_0 at which $m(\sigma, \varepsilon)$ increases for some $\varepsilon < \varepsilon_0$ is correlated with an α-recursively enumerable subset of $(\varepsilon_0 + 1)^2$. That set is α-finite, since $\varepsilon_0 < \alpha^*$.
Case 2: α^* is a regular α-cardinal. If $\varepsilon < \varepsilon_0$ and $\sigma > \sigma_0$, then changes in $m(\sigma, \varepsilon)$ at stage σ occur as in case 1 save that $n(\sigma, \varepsilon)$ is now bounded by $\alpha^* + 1$ instead of $\varepsilon + 1$. Let

$$Z = \{\varepsilon \mid \varepsilon < \varepsilon_0 \quad \& \quad \lim_{\sigma} n(\sigma, \varepsilon) = \alpha^*\}.$$

Z is α-recursively enumerable because $\alpha^* < \alpha$. Hence Z is α-finite, since $\varepsilon_0 < \alpha^*$. Choose $\sigma_1 > \sigma_0$ so that

$$(\varepsilon)\left[\varepsilon \in Z \to n(\sigma_1, \varepsilon) = \lim_{\sigma} n(\sigma, \varepsilon)\right].$$

Then $m(\sigma_1, \varepsilon) = \lim_{\sigma} m(\sigma, \varepsilon)$ for all $\varepsilon \in Z$.

For each $\varepsilon \in \varepsilon_0 - Z$, let K_ε be the set of stages after σ_0 at which $m(\sigma, \varepsilon)$ increases. K_ε is α-finite and has α-cardinality less than α^*. It follows from Lemma 2.3 that

$$\cup \{K_\varepsilon \mid \varepsilon \in (\varepsilon_0 - Z)\}$$

is α-finite, hence bounded by some σ_2. Then $m(\sigma_2, \varepsilon) = \lim_{\sigma} m(\sigma, \varepsilon)$ for all $\varepsilon \in (\varepsilon_0 - Z)$. □

Exercises 5.6 and 5.7 are intended to clarify the proof of Theorem 5.3, particularly the nature of the bounds on m and r obtained in the proof of Lemma 5.5.

These matters are discussed further in Chapter VIII. For now note that some of the functions developed in the course of a priority argument are Σ_2^α, for example, $\lambda\varepsilon\,|\,r(\varepsilon)$ in the proof of Lemma 5.5. Bounds on these Σ_2^α functions have to be derived somehow from the Σ_1 admissibility of $L(\alpha)$. For Theorem 5.5 the job was done by α^* and by Lemma 2.3 (cf. Exercise 2.12).

5.6–5.9 Exercises

5.6. α is said to be Σ_2 *admissible* if for all $f\in\Sigma_2^\alpha$,

$$(\gamma)_{\gamma<\alpha}[f\restriction\gamma \text{ is } \alpha\text{-finite}].$$

Show α is Σ_2 admissible iff every α-recursively enumerable set is hyperregular.

5.7. Let $m(\sigma,\varepsilon)$ be the α-recursive function defined in the construction of Theorem 5.5. Let $m(\varepsilon)=\lim_\sigma m(\sigma,\varepsilon)$. Show $\lambda\varepsilon\,|\,m(\varepsilon)$ is Σ_2^α. Show that $(\lambda\varepsilon|\,m(\varepsilon))\restriction\varepsilon_0$ is α-finite when $\varepsilon_0<\alpha^*$.

5.8. Suppose A is regular and hyperregular. Show

$$(B)[B\leq_{w\alpha}A\leftrightarrow B\leq_\alpha A\leftrightarrow B\in\Delta_1^{\alpha,A}].$$

Show $\langle L[A,\alpha],\varepsilon,A\rangle$ is Σ_1 admissible.

5.9. Assume $V=L$. Let α be ω_ω and A be $\alpha-\{\omega_n\,|\,n<\omega\}$. Verify that A is α-recursively enumerable and regular, but not hyperregular. Show A is complete, that is, $B\leq_\alpha A$ for every α-recursively enumerable B. Show C is hyperregular iff $A\not\leq_\alpha C$, for every $C\subseteq\alpha$.

Chapter VIII
Priority Arguments

In this chapter some standard results of classical recursion theory are lifted to every Σ_1 admissible ordinal α by techniques hinted at in earlier chapters. Since the classical proofs use the Σ_2 admissibility of $L(\omega)$, the proofs to come may be regarded as instances of the austere art of making Σ_1 admissibility do the work of Σ_2. The initial technique relies strongly on the L-ness of L, the ability of L to support downward Skolem–Löwenheim arguments. The later technique depends on combinatoric consequences of Σ_1 admissibility. Its dynamic nature makes it applicable to Σ_1 admissible structures for which hull-collapsing arguments fail.

1. α-Finite Injury via α^*

In this section and the next it will be shown that there exist α-recursively enumerable sets A and B such that neither is α-recursive in the other. The method extends that applied in Section 5.VII, to construct a hyperregular, non-α-recursive, α-recursively enumerable set. The injury sets become more complex. Back in Chapter VII each negative requirement was injured at most once for the sake of each positive requirement of higher priority. Such simplicity is rare in the present chapter. Consequently the α-recursive projection of α into α^*, which arranges that each negative requirement be opposed by less than-α^* positive requirements of higher priority, does not always compel the injury sets to be α-finite. In Section 2 below, when $\alpha^* = \alpha$ and there is a greatest α-cardinal, it will be necessary to project α downward by means of a carefully chosen Σ_2^α function.

1.1 Strategy. Define $\{p^{-1}\varepsilon\}^B$ for each $\varepsilon < \alpha^*$ as in the beginning of the proof of Theorem 5.5.VII. Thus $A \leq_{w\alpha} B$ iff $A = \{p^{-1}\varepsilon\}^B$ for some $\varepsilon < \alpha^*$. The requirements on A and B are as follows.

Requirement 2ε: If $\{p^{-1}\varepsilon\}^B$ is a total function, then $A \neq \{p^{-1}\varepsilon\}^B$.

Requirement $2\varepsilon + 1$: Same as req 2ε with A and B interchanged.

Let $\{Z_\varepsilon | \varepsilon < \alpha^*\}$ be a collection of simultaneously α-recursive, pairwise disjoint, unbounded subsets of α. The strategy for satisfying req 2ε consists of finding an $x \in Z_\varepsilon$ such that:

(1) $$\text{if } \{p^{-1}\dot{\varepsilon}\}^B(x) = 0, \text{ then } x \in A.$$

(Recall that $A(x)$, the characteristic function of A, is 1 when $x \in A$.) As in the proof of Theorem 5.5.VII, $\{p^{-1}\varepsilon\}^B(x)$ is approximated at stage σ of the construction of A and B by $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$, an α-recursive function of ε, σ and x. $B^{<\sigma}$ is that part of B enumerated prior to stage σ.

Suppose $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x) = 0$ and $x \notin A^{<\sigma}$ for some $x \in Z_\varepsilon$. Then at stage σ an attempt to satisfy req 2ε can be made by adding x to A and making a commitment to preserve the computation of $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$ through all future stages. The commitment reduces to: exclude all the elements of some α-finite K from B, where K is the set of negative facts about $B^{<\sigma}$ used in the computation of $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$. If the attempt is made at stage σ, and if the resulting commitment is honored at all future stages, then $x \in A$, $\{p^{-1}\varepsilon\}^B(x) = 0$ and req 2ε is satisfied.

The commitment is broken the first time some member of K is added to B for the sake of some req $2\varepsilon_0 + 1$ at some stage $\tau > \sigma$. In that event req 2ε is said to be *injured* at stage τ for the sake of req $2\varepsilon_0 + 1$. Let

$$I_{2\varepsilon} = \{\tau | \text{req } 2\varepsilon \text{ is injured at stage } \tau\}.$$

$I_{2\varepsilon}$ is called an injury set. Define $I_{2\varepsilon+1}$ similarly. The injury sets are simultaneously α-recursively enumerable. Once it is seen that all injury sets are α-finite, it is not difficult to show all requirements are met.

Two devices are employed to limit the extent of injury sets. The first assigns priorities. Requirement x has higher priority than requirement y if $x < y$. The priority method consists of allowing requirement y to be injured for the sake of requirement x only if x has higher priority than y. Thus req 2ε can be injured by adding some element to B for the sake of req $2\varepsilon_0 + 1$ only when $2\varepsilon_0 + 1 < 2\varepsilon$. The second device enhances the first. It indexes the requirements so as to form the shortest possible list compatible with a Σ_1^α construction. The shorter the list, the smaller the set of requirements of higher priority than a given requirement. In this section the requirements are indexed by ordinals less than α^* with the aid of p, an α-recursive projection of α into α^*. In the next section they are indexed by ordinals less than a certain ordinal below α^* with the aid of a Σ_2^α projection.

1.2 Construction of A and B. At each stage attention is paid to only one requirement, but every requirement is paid attention unboundedly often. Let σ be a stage at which attention is paid to req 2ε.

First suppose an attempt was made to satisfy req 2ε prior to stage σ and no subsequent injury was inflicted on req 2ε prior to stage σ. To elaborate: there was a stage $\sigma_0 < \sigma$ at which some element of Z_ε was put in A and a commitment was made to keep some α-finite K from touching B; that commitment has been honored up to now, in short, $K \subseteq cB^{<\sigma}$. There is clearly no need to make a new attempt at stage σ to satisfy req 2ε. Let $A^\sigma = A^{<\sigma}$ and $B^\sigma = B^{<\sigma}$.

Now suppose every attempt to satisfy req 2ε made prior to stage σ was injured prior to stage σ. Let $m_B(\sigma, \varepsilon)$ be the supremum of all y such that prior to stage σ, a commitment was made to keep y out of A for the sake of req $2\varepsilon_0 + 1$ for some $\varepsilon_0 < \varepsilon$. Then any $y > m_B(\sigma, \varepsilon)$ can be added to A at stage σ without injuring any

requirement of higher priority than req 2ε. Define

$$w(\sigma, \varepsilon) = \mu y[y > m_B(\sigma, \varepsilon) \quad \& \quad y \in Z_\varepsilon - A^{<\sigma}].$$

(1) If $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(w(\sigma, \varepsilon)) = 0$, then

$$A^\sigma = A^{<\sigma} \cup \{w(\sigma, \varepsilon)\} \quad \text{and} \quad B^\sigma = B^{<\sigma};$$

in addition a commitment is made to keep all of K out of B, where K is an α-finite set of negative membership facts about $B^{<\sigma}$ needed for the computation of (1). If (1) is false, then $A^\sigma = A^{<\sigma}$ and $B^\sigma = B^{<\sigma}$.

Req $2\varepsilon + 1$ is handled similarly. End of construction.

The next lemma captures the so-called *α-finite injury method.* The analogous Friedberg–Muchnik lemma says that the n-th requirement is injured less than 2^n times.

1.3 Lemma. *Suppose β is an infinite α-cardinal and $\delta < \beta$. Then injury set I_δ is α-finite and has α-cardinality $< \beta$.*

Proof. Let $\beta_0 \leq \beta$ be a regular α-cardinal such that $\delta < \beta_0$. For the sake of an induction, assume for $\gamma < \delta$,

$$I_\gamma \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality } < \beta_0.$$

the I_γ's are simultaneously α-recursively enumerable, and so by Lemma 2.3.VII,

$$\cup \{I_\gamma | \gamma < \delta\} \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality } < \beta_0.$$

Now define

$$A_\gamma = \{\sigma | \text{an attempt is made at stage } \sigma \text{ to satisfy req. } \gamma\}.$$

Observe that A_γ and I_γ are *interlaced*: between any two elements of either set lies a member of the other. Hence if either set is α-finite, then so is the other, and in addition their α-cardinalities differ by at most 1. Consequently

(1) $$\cup \{A_\gamma | \gamma < \delta\} \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality } < \beta_0.$$

I_δ is a subset of $\cup \{A_\gamma | \gamma < \delta\}$, and so can be viewed as an α-recursively enumerable subset of some ordinal less than β_0. Since $\beta_0 \leq \alpha^*$, I_δ is α-finite. □

Theorem 1.4 solves half of Post's problem. The other half is in the next section.

1.4 Theorem. *Suppose $\alpha^* < \alpha$, or $\alpha^* = \alpha$ and there is no greatest α-cardinal. Then there exist α-recursively enumerable sets A and B such that $A \not\leq_{w\alpha} B$ and $B \not\leq_{w\alpha} A$.*

Proof. Let A and B be the sets enumerated in subsection 1.2. Assume $\{p\varepsilon^{-1}\}^B$ is total with the intent of showing $A \neq \{p\varepsilon^{-1}\}^B$. According to formula (1) of the proof of Lemma 1.3, there exists a σ_0 such that for $\delta \leq 2\varepsilon$, all attempts to satisfy req 2ε occur prior to stage σ_0.

Case 1: there was an attempt at some stage σ to satisfy req 2ε, and the associated commitment was honored at all subsequent stages. Then

$$\{p^{-1}\varepsilon\}_{\sigma}^{B<\sigma}(w(\sigma, \varepsilon)) = 1 = \{p^{-1}\varepsilon\}^B(w(\sigma, \varepsilon))$$

and $A^{\sigma}(w(\sigma, \varepsilon)) = 0 = A(w(\sigma, \varepsilon))$.

Case 2: otherwise. Let $\sigma \geq \sigma_1 \geq \sigma_0$ and suppose σ and σ_1 are stages at which attention is paid to req 2ε. Then

$$m_B(\sigma, \varepsilon) = m_B(\sigma_1, \varepsilon) \quad \text{and} \quad w_B(\sigma, \varepsilon) = w_B(\sigma_1, \varepsilon) \notin A^{<\sigma}.$$

No attempt is made to satisfy req 2ε at stage σ, so

$$\{p^{-1}\varepsilon\}_{\sigma}^{B<\sigma}(w_B(\sigma, \varepsilon)) \neq 1.$$

Since $\{p^{-1}\varepsilon\}^B$ is total, it follows that

$$\{p^{-1}\varepsilon\}^B(w_B(\sigma_1, \varepsilon)) \neq 1.$$

Thus $w_B(\sigma_1, \varepsilon)$ is a witness to the inequality of A and $\{p^{-1}\varepsilon\}^B$. □

The technology of the next section is more than is needed to solve Post's problem. Exercise 1.5 describes a much less technical solution. Sections 1 and 2 are intended to introduce ideas needed for further results such as the uniform solution of subsection 3.7.

1.5 Exercise. Assume $\alpha^* = \alpha$. Construct A and B as in subsection 1.2 with one additional proviso: suppose req x is injured at stage σ for the sake of req y; then req x can be injured at stage $\tau > \sigma$ for the sake of req z only if $z < y$. Show each injury set is finite. Show A and B are α-recursively enumerable, $A \not\leq_{w\alpha} B$, and $B \not\leq_{w\alpha} A$.

2. *α-Finite Injury and Tameness*

In this section it is assumed that $\alpha^* = \alpha$ and there exists a greatest α-cardinal, call it $\aleph$. The solution to Post's problem given in the previous section now fails, because α^* is no longer the limit of regular α-cardinals, and consequently Lemma 1.3 is not applicable to all $\delta < \alpha^*$. One way around the difficulty is to project α into

something potentially smaller than α^*. The new mode of projection will be Σ_2^α rather than Σ_1^α, and the range of the projection will be some ordinal multiple of $\aleph$. Lemma 1.3 will apply to each block of requirements of length $\aleph$, and a further idea will be involved to control the injuries within a short union of blocks. The arguments of this section rely strongly on the fine structure of L and are as concrete as possible. Lemma 2.7 provides rare bounds on α-finite injury sets. In succeeding sections the method is dealt with more abstractly so as to be applicable to a wide range of Σ_1 admissible sets.

2.1 Tame Σ_2 Maps. The notion of tameness was invented by M. Lerman to clarify the proof of Theorem 2.6. It has many applications. Let $f: \alpha \to \alpha$ be Σ_2^α. f has an α-recursive approximation that will not surprise students of classical recursion theory. Since $f \in \Sigma_2^\alpha$, there is a $D \in \Delta_0^\alpha$ such that

$$f(x) = y \leftrightarrow L(\alpha) \models (\mathrm{Ev})(u) D(u, v, \underline{x}, \underline{y})$$

for all $x, y < \alpha$. Define α-recursive w and g by:

$$w(\sigma, x) = \mu w_{w<\sigma}(u)_{u<\sigma} D(u, (w)_0, x, (w)_1); \tag{1}$$

$$g(\sigma, x) = (w(\sigma, x))_1. \tag{2}$$

2.2 Proposition. *$f \in \Sigma_2^\alpha$ iff there exists an α-recursive g such that $f(x) = \lim_\sigma g(\sigma, x)$.*

Proof. Suppose $f(x) = \lim_\sigma g(\sigma, x)$. Then

$$f(x) = y \leftrightarrow (E\sigma)(\tau)_{\tau \geq \sigma}[g(\sigma, x) = y].$$

Now suppose f is Σ_2^α. Define $w(\sigma, x)$ and $g(\sigma, x)$ as in (1)–(2) of subsection 2.1. Fix x and let w be the least $\langle v, y \rangle$ such that $(u)D(u, v, x, y)$ holds in $L(\alpha)$. Then $(w)_1 = f(x)$. Define

$$h(z) \simeq \mu u \sim D(u, (z)_0, x, (z)_1).$$

h is partial α-recursive and defined for all $z < w$. Let

$$\tau = \sup\{h(z) | z < w\}.$$

$\tau < \alpha$ thanks to the Σ_1 admissibility of $L(\alpha)$. For all $\sigma > \tau$, $w(\sigma, x) = w$ and $g(\sigma, x) = f(x)$. □

Let $f: \delta \to \alpha$ for some $\delta \leq \alpha$. f is said to be *tame* Σ_2^α if there exists an α-recursive g such that

$$(\gamma)_{\gamma<\delta}(E\tau)(\sigma)_{\sigma>\tau}(x)_{x<\gamma}[g(\sigma, x) = f(x)].$$

Thus $\lim_{\sigma} g(\sigma, x) = f(x)$, and so $f \in \Sigma_2^\alpha$ as in the first part of the proof of Proposition 2.2. The tameness of f refers to the way g approximates f on proper initial segments of the domain of f. Let g_σ denote $\lambda x | g(\sigma, x)$. Then

$$(\gamma)_{\gamma<\delta}(E\tau)(\sigma)_{\sigma>\tau}[g_\sigma \restriction \gamma = f \restriction \gamma].$$

A Σ_2^α function need not be tame Σ_2^α (cf. Exercise 2.14).

The *tame* Σ_2^α *projectum* of α, denoted by $t\sigma p2(\alpha)$, is

$$\mu\gamma(\mathrm{Ef})[f \in \text{tame}\,\Sigma_2^\alpha \ \& \ f \text{ maps } \gamma \text{ onto } \alpha].$$

Warning: it can happen that $\alpha > t\sigma 2p(\alpha)$ and $t\sigma 2p(\alpha)$ is not an α-cardinal.

2.3 Lemma (Lerman). *Let $g(\sigma, x)$ be an α-recursive function such that $\lim_{\sigma} g(\sigma, x)$ exists for all x. Suppose $\lim_{\sigma} g(\sigma, x)$ maps α one-one into δ. Assume: $(\gamma)_{\gamma<\delta}(E\tau)(y)_{y<\gamma}$ either*

$$(x)(\sigma)_{\sigma>\tau}[g(\sigma, x) \neq y] \quad \textit{or} \quad (\mathrm{Ex})(\sigma)_{\sigma>\tau}[g(\sigma, x) = y].$$

Then $t\sigma 2p(\alpha) \leq \delta$.

Proof. The map $\lim_{\sigma} g(\sigma, x)$ is collapsed and inverted. Let $h(\sigma, y)$ be the least member of

$$\{g(\sigma, w) | w \leq \sigma\} - \{h(\sigma, z) | z < y\}$$

if there is one, and zero otherwise. Let δ_0 be the ordertype of the range of $\lambda x | \lim_{\sigma} g(\sigma, x)$. Then for each $y < \delta_0$, $\lim_{\sigma} h(\sigma, y)$ exists and is equal to y-th smallest member of the range of $\lambda x | \lim_{\sigma} g(\sigma, x)$. For $y < \delta_0$, define

$$k(\sigma, y) = \mu x_{x \leq \sigma}[g(\sigma, x) = h(\sigma, y)].$$

Then $k(\sigma, y)$ is α-recursive, and $\lim_{\sigma} k(\sigma, y)$ is a tame Σ_2^α map of δ_0 onto α. □

2.4 Corollary. $t\sigma 2p(\alpha) \leq \alpha^*$.

Proof. Let $g: \alpha \to \alpha^*$ be one-one, into, and α-recursive. Define $g(\sigma, x) = g(x)$ for all σ, $x < \alpha$. $g(\sigma, x)$ satisfies the hypotheses of Lemma 2.3 ($\delta = \alpha^*$) with the aid of Proposition 2.1.VII. □

The Σ_2^α *cofinality of* α, denoted by $\sigma 2\mathrm{cf}(\alpha)$, is

$$\mu\beta(\mathrm{Ef})[\, f \in \Sigma_2^\alpha \quad \& \quad f \text{ is strictly increasing} \quad \& \quad \mathrm{dom} f = \beta \quad \& \quad \text{sup range } f = \alpha].$$

$\sigma 2\mathrm{cf}(\alpha)$ measures the failure of $L(\alpha)$ to be Σ_2 admissible. Thus $L(\alpha)$ is Σ_2 admissible iff $\sigma 2\mathrm{cf}(\alpha) = \alpha$.

The *greatest cardinal of* α, denoted by $\mathrm{gc}(\alpha)$, is the greatest α-cardinal, if there is one, and α otherwise.

2.5 Lemma. If $t\sigma 2p(\alpha) > \mathrm{gc}(\alpha)$, *then*

$$t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha).$$

Proof. Let $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma + \lambda$, where $0 < \gamma \leq \alpha$ and $\lambda < \mathrm{gc}(\alpha)$; and let f be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. Suppose $\lambda > 0$. Then $f[\mathrm{gc}(\alpha) \cdot \gamma]$ is α-finite by the tameness of f. Let t be a one-one, α-recursive map of $\alpha - f[\mathrm{gc}(\alpha) \cdot \gamma]$ onto α. Then f_0, defined by

$$f_0(x) = t(f(\mathrm{gc}(\alpha) \cdot \gamma + x)),$$

is a tame Σ_2^α map from λ onto α. But that is impossible because $\lambda < t\sigma 2p(\alpha)$.

Thus $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma$. Suppose γ is not a limit ordinal. Then $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot (\gamma - 1) + \mathrm{gc}(\alpha)$, and the argument of the previous paragraph can be repeated with $\mathrm{gc}(\alpha)$ in place of λ to obtain a tame Σ_2^α map from $\mathrm{gc}(\alpha)$ onto α. An impossibility since $\mathrm{gc}(\alpha) < t\sigma 2p(\alpha)$.

Thus $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma$ for some limit γ. For each $\delta < \gamma$, let

$$h(\delta) = \sup\{ f(x) | x < \mathrm{gc}(\alpha) \cdot \delta \} + \delta.$$

The tameness of f implies $h(\delta) < \alpha$. h is a strictly increasing Σ_2^α map from γ into an unbounded subset of α. Hence $\sigma 2\mathrm{cf}(\alpha) \leq \gamma$.

It remains only to construct a tame Σ_2^α map j from $\mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$ onto α. Let k be a strictly increasing Σ_2^α map from $\sigma 2\mathrm{cf}(\alpha)$ onto an unbounded subset of α. Assume $k(0) = 0$. The range of k divides α into blocks. For each $\delta < \sigma 2\mathrm{cf}(\alpha)$, the δ-th block is $[k(\delta), k(\delta + 1))$. Let z_δ be the least α-finite map of $\mathrm{gc}(\alpha)$ onto the δ-th block ("least" is defined by Proposition 1.8.VII). Define

$$j(x) = z_\delta(x - (\mathrm{gc}(\alpha) \cdot \delta)) \quad \text{if} \quad \mathrm{gc}(\alpha) \cdot \delta \leq x < \mathrm{gc}(\alpha)$$

for all $x < \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$. j is tame Σ_2^α because k is. k is tame because every Σ_2^α function with domain $\leq \sigma 2\mathrm{cf}(\alpha)$ is tame (cf. Exercise 2.15). □

The ideas behind Lemma 2.5 are helpful when studying Σ_1 admissible structures that are not L-like. The present section concludes with the original solution to

Post's problem for Σ_1 admissible ordinals. It relies on a tame Σ_2 phenomenon derived from stability properties in L.

2.6 Theorem (Sacks & Simpson 1972). *There exist two α-recursively enumerable sets such that neither is weakly α-recursive in the other (Post's problem).*

Proof. By Theorem 1.4 it is safe to assume $\alpha^* = \alpha$ and there is a greatest α-cardinal, call it $\aleph$. By Lemma 2.6.VII, α is the limit of α-stable ordinals. Let

$$0 = \delta_0 < \aleph = < \delta_1 < \delta_2 < \cdots < \delta_\gamma < \cdots (\gamma < \lambda_0)$$

be a listing of all α-stable ordinals beyond $\aleph$. A tame Σ_2^α map f from $\aleph \cdot \lambda_0$ onto α will be defined in a moment. (It is possible that $t\sigma 2p(\alpha) < \aleph \cdot \lambda_0$.) The α-stable ordinals divide α into λ_0 blocks. Block γ is $[\delta_\gamma, \delta_{\gamma+1})$. Let h_γ be the least α-finite map of $\aleph$ onto block γ. Define

$$f(x) = h_\gamma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < \aleph \cdot (\gamma + 1).$$

f maps $\aleph \cdot \lambda_0$ onto α.

Guessing tamely at f proceeds as follows. At stage σ let

$$\aleph = \delta_1^\sigma < \delta_2^\sigma < \cdots < \delta_\gamma^\sigma < \cdots (\gamma < \lambda_0^\sigma)$$

be a listing of all σ-stable ordinals beyond $\aleph$. (β is σ-stable if $\beta < \sigma$ and $L(\beta) \prec_1 L(\sigma)$.) If β is σ-stable and $\sigma \geq$ some α-stable ordinal $\geq \beta$, then β is α-stable. Hence

$$\delta_\gamma^\sigma = \delta_\gamma \quad \text{for all} \quad \sigma \geq \delta_{\gamma+1}.$$

Thus δ_γ is a tame Σ_2^α function of γ.

Let h_γ^σ be the least α-finite map of $\aleph$ onto $[\delta_\gamma^\sigma, \delta_{\gamma+1}^\sigma)$ in $L(\sigma)$, if there is one; otherwise $h_\gamma^\sigma = 0$. Then $h_\gamma^\sigma = h_\gamma$ for all $\sigma \geq \delta_{\gamma+2}$. Let

$$g(\sigma, x) = h_\gamma^\sigma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < (\gamma + 1)$$

for all $x < \aleph \cdot \lambda_0^\sigma$. Then f is tame Σ_2^α via the α-recursive approximation g. In essence f is tame because a correct guess of one α-stable ordinal implies a correct guess of all lesser α-stable ordinals.

The construction of A and B proceeds as in subsections 1.1–1.2 save that requirements are indexed by ordinals less than $\aleph \cdot \lambda_0$. For each $\varepsilon < \aleph \cdot \lambda_0$, req 2ε is: if $\{f\varepsilon\}^B$ is total, then $A \neq \{f\varepsilon\}^B$. req $2\varepsilon + 1$ is similar with A and B exchanged. Stage σ unfolds as it did in subsection 1.2 with $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}$ replaced by $\{g(\sigma, \varepsilon)\}_\sigma^{B<\sigma}$. Consequently there is a new reason for making repeated attempts to satisfy req 2ε. Reduction procedure $\{g(\sigma, \varepsilon)\}$ varies with σ. The resulting disturbance dies down

quickly. By the Σ_2 tameness of f via g,

$$(x)_{x<\aleph\cdot\lambda_0}(E\tau)(\sigma)_{\sigma\geq\tau}(\varepsilon)_{\varepsilon<x}[g(\sigma,\varepsilon)=f(\varepsilon)].$$

For $\varepsilon<\aleph\cdot\lambda_0$ and $i<2$, define injury set

$$I^i_\varepsilon=\{\sigma|\text{req}(2\varepsilon+i)\text{ is injured at stage }\sigma\}.$$

The next lemma should be compared with Lemma 1.3.

2.7 Lemma. $I^i_{\aleph\cdot\gamma+\beta}\in L(\delta_{\gamma+2})$ $(\gamma<\lambda_0\ \&\ \beta<\aleph)$.

Proof. Req$(2\varepsilon+i)$ is said to be *active* at stage σ if it is injured at stage σ, or if an attempt to satisfy it is made at stage σ, or if

$$(\tau)_{\tau<\sigma}(E\eta)_{\tau\leq\eta<\sigma}[g(\eta,\varepsilon)\neq g(\sigma,\varepsilon)].$$

By induction on γ the following is proved: all activity occasional by req $2(\aleph\cdot\gamma+\beta)+i$ takes place before stage τ for some $\tau<\delta_{\gamma+2}$.

Req$(2\varepsilon+i)$ is said to be in block y if

$$\aleph\cdot y\leq\varepsilon<\aleph\cdot(y+1).$$

Fix γ. By induction all activity with respect to requirements in block y, for all $y<\gamma$, takes place before stage $\delta_{\gamma+1}$. $h_\gamma\in L(\delta_{\gamma+2})$, so there is a $\tau_0<\delta_{\gamma+2}$ such that

$$(\sigma)_{\sigma\geq\tau_0}(\varepsilon)_{\varepsilon<\aleph\cdot(\gamma+1)}[g(\sigma,\varepsilon)=f(\varepsilon)].$$

Now an induction on β shows that req $z(\aleph\cdot\gamma+\beta)+i$ is inactive after stage τ for some $\tau<\delta_{\gamma+2}$. The induction on β proceeds in the same fashion as the induction on δ in the proof of Lemma 1.3. After stage τ_0, injuries to, and attempts to satisfy, req $2(\aleph\cdot\gamma+\beta)+i$ are interlaced. As in the proof 1.3, Lemma 2.3.VII is applied to show $I^i_{\aleph\cdot\gamma+\beta}$ is α-finite and has α-cardinality less than $\aleph$. In short, the argument of 1.3 works within block β. Let

$$\tau_0\leq\sigma_0<\sigma_1<\ldots<\sigma_i<\ldots\ (i<\rho)$$

be a listing of $I^i_{\aleph\cdot\gamma+\beta}-\tau_0$. $\{\sigma_i|i<\rho\}$ is an α-recursively enumerable set defined by a Σ^α_1 formula whose parameters are i, γ, β, τ_0 and $\aleph$. ($\aleph$ is the only parameter needed for the enumeration of A and B.) Since $\rho<\aleph<\delta_{\gamma+2}$, the α-stability of $\delta_{\gamma+2}$ pins down the σ_i's. Suppose $\{\sigma_i|i<j\}\subseteq L(\delta_{\gamma+2})$. Proceed by induction on j. If $j\leq\rho$, then $\{\sigma_i|i<j\}\in L(\delta_{\gamma+2})$, because $\delta_{\gamma+2}$ is Σ_1 admissible. Hence $\{\sigma_i|i<\rho\}\in L(\delta_{\gamma+2})$.

The proof of Theorem 2.6 is completed as in the proof of Theorem 1.4. □

2.8–2.15 Exercises

2.8. Assume $A, B \subseteq \alpha$. $L[B, \alpha]$ was defined in subsection 3.5.VII. $L[B, \alpha]^+$ is the least $C \supseteq L[B, \alpha]$ such that $\langle C, B\rangle$ is Σ_1 admissible. A is said to be α-computable from B (in symbols $A \leq_{\alpha c} B$) iff A is Δ_1 over $\langle L[B, \alpha]^+, B\rangle$. Suppose B is α-recursively enumerable and hyperregular. Show

$$A \leq_{\alpha c} B \leftrightarrow A \leq_\alpha B.$$

2.9. Show there exist α-recursively enumerable sets A and B such that $A \not\leq_{\alpha c} B$ and $B \not\leq_{\alpha c} A$.

2.10. Show that $\sigma 2\mathrm{cf}(\alpha)$ equals

$$\mu\beta(\mathrm{Ef})[f \in \Sigma_2^\alpha \ \&\ \mathrm{dom}\, f = \beta \ \&\ \sup \mathrm{range}\, f = \alpha].$$

(In other words the clause "f is strictly increasing" can be dropped from the definition of $\sigma 2\mathrm{cf}(\alpha)$.)

2.11. Show that $t\sigma 2p(\alpha)$ equals

$$\mu\gamma(\mathrm{Ef})\left[f \in \mathrm{tame}\, \Sigma_2^\alpha \ \&\ f : \gamma \xrightarrow[\text{onto}]{\text{1-1}} \alpha\right].$$

2.12. Find an α such that $t\sigma 2p(\alpha) < \alpha$ and $t\sigma 2p(\alpha)$ is not an α-cardinal.

2.13. Find an α such that $t\sigma 2p(\alpha) < \aleph \cdot \lambda_0$, where $\aleph$ and λ_0 are as in the proof of Theorem 2.6.

2.14. Show α is Σ_2 admissible iff every Σ_2^α function is tame Σ_2^α.

2.15. Suppose $f \in \Sigma_2^\alpha$ and $\mathrm{dom} f \leq \sigma 2\mathrm{cf}(\alpha)$. Show f is tame Σ_2^α.

3. *Dynamic Versus Fine-Structure*

An argument in higher recursion theory, particularly an argument about "recursively enumerable" sets, is said to be fine-structure in character if it relies on the collapsing (or condensation) method associated with L. The proof of Theorem 2.6 is such an argument, because the stable ordinals δ_γ ($\gamma < \lambda_0$) owe their existence to Lemma 2.6.VII, whose proof makes explicit use of Mostowski's collapsing map. A dynamic argument relies on combinatoric reasoning about cofinalities and projecta. It may form hulls, but it does not collapse them. The regular sets theorem, 4.2.VII, is proved dynamically. A less obvious example is Theorem 5.3.VII, the existence of a non-α-recursive, hyperregular set. It makes use of Lemma 2.3.VII and Proposition 2.1.VII, both proved dynamically. Both 2.3 and 2.1 have a dependence on admissibility. If α is not Σ_1 admissible, then 2.3 can fail, but 2.1 remains true by a fine-structure argument.

Dynamic methods possess the force needed to operate outside L and are in harmony with classical recursion theory. Fine structure techniques are more delicate. They keep careful count of injuries to requirements. Shore's density theorem, Chapter IX, is a powerful combination of both approaches.

The purpose of this section is to enlarge the repertoire of dynamic methods.

3.1 Lerman's Tame Σ_2 Approach. The fine-structure aspects of the proof of Theorem 2.6 (Post's problem) can be eliminated systematically as follows. (The proof of 2.6 employed a tame Σ_2^{α} map from $\aleph \cdot \lambda_0$ onto α when $\alpha^* = \alpha$ and $\aleph = \mathrm{gc}(\alpha) < \alpha$.) Requirements are indexed by ordinals less than $t\sigma 2p(\alpha)$.

The treatment of requirements breaks into two cases.

(i) $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha) (\leq \alpha)$.
(ii) $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$.

By Lemma 2.5 either (i) or (ii) holds for all α. If (i) holds, then ε is less than some α-cardinal. If (ii) holds, then ε sits in block γ, that is,

$$\mathrm{gc}(\alpha) \cdot \gamma \leq \varepsilon < \mathrm{gc}(\alpha) \cdot (\gamma + 1),$$

and can be viewed as less than some α-cardinal modulo $\mathrm{gc}(\alpha)$.

The reasoning behind Lemma 1.3 applies to both cases. Fix ε and choose τ so that the α-recursive approximation of $f \restriction (\varepsilon + 1)$ is correct from stage τ onward. Assume I_x, the x-th injury set, is α-finite for all $x < \varepsilon$. If (i) holds, then I_ε is α-finite, as in 1.3, by the combinatoric lemma (2.3.VII). Note the use of Proposition 2.1.VII in the proof of 1.3.

Suppose (ii) holds. Let

$$J_\delta = \cup \{I_x | \mathrm{gc}(\alpha) \cdot \delta \leq x < \mathrm{gc}(\alpha) \cdot (\delta + 1)\}.$$

Assume J_δ is α-finite for all $\delta < \gamma$. Since J_δ is a Σ_2^{α} function of δ, and $\gamma < \sigma 2\mathrm{cf}(\alpha)$, it follows that $\cup \{J_\delta | \delta < \gamma\}$ is α-finite. Hence there is a stage $\sigma_\gamma (\geq \tau)$ after which all activity with respect to requirements in block δ, for all $\delta < \gamma$, ceases. Then requirements in $[\mathrm{gc}(\alpha) \cdot \gamma, \varepsilon)$ can be handled the same way as requirements in $[0, \varepsilon)$ were handled in case (i). In short: $\gamma < \sigma 2\mathrm{cf}(\alpha)$ implied a bound on activity in the first γ blocks; and the combinatoric lemma implies a bound on activity up to any point within a block, since the length of a block is an α-cardinal.

It still has to be shown that J_γ is α-finite. Let I'_x be the set of stages after σ_γ at which req x is injured. It suffices to show

$$(1) \qquad \cup \{I'_x | \mathrm{gc}(\alpha) \cdot \gamma \leq x < \mathrm{gc}(\alpha) \cdot (\gamma + 1)\}$$

is α-finite. The combinatoric lemma implies (by induction on x in block γ) that each I'_x is α-finite and has α-cardinality less than $\mathrm{gc}(\alpha)$. The simultaneous enumeration of the I'_x's gives rise to a partial α-recursive, 1-1 map of (1) into $\mathrm{gc}(\alpha) \cdot \mathrm{gc}(\alpha)$. If b is the β-th member of I'_x to be enumerated, then b is mapped to $\langle \beta, x - \mathrm{gc}(\alpha) \rangle$. Suppose (1) is not α-finite. Then there is an α-recursive, one-one map from α into (1), and

from (1) into $gc(\alpha) \cdot gc(\alpha)$. Hence $\alpha^* \leq gc(\alpha)$, and so $t\sigma 2p(\alpha) \leq gc(\alpha)$ by Corollary 2.4. But that contradicts the hypothesis of case (ii). □

3.2 Shore's Blocking Method. The notion of block is carried a step further. For Shore there is no conflict between requirements in the same block. For example, the requirements in a given block might all be of the form $A \neq \{\varepsilon\}^B$. The success of the method turns on a lemma concerning cofinalities. Define $\sigma 2\mathrm{cf}^{\alpha}(\delta)$, the Σ_2^{α} cofinality of δ, to be

$$\mu\beta(\mathrm{Eh})[h \in \Sigma_2^{\alpha} \ \& \ h \text{ is strictly increasing} \ \& \ \mathrm{dom}\, h = \beta \ \& \ \sup \mathrm{range}\ h = \delta].$$

Thus $\sigma 2\mathrm{cf}^{\alpha}(\alpha) = \sigma 2\mathrm{cf}(\alpha)$.

3.3 Lemma (Shore). $\sigma 2\mathrm{cf}^{\alpha}(\alpha^*) = \sigma 2\mathrm{cf}(\alpha)$.

Proof. Let f: $\alpha \to \alpha^*$ be α-recursive, one-one and into. Suppose g is a strictly increasing Σ_2^{α} map from β into α^* with range unbounded in α^*. Then there exists a strictly increasing Σ_2^{α} map h from β into α with unbounded range. A rough approximation of h is $f^{-1}g$. The details are as follows. Define

$$h(x) = \sup\{f^{-1}(y) | y < g(x) \ \& \ y \in \mathrm{range}\ f\} + x.$$

The "$+x$" insures that h is strictly increasing. Proposition 2.1.VII implies h is Σ_2^{α}; for each $x < \beta$ there exists α-finite sets w_1 and w_2 such that

$$w_1 \cup w_2 = g(x) \ \& \ w_1 \subseteq \mathrm{range}\, f \ \& \ w_2 \cap \mathrm{range}\, f = \phi.$$

Thus $h(x) = z$ iff

$$(1) \qquad (\mathrm{Ew})(\mathrm{Ew}_1)(\mathrm{Ew}_2)[w = g(x) \ \& \ w = w_1 \cup w_2 \ \& \ w \subseteq \mathrm{range}\ f \ \& \ w_2 \cap \mathrm{range}\, f = \phi \ \& \ z = \sup\{f^{-1}(y) | y \in w_1\} + x].$$

(1) is easily seen to be Σ_2^{α} save for perhaps one detail. The formula, $w \subseteq \mathrm{rg}\, f$, rendered as

$$(2) \qquad (u)_{u \in w}(\mathrm{Ex})(f(x) = u),$$

appears to be Π_2^{α}, but is in fact Σ_1^{α}. Let $P(x, u, v)$ be a Δ_0^{α} formula such that

$$f(x) = u \leftrightarrow (\mathrm{Ev})P(x, u, v).$$

Then (2) is equivalent to

$$(\mathrm{Eq})(u)_{u \in w}(\mathrm{Ex})_{x \in q}(\mathrm{Ev})_{v \in q} P(x, u, v).$$

Thus the Σ_1 admissibility of $L(\alpha)$ is needed to show h is Σ_2^α. Hence $\sigma 2\mathrm{cf}(\alpha) \leq \sigma 2\mathrm{cf}^\alpha(\alpha^*)$.

Now suppose a strictly increasing Σ_2^α map h from $\sigma 2\mathrm{cf}(\alpha)$ into α is given. A good approximation of the desired cofinality map g from $\sigma 2\mathrm{cf}(\alpha)$ into α^* is fh. Clearly fh is Σ_2^α. The range of fh is unbounded in α^*; otherwise range $fh \subseteq \delta$ for some $\delta < \alpha^*$, and then range h is bounded by $\sup f^{-1}[\delta] < \alpha$. The only difficulty is that fh may not be strictly increasing. Define

$$\beta = \mu\gamma(\mathrm{Et})[t \in \Sigma_2^\alpha \ \&\ \mathrm{dom}\, t = \gamma \ \&\ \sup \mathrm{range}\ t = \alpha].$$

Let t be a Σ_2^α map from β into α with unbounded range. ft is Σ_2^α and its range is unbounded in α^*. Define β_0 to be

$$\mu\gamma(\mathrm{Ep})[p \in \Sigma_2^\alpha \ \&\ \mathrm{dom}\, p = \gamma \ \&\ \sup \mathrm{range} = \alpha^*].$$

then $\beta_0 \leq \beta \leq \sigma 2\mathrm{cf}(\alpha)$. Let g_0 be a Σ_2^α map from β_0 into α^* with range unbounded in α^*. To show $\sigma 2\mathrm{cf}^\alpha(\alpha^*) \leq \sigma 2\mathrm{cf}(\alpha)$, it suffices to transform g_0 into a strictly increasing function g with the same domain $(=\beta_0)$.

For $x < \beta_0$, let

$$g(x) = \sup\{g_0(y) | y \leq x\} + x.$$

To see g is Σ_2^α, let $k(\alpha, x)$ be an α-recursive function such that

$$g_0(x) = \lim_\sigma k(\sigma, x) \quad (x < \dot{\beta}_0).$$

k exists by Proposition 2.2. Let

$$m(x) = \mu\sigma(\tau)_{\tau \geq \sigma}[k(\sigma, x) = k(\tau, x)].$$

m is Σ_2^α, since $m(x) = \sigma$ iff

$$(\tau)_{\tau \geq \sigma}[k(\sigma, x) = k(\tau, x)] \tag{3a}$$

$$\&\ (\rho)_{\rho < \sigma}(E\gamma)_{\rho < \gamma < \sigma}[k(\rho, x) \neq k(\gamma, x)], \tag{3b}$$

Hence for each $x < \beta_0 \leq \beta$,

$$\{m(y) | y \leq x\}$$

is bounded below α. In other words, for each $x < \beta_0$

$$(E\sigma)(\tau)_{\tau \geq \sigma}(y)_{y \leq x}[k(\sigma, y) = k(\tau, y)]. \tag{4}$$

(4) is Σ_2^α and serves as the principal part of a Σ_2^α definition of g. Note that $\sup\{g_0(y) | y \leq x\}$ equals $\sup\{k(\sigma, y) | y \leq x\}$ for all sufficiently large σ. □

Note well the use of Σ_1 admissibility in the above proof to show a formula is Σ_2^α.

3.4 Blocking. As in subsection 1.1 the requirements for Post's problem are indexed by ordinals less than α^*. Let g be a strictly increasing, Σ_2^α map from $\sigma 2\text{cf}(\alpha^*)$ into α^* with range unbounded in α^*. Shore's blocking method uses g to distribute requirements as follows. p is an α-recursive, 1-1 map of α into α^*.

Block 2δ: all requirements of the form $A \neq \{p^{-1}\varepsilon\}^B$, where $g(\delta) \leq \varepsilon < g(\delta+1)$.
Block $2\delta+1$: all requirements of the form $B \neq \{p^{-1}\varepsilon\}^A$, $g(\delta) \leq \varepsilon < g(\delta+1)$.

The procedure for trying to meet requirements is the one established in subsection 1.1. Hence there is no conflict between requirements in the same Shore block. Block γ is said to have a higher priority than block ρ if $\gamma < \rho$. Thus a requirement in block 2δ can be injured only by an attempt to satisfy a requirement in block $2\eta+1$ for some $\eta < \delta$. In order to proceed effectively at stage σ, the Σ_2^α cofinality map $g(\delta)$ is replaced by an α-recursive approximation $g(\sigma,\delta)$ supplied by Proposition 2.2.

Define $\lim_{\tau<\sigma} g(\tau,x)$ to be z iff

$$(E\gamma)_{\gamma<\sigma}(\tau)_{\gamma\leq\tau<\sigma}[g(\tau,x)=z].$$

For each $\rho < \sigma 2\text{cf}^\alpha(\alpha^*)$, let J_ρ be the set of all σ such that:

(i) an attempt is made to satisfy, or an injury occurs to, a requirement in $\cup\{\text{block } x \mid x \leq \rho\}$ at stage σ; or

(ii) $g(\sigma,x) \neq \lim_{\tau<\sigma} g(\tau,x)$ for some $x \leq \rho$.

As in Section 1 it suffices to see J_ρ is α-finite.

Assume J_x is α-finite for $x < \rho$. Then J_x is a Σ_2^α function of x below ρ. Since $\rho < \sigma 2\text{cf}^\alpha(\alpha^*)$, Lemma 3.3 implies

$$\sup\{J_x \mid x < \rho\} = \sigma_0 < \alpha.$$

Choose $\sigma_1 > \sigma_0$ so that $(\tau)_{\tau\geq\sigma_1}[g(\tau,\rho) = g(\sigma_1,\rho)]$. Let $J_\rho^1 = J_\rho - \sigma_1$. Suppose $\sigma \in J_\rho^1$. The *only* activity at stage σ with respect to block δ is an attempt to meet one requirement in that block. (Recall that the procedure for meeting requirements established in subsection 1.1 allows an attempt on at most one requirement at each stage.) Each requirement in block δ will be attempted at at most one stage in J_ρ^1. Thus J_ρ^1 is in α-recursive, 1-1 correspondence with an α-recursively enumerable subset of block ρ. That subset is α-finite, since block ρ is shorter than α^*. Hence J_ρ^1 is α-finite.

3.5 Post's Problem for Admissible Sets. Let A be a Σ_1 admissible set as defined in Section 1.VII. Recall that $B \subseteq A$ is said to be A-recursively enumerable if B is Σ_1^A and A-recursive if A is Δ_1^A. Also z is said to be A-finite if $z \in A$. Many, but not all, of

the notions of α-recursion theory extend to A. For example, regularity can only mean: $B \subseteq A$ is regular iff $(B \cap z) \in A$ for all $z \in A$. On the other hand, the notion of Σ_1 projectum does not seem to make sense for all Σ_1 admisible A's. If A is $(Ad(2^\omega))$, the least Σ_1 admissible set with 2^ω as an element, then there does not appear to be any useful way of defining the Σ_1 projectum of A.

To formulate Post's problem for all A, all that is needed is an extension of "α-recursive in" to "A-recursive in". Suppose $B, C \subseteq A$. B is A-recursive in C, symbolically $B \leq_A C$, if there exist partial A-recursive functions ϕ and ψ such that

$$x \subseteq B \leftrightarrow (\mathrm{Ey})(\mathrm{Ez})[\phi(x,y,z) = 0 \quad \& \quad y \subseteq C \quad \& \quad z \subseteq A - C],$$

$$x \subseteq A - B \leftrightarrow (\mathrm{Ey})(\mathrm{Ez})[\psi(x,y,z) = 0 \quad \& \quad y \subseteq C \quad \& \quad z \subseteq A - C],$$

for all $x \in A$. (y and z range over A.) Clearly $\leq_A$ is reflexive and transitive. Two sets have the same A-degree if each is A-recursive in the other. The notion of "weakly A-recursive in" ($\leq_{wA}$) is defined by substituting "$x \in$" for "$x \subseteq$" in the definition of $\leq_A$.

One formulation of Post's problem for A is: do there exist A-recursively enumerable sets B and C such that $B \not\leq_A C$ and $C \not\leq B$? Another formulation that asks for more: $B \not\leq_{wA} C$ and $C \not\leq_{wA} B$? Still another asks for less: does there exist an A-recursively enumerable set B such that B is neither A-recursive nor complete? (B is complete if every A-recursively enumerable set is A-recursive in B.) Simpson showed that HC, the set of all hereditarily countable sets, yields a negative answer to the third formulation above, if the axiom of determinateness is assumed. Later Harrington showed, as a theorem of ZF, that there exists a countable Σ_1 admissible set for which the third formulation has a negative answer. His proof can be found in Chong 1984.

On the other hand, Simpson 1974b and Stoltenberg–Hansen 1977 showed that the second (and strongest) formulation has a positive answer if A can be suitably well-ordered. The condition to be considered here is: A is *effectively well-orderable*, that is, there exists a one-one A-recursive map of A onto $\mathrm{ord}(A)$, the least ordinal not in A. If A is effectively well-orderable, then the notion of Σ_1 projectum is meaningful for A, and the dynamic solutions of Post's problem given above via Lerman's tame Σ_2 projectum (subsection 3.1) or Shore's blocking method (subsection 3.2) apply to A. The fine structure approach (Theorem 2.6) fails in A, because it is based on collapsing arguments that need initial segments of L to succeed. The next proposition is an aid to understanding why dynamic methods succeed in A when A is effectively well-orderable.

3.6 Proposition. *Assume A is a Σ_1 admissible set. Let $\alpha = \mathrm{ord}(A)$. Then (i) iff (ii).*

(*i*) *A is effectively well-orderable.*
(*ii*) *There exists a Δ_1^A $B \subseteq \alpha$ such that $A = L[B, \alpha]$.*

Proof. $L[B,\alpha]$ was defined in subsection 3.5.VII. If (ii) holds, then the natural enumeration of $L(\alpha)$, described in subsection 1.7.VII, extends to one of $L[B,\alpha]$, and yields an effective well-ordering of A.

Suppose (i) holds. Let $f\colon A \to \alpha$ be a one-one, onto, Σ_1^A map. B will encode all sets in A by relations on ordinals. Let $x \in A$, and define $tc(x)$, the transitive closure of x, to be the least transitive $y \supseteq x$. (Note that $tc(x)$, as a function of x, is Σ_1^A.) Define

$$\delta \varepsilon_f \gamma \text{ by } f^{-1}(\delta) \in f^{-1}(\gamma).$$

then $\langle tc(x), \in \rangle \approx \langle f[tc(x)], \varepsilon_f \rangle$. Let

$$r = \langle f[x], f[tc(x)], \varepsilon_f \cap (f[tc(x)])^2 \rangle.$$

r is an effective code for x in that the passage from r to x can be accomplished inside $L(\alpha, r)$. Note that the collection of all such r is A-recursive. r can be further coded as a set of ordinals.

Thus it will be enough for B to encode all sets of ordinals in A. A-recursive functions $g, h\colon \alpha \to \alpha$ are defined by recursion. Let f_0 be a one-one, A-recursive map of α onto $A \cap 2^\alpha$. Define

$$h(\delta) = (\sup_{\gamma < \delta} h(\gamma)) + \sup{}^+ f_0(\delta),$$

$$g(h(\delta) + x) = \begin{cases} 0 & \text{if } x \in f_0(\delta) \\ 1 & \text{if } x \notin f_0(\delta) \end{cases} \quad (x < \sup{}^+ f_0(\delta)).$$

Each $y \in A \cap 2^\alpha$ can be recovered from a sufficiently long initial segment of h and g. Since $h, g \subseteq \alpha^2$, they can be encoded by a $B \subseteq \alpha$. □

3.7 Theorem. *Let A be an effectively well-orderable, Σ_1 admissible set. Then there exist two A-recursively enumerable sets such that neither is weakly A-recursive in the other.*

Proof. By Shore's blocking method. Define

$$\alpha_A^* = \mu\gamma(\mathrm{Eg})\left[g \in \Sigma_1^A \ \&\ g\colon \alpha \xrightarrow[\text{into}]{\text{1-1}} \gamma \right].$$

α_A^* is the Σ_1 projectum of A. The proof of Proposition 2.1.VII was entirely dynamic in nature and so applies to α_A^*. Suppose $X \subseteq \delta \subseteq \alpha_A^*$ and X is A-recursively enumerable. Let f be a one-one, A-recursive map of A onto ord(A). With the aid of f, X becomes the range of a partial, one-one, A-recursive map g with domain equal to an initial segment of ord(A). The domain of g cannot be ord(A) because $\delta \subseteq \alpha_A^*$. Thus the domain of g, and hence X, is α-finite.

Similarly the proof of Proposition 2.2 shows that each Σ_2^A function from ord(A) into ord(A) is the limit of an A-recursive approximation. And the proof of Lemma 3.3 becomes a proof of

$$\sigma 2\mathrm{cf}^A(\alpha_A^*) = \sigma 2\mathrm{cf}^A(\mathrm{ord}(A)).$$

The arguments of subsection 3.4 transfer to A straightforwardly. The two "incomparable" sets obtained are subsets of α. □

3.8 Uniform Solutions to Post's Problem. Call a Σ_1^α subset of $L(\alpha)$ *lightface* if its Σ_1 definition has all its parameters in ω, and *boldface* otherwise. Thus the set of ordinals less than α^* in $L(\alpha)$ is boldface, but not lightface, Σ_1^α, because its definition needs α^* as a parameter. All the solutions of Post's problem given prior to this section yield "incomparable" sets that are boldface Σ_1^α. The parameters needed arise out of various definable projecta and confinalities. For example, the blocking argument of subsection 3.4 needs the parameters occurring in the Σ_1^α definition of p and the Σ_2^α definition of g. The solution given below avoids all such parameters and is uniform in α.

3.8 Theorem (R. Shore 1974). *There exist integers m and n such that for all α: the m-th and n-th lightface Σ_1^α sets have the property that neither is weakly α-recursive in the other.*

Proof. Curiously the uniform construction, viewed locally, is similar to that given in Section 1 when $\alpha^* < \alpha$. The idea is to divide α into intervals that are independent in the sense that requirements in different intervals do not conflict. The top of each interval is to be a Σ_1 admissible ordinal with strictly smaller Σ_1 projectum. The α-stable ordinals are Σ_1 admissible and, according to Lemma 2.7, define somewhat independent intervals.

Let $\{t(\sigma,\delta) \mid 0 < \delta < \sigma_0\}$ be a list in ascending order of the σ-stable ordinals. Set $t(\sigma,0) = \omega$ and $t(\sigma,\sigma_0) = \sigma$.

Define $t_\delta = t(\alpha,\delta)$. Thus t_δ is the δ-th α-stable ordinal. $t(\sigma,\delta) = t_\delta$ if $\sigma \geq t_\delta$, and $t(\sigma,\delta)$ is a lightface α-recursive function of σ and δ. Hence t_δ is a lightface, tame Σ_2^α function of δ.

Let h be the universal, partial α-recursive function central to the proof of Lemma 2.6.VII. Recall that h is lightface Σ_1^α, uniformly in α, and that $h[\gamma]$ is a Σ_1 substructure, and an initial segment, of $L(\alpha)$ for every infinite $\gamma < \alpha$. It follows that

$$(0) \qquad \text{(i) } h[t_\delta + 1] = L(t_{\delta+1}) \quad \text{and} \quad \text{(ii) } t^*_{\delta+1} \leq t_\delta$$

for all $\delta < \sigma_0$. (ii) is obtained from (i) by inverting h.

Since $t^*_{\delta+1} < t_{\delta+1} (\delta < \alpha_0)$, the local strategy of the uniform construction can be modeled on that of subsection 1.1. Suppose

$$t_\delta \leq \rho < \sigma < t_{\delta+1}.$$

At stage σ the intention is to consider only requirements of the form $A \neq \{\rho\}^B$ (req 2ρ) or $B \neq \{\rho\}^A$ (req $2\rho + 1$). The priorities are governed by f_σ, a partial, one-one $\Sigma_1^{t_{\delta+1}}$ map from $t^*_{\delta+1}$ onto $t_{\delta+1}$ defined below. The local strategy yields the desired global result: req ρ is satisfied prior to stage $t_{\delta+1}$, and that will be so thanks to the α-stability of $t_{\delta+1}$. Of course at stage σ the local strategy can only be guessed

at. The guesses must converge properly, and above all, must be lightface α-recursive uniformly in α.

Define $f_\delta = hg_\delta$, where g_δ is the first member of $L(t_{\delta+1})$ to be a one-one map of $t^*_{\delta+1}$ onto t_δ. $h \restriction t_\delta$ is $\Sigma_1^{t_{\delta+1}}$ because $t_{\delta+1}$ is stable. Thus f_δ is a partial $\Sigma_1^{t_{\delta+1}}$ function from $t^*_{\delta+1}$ onto $t_{\delta+1}$. It is safe to assume f_δ is one-one (cf. Exercise 1.16.VII). The only non-trivial parameter in the Σ_1 definition of f_δ is g_δ. g^σ_δ is the best guess in $L(\sigma)$ at a one-one map of the σ-cardinality of $t(\sigma,\delta)$ onto $t(\sigma,\delta)$. (Recall that for any Σ_1 admissible γ, if $\gamma^* < \gamma$, then γ^* is the greatest γ-cardinal.) As σ approaches $t_{\delta+1}$

$$t(\sigma,\delta) = t_\delta,$$

$$\sigma\text{-card}(t_\delta) = t^*_{\delta+1},$$

and $g^\sigma_\delta = g_\delta$. Let h^σ be the result of restricting the Σ^σ_1 definition of h to $L(\sigma)$. Then h^σ is lightface Σ^σ_1.

Define $f^\sigma_\delta = h^\sigma g^\sigma_\delta$. Then

$$z < t^*_{\delta+1} \to \lim_{\sigma \to t_{\delta+1}} f^\sigma_\delta \restriction z = f_\delta \restriction z. \tag{1}$$

Suppose $\omega \leq \rho < \sigma$. Let $t_{m(\sigma,\rho)}$ be the greatest σ-stable ordinal $\leq \rho$. Then

$$t_\delta \leq \rho < t_{\delta+1} \to \lim_{\sigma \to t_{\delta+1}} t_{m(\sigma,\rho)} = t_\delta. \tag{2}$$

(Remember that $\lim_{\sigma \to y} x_\sigma = x$ means $(E\tau)_{\tau < y}(\sigma)_{\tau \leq \sigma < y}(x_\sigma = x)$.)

The uniform solution.

Stage $\sigma < \omega$. Identical with the Friedberg–Muchnik solution to Post's problem.

Stage $\sigma \geq \omega$. Suppose $\rho < \sigma$ and an attempt was made at stage $\tau < \sigma$ to satisfy req ρ. That attempt, specifically the negative requirement imposed by that attempt on A or B, is now *discarded* if

$$t_{m(\tau,\rho)} > t_{m(\sigma,\rho)}.$$

If $r < \sigma$, then r belongs to a unique block of the form $[t(\sigma,\delta), t(\sigma,\delta+1))$. The local *priority* assigned to req r at stage σ is $(f^\sigma_\delta)^{-1}(r)$. As usual req x has *higher priority* than req y if the priority assigned to x is less than that assigned to y.

Req r needs *attention* at stage σ if:

(i) every attempt to satisfy req r prior to stage σ was injured prior to stage σ, or discarded prior to or at stage σ; and

(ii) there is an opportunity to satisfy req r at stage σ that does not threaten injury to any req of higher priority in the same block as r, or to any requirement in any block below that of r.

Go to the lowest block in which there is a requirement that needs attention. In that block go to the highest priority requirement that needs attention, and take the least opportunity to satisfy it.

End of uniform solution.

Fix $\delta < \alpha_0$ and $z < t^*_{\delta+1}$ in order to show by induction that J^δ_z (the set of all σ such that an injury to, or attempt to satisfy, req $f_\delta(z)$ occurs at stage σ) is $t_{\delta+1}$-finite. As in (1) and (2) there is a $\sigma_1 \in [t_\delta, t_{\delta+1})$ such that for all $\sigma \geq \sigma_1$:

$$t(\sigma, \delta) = t_\delta,$$
$$f^\sigma_\delta \restriction (z+1) = f_\delta \restriction (z+1)$$
$$t(\sigma, \delta+1) > \sup f_\delta[z+1].$$

Then for all $x \leq z$ at stage $\sigma \geq \sigma_1$:

(3) no prior attempt to satisfy req $f_\delta(x)$ is discarded;

(4) req $f_\delta(x)$ is injured only for the sake of $f_\delta(y)(y < x)$.

(4) follows from the induction hypothesis that $J^\gamma_v \in L(t_{\gamma+1})$; for all $\gamma < \delta$ and $v < t^*_{\gamma+1}$.

By induction the J^δ_x's $(x < z)$ are simultaneously $t_{\delta+1}$-recursively enumerable, and each is $t_{\delta+1}$-finite. The combinatoric lemma (2.3.VII) implies $\cup \{J^\delta_x \mid x < z\}$ is $t_{\delta+1}$-finite. After stage σ_1 and after $U\{J^\delta_x \mid x < z\}$ is enumerated, there is at most one attempt to satisfy req z. Hence J^δ_z is α-finite.

Fix $r \in [t_\delta, t_{\delta+1})$ to see req r is met. $r = f_\delta(z)$ for some $z < t^*_{\delta+1}$. Assume req r is of the form $A \neq \{r\}^B$. Choose $\sigma_2 \in (\sigma_1, t_{\delta+1})$ so that

$$\cup \{J^\delta_x \mid x \leq z\} \subseteq \sigma_2.$$

σ_1 is as above. Any attempt to satisfy req r involves a witness w and the inequation $A(w) \neq \{r\}^B(w)$. w is drawn from a witness set Z^δ_r. The witness sets are pairwise disjoint, unbounded in α, and simultaneously lightface α-recursive. The α-stability of $t_{\delta+1}$ implies that $Z^\delta_r - t_\delta$ is unbounded in $t_{\delta+1}$. Let w_0 be a member of $Z^\delta_r - t_\delta$ not put in A prior to stage σ_2. Then w_0 is never put in A, i.e. $A(w_0) = 0$. Suppose $\{r\}^B(w_0.) = 0$. Then there is a $\sigma < \alpha$ such that

$$\sigma > \sigma_2 \;\&\; A^{<\sigma}(w_0) = 0 \;\&\; \{r\}^{B^{<\sigma}}_\sigma(w_0) = 0. \tag{5}$$

Since A and B are lightface Σ^α_1, there is a $\sigma < t_{\delta+1}$ that satisfies (5). No requirement in any block below block δ, or in block δ and of higher priority than r, receives attention at stage σ. If r does not need attention at stage σ, then it was satisfied at an earlier stage by an attempt not yet discarded or injured, and hence never to be discarded or injured. If r does need attention at stage σ, then it will be satisfied at stage σ and remain so forever. □

The proof of Theorem 3.8 was dynamic in nature save for one fine-structure fact:

$$h[t_\delta + 1] = L(t_{\delta+1}). \tag{6}$$

The proof of (6) is a collapsing argument. Nonetheless the proof of Theorem 3.8 can be adjusted so as to avoid (6) (cf. Exercise 3.10). The result is an entirely dynamic, lightface solution to Post's problem that is uniform with respect to a wide class of structures.

3.9–3.12 Exercises

3.9. Suppose $\alpha^* < \alpha$. Show there exists a lightface, tame Σ_2^α map with range α and domain less than α.

3.10. Let A be a Σ_1 admissible set. Suppose A is effectively wellorderable via a lightface one-one, A-recursive map of A onto ord(A). Show Post's problem for A has a lightface solution.

3.11. (R. Shore 1974). Suppose $L(\alpha)$ is Σ_n admissible, that is, it satisfies Σ_n^α replacement. Find sets A_i $(i \leq 1)$ such that A_i is Σ_n^α, but is not Δ_n^α, over $\langle L[A_{1-i}, \alpha], \varepsilon, A_{1-i}\rangle$.

3.12. Let A be a Σ_1 admissible structure of the form $\langle L[B,\alpha], \varepsilon, B\rangle$. Solve Post's problem for A.

4. Σ_1 *Doing the Work of* Σ_2

Chapter VIII ends on the same note with which it began. Certain Σ_2 constructions that occur in classical recursion theory can, after some modification, be carried out for every Σ_1 admissible ordinal. A Σ_2 construction is one that succeeds by appeal to Σ_2 replacement. Although the Friedberg–Muchnick (F–M) construction is a Σ_1 recursion, Σ_2 replacement is needed to check that every requirement is met. $\sigma(e)$, the stage by which the e-th requirement is met, is a Σ_2 function of e; and the proof that the e-th requirement is met begins with finding a bound on $\{\sigma(c) | c < e\}$. Thus the proof that the F–M construction works can be lifted, with no conceptual change, from $L(\omega)$ to every Σ_2 admissible $L(\alpha)$.

The previous sections of this chapter studied various methods of lifting the F–M construction to every Σ_1 admissible ordinal. All the methods had in common the idea of projecting α downward to an ordinal with combinatoric properties reminiscent of Σ_2 replacement. The present section focuses on Lerman's tame Σ_2 approach and uses it to lift tame Σ_2 recursions, in a systematic fashion, from ω to α. The simplest example of a tame Σ_2 recursion in classical recursion theory is the construction of a 1-generic Δ_2 set. After the combinatoric facts about tame Σ_2^α recursion are established, they will be applied to obtain a 1-generic Δ_2^α set, and finally to prove Simpson's inversion theorem for the α-jump.

4.1 Tame Σ_2 Recursion. Let R be an α-recursively enumerable predicate, and S an α-recursive function. A *tame* Σ_2^α *recursion* is defined by

$$f(\gamma) = \begin{cases} \mu y R(f\restriction\gamma, \gamma, y) & \text{if } (\mathrm{E}y)R(f\restriction\gamma, \gamma, y) \\ S(f\restriction\gamma) & \text{otherwise.} \end{cases}$$

If α is Σ_2 admissible, then there exists a unique $\Sigma_2^\alpha f$ from α into α satisfying the above recursion equations. If α is merely Σ_1 admissible, then the recursion may break down at some δ such that $(f \restriction \delta) \notin L(\alpha)$. The first such δ is at least $t\sigma 2p(\alpha)$ according to Lemma 4.2.

A tame Σ_2^α recursion arises out of attempts to meet requirements indexed by ordinals less than α. The requirements are simultaneously α-recursively enumerable sets. A typical member of a requirement is an ordered pair of disjoint α-finite sets. $f(\gamma)$ might be the least extension of $f \restriction \gamma$ that satisfies requirement γ. Since the desired extension may not exist, f is Σ_2 in character rather than Σ_1.

4.2 Lemma. *The equations for a tame Σ_2^α recursion define a unique tame $\Sigma_2^\alpha f$ from $t\sigma 2p(\alpha)$ into α.*

Proof. The idea is to construe f as the limit of an α-recursive sequence f_σ. $f(\gamma)$ is approximated at the beginning of stage σ by $f_{<\sigma}(\gamma)$, and at the end of stage σ by $f_\sigma(\gamma)$. Convergence of f_σ to f is assured by a system of priorities. At stage σ the best guess for $f(\gamma_1)$ may be inconsistent with the best guess for $f(\gamma_2)$. If $\gamma_1 < \gamma_2$ then γ_1 is given preference. Thus a guess made for $f(\gamma_2)$ at stage σ may be discarded at a later stage for the sake of a new guess for $f(\gamma_1)$. Recall that

$$\lim_{\tau \to \sigma} k(\tau) = z \text{ means } (E\rho)_{\rho < \sigma}(\tau)_{\rho \leq \tau < \sigma}[k(\tau) = z].$$

Define (by induction on γ):

$$f_{<\sigma}(\gamma) = \begin{cases} \lim\limits_{\tau \to \sigma} f_\tau(\gamma) & \text{if the limit exists,} \\ S(f_{<\sigma} \restriction \gamma) & \text{otherwise.} \end{cases}$$

γ needs attention at stage σ if:

$$L(\sigma) \vDash (\mathrm{Ey}) R(f_{<\sigma} \restriction \gamma, \gamma, y)]; \text{ and}$$

$$f_{<\sigma}(\gamma) \neq \mu y [L(\sigma) \vDash R(f_{<\sigma} \restriction \gamma, \gamma, y)]. \tag{1}$$

Let γ^σ be the least γ that needs attention at stage σ, and y_0 the least y alluded to in (1) when $\gamma = \gamma^\sigma$. Define

$$f_\sigma(\gamma) = \begin{cases} f_{<\sigma}(\gamma) & \text{if } \gamma < \gamma^\sigma \\ y_0 & \text{if } \gamma = \gamma^\sigma \\ S(f_\sigma \restriction \gamma) & \text{if } \gamma > \gamma^\sigma. \end{cases}$$

Let I_γ be $\{\sigma \mid \gamma \geq \gamma^\sigma\}$. It need only be shown that I_γ is α-finite for all $\gamma < t\sigma 2p(\alpha)$. It then follows from the definition of f_σ that f_σ converges tamely to some f with domain $t\sigma 2p(\alpha)$. Then f, by induction on γ, is a solution of the tame Σ_2^α recursion equation.

Fix γ to see I_γ is α-finite. The argument splits into two cases, as it did in Lerman's tame Σ_1 approach to Post's problem in subsection 3.1.

Case 1: $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha) \leq \alpha$. Let κ be a regular α-cardinal such that $\gamma < \kappa$. Assume I_x $(x < \gamma)$ is α-finite and of α-cardinality less than κ. Then the combinatoric lemma (2.3.VII) implies $\cup \{I_x | x < \gamma\}$ is α-finite and of α-cardinality less than κ. It follows from the manner in which $I_\gamma - \cup \{I_x | x < \gamma\}$ is interlaced with $\cup \{I_x | x < \gamma\}$ that I_γ is α-finite with α-cardinality less than κ. The interlacing effect is similar to the one discussed in the proof of Lemma 1.3 with one change. Finite sequences of elements of $I_\gamma - \cup \{I_x | x < \gamma\}$, rather than single elements, are separated by elements of $\cup \{I_x | x < \gamma\}$. The change is caused by the insistence that $f(\gamma)$ be the *least* y that satisfies R.

Case 2: $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$. Hence

$$\gamma = \mathrm{gc}(\alpha) \cdot \delta + w$$

for some $\delta < \sigma 2\mathrm{cf}(\alpha)$ and $w < \mathrm{gc}(\alpha)$. Assume

$$J_z = \cup \{I_x | \mathrm{gc}(\alpha) \cdot z \leq x < \mathrm{gc}(\alpha) \cdot (z+1)\}$$

is α-finite for each $z < \delta$. Then J_z is a Σ_2^α function of z $(z < \delta)$, and so $\cup \{J_z | z < \delta\}$ is α-finite, since $\delta < \sigma 2\mathrm{cf}(\alpha)$. Let its supremum be σ_2. The argument of Case 1 shows, for each $x \in (\mathrm{gc}(\alpha) \cdot \delta, \mathrm{gc}(\alpha) \cdot (\delta + 1))$, that $I_x - \sigma_2$ is α-finite and of α-cardinality less than some regular α-cardinal. Hence I_γ is α-finite.

To complete case 2 it must be shown that J_δ is α-finite. Consider the simultaneous α-recursive enumeration of $\{I_x - \sigma_2 | x \in [\mathrm{gc}(\alpha) \cdot \delta, \mathrm{gc}(\alpha) \cdot (\delta + 1))\}$. It gives rise to a one-one, α-recursive map of $J_\delta - \sigma_2$ into $\mathrm{gc}(\alpha) \cdot \mathrm{gc}(\alpha)$. When the ρ-th entry in the enumeration of $I_x - \sigma_2$ appears, it is mapped to $\langle x - \mathrm{gc}(\alpha) \cdot \delta, \rho \rangle$. $\rho < \mathrm{gc}(\alpha)$ because the α-cardinality of $I_x - \sigma_2$ is less than some regular α-cardinal. If $J_\sigma - \sigma_2$ is not α-finite, then there is one-one, α-recursive map of α onto $J_\delta - \sigma_2$. But then $\alpha^* \leq \mathrm{gc}(\alpha) < t\sigma 2p(\alpha)$, an impossibility by Corollary 2.4.VII. □

4.3 Reduced Tame Σ_2 Recursion. The equations for a tame Σ_2 recursion were formulated with the idea that $f(\gamma)$ would be defined so as to satisfy requirement γ in some construction involving α requirements. Since a tame Σ_2 recursion may break down long before α is reached, it is necessary to re-index requirements. Let t be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. The equations for tame Σ_2 recursion, *reduced* by t, are

$$f(\gamma) = \begin{cases} \mu y\, R(f \restriction \gamma, t(\gamma), y) & \text{if } (\mathrm{E}y)\, R(f \restriction \gamma, t(\gamma), y) \\ S(f \restriction y) & \text{otherwise.} \end{cases}$$

The intention now is that $f(\gamma)$ be defined to satisfy requirement $t(\gamma)$. Thus there will be time to satisfy all requirements if the recursion does not break down before stage $t\sigma 2p(\alpha)$. And it does not, according to the next result.

4.4 Theorem. *Let t be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. Then the equations for a tame Σ_2^α recursion, reduced by t, define a unique function f from $t\sigma 2p(\alpha)$ into α. Furthermore f is tame Σ_2^α.*

Proof. Same as that of Lemma 4.2. The presence of t makes very little difference. At stage σ, $t(\gamma)$ is guessed at by $t(\sigma, \gamma)$, an α-recursive function that converges tamely to $t(\gamma)$. Each of the arguments of 4.3 is altered in the same fashion. First wait for $t(\sigma, \gamma)$ to settle down on the appropriate proper initial segment of $t\sigma 2p(\sigma)$, and then proceed as in 4.2. □

Theorem 4.4 is a precise interpretation of the phrase: "making Σ_1 do the work of Σ_2". It will be applied in the proof of Theorem 4.5 to obtain a 1-generic subset of α for every Σ_1 admissible α. More generally, and less precisely, if a Σ_2 construction is tame, then it can be executed using only Σ_1 admissibility.

4.5 1-Genericity. A set $A \subseteq \alpha$ is 1-generic if it is generic with respect to certain bounded Π_2^α sentences in the sense defined below. ("Bounded" means the universal quantifier is bounded.)

$p, q, r \ldots$ are forcing conditions. A condition p is a pair (p^+, p^-) of disjoint α-finite sets. $p \geq q$ (p is extended by q) if $p^+ \subseteq q^+$ and $p^- \subseteq q^-$. $A \in p$ (A satisfies p) if $p^+ \subseteq A$ and $p^- \subseteq \alpha - A$. Define

$$\{\gamma\}^p(\delta) = x \quad \text{by} \quad (p^+, p^-, \delta, x) \in R_\gamma.$$

R_γ is the γ-th reduction procedure as in subsection 3.2.VII. Let $\uparrow$ mean undefined, and $\downarrow$ defined. Thus

$$\{\gamma\}^p(\delta)\downarrow \quad \text{if} \quad (\mathrm{Ex})[\{\gamma\}^p(\delta) = x].$$

Note that $\{\gamma\}^p(\delta)\downarrow$ is an α-recursively enumerable relation on γ, p and δ.

A is *1-generic* if for all γ, δ, $x < \alpha$:

(1) $$[\{\gamma\}^A(\delta)\uparrow] \to (\mathrm{Ep})_{A \in p}(q)_{p \geq q}[\{\gamma\}^q(\delta)\uparrow]; \text{ and}$$

(2) $$(\delta)_{\delta < x}[\{\gamma\}^A(\delta)\downarrow] \to (\mathrm{Ep})_{A \in p}(\delta)_{\delta < x}[\{\gamma\}^p(\delta)\downarrow].$$

A set is said to be tame Δ_2^α if its characteristic function is tame Σ_2^α.

4.6 Theorem. *There exists a tame Δ_2^α, 1-generic set.*

Proof. Let S be an α-recursive function such that

$$S(s) = \left(\bigcup_{\delta < \beta} s_\delta^+, \bigcup_{\delta < \beta} s_\delta^- \right),$$

where s is an α-finite sequence (of length β) of forcing conditions. Let $R(s, \gamma, p)$ be the α-recursively enumerable relation

$$S(s) \geq p \quad \& \quad \{(\gamma)_0\}^p((\gamma)_1)\downarrow.$$

According to Theorem 4.4.VII there is a tame Σ_2^α function f with domain $t\sigma 2p(\alpha)$ that satisfies

$$f(\gamma) = \begin{cases} \mu y\, R(f\restriction\gamma, t(\gamma), y) & \text{if } (\mathrm{E}y)R(f\restriction\gamma, t(\gamma), y) \\ S(f\restriction\gamma) & \text{otherwise.} \end{cases}$$

f is a sequence of forcing conditions. $f(\delta_1) \geq f(\delta_2)$ when $\delta_1 \leq \delta_2$. Define

$$A = \cup\{f(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Suppose $\{\gamma\}^A(\delta)\uparrow$ with the intent of checking clause (1) of the definition of 1-genericity. Choose x so that $(t(x))_0 = \gamma$ and $(t(x))_1 = \delta$, where t is as in subsection 4.3. Let $p = S(f\restriction x)$. Then $A \in p$ and there is no $q \leq p$ such that $\{\gamma\}^q(\delta)\downarrow$. If there were such a q, then $f(x)$ would be such a q, $A \in f(x)$, and $\{\gamma\}^A(\delta)$ would be defined.

The checking of clause (2) of the definition of 1-genericity is managed by a trick learned from Normann [1975]. It is based on standard manipulations with closed unbounded subsets of regular cardinals. Let f_σ be the α-recursive function in the proof of Theorem 4.4. Define

$$A^{<\sigma} = \cup\{f_{<\sigma}(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Thus $A^{<\sigma}$ is the α-recursive approximation to A provided by the proof of Theorem 4.4 at the beginning of stage σ.

Assume $(\delta)_{\delta<x}[\{\gamma\}^A(\delta)\downarrow]$. The "closed unbounded sets" trick is needed to show

$$(1) \qquad (\tau)(E\sigma)_{\sigma\geq\tau}(L(\sigma) \vDash (\delta)_{\delta<x}[\{\gamma\}^{A^{<\sigma}}(\delta)\downarrow]).$$

Suppose for the moment that (1) holds. Let ε be an index such that for all $B \subseteq \alpha$,

$$\{\varepsilon\}^B(0)\downarrow \leftrightarrow (\mathrm{Eq})_{B\in q}(\delta)_{\delta<x}[\{\gamma\}^q(\delta)\downarrow].$$

Choose y so that $(t(y))_0 = \varepsilon$ and $(t(y))_1 = 0$. Then (1), and the tame Σ_2 convergence of f_σ to f, imply $\{\gamma\}^{f(y)}(\delta)\downarrow$ for all $\delta < x$. $A \in f(y)$, so clause (2) of the definition of 1-genericity holds.

It remains to prove (1). Let ρ be the α-cardinality of x, and m an α-finite map of ρ onto x. γ_m is an index such that

$$\{\gamma_m\}^B(z)\downarrow \leftrightarrow \{\gamma\}^B(m(z))\downarrow$$

for all $B \subseteq \alpha$. Hence

$$(z)_{z<\rho}[\{\gamma_m\}^B(z)\downarrow] \leftrightarrow (\delta)_{\delta<x}[\{\gamma\}^B(\delta)\downarrow].$$

Thus it suffices to prove (1) when x is an α-cardinal. Proving (1) by induction on the α-cardinality of x makes it safe to assume x is a regular α-cardinal. (Replace x by its cofinality.) According to Lemma 2.5 there are only two cases.

Case 1: $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha) \leq \alpha$. By Exercise 4.14 there is a regular α-cardinal β and an α-finite sequence $y_\delta(\delta < x)$ such that

$$y_\delta < \beta, (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.$$

Define

$$k(\sigma, \delta) = \mu\tau_{\tau \geq \sigma}[\{\gamma\}^{f_\tau(y_\delta)}(\delta)\downarrow].$$

$k(\sigma, \delta)$ is defined for all $\sigma < \alpha$ and $\delta < x$, since

$$\{\gamma\}^{f(y_\delta)}(\delta) \simeq \{\gamma\}^A(\delta) \text{ is defined.}$$

Let ν be an α-finite map of β onto x such that $\nu^{-1}(\delta)$ is an unbounded subset of β for every $\delta < x$.

Fix τ in order to generate a σ that satisfies the matrix of (1). Let

$$\begin{aligned} \sigma_0 &= \tau, \\ \sigma_{w+1} &= k(\sigma_w, \nu(w)) \quad (w < \beta), \\ \sigma_\lambda &= \sup\{\sigma_w | w < \lambda\} \quad (\lambda \text{ a limit} \leq \beta). \end{aligned}$$

Fix δ and consider the behavior of $\{\gamma\}^{f_\sigma(y_\delta)}(\delta)$ as $\sigma \to \sigma_\beta$. The choice of ν implies $\{\gamma\}^{f_\sigma(y_\delta)}(\delta)\downarrow$ unboundedly often as $\sigma \to \sigma_\beta$. Since $y_\delta < \beta$ and β is α-regular, $f_\sigma(y_\delta)$ suffers fewer than β changes as $\sigma \to \alpha$. (This point is discussed in Case 1 of the proof of Lemma 4.2. The argument there uses Lemma 2.3.VII to show $f_\sigma(\gamma)$ changes less than κ times, where κ is a regular α-cardinal and $\gamma < \kappa$.) Consequently $f_\sigma(y_\delta)$ changes less than β times as $\sigma \to \sigma_\beta$. Since the cofinality of σ_β is β, it follows that $f_\sigma(y_\delta)$ is constant for all sufficiently large σ as $\sigma \to \sigma_\beta$. That constant value is $f_{<\sigma_\beta}(y_\delta)$, and so $\{\gamma\}^{A^{<\sigma_\beta}}(\delta)\downarrow$.

Case 2: $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$. Similar to Case 1. The idea is to repeat the argument of Case 1 inside the block occupied by γ. Define σ_2 as in Case 2 of the proof of Lemma 4.2. Then consider only what happens after stage σ_2. For example, $f_\sigma(y_\delta)$ suffers fewer than β changes after stage σ_2 as $\sigma \to \alpha$. □

4.7 Proposition. *If A is 1-generic, then A is regular and hyperregular.*

Proof. Suppose $f \leq_{w\alpha} A$ and $x < \alpha$. By Lemma 5.2.VII it suffices to show $f \restriction x$ is α-finite. If f is $\{\gamma\}^A$, then there exists a p such that

$$A \in p \quad \text{and} \quad (\delta)_{\delta < x}[\{\gamma\}^p(\delta)\downarrow].$$

Then $f \restriction x$ is $\{\gamma\}^p \restriction x$. □

4.8 The α-Jump. Let $A \subseteq \alpha$. By analogy with the Turing jump of classical recursion theory, the α-jump of A should be a universal, α-recursively-enumerable-in-A set. To be precise, the α-jump of A should be a set B such that B is α-recursively

enumerable in A, and such that $C \leq_\alpha B$ for every C α-recursively enumerable in A. Three candidates for the definition of "B is α-recursively enumerable in A" come to mind.

(i) There is a $\gamma < \alpha$ such that

$$B = \{\delta \mid \{\gamma\}^A(\delta)\downarrow\}.$$

(Recall that "$\downarrow$" means "defined".)

(ii) There is a $\gamma < \alpha$ such that for all $\delta < \alpha$,

$$K_\delta \subseteq B \leftrightarrow \{\gamma\}^A(\delta)\downarrow.$$

($\{K_\delta \mid \delta < \alpha\}$ is the strong enumeration of α-finite sets, cf. 3.1.VII.)

(iii) B is $\Sigma_1^{\alpha, A}$. That is, B is definable over $\langle L[A,\alpha], \varepsilon, A\rangle$ by means of a Σ_1 formula with parameters in $L[A, \alpha]$ with $x \in A$ regarded as a Δ_0 formula.

Clearly (ii) $\rightarrow$ (i) $\rightarrow$ (iii). ($x \rightarrow y$ means if B is enumerable by definition x, then it is enumerable by definition y.) If A is regular, then (iii) $\rightarrow$ (i). If A is regular and hyperregular, then all three notions agree by Lemma 5.2.VII. Hence if A is α-recursively enumerable and hyperregular, then all three notions coincide. There exists an α and an α-recursively enumerable A on which all three differ.

One objection to (iii) is its dependence on non-α-finite computations. If (iii) holds, then an element is added to B because some bounded, but not necessarily α-finite, set of membership statements is satisfied by A. (ii) and (i) are based on α-finite computations. (ii) has the virtue of symmetry over (i). If (ii) holds, then an α-finite set of positive facts about B follows from an α-finite set of facts about A. Symmetry considerations led to the rejection of $\leq_{w\alpha}$ in favor of $\leq_\alpha$. On the other hand there is in general no universal set as defined above in the sense of (ii).

Thus (i) is the preferred choice for the definition of relative α-recursive enumerability on which to build the definition of α-jump. Let

$$A' \text{ be } \{\langle \gamma, \delta\rangle \mid \{\gamma\}^A(\delta)\downarrow\}.$$

Then A' is α-recursively enumerable in A (in the sense of (i)). Suppose B is α-r.e. in A. Thus

$$B = \{\delta \mid \{\gamma_0\}^A(\delta)\downarrow\}.$$

Then

$$H \subseteq B \leftrightarrow \{\gamma_0\} \times H \subseteq A', \text{ and}$$

$$J \subseteq \alpha - B \leftrightarrow \{\gamma_0\} \times J \subseteq \alpha - A'.$$

Hence $B \leq_\alpha A'$. If $A_1 \leq_\alpha A_2$, then A_1' is α-recursively enumerable in A_2, consequently α-recursive in A_2'. Thus the α-jump is well defined on α-degrees. For any α-degree, d, let d', the *α-jump* of d, be the α-degree of D, where D is any set of degree d.

Define $A^{(n+1)}$ to be $(A^{(n)})'$, and $A^{(0)}$ to be A.

Warning: ϕ' has the same α-degree as some complete Σ_1^α set, but in general $\phi^{(2)}$ does not have the same α-degree as some complete Σ_2^α set. Thus the familiar connection between $\phi^{(n)}$ and Σ_n is broken.

4.9 Theorem. *If A is regular and hyperregular, then A' has the same α-degree as some regular set α-recursively enumerable in A.*

Proof. Since A is regular and hyperregular, the structure $\langle L[A,\alpha], \varepsilon, A\rangle$ is Σ_1 admissible (cf. Exercise 5.8.VII). In addition $L[A,\alpha]$ equals $L(\alpha)$. The proof of the regular sets theorem (4.2.VII) is entirely dynamic in nature, and consequently can be extended from $L(\alpha)$ to $\langle L[A,\alpha], \varepsilon, A\rangle$ without significant change. The extended version states: if $B \in \Sigma_1^{\alpha, A}$, then there is a $C \in \Sigma_1^{\alpha, A}$ such that

(i) C is regular in the sense of $L[A,\alpha]$, and
(ii) $C \equiv_{\alpha, A} B$.

(i) means $(C \cap x) \in L[A,\alpha]$ for all $x \in L[A,\alpha]$. In (ii), $\equiv_{\alpha, A}$ refers to reduction procedures that are $\Sigma_1^{\alpha, A}$ sets of 4-tuples $\langle H, J, \delta, \gamma\rangle$ from $L[A,\alpha]$. To visualize the extended proof, extend the natural enumeration of $L(\alpha)$ to one of $L[A,\alpha]$.

Since $L[A,\alpha] = L(\alpha)$, it follows that C is regular, $C \oplus A \equiv_\alpha B \oplus A$, and C is α-recursively enumerable in A. If B is A', then $C \oplus A$ is the desired regular set. □

The Simpson jump theorem is a lifting of the Friedberg jump theorem of classical recursion theory. The latter states: $\phi' \leq d$ iff $c' = d$ for some c.

4.10 Theorem (Simpson 1974a). *(i) and (ii)' are equivalent.*
(i) $\phi' \leq_\alpha D$ and D has the same α-degree as some regular set.
(ii) $C' \equiv_\alpha D$ for some regular, hyperregular C.

Proof. (ii) implies (i) by Theorem 4.9. Now assume D satisfies (i). C is constructed from D so that

(0) (a) $C' \leq_\alpha \phi', D$ and (b) $D \leq_\alpha C, \phi'$.

The idea is to make C 1-generic so that (a) holds, and to code D into C so that (b) holds. In addition, the 1-genericity of C implies C is regular and hyperregular by Proposition 4.7.

Note that 4.7 makes it safe to assume that all forcing conditions are equivalent to initial segments of characteristic functions. If p and q are forcing conditions then $p^\frown q$ is the condition that begins with p and continues with q.

$$(p^\frown q)(x) = \begin{cases} p(x) & \text{if } x < lh(p) \\ q(x - lh(p)) & \text{if } lh(p) \leq x < lh(p) + lh(q). \end{cases}$$

Assume D is regular. Let s be an α-finite sequence of length γ of forcing conditions. Let $S(s)$ be as in the proof of Theorem 4.6. Define

$$S_0(s) = S(s)^\frown D\restriction(t(\gamma)),$$
$$R_0(s, \gamma, q) \leftrightarrow S_0(s) \geq q \quad \& \quad \{(\gamma)_0\}^q((\gamma)_1)\downarrow.$$

Consider the reduced recursion equations

$$f(\gamma) = \begin{cases} \mu y R_0(f\restriction\gamma, t(\gamma), y) & \text{if } (\mathrm{Ey})R_0(f\restriction\gamma, t(\gamma), y) \\ S_0(f\restriction\gamma) & \text{otherwise.} \end{cases} \tag{1}$$

t is a tame Σ^α_2 map from $t\sigma 2p(\alpha)$ onto α.

Fix $z < t\sigma 2p(\alpha)$ in order to study the recursion defined by (1). As the recursion progresses through γ's less than z, the only information about D that is needed is

$$D_z = D\restriction\left(\sup_{\gamma<z} t(\gamma)\right).$$

The regularity of D and tameness of t imply D_z α-finite. Let $(1)_2$ be the result of replacing D by D_z in (1). By Theorem 4.4, $(1)_2$ has a tame Σ^α_2 solution f_2 from $t\sigma 2p(\alpha)$ into α. The α-finite function $f_z\restriction z$ is a solution of (1) for $\gamma < z$. Hence

$$\cup\{f_z\restriction z \mid z < t\sigma 2p(\alpha)\}$$

is a solution of (1). In short, (1) does have a solution f, necessarily unique, from $t\sigma 2p(\alpha)$ into α. Define

$$C = \cup\{f(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Initial segments of the characteristic function of C correspond to initial segments of f. The value of $f\restriction\gamma$ is determined by (1) from an α-finite set of facts about D, t and ϕ'. ϕ' is needed to decide when there is a y that satisfies R_0. R_0 was defined so that

$$\gamma \in C' \leftrightarrow (\mathrm{Ey})R_0(f\restriction\gamma, t(\gamma), y).$$

It follows that $C' \leq_\alpha \phi', D, t$. The Σ^α_2 tameness of t implies $t \leq_\alpha \phi'$ (cf. Exercise 4.13). Thus (0) (a) is proved.

A simultaneous recursion on γ shows $\lambda\gamma\,|\,f\restriction\gamma$ and $\lambda\gamma\,|\,D\restriction t(\gamma)$ are weakly α-recursive in C, ϕ'. Suppose $f\restriction\gamma$ has been computed α-finitely from C, ϕ'. $I(f\restriction\gamma)^\frown D\restriction t(\gamma)$ is an initial segment of $f(\gamma)$, hence an initial segment of C. So $D\restriction b(\gamma)$ can be extracted from C, $f\restriction\gamma$, $t(\gamma)$. Then $f(\gamma)$ can be computed α-finitely from $f\restriction\gamma$, $D\restriction t(\gamma)$, ϕ'. Thus (0) (b) is proved. □

A jump theorem (Sacks 1963) of classical recursion theory that relates recursive enumerability and the Turing jump states: D is recursively enumerable in ϕ' and

$D \geq_\omega \phi'$ iff $D \equiv_\omega C'$ for some incomplete, recursively enumerable C. Maass 1977 has shown this statement holds for α (in place of ω) iff α is Σ_2 admissible.

4.11–4.15 Exercises

4.11. Use Shore's blocking method to prove Theorem 4.10.

4.12. Show a 1-generic set cannot be α-recursively enumerable.

4.13. Let $A \subseteq \alpha$. Show A is α-recursively enumerable in ϕ' iff A is Σ_2^α. Show A is tame Δ_2^α iff $A \leq_\alpha \phi'$ and A is regular.

4.14. Suppose $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha)$. Find a tame Σ_2^α function t from $t\sigma 2p(\alpha)$ onto α with the following properties. For each $\gamma < \alpha$ and each regular α-cardinal x, there exists a regular α-cardinal β and an α-finite sequence $y_\delta (\delta < x)$ such that

$$y_\delta < \beta, (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.$$

4.15. Provide the details of Case 2 of the proof of Theorem 4.6.

Chapter IX
Splitting, Density and Beyond

This last chapter on α-recursion theory focuses on priority arguments more difficult than those of Chapters VII and VIII. Shore's splitting theorem relies heavily on his method of Σ_2 blocking. His density theorem, the first instance of α-infinite injury, requires further fine structure results, and consequently its proof is not entirely dynamic. Its nondynamic features support consideration of recursion theory on inadmissible structures, the concluding topic of the chapter.

1. Shore's Splitting Theorem

Let A be a regular α-recursively enumerable set, not α-recursive. The object is to split A into two sets, B_0 and B_1, so that each is of lower α-degree than A. Thus

$$A = B_0 \cup B_1,\ B_0 \cap B_1 = \varnothing, \quad \text{and} \quad A \not\leq_\alpha B_i (i < 2).$$

Superficially the strategy is the same as that followed by Sacks 1963b when $\alpha = \omega$. In the ω-case the splitting theorem makes stronger use of Σ_2 replacement than the Friedman–Muchnik theorem does. In general terms the former is a full-blown Σ_2 recursion while the latter is tame in the sense of Theorem 4.4.VIII. In specific terms the difference arises from the urgency of splitting. At stage σ some x is enumerated in A. That x must be put in either B_0 or B_1 immediately. The force of the positive requirements is so great that numerous negative requirements are unavoidably injured.

The negative requirements are indexed by ordinals less than α^*:

$$\text{Req } 2\varepsilon: \quad A \neq \{\varepsilon\}^{B_0},$$

$$\text{Req } 2\varepsilon + 1: \quad B \neq \{\varepsilon\}^{B_1}.$$

$\{\varepsilon\}$ means $\{f^{-1}\varepsilon\}$ for some one-one α-recursive f from α into α^*.
Req u has higher priority than req v if $u < v$. Thus req 0 is never injured. As in the ω-case, if A and $\{\varepsilon\}^{B_i}$ agree on an initial segment of α at stage σ, then $\{\varepsilon\}^{B_i}$ is commmited to preservation on that initial segment, if the priorities allow it. The preservations associated with req 0 must be bounded in both time and space, since otherwise A would be α-recursive. To compute $A(z)$, unfold the construction until

$\{0\}^{B_0}(z)$ is committed to preservation. At that stage $A(z) = \{0\}^{B_0}(z)$ and the latter equation holds forever after. So there must be a z such that $A(x) = \{0\}^{B_0}(x)$ for all $x < z$, and either $\{0\}^{B_0}(z)$ is undefined, or is unequal to $A(z)$, at the end of the construction. Since the preservations associated with req 0 are bounded, and since A is regular, eventually it will be safe to put every element of A that might injure req 1 into B_0. Thus req 1 is injured only α-finitely often, and so req 1 is met. And so on. Let $f(r)$ be the least σ after which no new preservation commitments are made for the sake of req r. Since f is Σ_2^α, and not tame, it is unlikely that there will be time to meet all requirements.

If $\alpha^* \leq \sigma 2\mathrm{cf}(\alpha)$ then all is well. If not, then Shore's idea is to divide α^* into $\sigma 2\mathrm{cf}(\alpha^*)$ many blocks. The negative requirements are divided as follows.

g: $\sigma 2\mathrm{cf}(\alpha) \to \alpha$ is strictly increasing, Σ_2^α and unbounded.

Block 2δ: all requirements of the form $A \neq \{\varepsilon\}^{B_0}$ for all $\varepsilon \in [g(\delta), g(\delta+1))$.

Block $(2\delta+1)$: all requirements of the form $A \neq \{\varepsilon\}^{B_1}$ for all $\varepsilon \in [g(\delta), g(\delta+1))$.

Within each block there is no conflict between requirements. Hence each block can be regarded as a single requirement. It turns out that the supremum of all stages at which block b is active is a Σ_2^α function of b. Lemma 3.3.VIII says $\sigma 2\mathrm{cf}(\alpha^*) = \sigma 2\mathrm{cf}(\alpha)$, so there is enough time to meet all requirements.

1.1 Theorem (Shore 1975). *Let A be α-recursively enumerable and regular. Then there exists α-recursively enumerable B_0 and B_1 such that $A = B_0 \cup B_1$, $B_0 \cap B_1 = \varnothing$ and $A \not\leq_\alpha B_i$ $(i < 2)$.*

Proof. Assume the terminology of the preliminary discussion of splitting immediately above. Let $g(\sigma, \delta)$ be an α-recursive approximation of the Σ_2^α cofinality function $g(\delta)$. Thus

$$\lim_\sigma g(\sigma, w) = g(\delta)$$

for all $\delta < \sigma 2\mathrm{cf}(\alpha)$. Note that $g \restriction \sigma 2\mathrm{cf}(\alpha)$ is tame Σ_2^α. The key function in the creation of negative requirements will be

$$t_i(\sigma, \varepsilon) = \mu y [A^{<\sigma}(y) \neq \{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)].$$

In order to construe each block of requirements as a single requirement, let

$$t(\sigma, 2\delta + i) = \sup{}^+ \{t_i(\sigma, \varepsilon) \mid g(\sigma, \delta) \leq \varepsilon < g(\sigma, \delta+1)\}.$$

($\sup^+$ is the strict supremum; thus $\sup^+ \{\beta\} = \beta + 1$.) t measures the length of initial segments to be preserved. m (below) measures the restraint on B_i needed to preserve those segments. The definition of $m(\sigma, 2\delta + i)$ has an all-important monotonicity clause, (1a). (1a) insures that increases in m are accompanied by increases in

t. It would be senseless to impose greater and greater negative restraints on B_i merely to preserve a fixed initial segment.

The definition of $m(\sigma, 2\delta+i)$ has two cases.

(1a) If $t(\sigma, 2\delta+i) > \sup^+_{\gamma<\sigma} t(\tau, 2\delta+i)$, then $m(\sigma, 2\delta+i)$ is the max of the right side of (2) and $\sup^+ \{z | "z \notin B_i^{<\sigma}"$ is needed for the computation of $\{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)$ for some $\varepsilon \in [g(\sigma, \delta), g(\sigma, \delta+1))$ and some $y < t_i(\sigma, \varepsilon)\}$.

(1b) If $t(\sigma, 2\delta+i) \leq \sup_{\tau<\sigma} t(\tau, 2\delta+1)$, then

$$m(\sigma, 2\delta+i) = \sup_{\tau<\sigma} m(\tau, 2\delta+i). \tag{2}$$

The equality $A^{<\sigma}(y) = \{\varepsilon\}^{B_i^{<\sigma}}(y)$ is *committed to preservation* at stage σ if (1a) holds and $y < t_i(\sigma, \varepsilon)$. Let x be an element of A enumerated at stage σ. Block $2\delta+i$ is *injured* at stage σ if x is put in B_i and $x < m(\sigma, 2\delta+i)$. Let $2u_j+j$ be the least w such that block w would be injured if x were put in B_j at stage σ.

Put x in B_1 if $u_0 \leq u_1$, and put x in B_0 if $u_1 < u_0$.

Block $2\delta+1$ is *stable* at stage τ if

$$(\sigma)_{\sigma \geq \tau}[m(\sigma, 2\delta+i) = m(\tau, 2\delta+i)].$$

1.2 Proposition. *Fix $\gamma < \sigma 2\mathrm{cf}(\alpha^*)$. Suppose for each $w < \gamma$, there is a τ such that block w is stable at stage τ. Then there is a τ such that for all $w < \gamma$, block w is stable at stage τ.*

Proof. Let $f(w)$ be the least τ such that w is stable at stage τ. Then f is $\mu - \Pi_1^\alpha$, hence Σ_2^α as in Exercise 2.12.VII. According to Lemma 3.3.VII, $\sigma 2\mathrm{cf}(\alpha^*) = \sigma 2\mathrm{cf}(\alpha)$. Hence $\sup f[\gamma] < \alpha$. □

The next lemma is the combinatoric essence of the proof of Theorem 1.1.

1.3 Lemma. *Fix δ and i. Suppose there is a τ such that block w is stable at stage τ for all $w < 2\delta+i$. Then there is a ρ such that block $2\delta+i$ is stable at stage ρ.*

Proof. Let $m = \sup\{m(\tau, w) | w < 2\delta+i\}$. Since A is regular, there is a $\sigma_0 > \tau$ such that $(A - A^{<\sigma_0}) \cap m$ is empty. Thus from stage σ_0 on, it is impossible to injure block w for any $w < 2\delta+i$. Hence block $2\delta+i$ will not be injured at, or after, stage σ_0. Assume σ_0 is so large that

$$(\sigma)_{\sigma \geq \sigma_0}[g(\sigma, \delta) = g(\delta) \quad \& \quad g(\sigma, \delta+1) = g(\delta+1)].$$

Let K be the set of all $\varepsilon \in [g(\delta), g(\delta+1)]$ such that:

$$(\mathrm{Ep})_{\sigma_0 \leq p}(\mathrm{Eq})_{p<q}(\mathrm{Ey})\ [\text{the equality } A^{<p}(y) = \{\varepsilon\}_p^{B_i^{<p}}(y)$$

is committed to preservation at stage p & $A^{<p}(y) = 0$ & $A^{<q}(y) = 1$]. Since $\varepsilon \in K$, the choice of σ_0 implies for some y, $\{\varepsilon\}^{B_i}(y)$ is defined but unequal to $A(y)$. K is α-finite, since K is an α-recursively enumerable set bounded by an ordinal less than α^*. Let $\sigma_1 > \sigma_0$ be so large that every $\varepsilon \in K$ is established by the appearance of a suitable p, q and y prior to stage σ_1. Then for each $\varepsilon \in K$, there is a permanent inequality that prevents any increase in $t_i(\sigma, \varepsilon)$ after stage σ_1. Thus $t_i(\sigma, \varepsilon) \leq t_i(\sigma_1, \varepsilon)$ for all $\sigma \geq \sigma_1$ and $\varepsilon \in K$.

Consider $\varepsilon \in [g(\delta),\ g(\delta)] - K$. Any equality of the form $A^{<\sigma}(y) = \{\varepsilon\}_{\sigma}^{B_i^{<\sigma}}(y)$ committed to preservation at stage $\sigma \geq \sigma_1$ is permanent. For such a σ and y,

$$A(y) = \{\varepsilon\}^{B_i}(y).$$

It follows that the set of all such y's, as ε ranges over $[g(\delta),\ g(\delta+1)] - K$, is bounded below α. Otherwise A would be α-recursive. Let b bound all such y's.

For each $\sigma \geq \sigma_1$ let b_σ be the sup of all y such that $A^{<\sigma}(y) = \{\varepsilon\}_{\sigma}^{B_i^{<\sigma}}(y)$ is committed to preservation at stage σ for some $\varepsilon \in [g(\delta),\ g(\delta+1)]$. b_σ is a non-decreasing function of σ bounded by b. The set of stages at which b_σ increases is α-recursively enumerable. The enumeration of an α-finite set in increasing order must finish in α-finitely many steps. Thus $b = b_{\sigma_2}$ for some $\sigma_2 \geq \sigma_1$. Now the monotonicity hypothesis of clause (1a) exerts its power. After stage σ_2, clause (1b) holds, and so block $2\delta + i$ is stable. □

End of proof of Theorem 1.1: By 1.2 and 1.3 every block is eventually stable. Suppose $A = \{\varepsilon\}^{B_i}$ for some $\varepsilon \in [g(\delta),\ g(\delta+1))$. Then $t_i(\delta, \varepsilon)$ converges to α, and block $2\delta + 1$ is never stable.

2. *Further Fine Structure*

The proof of Shore's splitting theorem was entirely dynamic and so holds in a variety of Σ_1 admissible structures that are not L-like (cf. Exercise 2.13). Shore's density theorem, IX.5.1, is tied strongly to $L(\alpha)$ by some fine structure facts based on collapsing arguments, in particular Lemma 2.2 below.

2.1 Proposition. *Assume $P(x, y)$ is Σ_2^α. Then there exists a partial Σ_2^α function f such that*

$$(x)[(\mathrm{E}y)P(x, y) \leftrightarrow f(x) \text{ is defined } \&\ P(x, f(x))].$$

(Uniformization of Σ_2^α by Σ_2^α.)

Proof. Let $P(x, y)$ be $(\mathrm{Eu})(v)Q(u, v, x, y)$ for some $\Delta_0^\alpha Q$. It suffices to uniformize $(v)Q((y)_0, v, x, (y)_1)$ by some partial Σ_2^α function f. Thus it is safe to assume P is Π_1^α. Say $P(x, y)$ is $(v)R(v, x, y)$ for some $\Delta_0^\alpha R$. Define

$$P_1(x, y) \text{ by } P(x, y) \quad \&\quad (z)_{z<y}(\mathrm{Ev}) \sim R(v, x, z).$$

$P_1(x, y)$ defines the graph of f. The predicate $z < y$ is α-recursive since it refers to the natural enumeration of $L(\alpha)$. The Σ_1 admissibility of $L(\alpha)$ implies

$$(z)_{z<y}(\mathrm{Ev}) \sim R(v, x, z) \leftrightarrow (\mathrm{Ew})(z)_{z<y}(\mathrm{Ev})_{v<w} \sim R(v, x, z).$$

Thus P_1 and f are Σ_2^α. □

The Σ_2 projectum of α, denoted by $\sigma 2p(\alpha)$, is the least $\gamma \leq \alpha$ such that some partial Σ_2^α function maps γ onto α. The next lemma is analogous to Proposition 2.1.VII, and can be proved in a similar fashion, that is, dynamically, when α is Σ_2 admissible. Otherwise a collapsing argument is necessary. Jensen has proved Lemma 2.2 with n in place of 2 for all $n \geq 1$ and all α without any admissibility assumptions.

2.2 Lemma. *Assume α is Σ_1 admissible. If $\gamma < \sigma 2p(\alpha)$ and $Y \subseteq \gamma$ is Σ_2^α, then Y is α-finite.*

Proof. As in the proof of the enumeration theorem (1.9.VII), the natural enumeration of $L(\alpha)$ gives rise to an α-recursive enumeration of all Δ_0 facts about elements of $L(\alpha)$. Thus there is a Δ_1^α formula $Q(u, v, e, x, y)$ such that

$$L(\alpha) \models Q(u, v, e, x, y) \quad \text{iff} \quad L(\alpha) \models F_e(u, v, x, y).$$

F_e is the e-th Δ_0^{ZF} formula. Q is lightface Δ_1^α because the definition of the natural enumeration of $L(\alpha)$ does not require any parameters from $L(\alpha)$.

It follows that $(\mathrm{Eu})(v)Q(u, v, e, x, y)$ is a universal, lightface Σ_2^α formula. By Proposition 2.1 $(\mathrm{Eu})(v)Q(u, v, e, x, y)$ can be uniformized by a partial Σ_2^α function $h(e, x)$. If $P(x, y)$ is the e-th Σ_2^α formula with free variables x and y, and if

$$L(\alpha) \models (\mathrm{Ey})P(\underline{a}, y) \text{ for some } a \in L(\alpha),$$

then $h(e, a)$ is defined and $L(\alpha) \models P(a, \underline{h(e, a)})$.

If e is the Gödel number of a Σ_2^α formula $P(x_1, \ldots, x_n)$, let $\langle e \rangle$ be the Gödel number of a Σ_2^α formula $Q(x)$ with the property that,

$$P(x_1, \ldots, x_n) \leftrightarrow Q(\langle x_1, \ldots, x_n \rangle).$$

Let $h_{\langle\rangle}(e, x_1, \ldots, x_n)$ be $h(\langle e \rangle, \langle x_1, \ldots, x_n \rangle)$.

$z^{<\omega}$ is the set of all finite sequences of elements of z. Define $H_2(z)$, the Σ_2^α Skolem hull of z, to be $h[\omega \times z^{<\omega}]$. Then $H_2(z) \prec_2 L(\alpha)$ by Exercise 2.8. Hence $[V = L]$, as described in the proof of Lemma 2.5.VII, is true in $H_2(z)$. Define the collapse of $H_2(z)$ as in the proof of Lemma 2.6.VII. For each $x \in H_2(z)$,

$$t(x) = \{t(y) \mid y \in x \quad \& \quad y \in H_2(z)\}.$$

$t[H_2(z)]$ is transitive, and t maps $H_2(z)$ isomorphically onto $t[H_2(z)]$.

$Y \subseteq \gamma$ has a Σ_2 definition over $L(\alpha)$ with parameter $p \in L(\alpha)$. Let $z = \gamma \cup \{p\}$. Then $Y \subseteq \gamma$ has a Σ_2 definition over $H_2(z)$ with parameter $p \in H_2(z)$, since $H_2(z) \prec_2 L(\alpha)$. Now collapse the hull. $t[Y] \subseteq t[\gamma]$ is Σ_2 definable over $t[H_2(z)]$ with parameter $t(p) \in t[H_2(z)]$. By Lemma 2.5.VII, $t[H_2(z)] = L(\gamma_0)$ for some $\gamma_0 \leq \alpha$. Thus Y is Σ_2 definable over $L(\gamma_0)$. If $\gamma_0 < \alpha$, then Y must be α-finite.

Suppose $\gamma_0 = \alpha$. Then h gives rise to a partial map of $\omega \times z^{<\omega}$ onto $H_2(z)$; in addition the map is Σ_2 over $H_2(z)$. Since $\gamma < \sigma 2p(\alpha)$, the α-cardinality of $\omega \times z^{<\omega}$ equals some $\rho < \sigma 2\mathrm{cf}(\alpha)$. But then there is a partial map Σ_2^α from ρ onto $L(\alpha)$, an impossibility. □

Assume $A \subseteq L(\alpha)$. The Σ_1 projectum of α relative to A, denoted by $\sigma 1 p_A(\alpha)$, is the least $\gamma \leq \alpha$ such that there exists a partial map Σ_1 over $\langle L[A, \alpha], A \rangle$ from γ onto $L(\alpha)$. In short

$$\alpha_A^* = \mu\gamma(\mathrm{Ef})\ [\mathrm{dom} f \subseteq \gamma \quad \& \quad f \in \Sigma_1^{\alpha, A} \quad \& \quad \mathrm{rng} f = L(\alpha)].$$

In general "$f \in \Sigma_1^{\alpha, A}$" means the graph of f is defined by a Σ_1 formula whose existential quantifier ranges over $L[A, \alpha]$, whose parameters belong to $L[\alpha, A]$, and whose atomic subformulas may include "$x \in A$". If A is regular, then $L[A, \alpha] = L(\alpha)$, and "$f \in \Sigma_1^{\alpha, A}$" is equivalent to "$f \leq_{w\alpha} A$".

The next theorem is a rare combination of fine structure and recursive approximation.

2.3 Theorem (Shore 1976). *Let A be regular and α-recursively enumerable. If $\gamma < \alpha_A^*$ and $Y \subseteq \gamma$ is $\Sigma_1^{\sigma, A}$, then Y is α-finite.*

Proof. Since A is regular the natural enumeration of $L(\alpha)$ leads to a $\Delta_1^{\alpha, A}$ enumeration of T^A, the set of all Δ_0 sentences true in $\langle L(\alpha), A \rangle$. The truth value of each such sentence depends not on A, but only on $A \cap L(\delta)$ for some $\delta < \alpha$. Regularity was assumed so that $A \cap L(\delta)$ would be an element of A. Thus $T^A \subseteq L(\alpha)$; and

$$\langle e, \langle b \rangle \rangle \in T^A \leftrightarrow \langle L(\alpha), A \rangle \models F_e(\langle \underline{b} \rangle),$$

where $F_e(\langle x \rangle)$ is the e-th Δ_0 formula of ZF with $x \in A$ as an additional atomic formula, and $\langle b \rangle \in L(\alpha)$.

A universal $\Sigma_1^{\alpha, A}$ predicate can be obtained from T^A. That predicate can be uniformized by a partial $\Sigma_1^{\alpha, A}$ function thanks to the regularity of A. Thus there exists a universal partial $\Sigma_1^{\alpha, A}$ function h. If $P(\langle x \rangle, y)$ is the e-th Δ_0 formula of ZF with $x \in A$ as an extra atomic formula and $\langle x \rangle, y$ as free variables, and

$$L(\alpha) \models (\mathrm{Ey}) P(\langle \underline{a} \rangle, y)$$

for some $\langle a \rangle \in L(\alpha)$, then $h(e, \langle a \rangle)$ is defined and

$$L(\alpha) \models P(\langle \underline{a} \rangle, \underline{h(e, \langle a \rangle)}).$$

Let $z = \gamma \cup \{p\}$, where p encodes the parameters needed for the $\Sigma_1^{\alpha, A}$ definition of $Y \subseteq \gamma$, and the Σ_1^{α} definition of g_0 below. Form $H = h[\omega \times z^{<\omega}]$, the $\Sigma_1^{\alpha, A}$ hull of z. Then $\langle H, A\rangle \prec_1 \langle L(\alpha), A\rangle$. The last assertion means that every $\Sigma_1^{\alpha, A}$ sentence with parameters in H is true in $\langle L(\alpha), A\rangle$ iff it is true in $\langle H, A\rangle$.

Suppose H is bounded, that is, $H \subseteq L(\beta)$ for some $\beta < \alpha$. Then

$$\langle H, A\rangle \prec_1 \langle L(\beta), L(\beta) \cap A\rangle.$$

Y is Σ_1 over $\langle H, A\rangle$, hence Σ_1 over $\langle L(\beta), L(\beta) \cap A\rangle$, hence α-finite, since $L(\beta) \cap A$ is α-finite.

Suppose $\gamma < \sigma 2p(\alpha)$. Since Y is $\Sigma_1^{\alpha, A}$, and A is regular and Σ_1^{α}, it follows that Y is Σ_2^{α} (cf. Exercise 2.9). Then Y is α-finite by Lemma 2.2.

Assume H is unbounded and $\gamma \geq \sigma 2p(\alpha)$ with the intention of showing $H = L(\alpha)$. It suffices to show $\alpha \subseteq H$ since the f of 1.8.VII is lightface. Let

$$O(c) = \mu\beta[c \in L(\beta)] \quad (c \in L(\alpha)).$$

$O(c)$ is Σ_1^{α}, hence $O[H] \subseteq H$, and so H contains arbitrarily large ordinals less than α. Let g be a partial Σ_2^{α} function from $\sigma 2p(\alpha)$ onto α. By Proposition 2.2.VIII there is an α-recursive $g_0(\sigma, x)$ such that

$$\lim_{\sigma} g_0(\sigma, x) = g(x)$$

for all $x \in \text{dom } g$. Fix $u < \alpha$. Choose $x < \sigma 2p(\alpha)$ so that $g(x) = u$. Then for a sufficiently large σ in H, $g_0(\sigma, x) = u$; and so $u \in H$, since $\sigma 2p(\alpha) \leq \gamma$.

The equality of H and $L(\alpha)$ implies there is a partial $\Sigma_1^{\alpha, A}$ map, namely h, from $\omega \times z^{<\omega}$ onto $L(\alpha)$. An impossibility since the α-cardinality of $\omega \times z^{<\omega}$ is less than $\sigma 1p_A(\alpha)$. □

2.4 Corollary (R. Shore 1976). *Suppose A is α-recursively enumerable, regular and incomplete. Then $\sigma 1\text{cf}_A(\alpha) \geq \sigma 1p_A(\alpha)$.*

Proof. Assume $\sigma 1\text{cf}_A(\alpha) < \sigma 1p_A(\alpha)$ with the intention of showing A complete. Let C be a regular, complete, α-recursively enumerable set, and $\{C^{\sigma} \mid \sigma < \alpha\}$ an α-recursive enumeration of C. Let f: $\sigma 1\text{cf}_A(\alpha) \to \alpha$ be an unbounded $\Sigma_1^{\alpha, A}$ function. Since C is regular, for each x there is a y such that

$$C \cap f(x) \subseteq C^{f(y)}. \tag{1}$$

As a relation on x and y, (1) is a $\Pi_1^{\alpha, A}$ subset of $(\sigma 1\text{cf}_A(\alpha))^2$, hence α-finite by Theorem 2.3. Let $g(x)$ be the least y that satisfies (1). It follows that g: $\sigma 1\text{cf}_A(\alpha) \to \sigma 1\text{cf}_A(\alpha)$ is α-finite. Then

$$H \subseteq cC \leftrightarrow (\text{Ex})[H \subseteq f(x) \quad \& \quad H \cap C^{f(g(x))} = \phi]$$

for all α-finite H, and so $C \leq_\alpha A$. (Keep in mind that the regularity of A implies every $\Sigma_1^{\alpha, A}$ function, in particular f, is weakly α-recursive in A.) $\square$

2.5 Weak Σ_1 Admissibility. Let β be a limit ordinal. $L(\beta)$ need not be Σ_1 admissible, but it is closed under pairing, and consequently the fundamentals of recursion theory lift to $L(\beta)$. β is said to be *weakly Σ_1 admissible* if $\sigma 1\mathrm{cf}(\beta) \geq \sigma 1 p(\beta)$. Thus Corollary 2.4 becomes: if A is α-recursively enumerable, regular and incomplete, then the structure $\langle L(\alpha), A\rangle$ is weakly Σ_1 admissible. Some of the solutions to Post's problem given in Chapter VIII can be adapted to weakly Σ_1 admissible structures. The main obstacle is the limited time in which to meet incomparability requirements. There are α requirements but only $\sigma 1\mathrm{cf}_A(\alpha)$ stages of construction. This difficulty is overcome by the next lemma, which is essential to the proof of density in Section 5.

2.6 Lemma. *Suppose $\langle L[A, \alpha], A\rangle$ is weakly Σ_1 admissible. Then there exists a one-one $\Sigma_1^{\alpha, A}$ map from $\sigma 1\mathrm{cf}_A(\alpha)$ onto α.*

Proof. $c = \sigma 1\mathrm{cf}_A(\alpha)$ and $p = \sigma 1 p_A(\alpha)$. Let f be a strictly increasing $\Sigma_1^{\alpha, A}$ map from c into α, unbounded in α. Let g be a one-one $\Sigma_1^{\alpha, A}$ map from α into p.

Define h, a partial, one-one $\Sigma_1^{\alpha, A}$ map from $c \times c$ onto α as follows: $h(u, v) = w$ if:

(a) $g(w) = u$; and
(b) the existential witness needed to show "$g(w) = u$", and the parameters in the $\Sigma_1^{\alpha, A}$ definition of g belong to $L[A, f(v)]$; and
(c) (b) is false when v is replaced by $v_0 < v$.

To check that h is $\Sigma_1^{\alpha, A}$, note that the existential witnesses needed to define $f(v_0)$ for all $v_0 \leq v$ are bounded below α, since $v < c$. Range $h = \alpha$, because $\alpha =$ range g^{-1} and domain $g^{-1} \subseteq p \subseteq c$. The domain of h is a $\Delta_1^{\alpha, A}$ subset of $c \times c$. Extend the domain of h to all of $c \times c$ by setting $h(u, v) = 0$ for those $\langle u, v\rangle$'s not covered by (a), (b), (c).

Thus h is a $\Sigma_1^{\alpha, A}$ map from $c \times c$ onto α. Let $t \in L[\alpha, A]$ map c onto $c \times c$, and set $h_1 = h \circ t$. Define $h_2(x)$ by recursion on $x < c$.

$$m(x) = \mu z[h_1(z) \notin h_2[x]].$$
$$h_2(x) = h_1(m(x)).$$

Since $x < c$, the existential witnesses needed to establish $m(x)$ are bounded below α. Thus h_2 is a one-one, $\Sigma_2^{\alpha, A}$ map from c onto α. $\square$

Note that the Σ_1 admissibility of $L(\alpha)$ was not used in the proof of Lemma 2.6.

2.7 Corollary (Shore 1976). *Suppose A is α-recursively enumerable, regular and incomplete. Then there exists a one-one, $\Sigma_1^{\alpha, A}$ map from $\sigma 1\mathrm{cf}_A(\alpha)$ onto α.*

Proof. By Corollary 2.4 and Lemma 2.6.

An early version of Corollary 2.7, for $\alpha = \omega_1^{CK}$, occurs in the proof of Driscoll's (1968) density theorem for metarecursion theory.

2.8–2.12 Exercises

2.8. Let $H_2(z)$ be the Σ_2^α Skolem hull of z as defined in Lemma 2.2. Show $H_2(z) \prec_2 L(\alpha)$, that is, each Σ_2^{ZF} sentence with parameters in $H_2(z)$ is true in $H_2(z)$ if it is true in $L(\alpha)$.

2.9. Let A be a regular, α-recursively enumerable set. Suppose Y is $\Sigma_1^{\alpha, A}$. Show Y is Σ_2^α.

2.10. Let β be a weakly Σ_1 admissible ordinal. Show there exists a one-one Σ_1^β map from $\sigma 1 \mathrm{cf}(\beta)$ onto β.

2.11. Let β be weakly Σ_1 admissible. Reformulate and prove the combinatoric lemma, 2.3.VII, for $L(\beta)$.

2.12. Let β be weakly Σ_1 admissible. $A \subseteq L(\beta)$ is said to be β-recursively enumerable if A is Σ_1^β. Define "weakly β-recursive in" as in subsection 3.2.VII with β in place of α. Show there exist two β-recursively enumerable sets such that neither is weakly β-recursive in the other.

2.13. Let A be a Σ_1 admissible structure of the form $\langle L[B, \alpha], \varepsilon, B\rangle$. Prove Shore's splitting theorem for A.

3. *Density for ω*

The following sketch of the original proof of the density of the recursively enumerable degrees will prove helpful in the proof of density for all α given in Sections 4 and 5.

Let A and C be recursively enumerable subsets of ω such that $A <_T C$. The objective is a recursively enumerable B such that $A <_T B <_T C$. $A \leq_T B$ is accomplished by planting A in the even coordinates of B. The remaining action takes place on the odd coordinates of B. The strategy for realizing $B \not\leq_T A$ is positive in nature; bits of C are planted in B. If the strategy fails, then $C \leq_T A$. If it succeeds, then the bits planted in B add up to something infinite but manageable with respect to the negative requirements.

The strategy for realizing $C \not\leq_T B$ is negative in nature; initial segments of $\{e\}^B$ are preserved. If the strategy fails, then $C \leq_T A$.

Positive requirements associated with $B \neq \{e_1\}^A$ have higher priority than negative requirements associated with $C \neq \{e_2\}^B$ if $e_1 < e_2$. To meet positive requirement e_1, it must be shown that the obstacles raised by negative requirement e ($e \leq e_1$) drop back simultaneously, and infinitely often, to some fixed $\ell(e_1)$. To

meet negative requirement e_2, it must be shown that injury set I_{e_2} is recursive in A. I_{e_2} is the set of all elements added to B for the sake of positive requirement e ($e < e_2$) together with A.

The term "infinite injury" refers to the fact that I_{e_2} is infinite (rather than finite). The operation "lim inf" is also helpful, because recursive sequences generated during the construction tend to have limit infimums rather than limits.

The planting strategy for $B \neq \{e\}^A$ is as follows. Let A^s be that part of A enumerated by the end of stage s. Let $\{e\}_s$ be $\{e\}$ restricted to the first s computations. (The complexity of planting arises from the possibility that both (i) and (ii) may be true.

(i) $\{e\}^A(n)$ is undefined.
(ii) $\{e\}_s^{A^s}(n)$ is defined for infinitely many s.

In that event there is no hope of finding an s such that

$$B^s(n) \neq \{e\}_s^{A^s}(n),$$

and then preserving the above inequality forever after.) Assume that the odd part of B is divided, in an effective manner depending on e, into infinitely many infinite rows. Let $\langle e, n, i\rangle$ be the i-th space in the n-th row. The idea is to plant $C(n)$ in the n-th row of B. $C \not\leq_T A$, so if enough of C is planted in B, then $B \not\leq_T A$. On the other hand, the planting must not go too far, because $C \not\leq_T B$ is also desired.

$\ell(e, n, s)$ is the proposed location for $C(n)$ at stage s. If $\ell(e, n, s) = -1$, then there is no location.

$\ell(e, n, 0) = -1$.

Case 1: $\ell(e, n, s-1) = -1$. If

$$(x)_{x \leq n}\,[B^{s-1}(x) = \{e\}_s^{A^{s-1}}(x)], \tag{1}$$

then $\ell(e, n, s) = s$. Otherwise $\ell(e, n, s) = -1$.

Case 2: $\ell(e, n, s-1) = t \neq -1$. The value t was chosen at stage $t < s$ when (1) held with t in place of s. If for all $x \leq n$, $B^{s-1}(x) = B^{t-1}(x)$,

$$\{e\}_s^{A^{s-1}}(x) = \{e\}_s^{A^{t-1}}(x), \tag{2}$$

and the *same* computation is used for both sides of (2), then $\ell(e, n, s) = \ell(e, n, s-1)$. Otherwise $\ell(e, n, s) = -1$.

Initially there is no location for $C(n)$. A location is defined at the first stage (1) is true. That location remains fixed unless a relevant change in B occurs or some change in A occurs that invalidates a computation needed for (1). In that event there is again no location. Thus as s increases, the location may come and go. Each time it returns, it is a bit further to the right. If B and $\{e\}^A$ agree on $[0, n]$, then eventually a permanent location develops, and conversely. If eventually there is no stage at which B and $\{e\}^A$ appear to agree on $[0, n]$, then the location eventually disappears forever. If they appear to agree infinitely often, without agreeing in the limit, then the location moves steadily off to infinity on the right.

P_e^s, the e-th set of positive requirements at stage s, consists of all sentences of the form:

$$\text{if} \quad n \in C, \quad \text{then} \quad \langle e, n, \ell(e, n, s)\rangle \in B.$$

Of course the above makes sense only if $\ell(e, n, s) \neq -1$.

If $\lim_s \ell(e, n, s)$ exists and $\neq -1$, call it $\ell(e, n)$.

P_e, the e-th set of positive requirements, consists of all sentences of the form:

$$\text{if} \quad n \in C, \quad \text{then} \quad \langle e, n, \ell(e, n)\rangle \in B. \tag{3}$$

Matters are arranged so that the only way $\langle e, n, \ell(e, n)\rangle$ can land in B is via (3).

Suppose all but finitely much of P_e is met with the intent of showing $B \neq \{e\}^A$. For a reductio ad absurdum, assume $B = \{e\}^A$. Then $\ell(e, n)$ exists for all n and is computable from A. But then $C \leq_T A$, since

$$\langle e, n, \ell(e, n)\rangle \in B \leftrightarrow n \in C$$

for all but finitely many n.

Let P_e^* be the set added to B for the sake of P_e^s. Again assume all but finitely much of P_e is met in order to determine P_e^*. As above $B \neq \{e\}^A$. Let n_0 be the least n such that $\ell(e, n)$ does not exist. For each $n < n_0$, there is only a finite amount of activity on row n. If $n \geq n_0$, the location either (i) eventually disappears forever, or (ii) moves off to infinity on the right.

Consequently P_e^* is recursive. Suppose $n \geq n_0$. To decide if $\langle e, n, i\rangle$ is in B, run the construction until a stage s is found such that $s > i$, and $\ell(e, n, s)$ is either -1 or greater than i. Then $\ell(e, n, t) \neq i$ for all $t \geq s$, and so $\langle e, n, i\rangle$ was put in B only if it was put in before stage s.

If the recursive determination of P_e^* is valid for all $e < e_2$, then the injury set

$$I_{e_2} = A \cup \bigcup \{P_e^* \mid e < e_2\}$$

is recursive in A.

The e-th set of *negative requirements*: Define

$$r(e, s) = \mu z_{z < s}[C^{s-1}(z) \neq \{e\}_s^{B^{s-1}}(z)].$$

Let $p(e, x, s)$ be the $\sup^+$ of all negative facts about B^{s-1} used in the computation of $\{e\}_s^{B^{s-1}}(x)$. Define

$$p(e, s) = \sup \{p(e, x, s) \mid x < r(e, s)\}.$$

Keeping numbers less than $p(e, s)$ out of B will preserve the value of $\{e\}_s^{B^{s-1}}(z)$ for all $z < r(e, s)$.

Stage $s > 0$ begins with the definition of $r(e, s)$ and $p(e, s)$ for all e followed by the addition of the s-th member of A to B. Then attempts to satisfy P_e^s alternate with

revisions of r and p. Fix e. Let w_e^s be the least element added to B for the sake of A or $P_{e'}^s(e' < e)$. Define

$$r_0(e, s) = \mu z_{z < r(e, s)} [w_e^s < p(e, z, s)].$$
$$p_0(e, s) = \sup \{p(e, x, s) \mid x < r_0(e, s)\}.$$

Now add to B all elements not less than $\inf\{p_0(e', s) \mid e' < e\}$ required by P_e^s.

Suppose I_e is recursive in A to show

$$\liminf_s r_0(e, s) < \infty. \tag{4}$$

The negation of (4) implies $C \leq_T A$. Fix n to see how $C\restriction n$ is computed from A. If (4) fails, then

$$(\mathrm{Et})(s)_{s \geq t}[r_0(e, s) \geq n]. \tag{5}$$

At the beginning of stage t,

$$(x)_{x \leq n}[C^{t-1}(x) = \{e\}_t^{B^{t-1}}(x)],$$

since $r(e, s) \geq r_0(e, s)$. It follows from (5) that the computation of $\{e\}_t^{B^{t-1}})\restriction n$ is permanent, and that

$$C\restriction n = \{e\}^B \restriction n = \{e\}_t^{B^{t-1}})\restriction n.$$

To find t run the construction until a stage t is reached where $r(e, t) \geq n$ and an appeal to I_e makes clear that the t-th approximation of $\{e\}^B\restriction n$ is permanent.

Similarly $I_e \leq_T A$ implies $\lim_s \inf p_0(e,s)$ exists. $\lambda s \mid p_0(e, s)$ behaves as follows. There is an n_e and an s_e such that the computation of

$$\{e\}_{s_e}^{B^{s_e}}\restriction n_e \tag{6}$$

is permanent. In addition (6) equals $C\restriction n_e$ and $\{e\}^B(n_e)$ is either undefined or unequal to $C(n_e)$. Hence $\lim_s \inf p_0(e, s)$ is the $\sup^+$ of the negative facts about B used in the computation of (6).

The barrier to meeting P_{e_0} is defined by

$$\sup_{e < e_0} \lim_s \inf p_0(e, s). \tag{7}$$

The drama of infinite injury is at its highest pitch when it is revealed that

$$(E_\infty s)\,(e)_{e \leq e_0}\,[p_0(e, s) = \lim_s \inf p_0(e, s)]. \tag{8}$$

("$(E_\infty s)$" means "there exist infinitely many s".)

The most direct proof of (8) is due to Lachlan. For each s let x_s be the least element put in B at stage s. Let

$$D = \{s \mid (t)_{t>s}\,(x_t > x_s)\}.$$

The value of $p_0(e, s)$ is a downward revision of the value of $p(e, s)$ caused by the addition of w_e^s to B. $x_s \leq w_e^s$. Hence for all sufficiently large s in D, $p_0(e, s)$ is a non-decreasing function of s, and

$$\liminf_s p_0(e, s) = \lim \{p_0(e, s) \mid s \in D\}. \tag{9}$$

(8) is a consequence of (9). (8) implies I_{e_0} is recursive.

$B \leq_T C$ is yet to be shown. B is the disjoint recursive sum of A and $P_e^*(e \geq 0)$. $A \leq_T C$ by hypothesis. P_e^* is recursive in C uniformly in e. There is a recursive function z such that $P_e^* = \{z(e)\}^C$ for all e. Procedure $\{z(e)\}$ is defined by recursion on e. Fix $\langle e, n, i\rangle$. Run the construction and keep an eye on $\ell(e, n, s)$. If $\ell(e, n)$ does not exist, then there is a stage $s \geq i$ such that $\ell(e, n, s) > i$ or $\ell(e, n, s) = -1$. Either way $\langle e, n, i\rangle$ gets into B only if it does before stage s. Suppose $\ell(e, n)$ does exist. Then $\ell(e, n) = \ell(e, n, t)$ for some t that can be recognized by referring to A; at stage t the computation from A underlying the value of $\ell(e, n, t)$ was correct.

If $\ell(e, n) < i$, then $\langle y, n, i\rangle \notin B$. If $\ell(e, n) > i$, then running the construction will settle "$\langle e, n, i\rangle \in B$?". Finally suppose $\ell(e, n) = i$. If $n \notin C$, then $\langle e, n, i\rangle \notin B$. Assume $n \in C$. Then $\langle e, n, i\rangle$ is not put in B iff some permanent negative requirement keeps it out. Such a requirement, if it exists, can be found by running the construction. Its permanence is established by some negative facts about $A \cup \{P_{e'}^* \mid e' < e\}$ computed from C via $\{z(e')\}$ $(e' < e)$. □

4. *Preliminaries to α-Density*

Suppose A and C are regular α-r.e., sets such that $A <_\alpha C$. The following parameters will be used in the construction of an α-recursively enumerable B such that $A <_\alpha B <_\alpha C$:

$$c_1^A = \sigma 1\mathrm{cf}_A(\alpha);$$

$$c_2^A = \sigma 2\mathrm{cf}_A(\alpha);$$

$$\alpha_A^* = \sigma 1 p_A(\alpha).$$

Define $c_2^A(x) = \sigma 2\mathrm{cf}_A^\alpha(x)$, that is, the Σ_2 cofinality of x in the structure $\langle L[A, \alpha], A\rangle$.

4.1 Lemma

(*i*) $c_1^A \geq \alpha_A^*$.

(*ii*) (Ek) [*k*: $c_1^A \to \alpha$ *is one-one, onto and* $\Sigma_1^{\alpha, A}$].

(iii) *If* $Z \subseteq \delta < \alpha_A^*$ *and* Z *is* $\Sigma_1^{\alpha, A}$, *then* Z *is* α*-finite.*
(iv) $c_2^A = c_2^A(\alpha_A^*) = c_2^A(c_1^A)$.

Proof. (i) is Corollary 2.4. (ii) is Corollary 2.7. (iii) is Theorem 2.3. (iv) is a relativization of Lemma 3.3.VIII. The relativization succeeds with the aid of (i), (ii) and (iii). □

4.2 Cofinality Function *g*. According to Lemma 4.1(iv) there exists a $\Sigma_2^{\alpha, A}$ function h: $c_2^A \to c_1^A$ with range unbounded in c_1^A. Thus

$$h(x) = y \leftrightarrow (\mathrm{Eu})\,(v) D^A(u, v, x, y)$$

for some $\Delta_0^{\alpha, A}$ formula D^A. u can be construed as less than c_1^A with the aid of a $\Sigma_1^{\alpha, A}$ map k from c_1^A onto α provided by Lemma 4.1(ii).

$$h(x) = y \leftrightarrow (\mathrm{Eu})_{u < c_A}\,(v) D^A\,(k(u), v, x, y).$$

Define

$$g(x) = \mu z(v) D^A(k(z_0), v, x, z_1).$$

(For simplicity, z_i instead of $(z)_i$.)
Note that $D^A(k(z_0), v, x, z_1)$ is $\Pi_1^{A, \alpha}$ because it is equivalent to

$$(t)[k(z_0) = t \to D^A(t, v, x, z_1)].$$

Thus g: $c_2^A \to c_1^A$ is $\Sigma_2^{A, \alpha}$ and has range unbounded in c_1^A. g is more suitable than h for approximation. Let

$$g_\sigma^A(x) = \mu z_{z < \sigma}\,(v)_{v < \sigma}\, D^A(k(z_0), v, x, z_1).$$

Then g_σ^A is $\Sigma_1^{\alpha, A}$. Consider

(1) $\quad g(x) = z \leftrightarrow (v) D^A(k(z_0), v, x, z_1) \quad \& \quad (w)_{w < z}(\mathrm{Ev}) \sim D^A(k(w_0), v, x, w_1).$

For each $w < g(x)$ let $\ell(x, w)$ bound x, w and all the quantifiers in the $\Sigma_1^{\alpha, A}$ formula

(2) $$(\mathrm{Ev}) \sim D^A(k(w_0), v, x, w_1).$$

Thus $L[A, \alpha] \models (2)$ iff $L[A \cap \ell(x, w),\ \ell(x, w)] \models (2)$. Let $\ell(x)$ be the $\sup^+$ of $\{\ell(x, w) \mid w < g(x)\}$. $\ell(x) < \alpha$ because $g(x) < c_1^A$. Then

$$g(x) = \mu z(v)_{v < \ell(x)}\, D^A(k(z_0), v, x, z_1),$$
$$\sigma \geq \ell(x) \to g_\sigma^A(x) = g(x).$$

Define g_σ by replacing A by $A^{<\sigma}$ in the definition of g_σ^A. g_σ is Σ_1^α.
Call σ *A-correct* if $A^{<\sigma} = A \cap \sigma$.

4.3 Proposition. *If $c_1^A > \omega$, then there exist arbitrarily large A-correct stages.*

Proof. Choose σ_0. Let σ_{n+1} be the least $\sigma \geq \sigma_n$ such that $A \cap \sigma_n = A^{<\sigma}$. Then $\lambda_n \mid \sigma_n$ is $\Sigma_1^{\alpha, A}$. Let $\tau = \lim_n \sigma_n$. $\tau < \alpha$, since $\omega < c_1^A$. $A^{<\tau} = A \cap \tau$. □

4.4 Lemma. *Assume $b < c_2^A$.*
(i) $(x)_{x<b}\,[g_\sigma(x) \geq g(x)]$ *for all sufficiently large* σ.

(ii) $g_\sigma \restriction b = g \restriction b$ *for all sufficiently large A-correct* σ.

Proof. (1) The function ℓ: $c_2^A \to \alpha$ of subsection 4.2 is $\Sigma_2^{\alpha, A}$. Hence $\ell[b]$ is bounded by some $\ell_b < \alpha$. Consider $x < b$.

$$g(x) = \mu z(v)_{v<\ell_b}\, D^A(k(z_0), v, x, z_1). \tag{1}$$

The right side of (1) is evaluated using only $A \cap \ell_b$, hence $g \restriction b$ is α-finite thanks to the regularity of A. Suppose $A \cap \ell_b \subseteq A^{<\sigma_0}$ for some $\sigma_0 \geq \ell_b$. Let $\sigma \geq \sigma_0$. Then

$$(z)_{z<g(x)}\,(\mathrm{Ev})_{v<\ell_b} \sim D^{A^{<\sigma}}(k(z_0), v, x, z_1)$$

because D is Δ_0 and

$$(z)_{z<g(x)}\,(\mathrm{Ev})_{v<\ell_b} \sim D^{A\cap\ell_b}(k(z_0), v, x, z_1).$$

Hence $g_\sigma(x) \geq g(x)$.

(ii) Suppose $\sigma \geq \sigma_0$ is A-correct. Then for $x < b$,

$$g_\sigma(x) = g_\sigma^{A^{<\sigma}}(x) = g_\sigma^{A\cap\sigma}(x) = g_\sigma^A(x). \qquad \square$$

g will be used in the next section to define blocks of requirements, It will follow from Lemma 4.4(i) that for all sufficiently large σ, if a negative requirement lands in block x at stage σ, then it is never discarded. g will be approximated by g_σ. On the surface g is Σ_3^α, and in general Σ_3^α blocking functions are intractable in the presence of Σ_2 inadmissibility. But g is workable because it is $\Sigma_2^{A,\alpha}$ for an A such that $\langle L[\alpha, A], A\rangle$ is weakly Σ_1 admissible, and Σ_2 blocking needs only Σ_1 admissibility to succeed.

5. *Shore's Density Theorem*

The density theorem for ω, as sketched in Section 3, appears to rely on the fact that $L(\omega)$ satisfies Σ_3 replacement. In fact it uses only Σ_2^A replacement, where A is an incomplete, recursively enumerable set. Very little is known about making a Σ_1 admissible α do the work of Σ_3 replacement. The proof of the density theorem for α makes α do the work of Σ_2^A replacement for an incomplete, regular, α-recursively

enumerable set. For such an A, $\langle L[A, \alpha], A\rangle$ is weakly Σ_1 admissible, and so some of the thinking behind the α-finite injury method is applicable. New difficulties arise because A has to be guessed at.

5.1 Theorem (Shore 1976). *Let A and C be α-recursively enumerable sets such that $A <_\alpha C$. Then there exists an α-recursively enumerable B such that $A <_\alpha B <_\alpha C$.*

Proof. As usual A and C are assumed to be regular. The following construction yields a regular B. The density sketch given in Section 3 will be relied on heavily. The principal difference between the argument below and that of Section 3 is the use of blocking. Let g and g_σ be as in Section 4. Let f^A be a one-one $\Sigma_1^{\alpha, A}$ map of α into α_A^*. *Block* y is $[0, f^A g(y))$ for each $y < c_2^A$. Each ε in block y is associated with a reduction procedure $\{(f^A)^{-1}(\varepsilon)\}$, written simply as $\{\varepsilon\}$. $f^A g[c_2^A]$ is unbounded in α_A^* by Theorem 2.3. During the construction $(f^A)^{-1}(\varepsilon)$ is approximated by $f_\sigma^{-1}(\varepsilon)$, abbreviated as ε_σ. f_σ is the result of replacing A by $A^{<\sigma}$ in the $\Sigma_1^{\alpha, A}$ definition of f. *Note Well*: Let $D^A(a, b, z)$ be a $\Delta_0^{\alpha, A}$ formula such that

$$f^A(a) = b \leftrightarrow (\mathrm{Ez}) D^A(a, b, z).$$

Define $f_\sigma(a)$ to be $\mu b_{b<\sigma}(\mathrm{Ez})_{z<\sigma} D^{A^{<\sigma}}(a, b, z)$. If $f_\sigma(a) < \sigma$ for some A-correct σ, then $f_\sigma(a) = f^A(a)$. ε_σ is not defined at stage σ unless $\varepsilon < \sigma$ and $f_\sigma^{-1}(\varepsilon) < \sigma$. If $f_\sigma^{-1}(\varepsilon)$ has more than one value, then the least is used.

The even part of B is reserved for A. All the remaining action takes place on the odd part of B. *Positive requirements* are elements of

$$\{\langle y, n, i\rangle \mid y < c_2^A \quad \& \quad n < c_1^A \quad \& \quad i < \alpha\}.$$

They are added to B in order to insure that $B \neq \{\varepsilon\}^A$ for all ε in block y.

Let $k^A: c_1^A \to \alpha$ be an onto, $\Sigma_1^{\alpha, A}$ map as in Lemma 4.1 (ii). Define k_σ by replacing A by $A^{<\sigma}$ in the $\Sigma_1^{\alpha, A}$ definition of k^A. Then

$$k^A(x) = \lim_\sigma k^\sigma(x)$$

for all $x < c_1^A$, and k^A is tame Σ_2^α (tame via the α-recursive approximation k_σ).

The planting of C in B is less troublesome if C is replaced by C_0 provided by Proposition 3.4.VII. The proof of 3.4 implies C_0 is regular if C is. C_0 is α-recursively enumerable, $C_0 \equiv_\alpha C$ and

$$(1) \qquad (X)[C_0 \leq_{w\alpha} X \leftrightarrow C \leq_\alpha X].$$

$\ell(y, n, \sigma)$ is the proposed location of $C_0(k^A(n))$ in B at the beginning of stage σ. $\ell(y, n, 0) = -1$ (no location).

Case 1: $\sup^+ \{\tau \mid \tau < \sigma \,\&\, \ell(y, n, \tau) = -1\} = \sigma$. Then $\ell(y, n, \sigma) = \sigma$ if

$$(2)\ (E\varepsilon)_{\varepsilon < f_\sigma g_\sigma(y)} [B^{<\sigma} \upharpoonright k_\sigma[n+1] = \{\varepsilon_\sigma\}^{A^{<\sigma}}_\sigma \upharpoonright k_\sigma[n+1]]; \text{ otherwise } \ell(y, n, \sigma) = -1.$$

Case 2: $\lim_{\tau<\sigma} \ell(y,n,\tau) = \gamma$ for some $\gamma < \sigma$. The value γ was chosen at stage γ when (2) held with γ in place of σ. If

$$B^{<\gamma} \upharpoonright k_\gamma[n+1] = B^{<\sigma} \upharpoonright k_\sigma[n+1] \quad \text{and} \quad (\gamma - A^{<\gamma}) \cap A^{<\sigma} = \varnothing,$$

then $\ell(y,n,\sigma) = \gamma$; otherwise $\ell(y,n,\sigma) = -1$.

Initially there is no location. A location is created at stage σ if B appears to equal $\{\varepsilon\}^A$ on $k[n+1]$. It is lost (i.e., $= -1$) if σ turns out to be A-incorrect or if $B^{<\sigma} \upharpoonright k_\sigma[n+1]$ changes.

The y-th set of *positive requirements at stage* σ is

$$P_y^\sigma = \{\langle y,n,\ell(y,n,\sigma)\rangle \mid k_\sigma(n) \in C_0\}.$$

If $\lim_\sigma \ell(y,n,\sigma)$ exists and $\neq -1$, then it defines a permanent location denoted by $\ell(y,n)$.

The y-th set of *positive requirements* is

$$P_y = \{\langle y,n,\ell(y,n)\rangle \mid k^A(n) \in C_0\}.$$

From now on assume

$$c_1^A > \omega. \tag{3}$$

According to Proposition 4.3 there are arbitrarily large A-correct stages.

The first thing to show is (4).

(4) If P_y is met for all sufficiently large $n < c_1^A$, then $B \neq \{\varepsilon\}^A$ for every ε in block y.

To check (4) fix ε in block y and assume $B = \{\varepsilon\}^A$ for a contradiction. Then $\ell(y,n)$ exists for all $n < c_1^A$. $\lambda n\,|\,\ell(y,n)$ is weakly α-recursive in A as follows. Run the construction until an A-correct stage σ is found that satisfies the matrix of (2). (Note that

$$\{\varepsilon\}^A \upharpoonright k^A[n+1]$$

is determined by an α-finite initial segment of A because $n < c_1^A$). But then $C \leq_\alpha A$ by (1). (Similar to the argument following (3) of Section 3.)

A shade more intricate than (4) is:

(5) If P_y is met for all sufficiently large $n < c_1^A$, then $\ell(y,n)$ does not exist for some n.

The proof of (5) takes into account the details of the approximation of blocking. Suppose (5) fails with the intent of showing $C_0 \leq_{w\alpha} A$. As in the proof of (4) it suffices to check that $(\lambda n\,|\,\ell(y,n)) \leq_{w\alpha} A$ and $B \leq_{w\alpha} A$. Choose σ_0 with the aid of

Lemma 4.4 so that

(6a) $$g_\sigma(y) \geq g(y) \quad \text{for all} \quad \sigma \geq \sigma_0, \text{ and}$$
$$g_\sigma(y) = g(y) \quad \text{for all} \quad A\text{-correct } \sigma \geq \sigma_0.$$

Hence any ε that participates in (2) at an A-correct stage after σ_0 is less than $f^A g(y)$.

Put ε in J if: $\varepsilon < f^A g(y)$; and

(6b) at some A-correct stage σ after σ_0, ε participates in (2) and thereby compels the value of $\ell(y, n, \sigma)$ to be σ;

and $B^{<\sigma} \upharpoonright k[n+1] \neq B^{<\tau} \upharpoonright k[n+1]$ for some $\tau > \sigma$.

If ε belongs to J, then a permanent inequality between B and $\{\varepsilon\}^A$ develops at stage τ of clause (ii), and subsequent to τ, ε cannot participate in (2). J is $\Sigma_1^{\alpha, A}$, bounded below α_A^*, hence α-finite by Lemma 4.1 (iii). Furthermore all activity associated with the definition of J is α-finite. Consequently the associated values of $k[n+1]$ are bounded below α. Moreover the associated values of $n+1$ are bounded below c_1^A.

Let $\bar{J}$ be $f^A g(y) - J$. Eventually only ε's in $\bar{J}$ participate in (2). Thus for all sufficiently large $n < c_1^A$, the value of $\ell(y, n)$ is determined by some ε in $\bar{J}$. Hence $(\lambda n \,|\, \ell(y, n)) \leq_{w\alpha} A$. Also $B \leq_\alpha A$, because for all sufficiently large $n < c_1^A$, there is an ε in $\bar{J}$ such that $\{\varepsilon\}^A \upharpoonright k^A[n+1]$ is defined, and for all such ε, the computation from A equals $B \upharpoonright k^A[n+1]$. So (5) is proved.

The hypothesis of (5) has a further consequence. Let

$$P_y^* = \{\langle y, n, i\rangle \,|\, n < c_1^A \quad \& \quad i < \alpha\} \cap B.$$

Then P_y^* is α-recursive. Let n_0 be the least n that satisfies the conclusion of (5). The total of all activity on row n for all $n < n_0$ is α-finite, since $n_0 < c_1^A$ and $\ell(y, n)$ $(n < n_0)$ can be computed from A and $B \upharpoonright k[n_0]$. If $n \geq n_0$, then the location for row n either (i) eventually disappears or (ii) moves off to $\infty (= \alpha)$. If (i) holds for some $n \geq n_0$, then (i) holds for all $n' \geq n$, and the total of all activity on all rows from n on is α-finite.

Consequently P_y^* is α-recursive if the conclusion of (5) holds. Suppose $n \geq n_0$. To decide if $\langle y, n, i\rangle \in B$, run the construction until some σ is found such that $\sigma > i$, and $\ell(y, n, \sigma)$ is either -1 or greater than i. Then $\ell(y, n, \tau) \neq i$ for all $\tau \geq \sigma$, and so $\langle y, n, i\rangle$ is put in B only if it is put in before stage σ.

A *negative requirement* is an ordinal denoted by $p(\sigma, \varepsilon, x)$. Its purpose is to preserve the value of

(7) $$\{\varepsilon_\sigma\}_\sigma^{B^{<\sigma}}(k_\sigma(x));$$

its value is the supremum$^+$ of all negative facts about $B^{<\sigma}$ used in the computation of (7). It is added to the y-th block of negative requirements at stage σ if certain conditions hold. Once added to block y at stage σ it remains there forever or until

removed at stage $\tau \geq \sigma$. Removal is caused by injuries or changes in A. $p(\sigma, \varepsilon, x)$ is injured if some $w < p(\sigma, \varepsilon, x)$ is added to B at stage $\tau \geq \sigma$. σ is seen to be A-incorrect at stage $\tau > \sigma$ if

$$A^{<\sigma} \neq A^{<\tau} \cap \sigma.$$

Define

$$q(\sigma, \varepsilon) = \mu x\left[C_0^{<\sigma}(k_\sigma(x)) \neq \{\varepsilon_\sigma\}_\sigma^{B^{<\sigma}}(k_\sigma(x))\right],$$

$$r(\sigma, y) = \sup\{q(\sigma, \varepsilon) \mid \varepsilon < f_\sigma g_\sigma(y)\}.$$

Removal of negative requirements at the beginning of stage σ: If ρ is seen to be A-incorrect at stage σ, then remove all negative requirements added at stage ρ. If $p(\rho, \varepsilon, x)$ is removed, then also remove $p(\rho, \varepsilon, x')$ for all $x' > x$.

Addition of negative requirements at the beginning of stage σ: Suppose $\varepsilon < f_\sigma g_\sigma(y)$ and $x < q(\sigma, \varepsilon)$. Add $p(\sigma, \varepsilon, x)$ to block y if

(8) $r(\sigma, y) > \sup^+\{x' \mid (E\rho)_{\rho<\sigma}(E\varepsilon')[p(\rho, \varepsilon', x')$ added to block y at stage ρ and not yet removed$]\}$.

Clause (8) limits the addition of negative requirements to blocks. It is necessitated by blocking, and is similar to the monotonicity clause, (1a), in the proof of Shore splitting, Theorem 1.1.

Note: the above addition step is performed after the preceding removal step.

Construction of B. Add the σ-th member of A to B. Next comes a recursion on $y < c_2^A$ that alternates between removing negative requirements from block y and moving positive requirements from P_y^σ into B.

Fix $y < c_2^A$. Let w_y^σ be the least element of $A \cup \bigcup\{P_{y'}^\sigma \mid y' < y\}$ added to B at stage σ. If w_y^σ is less than some negative requirement in block y, then remove that requirement.

If $p(\rho, \varepsilon, x)$ is removed, then also remove $p(\rho, \varepsilon, x')$ for all $x' > x$. The removal of negative requirements injured by w_σ^y makes it easier to add elements of P_y^σ to B. Define

$$p(\sigma, y) = \text{sup of negative requirements still in block } y.$$

Add to B all members of P_y^σ not less than

$$\sup\{p(\sigma, y) \mid y' \leq y\}.$$

End of recursion on y and stage σ of construction of B.

Behavior of Negative Requirements. Suppose $p(\sigma, \varepsilon, x)$ is put in block y at the beginning of stage σ and is never removed. Thus σ is A-correct and all computations based on $A^{<\sigma}$, and performed at stage σ, are correct. In particular $k_\sigma \upharpoonright x = k \upharpoonright x$ and $\varepsilon_\sigma = (f^A)^{-1}(\varepsilon)$. Also $p(\sigma, \varepsilon, x')$ has not been removed for any $x' < x$. So

$$\{\varepsilon_\sigma\}^{B^{<\sigma}} \upharpoonright k_\sigma[x] = \{\varepsilon\}^B \upharpoonright k[x].$$

It may happen that

$$C_0^{<\sigma}\upharpoonright k[x] \neq C_0^{<\tau}\upharpoonright k[x]$$

for some $\tau > \sigma$. In that event a permanent inequality between C_0 and $\{\varepsilon\}^B$ arises at stage τ; and so at all subsequent stages, no new negative requirement associated with ε is added to any block. In the absence of a permanent inequality the situation is more complicated.

For $y < c_2^A$ define the y-th injury set to be

$$I_y = A \cup \bigcup\{P_{y'}^* \mid y' < y\}.$$

for any t: $\alpha \to \alpha$, define

$$\liminf_{\sigma} t(\sigma) \text{ to be } \mu\beta(\sigma)\,(E\tau)_{\tau>\sigma}\,[t(\tau) \leq \beta].$$

An induction on y shows:

(9a) $\liminf_{\sigma} p(\sigma, y) < c_1^A$;

(9b) $I_y \leq_\alpha A$ and I_y is regular.

The induction on y is organized as follows: (9b) is true when $y = 0$ because $I_0 = A$; (9b) implies (9a); if (9a), with y' in place of y, holds for all $y' < y$, then (9b) holds.

Assume (9b) to prove (9a). As in the proof of Proposition 4.3, the regularity of I_y and assumption (3) imply the existence of arbitrarily large I_y-correct σ, that is,

$$I_y \cap \sigma = I_y^{<\sigma} = \{z \mid z \in I_y \quad \& \quad z \text{ put in } B \text{ before stage } \sigma\}.$$

I_y-correctness entails A-correctness since $A \subseteq I_y$.

Let σ_0 be as in (6a). Then on all I_y-correct stages σ beyond $\sigma_0, f_\sigma g_\sigma(y) = f^A g(y)$. More precisely, the y-th block, $[0, f_\sigma g_\sigma(y))$, is constant on all sufficiently large A-correct stages. Beyond σ_0, if a negative requirement is added to block y at an I_y-correct stage, then it is never removed; if it is added before some I_y-correct stage τ, and is not removed before stage τ, then it is never removed.

Put ε in K if:

$\varepsilon < f^A g(y)$;
at some I_y-correct $\sigma > \sigma_0$, clause (8) holds;
and for some $x < q(\sigma, \varepsilon)$ & $\tau > \sigma$, $C_0^{<\sigma}(k(x)) \neq C_0^{<\tau}(k(x))$.

If ε is in K, then some permanent inequality develops between C_0 and $\{\varepsilon\}^B$ and $q(\sigma, \varepsilon)$ is permanently bounded. K is Σ_1^A because $I_y \leq_\alpha A$ by (9b). K, and all activity associated with the development of K, are α-finite, because K is bounded below α_A^*.

Let $\bar{K}$ be $K - f^A g(y)$. Eventually only ε's in $\bar{K}$ participate in the addition of negative requirements to block y at A-correct stages. Any such addition is per-

manent if it is made at an I_y-correct stage or if it is not removed prior to the next I_y-correct stage. Thus the set of negative requirements in block y, viewed only at all sufficiently large I_y-correct σ, is a non-decreasing function of σ. If (9a) is false, then $C_0 \leq_{w\alpha} A$ (hence $C \leq_\alpha A$) as follows. To compute $C_0(k^A(n))$ from A, look for ε in $\bar{K}$ and $\sigma > \sigma_0$ such that

$$n < q(\sigma, \varepsilon) \quad \& \quad \sigma \text{ is } I_y\text{-correct.} \tag{10}$$

Then $C_0(k^A(n)) = \{\varepsilon\}_\sigma^{B^{<\sigma}}(k_\sigma^A(n))$, since $\varepsilon \notin K$. The existence of ε and σ satisfying (10) follows from the monotonicity clause, (8). To verify the last claim, focus on the sufficiently large I_y-correct stages. On those stages $p(\sigma, y)$ is nondecreasing. The falsity of (9a) implies $p(\sigma, y)$ increases unboundedly often. All such increases are associated with ε's in $\bar{K}$. Clause (8) implies that $r(\sigma, y)$ is non-decreasing on all sufficiently large I_y-correct stages and increases unboundedly often on such stages. Each increase in r is the result of an increase in some $q(\sigma, \varepsilon)$ for some ε in $\bar{K}$, an increase beyond the previous value of r.

Thus (9b) implies (9a). The next task is to draw a further consequence of (9b), namely

$$\text{all sufficiently large members of } P_y \text{ are put in } B. \tag{11}$$

Consider the behavior of $p(\sigma, y)$ on all sufficiently large I_y-correct stages. As described immediately above, $p(\sigma, y)$ is nondecreasing and bounded. Thus

$$\lim_\sigma \inf p(\sigma, y) = \lim_\sigma \{p(\sigma, y)) \mid \sigma \ I_y\text{-correct}\} < \alpha. \tag{12}$$

The ineluctable barrier to adding elements of P_y to B is

$$\sup_{y' \leq y} \lim_\sigma \inf p(\sigma, y'). \tag{13}$$

Suppose $y' < y$. Then every $I_{y'}$-correct stage is also I_y-correct, since $I_{y'} \subseteq I_y$. Also (9b) implies $I_{y'} \leq_\alpha A$ and $I_{y'}$ is regular. Hence the derivation of (9a) from (9b) also shows

$$\lim_\sigma \inf p(\sigma, y') = \lim_\sigma \{p(\sigma, y') \mid \sigma \, I_y\text{-correct}\} < \alpha \tag{14}$$

for all $y' < y$. It follows from (14) that

$$\lim_\sigma \inf p(\sigma, y') \ (y' \leq y)$$

is $\Sigma_2^{\alpha, A}$, because $I_y \leq_\alpha A$. $y < c_2^A$, so (13) $< \alpha$. Thus any member of P_y larger than (13) can be added to B at any sufficiently large I_y-correct stage.

The last part of the induction on y is devoted to proving (9b) under the assumption that (9a), with y' in place of y, holds for all $y' < y$. By induction (9b), hence (11), holds with y' in place of y for all $y' < y$. It follows from (5) that

(15) $\qquad \ell(y', n)$ does not exist for some $n \quad (y' < y)$.

Let $n_0(y')$ be the least n that satisfies (15). Recall the proof of the α-recursiveness of $P^*_{y'}$ that immediately follows the proof of (5). A Gödel number for the α-recursive set $P^*_{y'}$ can be obtained effectively from the values of $n_0(y')$ and $\ell(y', n)$ $(n < n_0(y'))$. Hence $I_y \leq_\alpha A$ if the functions

(16)
$$\begin{array}{ll} n_0(y') & (y' < y) \\ \ell(y', n) & (y' < y \ \& \ n < n_0(y')) \end{array}$$

are α-finite. It suffices to show these functions are $\Sigma_2^{\alpha, A}$ because $y < c_2^A$. (If h is any $\Sigma_2^{\alpha, A}$ function on c_2^A, then h is tame $\Sigma_2^{\alpha, A}$, and so $h \restriction y$ is $\Sigma_1^{\alpha, A}$, hence α-finite by the regularity of A.) The definition of $\ell(y', n)$ is $\Sigma_2^{\alpha, A}$, because it says some computations from A exist for an initial segment of arguments shorter than c_1^A and the results agree with B (cf. "$\ell(x) < \alpha$" in Section 4.2). The definition of $n_0(y')$ has a clause concerning the non-existence of a computation from A, a $\Pi_1^{\alpha, A}$ statement.

So ends the proof of (9a) and (9b) by induction on y. All that remains is the proof of $B \leq_\alpha C$ and the disposal of assumption (3). The recovery of B from C is controlled by the permanent negative requirements. A simultaneous recursion on y defines α-recursive functions $\varepsilon_1(y)$ and $\varepsilon_2(y)$ such that for all $y < c_2^A$:

(17a) the set of permanent negative requirements in block y is α-recursively enumerable in C via Gödel number $\varepsilon_1(y)$;

(17b) $P^*_y \leq_{w\alpha} C$ via Gödel number $\varepsilon_2(y)$.

Fix $y < c_2^A$ and assume $\varepsilon_1(y')$ is already defined for all $y' \leq y$. Consider $\langle y, n, i \rangle$ in order to see how $\{\varepsilon_2(y)\}$ works. Suppose $\ell(y, n)$ does not exist. Then running the construction will produce a stage $\sigma > i$ at which $\ell(y, n, \sigma) > i$ or $= -1$. By then the question, $\langle y, n, i \rangle \in B$?, will have been resolved. Suppose $\ell(y, n)$ does exist. If $\ell(y, n) > i$, then running the construction will resolve the matter. If $\ell(y, n) < i$, then an appeal to A establishes that the computations from A in (2) are correct, and that consequently the location will never move out to i. If $\ell(y, n) = i$, then an appeal to C_0 is needed. If $k^A(n) \in C_0$, then $\langle y, n, i \rangle$ is kept out of B by a permanent negative requirement in block y' (for some $y' \leq y$) enumerated from C via Gödel number $\varepsilon_1(y')$.

Now fix y and assume $\varepsilon_2(y')$ is defined for all $y' < y$. To see how $\varepsilon_1(y)$ works, run the construction. A negative requirement put in block y at stage σ is permanent if σ is I_y-correct. (More precisely, a negative requirement put in block y at stage σ is permanent if none of the computations underlying that requirement use a negative membership fact contradicted by a positive fact about I_y.) A full account of the

I_y-correctness of σ can be ascertained from C via $\varepsilon_2(y')$ $(y' < y)$. This last claim is delicate and has to be supported by careful examination of the description of ε_2 given in the previous paragraph. Note that appeals to C_0 and A are made by $\{\varepsilon_2(y')\}$ only if $n < n_0(y')$. By (16) and remarks subsequent, $n_0(y')$ $(y' < y)$ is bounded below c_1^A. Consequently $k^A(n)$ $(n < n_0(y') \ \& \ y' < y)$ is bounded below α. Therefore the set of appeals to C_0 made by $\varepsilon_2(y')(y' < y)$ is bounded below α, hence is α-finite by the regularity of C_0. The appeals to A concern computations from A on an initial segment of arguments bounded by $\sup\{n_0(y') \mid y' < y\}$, which, as just noted, is less than c_1^A.

Thus the I_y-correctness of σ can be ascertained from C via $\varepsilon_2(y')$ $(y' < y)$, since the procedures $\{\varepsilon_2(y')\}$ $(y' < y)$ will draw only on α-finitely much of C and A as they compute P_y^* from C. That ends the definitions of ε_1 and ε_2. Something a bit stronger than (17a) and (17b) has been proved.

(18) The set of all permanent negative requirements is α-recursively enumerable in C. For each $y < c_2^A$, the enumeration of requirements in the blocks below block y draws only on a bounded part of C determined by y.

(18) is what is needed to see $B \leq_\alpha C$. Suppose H is an α-finite subset of $\alpha - B$. "$H \subseteq cB$" is established by an α-finite set of facts about C as follows. First a $y_0 < c_2^A$ has to be found so that

$$\sup_{y' < y_0} \lim_{\sigma} \inf p(\sigma, y') > \sup H.$$

By (18), y_0 can be established by α-finitely much of C. Suppose $\langle y, n, i\rangle \in H$. The question, $\langle y, n, i\rangle \in B$?, is dealt with as it was in the definition of ε_2. If $y \geq y_0$, then some permanent negative requirements involved in the definition of y_0 will serve, if needed, as the reason that $\langle y, n, i\rangle \notin B$. If $y < y_0$, then (18) implies a bound on the amount of C needed to establish a permanent negative requirement that keeps $\langle y, n, i\rangle$ out of B.

Assumption (3) is disposed of in the next subsection.

End of proof of density.

5.2 Assumption (3). The above account of Theorem 5.1 relied heavily on assumption (3), namely $c_1^A > \omega$. Now suppose $c_1^A = \omega$. Then

$$c_2^A = \alpha_A^* = c_1^A = \omega,$$

since $\alpha_A^* \leq c_1^A$ by Lemma 4.1. Many difficulties disappear. There is no blocking. The only complication left is the use of k^A, the $\Sigma_1^{\alpha, A}$ map from c_1^A onto α. The density construction given above is greatly simplified. It still works and is similar to one given by Driscoll (1968) for $\alpha = \omega_1^{CK}$. Proposition 4.3 is lost. There may not be any A-correct stages. Of course there is less need for them with blocking gone. Instead of A-correct stages, non-deficiency stages are used as in the ω-case. In short the

density argument for $c_1^A = \omega$ is close to that of classical recursion theory (cf. Exercise 5.5).

5.3 $\Sigma_2^{\alpha,A}$ versus Σ_3^α. A look backward reveals that the functions developed in the proof of density are at worst $\Sigma_2^{\alpha,A}$, hence Σ_3^α. The knowledge they are Σ_3^α would have been of little use with only Σ_1 admissibility available. But since they are $\Sigma_2^{\alpha,A}$, and since $\langle L(\alpha), A\rangle$ is weakly Σ_1 admissible, it was possible to apply the methods of Chapters VIII and IX for making Σ_1 admissibility do the work of Σ_2 admissibility. The idea of blocking was important. Blocking makes possible the full utilization of the limited Σ_2 admissibility properties possessed by every Σ_1 admissible $L(\alpha)$.

It is not clear how far Σ_1 admissibility can be stretched. α-recursively enumerable sets A and B are said to form a minimal pair if every set α-recursive i n both A and B is α-recursive. The existence of a minimal pair when $\alpha = \omega$ is a well known result of classical recursion theory (Lachlan 1966, Yates 1966). For $\alpha > \omega$ some positive partial results have been obtained by Lerman & Sacks 1972, Maass 1977a and Shore 1975. The problem remains open for most α because the classical proof, and its α-variations, appear to need strong forms of Σ_2 replacement that do not seem manageable by Σ_2 blocking.

It is quite possible that some of the constructions of the classical theory of recursively enumerable sets make essential use of Σ_2 replacement. Evidence for this view is provided by a result of Shore 1976. He showed that Lachlan's non-splitting theorem 1975 fails when $\alpha = \omega_\omega$.

5.5–5.6 Exercises

5.5. Prove Shore's density theorem when $\sigma 1\mathrm{cf}_A(\alpha) = \omega$.

5.6. Assume A is α-recursively enumerable and incomplete. Prove

$$\sigma 2\mathrm{cf}_A^\alpha(\alpha) = \sigma 2\mathrm{cf}_A^\alpha(\alpha_A^*) = \sigma 2\mathrm{cf}_A^\alpha(\sigma 1\mathrm{cf}_A(\alpha)).$$

6. *β-Recursion Theory*

β-recursion theory was introduced by S. Friedman and Sacks 1977. Its purpose is to see how far recursion theory can be developed without Σ_1 admissibility. Some of the technical problems that arise are similar to those discussed above when α is Σ_1 admissible, but not Σ_2 admissible, and a Σ_2 construction is attempted. The proper setting for β-recursion theory is Jensen's J hierarchy, a reformulation of Gödel's L hierarchy. For the brief sketch given here L will suffice. From now on let β be a limit ordinal. Thus $L(\beta)$ need not be Σ_1 admissible, but it will be closed under the operations of pairing and union, and it will satisfy Δ_0-separation.

The fundamental definitions of β-recursion are in essence the same as those of α-recursion. Let $A \subseteq L(\beta)$. A is β-recursively enumerable if A is Σ_1^β. A is β-recursive

if A is Δ_1^β. The β-finite sets are simply the sets in $L(\beta)$. $\leq_{w\beta}$ (weakly β-recursive in) and $\leq_\beta$ (β-recursive in) are defined in precisely the same manner as their counterparts in α-recursion theory. The failure of Σ_1 admissibility makes possible new distinctions among the β-recursive sets. For example it can happen that A is β-recursive, but not β-recursive in $\varnothing$, the empty set.

For each $\gamma \leq \beta$, the Σ_n^β cofinality of γ, and the Σ_n^β projection of γ, are defined as in α-recursion theory. β^*, the Σ_1^β projection of β, is of special interest. The loss of Σ_1 admissibility shifts the burden of many proofs from the dynamic approach to that of fine structure. An example is the proof of: if A is Σ_1^β and $A \subseteq \delta < \beta^*$, then A is β-finite (cf. Exercise VII.2.10). Let $\hat{\beta}$ be the least γ such that there exists a one-one, β-recursive map of γ onto β. $\hat{\beta} < \beta$ iff β is not Σ_1 admissible. S. Friedman observed that $\hat{\beta} = \max(\beta^*, \sigma 1\mathrm{cf}^\beta(\beta))$. If β is not Σ_1 admissible, then there is a greatest β-cardinal; also $\beta^* < \beta$.

An extremely useful distinction made by Maass is: call β weakly admissible if $\sigma 1\mathrm{cf}^\beta(\beta) \geq \beta^*$; otherwise call β strongly inadmissible. It turns out that some, but not all, of the ideas and results of α-recursion theory carry over to β when β is weakly admissible. The truth of this was evident in the proof of Shore's density theorem, which exploited the weak admissibility of $\langle L(\alpha), A\rangle$, a consequence of the α-recursive enumerability and incompleteness of A. If β is weakly admissible, then there is a one-one, β-recursive correspondence between $\sigma 1\mathrm{cf}^\beta(\beta)$ and β, and β^* behaves in a familiar manner. It is then not surprising that the solution to Post's problem comes over from α-recursion theory. There exist β-recursively enumerable sets B and C such that $B \not\leq_{w\beta} C$ and $C \not\leq_{w\beta} B$. On the other hand the regular sets theorem can fail. Maass 1977b gives a complete description of those β-recursively enumerable sets that have the same degree as some regular β-recursively enumerable set when β is weakly admissible.

A stronger assumption than weak admissibility is: $\langle L(\beta), A\rangle$ is weakly admissible for every regular β-recursive A. Another way to put it is: $\sigma 1\mathrm{cf}^\beta(\beta) \geq \beta^*$ and $\sigma 2\mathrm{cf}^\beta(\beta) \geq \sigma 2p^\beta(\beta)$, the Σ_2^β projectum of β. If β satisfies the stronger assumption, then the regular sets theorem holds (Maass 1977b) and the density theorem (for β-recursively enumerable sets) holds (Homer & Sacks 1983). It is not known if density holds for every weakly admissible β. Some complex partial results have been obtained by Bailey 1984.

The central problem of the subject is Post's. A strong solution to Post's problem consists of two β-recursively enumerable sets such that neither is weakly β-recursive in the other. S. Friedman 1979 has shown: if β^* is regular (in the sense that cardinals are regular) with respect to all Σ_1^β functions, then β has a strong solution to Post's problem. His argument is based on an effective version of Jensen's diamond principle. He has also found a β such that β does not have a strong solution to Post's problem. It is still possible that every β has a weak solution, *a pair of β-recursively enumerable sets such that neither is β-recursive in* the other.

Maass 1977 points out that the methods of β-recursion theory are applicable to α-recursion theory. An excellent example is his proof that: α is Σ_2 admissible iff every Σ_2^α set, in which ϕ' is α-recursive, is of the same α-degree as the α-jump of some incomplete α-recursively enumerable set.

6.1–6.2 Exercises

6.1. Suppose $\lambda \leq \beta^*$ and λ is a successor β-cardinal. Show λ is regular with respect to all Σ_1^β functions.

6.2. Suppose $\sigma 1\text{cf}^\beta(\beta) \leq \beta^*$. Show there exists a one-one, α-recursive map of β^* onto β.

Part D

E-Recursion

E-recursion theory extends the notion of computation from hereditarily finite sets to sets of arbitrary rank. Selection theorems, forcing arguments and priority constructions are developed in the setting of *E*-recursive enumerability.

Chapter X
E-Closed Structures

The class of partial E-recursive functions is defined. A connection is made between E-closure and Gödel's L. Gandy selection is proved. Moschovachis witnesses are introduced and used to characterize divergence of computations in terms of reflection phenomena.

1. Partial E-Recursive Functions

The intent of E-recursion theory is to assign a meaning to $\{e\}(x)$ for every set x via an appropriate notion of computation. The Σ_1 admissibility approach of Part C is deemed inappropriate since it equates "computation" with "existential witness". Let $f: V \to V$ be a partial Σ_1 function; that is the graph of f is definable over V, the class of all sets, by some formula $(\mathrm{Ew})D(x, y, w)$, where D is a Δ_0 formula of ZF. It makes sense to speak of computing $f(x)$ by searching for a pair $\langle y, w\rangle$ that satisfies $D(x, y, w)$ in V, but it does not seem sensible to construe a wide open search through V as a computation procedure. Even if V is replaced by L, thereby allowing the search to proceed methodically through the ordinals, it is still difficult to view the search as the execution of a rule of computation.

Computation rules permit a computer to move in a direct way from argument to value by following an initial instruction and then other instructions developed along the way. Following the rules generates a tree-like object called a computation. An element of self-reference is to be expected since rules may talk about rules.

Kleene [1959] was the first to give schemes that assign a meaning to $\{e\}(x)$ when x is an object of finite type. (Integers are of type 0, and an object of type n is an arbitrary collection of objects of type less than n.) E-recursion generalizes the so-called normal version of his theory; "normal" indicates that equality is a recursive predicate. The Normann schemes for E-recursion [1978] extend the normal Kleene schemes to objects of arbitrary type, namely sets. They were devised by Normann, and subsequently by Moschovakis.

1.1 The Normann Schemes. Assume some standard method of encoding finite sequences. Let $\langle a_1, \ldots, a_n\rangle$ be the code for $a_1, \ldots, a_n$. Each Normann scheme

is a closure condition in the positive inductive definition of E, the class of all tuples $\langle e, \langle x_1, \ldots, x_n\rangle, y\rangle$ such that $\{e\}\ (x_1, \ldots, x_n)$ is defined and equal to y.

(1) $\{e\}\ (x_1, \ldots, x_n) = x_i$ if $e = \langle 1, n, i\rangle$ (projection).

(2) $\{e\}\ (x_1, \ldots, x_n) = x_i - x_j$ if $e = \langle 2, n, i, j\rangle$ (difference).

(3) $\{e\}\ (x_1, \ldots, x_n) = \{x_i, x_j\}$ if $e = \langle 3, n, i, j\rangle$ (pairing).

(4) $\{e\}\ (x_1, \ldots, x_n) \simeq \cup\{\{c\}\ (y, x_2, \ldots, x_n) \mid y \in x_1\}$ if $e = \langle 4, n, c\rangle$ (bounding with union).

The left side of (4) is not defined unless $\{c\}\ (y, x_2, \ldots, x_n)$ is defined for all $y \in x_1$.

(5) $\{e\}\ (x_1, \ldots, x_n) \simeq \{c\}\ (\{d_1\}\ (x_1, \ldots, x_n), \ldots, \{d_m\}\ (x_1, \ldots, x_n))$ if $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$ (composition).

(6) $\{e\}\ (c, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{c\}\ (x_1, \ldots, x_n)$ if $e = \langle 6, n, m\rangle$ (enumeration).

$\simeq$ is Kleene's symbol for strong equality. If f and g are partial functions, then $f(x) \simeq g(x)$ iff neither $f(x)$ nor $g(x)$ is defined, or $f(x)$ and $g(x)$ are defined and equal.

Let A be a class of sets. A is closed with respect to scheme (1) if $\langle e, \langle x_1, \ldots, x_n\rangle, x_i\rangle \in A$ whenever $x_1, \ldots, x_n \in V$ and $e = \langle 1, n, i\rangle$. The closure of A with respect to schemes (2) and (3) is defined similarly. A is closed with respect to scheme (4) if $\langle e, \langle x_1, \ldots, x_n\rangle, z\rangle \in A$ whenever

$$e = \langle 4, n, c\rangle \,\&\, x_1, \ldots, x_n \in V,$$
$$(y)_{y \in x_1}\ (\mathrm{Ew})\ [\langle c, \langle y, x_2, \ldots, x_n\rangle, w\rangle \in A], \text{ and}$$
$$z = \{w \mid (\mathrm{Ey})_{y \in x_1}\ [\langle c, \langle y, x_2, \ldots, x_n\rangle, w\rangle \in A]\}.$$

The closure of A with respect to schemes (5) and (6) is defined similarly.

E is defined to be the least class closed with respect to schemes (1)–(6). E is viewed most simply as the intersection of all classes closed under (1)–(6). This view makes good sense, since each scheme is equivalent to a positive closure condition. Something is put in E if something else was put in earlier. The "intersection" proof of the existence of E can be formalized in ZF by thinking of V as the union of many Σ_1 admissible sets and doing the "intersection" proof within each Σ_1 admissible set. Another approach to the existence of E is that of natural enumeration. First some definitions.

$$\{e\}\ (x_1, \ldots, x_n) \text{ is defined and equal to } y$$

if $\langle e, \langle x_1, \ldots, x_n\rangle, y\rangle \in E$. A partial function f from V into V is partial E-recursive if there exists an e such that $f(x) \simeq \{e\}\ (x)$ for all $x \in V$. $\{e\}\ (x)$ *converges* (in symbols $\{e\}\ (x)\downarrow$) if $\{e\}\ (x)$ is defined. Otherwise $\{e\}\ (x)$ diverges (in symbols $\{e\}\ (x)\uparrow$).

1.2 Natural Enumeration of E. σ is an arbitrary ordinal number. Proceed at stage σ as follows.

$\sigma = 0$: $\langle e, \langle x_1, \ldots, x_n \rangle, x_i \rangle$ is put in E if $e = \langle 1, n, i \rangle$. Schemes (2) and (3) are treated similarly.

$\sigma > 0$: $\langle e, \langle x_1, \ldots, x_n \rangle, z \rangle$ is put in E if

$e = \langle 4, n, c \rangle$,

$(y)_{y \in x_1}$ (Ew)[$\langle c, \langle y, x_2, \ldots, x_n \rangle w \rangle$ put in E prior to stage σ],

and $z = \{w \mid (\mathrm{Ey})_{y \in x_1}$ [$\langle c, \langle y, x_2, \ldots, x_n \rangle, w \rangle$ is already in E]$\}$.

Schemes (5) and (6) are treated similarly.

Note that scheme (4) is the only scheme that pushes the natural enumeration into infinite stages.

The natural enumeration makes it easy to check certain points about E. For example, if $\{e\}(x)$ is defined, then it has only one value (cf. Exercise 1.5). Thus for all e, $\lambda x \mid \{e\}(x)$ is a partial *function*. The natural enumeration is defined by a Σ_1 recursion, and so E is a Σ_1 class (cf. VII.1.6). To be precise there is a Σ_1 formula $\mathscr{F}(x)$ of ZF such that for all $a \in V$,

$$a \in E \leftrightarrow V \vDash \mathscr{F}(\underline{a}).$$

It follows that the graph of each partial E-recursive function is Σ_1 definable over V. The converse is false. Consider $\mathcal{O}(x)$, Gödel's order of constructibility function

$$\mathcal{O}(x) \simeq \mu\gamma[x \in L(\gamma + 1) - L(\gamma)].$$

$\mathcal{O}(x)$ is partial Σ_1 but not partial E-recursive. Roughly speaking, if $x \in L$, then $\mathcal{O}(x)$ can be found by an unbounded search but not by any computation procedure that moves in a direct manner from x to $\mathcal{O}(x)$. Some forcing arguments will be applied in Chapter XI to show the non-E-recursiveness of certain predicates. On the other hand many Σ_1 functions are E-recursive, and the reason for this is best expressed by van de Wiele's theorem in Chapter XIII.

Let $f\colon \omega \to \omega$ be a partial function. Then f is partial recursive in the sense of classical recursive theory iff f is partial E-recursive iff f is Σ_1 over HF, the hereditarily finite sets (cf. Exercise 1.8). Agreements between E-recursion and Σ_1 admissibility will figure prominently up ahead. They are often consequences of selection theorems that make it possible to E-recursively compute existential witnesses.

Let $P(x)$ be a predicate and $f(x)$ its representing function:

$$P(x) \to f(x) = 0; \sim P(x) \to f(x) = 1.$$

$P(x)$ is said to be E-recursive if f is E-recursive.

1.3 Proposition. *If $P(x)$ is Δ_0^{ZF}, then $P(x)$ is E-recursive.*

Proof (a sketch). By induction on the logical complexity of $P(x)$. Observe that

$$x \in y \leftrightarrow \{x\} - y = \varnothing.$$

Suppose $g(x, y)$ is the representing function of $Q(x, y)$. Then

$$(x)_{x \in z}\, Q(x, y) \leftrightarrow (x)_{x \in z}\, [g(x, y) = 0]$$
$$\leftrightarrow \cup \{g(x, y) \mid x \in z\} = \varnothing. \quad \square$$

A predicate $P(x)$ is said to be *E-recursively enumerable* if there exists a partial E-recursive function f such that for all x,

$$P(x) \leftrightarrow f(x)\downarrow.$$

A is E-recursively enumerable if $x \in A$ is E-recursively enumerable.

Warning: The graph of a function can be E-recursively enumerable, or even E-recursive, without the function being partial E-recursive. One example is Gödel's order of constructibility function $\mathcal{O}(x)$. The graph of $\mathcal{O}$ is E-recursive, but $\lambda x \mid \mathcal{O}(x)$ is not partial E-recursive. Another example is the predicate, "x is an ordinal but not a cardinal". It is Σ_1 but not E-recursively enumerable.

1.4 *E*-Closed Structures. Let A be a transitive set (that is, $x \in A \;\&\; y \in x \rightarrow y \in A$). A is said to be E-closed if

$$\vec{x} \in A \;\&\; f(\vec{x})\downarrow \rightarrow f(\vec{x}) \in A \quad (\vec{x} \text{ is } x_1, \ldots, x_n)$$

for every partial E-recursive f. It will be seen that every Σ_1 admissible set is E-closed, but not conversely. Part D is devoted largely to the study of forcing and priority arguments over E-closed structures. What does it mean to force a computation to converge?, to diverge? How is Post's problem formulated and solved in E-recursion? The answers bear little resemblance to their counterparts in α-recursion theory.

The E-closure of x, in symbols $E(x)$, is the least E-closed $y \supseteq TC(\{x\})$. Recall that $TC(z)$ is the least transitive $y \supseteq z$. $E(x)$ is transitive by an induction that parallels the natural enumeration of E. Suppose $z \in E(x)$ to see $z \subseteq E(x)$. Assume z is $\cup \{\{c\}(y) \mid y \in x\}$. Let e be $\langle 4, 1, c\rangle$. Suppose $\langle e, x, z\rangle$ is put in E at stage σ. But then

$$\langle c, \langle y\rangle, \{c\}(y)\rangle$$

is put in E prior to stage σ for all $y \in x$. By induction $\{c\}(y) \subseteq E(x)$ for all $y \in x$, and so $z \subseteq x$. Note that the natural enumeration of E induces a natural enumeration of $E(x)$.

$E(\varnothing)$ is HF. $E(\omega)$ is $L(\omega_1^{CK})$. Let $A_1(x)$ be the least Σ_1 admissible $y \supseteq TC(\{x\})$. Then $E(x) \subseteq A_1(x)$ (cf. Exercise 1.7). It will be shown that $E(2^\omega)$ is not Σ_1 admissible. Thus $E(2^\omega)$ is a proper subset of $A_1(2^\omega)$.

1.5–1.8 Exercises

1.5. Show: if $\{e\}(x)\downarrow$, then the value of $\{e\}(x)$ is unique.

1.6. Complete the proof of Proposition 1.3.

1.7. For any set x, let α^x be the least γ such that $L(\gamma, TC(\{x\}))$ is Σ_1 admissible. Show $L(\alpha^x, TC(\{x\}))$ is the least Σ_1 admissible $y \supseteq TC(\{x\})$, namely $A_1(x)$. Show $E(x) \subseteq A_1(x)$. ($L(\gamma, x)$ is defined just before 2.8.)

1.8. Let $f\colon \omega \to \omega$ be partial. Show f is partial recursive iff f is partial E-recursive.

2. Computations

A computation instruction is any $(n+1)$-tuple of the form $\langle e, x_1, \ldots, x_n\rangle$, or more simply $\langle e, x\rangle$. $\langle e, x\rangle$ is to be thought of as instruction to compute $\{e\}(x)$. Associated with computation instruction $\langle e, x\rangle$ is a tree $T_{\langle e, x\rangle}$. Every node of the tree is a computation instruction. The tree is to be visualized as starting at the top of the page and then branching downward. All the Normann schemes, save (4), give rise to finite branching. The process of forming $T_{\langle e, x\rangle}$ can be regarded as a reversal of the natural enumeration of E (cf. subsection 1.2). If $\{e\}(x)\downarrow$ and $\langle e, x, \{e\}(x)\rangle$ is put in E at stage σ, then $T_{\langle e, x\rangle}$ traces $\{e\}(x)$ back to its roots below stage σ. If $\{e\}(x)\uparrow$, then $T_{\langle e, x\rangle}$ witnesses the fact that $\langle e, x, w\rangle \notin E$ for any w.

2.1 The Universal Computation Tree. The nodes of U, the universal computation tree, are computation instructions. b is a subcomputation instruction of a, in symbols $a >_U b$, if there exists a finite sequence $b_0, \ldots, b_n$ such that $a = b_0$, $b_n = b$, and b_{i+1} is an *immediate* subcomputation instruction of b_i ($i < n$). The definition of immediate subcomputation instruction (immed. subcomp. instruc.) has six clauses corresponding to the Normann schemes and a seventh clause technical in nature.

(i) If $e = \langle 1, n, i\rangle$ $(1 \leq i \leq n)$, then $\langle e, x_1, \ldots, x_n\rangle$ has no immed. subcomp. instrucs. In other words $\langle e, x_1, \ldots, x_n\rangle$ is a terminal, or minimal, node of U.

(ii) Similar to (i) with $e = \langle 2, n, i, j\rangle$.

(iii) Similar to (ii) with $e = \langle 3, n, i, j\rangle$.

(iv) If $e = \langle 4, n, c\rangle$, then $\langle c, y, x_2, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$ for every $y \in x_1$.

(v) (a) If $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$, then $\langle d_i, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$ for $1 \leq i \leq m$.

(v) (b) If $e = \langle 5, n, m, c, d_1, \ldots, d_m\rangle$ and $\{d_i\}$ $(x_1, \ldots, x_n)$ converges and equals y_i $(1 \leq i \leq m)$, then $\langle c, y_1, \ldots, y_m\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$.

(vi) If $e = \langle 6, n, m\rangle$, then $\langle c, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, c, x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$.

(vii) If e is not an index of a scheme, or n is not the correct number of arguments, then $\langle e, x_1, \ldots, x_n\rangle$ is an immed. subcomp. instruc. of $\langle e, x_1, \ldots, x_n\rangle$.

Clause (vii) is needed in the proof of Proposition 2.2.

Clause (v)(b) is worthy of a moment's attention. It differs from the other clauses by requiring convergence information for the establishment of subcomputation instructions. If (v)(b) is overlooked, then it is easy to think (mistakenly) that the immediate subcomputation relation is E-recursive. In fact it is not E-recursive, but is E-recursively enumerable thanks to scheme (6), the enumeration scheme. Another trap is to think $>_U$ is E-recursively enumerable. Not so, it is merely Σ_1.

The tree-like object consisting of $\langle e, x\rangle$ and that part of $>_U$ below $\langle e, x\rangle$ is called the *computation* of $\{e\}(x)$, and is denoted by $T_{\langle e, x\rangle}$.

2.2 Proposition. *$\{e\}(x)\downarrow \leftrightarrow T_{\langle e, x\rangle}$ is wellfounded.*

Proof. First suppose $T_{\langle e, x\rangle}$ is wellfounded. By clause (vii) of the definition of immediate subcomputation, e is the index of a Normann scheme. A straightforward transfinite induction on $>_U$ restricted to $T_{\langle e, x\rangle}$ shows if $\langle c, z\rangle$ is a node on $T_{\langle e, x\rangle}$, then $\{c\}(z)\downarrow$.

Now suppose $\{e\}(x)\downarrow$. Then $\langle e, x, \{e\}(x)\rangle$ was put in E at some stage σ, as described in subsection 1.2. If $\langle c, u\rangle$ is an immed. subcomp. instruc. of $\langle e, x\rangle$, then $\langle c, u, \{c\}(u)\rangle$ was put in E prior to stage σ. By induction on σ, $T_{\langle c, u\rangle}$ is well-founded. But then $T_{\langle e, x\rangle}$ is well-founded. □

If $\{e\}(x)\downarrow$, then travelling down the tree $T_{\langle e, x\rangle}$ retraces the steps by which $\langle e, x, \{e\}(x)\rangle$ was put in E.

2.3 Effective Transfinite Recursion (ETR). ETR is an essential tool in the study of E-recursion. It originated in the work of Kleene and Church on notations for ordinals, and was clarified by Rogers. In this book it is presented as Theorem 3.2.I. It is a consequence of Theorem 3.1.I, Kleene's fixed point theorem. The proof of 3.1.I remains valid when $\{d\}$ is interpreted as the $\{d\}$-th partial E-recursive function. Consequently 3.2.1 is valid under a similar interpretation. The next proposition, 2.4, is a useful recursion principle, more comprehensible than ETR, but weaker.

2.4 Proposition. *Suppose H is a total E-recursive function. Then the unique f such that*

$$f(\gamma) = H(f \restriction \gamma) \text{ for every ordinal } \gamma$$

is E-recursive.

Proof. Let $I: \omega \to \omega$ be a recursive function such that for all $e < \omega$ and all γ,

$$\{I(e)\}(\gamma) \simeq H(\{e\} \restriction \gamma).$$

By the fixed point theorem (for partial E-recursive functions) there is a c such that $\{I(c)\} \simeq \{c\}$. Then f is $\{c\}$.

2.5 Lemma. *There exists a partial E-recursive function g such that for all $d < \omega$ and all x:*

(*i*) $\{d\}(x)\downarrow \leftrightarrow g(d,x)\downarrow$
(*ii*) $\{d\}(x)\downarrow \rightarrow g(d,x) = T_{\langle d,x\rangle}$.

Proof. By effective transfinite recursion. The definition of the recursive iterator I has several cases, one for each Normann scheme. From now on it will be convenient to restrict attention to an artificial Normannesque scheme that combines the important features of the Normann schemes. Regard scheme

(T) $$\{2^m\cdot 3^n\}(x) \simeq \{\{n\}(y) \mid y\in\{m\}(x)\}.$$

It is understood in (T) that $\{2^m\cdot 3^n\}(x)\downarrow$ iff $\{m\}(x)\downarrow$ and $\{n\}(y)\downarrow$ for all $y\in\{m\}(x)$.The immediate subcomputations of $\langle 2^m\cdot 3^n, x\rangle$ are:

(Ta) $\langle m,x\rangle$;

(Tb) $\langle n,y\rangle$ if $\{m\}(x)\downarrow$ and $y\in\{m\}(x)$.

A recursion or induction based on the Normann schemes has six cases, one for each scheme. From now on only one case will be considered, that of scheme T. (T stands for typical.)

Thus the recursive iterator $I(e)$ is defined explicitly only for case T. If $\{2^m\cdot 3^n\}(x)\downarrow$, then

(1) $$T_{\langle 2^m\cdot 3^n,x\rangle} = T_{\langle m,x\rangle} \cup \bigcup\{T_{\langle n,y\rangle} \mid y\in\{m\}(x)\}.$$

(The $\bigcup$ operation of (1) is the union operation of set theory slightly modified so as to produce trees from subtrees in an appropriate fashion.) Define $I(e)$ so that

(2). $$\{I(e)\}(2^m\cdot 3^n, x) \simeq \{e\}(m,x) \cup \bigcup\{\{e\}(n,y) \mid y\in\{m\}(x)\}.$$

Observe that $I(e)$ is defined even when $\{2^m\cdot 3^n\}(x)\uparrow$, because $I(e)$ is merely an instruction.

By the fixed point theorem, there is a c such that $\{I(c)\}\simeq\{c\}$. Let $g(d,x)$ be $\{c\}(d,x)$. If $\{d\}(x)\downarrow$, then $T_{\langle d,x\rangle}$ is well-founded according to Proposition 2.2. In that event an induction on $>_U$ restricted to $T_{\langle d,x\rangle}$ shows $g(b,u) = T_{\langle b,u\rangle}$ for all $\langle b,u\rangle \leq_U \langle d,x\rangle$. The induction succeeds because $\{I(c)\}\simeq\{c\}$. □

Let $\mathscr{E}$ be an E-closed structure. Lemma 2.5 implies $\mathscr{E}$ is closed with respect to the formation of convergent computation trees: if $x\in\mathscr{E}$ and $\{e\}(x)\downarrow$, then $T_{\langle e,x\rangle}\in\mathscr{E}$. A major question to be considered shortly is: Suppose $x\in\mathscr{E}$ and $\{e\}(x)\uparrow$; does $T_{\langle \varepsilon,x\rangle}$, or some significant part of $T_{\langle e,x\rangle}$, belong to $\mathscr{E}$? The answer is yes in many cases. For example, if $L(\kappa)$ is E-closed but not Σ_1 admissible, $x\in L(\kappa)$ and $\{e\}(x)\uparrow$, then some infinite descending path through $T_{\langle e,x\rangle}$ is an element of $L(\kappa)$ (cf. Theorem 5.7.)

Every computation tree $T_{\langle e,x\rangle}$ has a height denoted by $|T_{\langle e,x\rangle}|$. If $T_{\langle e,x\rangle}$ is not wellfounded, then $|T_{\langle e,x\rangle}|$ is defined to be ∞ with the understanding that $\infty > \gamma$ for every ordinal γ. If $T_{\langle e,x\rangle}$ is wellfounded, then $|T_{\langle e,x\rangle}|$ is $\sup^+$ (strict least upper bound) of

$$\{|T_{\langle c,y\rangle}| \mid \langle c,y\rangle \text{ is an immed. subcomp. instruc. of } \langle e,x\rangle\}.$$

Also $|\{e\}(x)|$, by definition, is $|T_{\langle e,x\rangle}|$ and is called the length of the computation of $\langle e,x\rangle$.

2.6 Lemma. *There exists a partial E-recursive function h such that for all e and x:*

(i) $\{e\}(x)\downarrow \leftrightarrow h(e,x)\downarrow$

(ii) $\{e\}(x)\downarrow \rightarrow h(e,x) = |\{e\}(x)|$.

Proof. By effective transfinite recursion. Only scheme T is considered, as in the proof of Lemma 2.5. Thus $e = 2^m \cdot 3^n$, and $|\{e\}(x)|$ is the strict least upper bound of

$$\{|\{m\}(x)|\} \cup \{|\{n\}(y)| \mid y \in \{m\}(x)\}. \tag{1}$$

The recursive iterator $I: \omega \rightarrow \omega$ is such that for all $c < \omega$ and all x,

$$\{I(c)\}(e,x) \simeq \sup^+((1)).$$

The supremum operation is effective thanks to Normann scheme (4) of 1.1. h is $\{d\}$, where $\{I(d)\} \simeq \{d\}$. □

2.7 Lemma. *The predicates, $|\{e\}(x)| < \gamma$ and $|\{e\}(x)| = \gamma$, are E-recursive.*

Proof. By effective transfinite recursion on γ. Similar to the proof of Lemma 2.6. □

Let x be any set. Define:

$L(0,x) = x$;

$L(\delta+1,x) =$ set of all first order definable subsets of $L(\delta,x)$;

$L(\lambda,x) = \bigcup\{L(\delta,x) \mid \delta < \lambda\}$ (limit λ).

If y is transitive, then $L(\delta,y)$ is transitive for every ordinal δ. $TC(\{x\})$ is the least transitive set with x as a member.

The proof of Proposition 2.8 relies on a *flat*, ordered pairing function f, an idea of Quine.

$$f(x,y) = \begin{cases} \langle x,y\rangle & \text{if } x,y \in L(\omega), \\ \{f(a,b) \mid a \in x \ \ \& \ \ b \in y\} & \text{otherwise.} \end{cases}$$

If x or y has infinite rank, then rank $(f(x, y)) = \max(\text{rank}(x), \text{rank}(y))$. This last says f is flat. f is relativized to x by replacing $L(\omega)$ by $L(\omega, TC(\{x\}))$.

2.8 Proposition. *If* $\{e\}(x)\downarrow$ *and* $|\{e\}(x)| \geq \omega$, *then* $\{e\}(x)$ *and* $T_{\langle e, x\rangle}$ *are first order definable over* $L(|\{e\}(x)|, TC(\{x\}))$.

Proof. Let e be $2^m \cdot 3^n$ (scheme (T) of Lemma 2.5). By induction on $|\{e\}(x)|$, $\{m\}(x)$ and $\{n\}(y)$ $(y \in \{m\}(x))$ belong to $L(|\{e\}(x)|, TC(\{x\}))$.

$$\{e\}(x) \quad \text{is} \quad \{\{n\}(y) \mid y \in \{m\}(x)\}.$$

View $T_{\langle e, x\rangle}$ as a set of ordered pairs defined by the flat function f immediately above. □

2.9 Recursive Ordinals. $y, x_1, \ldots, x_n$ are sets. y is said to be E-recursive in $x_1, \ldots, x_n$ (in symbols, $y \leq_E x_1, \ldots, x_n$) if $y = \{e\}(x_1, \ldots, x_n)$ for some e. If f is an E-recursive function, then $f(x) \leq_E x$ for all x. More precisely, $f(x) \leq_E x$ uniformly in x. Note that $TC(x) \leq_E x$ uniformly. $\leq_E$ is not a generalization of Turing reducibility. That will come later. $\leq_E$ can be viewed as a generalization of hyperarithmetic reducibility. If $x, y \subseteq \omega$, then

$$y \leq_h x \leftrightarrow y \leq_E x, \omega.$$

An ordinal γ is said to be E-recursive in x if $\gamma \leq_E x$. Thus the recursive ordinals of Part A are simply the ordinals E-recursive in ω. Define

$$\kappa_0^x = \sup\{\gamma \mid \gamma \leq_E x\}.$$

κ_0^x is always a limit ordinal of countable cofinality. In general the ordinals recursive in x do not constitute an initial segment. If $x \subseteq \omega$, then Church–Kleene ω_1^x is $\kappa_0^{x,\omega}$.

κ_0^x need not be a Σ_1 admissible ordinal. Consider $x = \omega_1$. $\omega_1 < \kappa_0^{\omega_1}$ since $\omega_1 \leq_E \omega_1$. But $\kappa_0^{\omega_1}$ is less than the first Σ_1 admissible greater than ω_1, as will be seen below (cf. Exercise 5.15.)

κ_0^x has a natural development akin to the enumeration of a complete Postian set. A is said to be *E-recursively enumerable in* x if

$$A = \{y \mid \{e\}(y, x)\downarrow\}.$$

The complete E-recursively enumerable-in-x subset of ω is

$$K^x = \{n \mid n \in \omega \quad \& \quad \{(n)_0\}((n)_1, x)\downarrow\}.$$

It follows from Lemma 2.6 that

$$\kappa_0^x = \sup\{|\{(n)_0\}((n)_1, x)| \mid n \in \kappa^x\}. \tag{1}$$

The ordinals occurring in the set on the right side of (1) are, by definition, the ordinals constructive in x. For each γ constructive in x, define

$$K_\gamma^x = \{n \mid n \in \omega \quad \& \quad |\{(n)_0\}((n)_1)| < \gamma\}.$$

Then $K^x = \bigcup\{K_\gamma^x \mid \gamma \text{ constructive in } x\}$. Let λ_0^x be the ordertype, necessarily countable, of the set of ordinals constructive in x. Then K^x can be enumerated in λ_0^x steps. Step γ produces K_γ^x, a set E-recursive in x. The natural development of K^x is sometimes described as an "enumeration with gaps", since consecutive, E-constructive-in-x ordinals can be quite far apart.

Warning: an ordinal can be E-recursive in x without being E-constructive in x (cf. Exercise 2.13).

Recall that the E-closure of x (in symbols, $E(x)$) is the least E-closed $y \supseteq TC(\{x\})$. Thus $x \in E(x)$ and $E(x)$ is transitive. In fact $E(x)$ is the least transitive, E-closed set with x as an element. Each member of $E(x)$ is of the form $\{e\}(x, a_0, \ldots, a_n)$, where $a_i \in TC(x)$. It follows that κ^x, the supremum of all ordinals in $E(x)$, is

$$\sup\{\gamma \leq_E x, a_0, \ldots, a_n \text{ for some } a_0, \ldots, a_n \in TC(x)\}.$$

With the aid of Proposition 2.8 a precise connection can be made between $E(x)$ and $L(x)$. First observe that $\kappa^\omega = \omega_1^{CK}$ and $E(\omega) = L(\omega_1^{CK})$.

2.10 Proposition. $E(x) = L(\kappa^x, TC(\{x\}))$.

Proof. By Proposition 2.8, $E(x) \subseteq L(\kappa^x, TC(\{x\}))$. Suppose $\gamma < \kappa^x$ and y is first order definable over $L(\gamma, TC(\{x\}))$ via parameter $p \in L(\gamma, TC(\{x\}))$. Then $y \leq_E L(\gamma, TC(\{x\})), p$. By induction on γ, $p \in E(x)$, so it suffices to show $L(\gamma, TC(\{x\})) \leq_E \gamma, x$. The usual set theoretic definition of $L(\gamma, TC(\{x\}))$ by transfinite recursion is in fact effective, hence $\lambda\gamma \mid L(\gamma, TC(\{x\}))$ is E-recursive in x. □

2.11–2.12 Exercises

2.11. What is the role of flat pairing in Proposition 2.8?

2.12. Verify that $\kappa^\omega = \omega_1^{CK}$.

2.13. Let λ be the limit of all countable ordinals E-constructive in ω_1. Show λ is E-recursive in ω_1.

3. *Reflection*

The notion of reflection is needed in the study of divergence in Section 4. If $\{e\}(x)\uparrow$, then any infinite, descending path through $T_{\langle e, x\rangle}$ is termed a Moschovakis witness

to the divergence of $\{e\}(x)$. The least ordinal that suffices to construct such a witness is, under favorable circumstances, the greatest x-reflecting ordinal. A precise connection between divergence and reflection was first made by Harrington [1973] in the setting of Kleene's theory of recursion in normal objects of finite type.

An ordinal δ is said to be x-reflecting if

$$L(\delta, TC(\{x\}) \models \mathscr{F} \rightarrow L(\kappa_0^x, TC(\{x\})) \models \mathscr{F}$$

for every Σ_1^{ZF} sentence $\mathscr{F}$ whose only parameter is x. Clearly the union of x-reflecting ordinals is x-reflecting. Define

$$\kappa_r^x = \text{the greatest } x\text{-reflecting ordinal.}$$

$\kappa_0^x \leq \kappa_r^x$, and in a moment it will be seen that $\kappa_r^x \leq \kappa^x$. The situation of greatest interest occurs when $\kappa_0^x < \kappa_r^x < \kappa^x$, because it is then that non-trivial reflection phenomena occur within $E(x)$. An important early result of Harrington [1973],

$$\kappa_r^{2^\omega} > \kappa_0^{2^\omega} ,$$

had key applications in forcing and degree-theoretic arguments. It was preceded by a result of Sacks [1974] needed for a forcing proof,

$$\kappa_r^{2^{2^\omega}} > \kappa_0^{2^{2^\omega}} .$$

For a dynamic view of κ_r let $A \subseteq TC(\{x\})$ be E-recursively enumerable in x. Suppose $b \in A$ and the height of a computation c that puts b in A is γ for some $\gamma < \kappa_r^x$. Thus

$$L(\gamma + 1, TC(\{x\})) \models (\mathrm{Ec})(\mathrm{Eb})[c \text{ puts } b \text{ in } A].$$

By reflection there is a c_0 and b_0 such that $|c_0| < \kappa_0^x$ and c_0 puts b_0 in A. Since κ_0^x is the limit of ordinals recursive in x, there is an infinite $\beta \leq_E x$ such that $|c_0| < \beta$. By Proposition 2.8, K_β, the set of all computations from x of height less than β, is first order definable over $L(\beta, TC\{x\}))$. Let γ_0 be the length of the shortest computation that puts some γ into A. $\gamma_0 < \beta$, and γ_0 is first order definable over $L(\beta, TC(\{x\}))$. Hence $\gamma_0 \leq_E x$. Let A_0 be the set of all y put in A via computations of height at most γ_0. Then A_0 is a non-empty subset of A E-recursive in x.

To sum up, if a Σ_1 fact about x is true in $L(\gamma, TC(\{x\}))$ for some $\gamma < \kappa_r^x$, then it is true in $L(\gamma_0, TC(\{x\}))$ for some $\gamma_0 \leq_E x$.

3.1 Proposition. *There exists a* Π_3^{ZF} sentence $\mathscr{F}$ such that for all transitive A,

$$A \text{ is } E\text{-closed} \leftrightarrow A \models \mathscr{F}.$$

Proof. Let $\mathscr{F}$ have Π_2^{ZF} clauses that require closure under pairing, difference and union operations. The remainder of $\mathscr{F}$ insures:

$$(1) \qquad (x)\,[(y)_{y\in x}(\mathrm{E}\gamma)P(c, y, \gamma) \to (\mathrm{E}\gamma)(y)_{y\in x}P(c, y, \gamma)],$$

where $P(c, y, \gamma)$ is

(2) $T_{\langle c, y\rangle}$ is first ord. def. via M over $L(\gamma, TC(\{x\}))$.

M is a recursion, hidden in the proof of Proposition 2.8, that tries to define $T_{\langle c, y\rangle}$ over $L(\gamma, TC(\{x\}))$. M succeeds iff $y\in x$ and $\gamma \geq \max(\omega, |\{c\}(y)|)$. It follows that (1) is Π_3^{ZF}. □

3.2 Proposition. $\kappa_r^{x, a} \leq \kappa^x$ *for all* $a\in TC(x)$.

Proof. Suppose $a\in TC(x)$ and $\kappa_r^{x,a} > \kappa^x$. Since $E(\langle a, x\rangle) = E(x)$, it follows from Proposition 2.10 that

$$(1) \qquad E(\langle a, x\rangle)\in L(\kappa_r^{x, a}, TC(\{x\})).$$

With the aid of Proposition 3.1, (1) can be "reflected down" to $\kappa_0^{x, a}$. Thus

$$E(\langle a, x\rangle)\in L(\kappa_0^{x, a}, TC(\{x\})).$$

But then $\kappa^{x, a} < \kappa_0^{x, a}$, an absurdity. □

In Section 4 it will be shown that $\kappa_r^{a, x} = \kappa^x$ for all $a\in TC(x)$ only if $E(x)$ is Σ_1 admissible.

4. *Gandy Selection*

Selection principles in E-recursion theory have the following form. If A is E-recursively enumerable in x and nonempty, then there exists a nonempty $b \leq_E x$ such that $b \subseteq A$. In addition there is a uniform method for computing b from x and an index for A. If $A \subseteq \omega$ and $x\in\omega$, then A is a classical recursively enumerable set and selection is proved by enumerating A until some element appears. This trivial proof is worth considering more closely. Let K_σ^x be the set of all computations from x of length less than σ for σ a limit ordinal. By Proposition 2.8, $\mathrm{K}_\sigma^x \leq_E \sigma$, x uniformly in σ and x. Let σ_A be the least σ such that some computation in K_σ^x puts an element in A. Let b be the set of all such elements of A.

Then $b \leq_E \sigma_A, x$. So to complete the proof of selection it need only be shown that $\sigma_A \leq_E x$. In general, this last is false. It is true if $A \subseteq \omega$ and $x \in \omega$ by arguments of classical recursion theory. If $A \subseteq \omega$ and x is an arbitrary set, then the arguments of this section show $\sigma_A \leq_E x$. Later it will be seen that there exists a nonempty $A \subseteq 2^\omega$, E-recursively enumerable in 2^ω, such that $\sigma_A \not\leq_E 2^\omega$. The study of forcing over E-closed structures in the next chapter owes some of its technical interest to the lack of general selection principles.

Gandy [1967] originally proved selection for $A \subseteq \omega$ and $x = \omega$. Moschovakis extended the result to x's of higher type. The proof given here is derived largely from Moldstad [1977].

4.1 Theorem (Gandy Selection). *There exists a partial E-recursive function* $\phi(e, x)$ *such that for all* $e < \omega$ *and all* x:

(*i*) $(\mathrm{En})_{n<\omega}\,[\{e\}(n, x)\downarrow] \leftrightarrow \phi(e, x)\downarrow$;

(*ii*) $\phi(e, x)\downarrow \rightarrow \{e\}(\phi(e, x), x)\downarrow$.

The proof of Theorem 4.1 requires a preparatory lemma originated, and dubbed "stage comparison", by Moschovakis.

4.2 Lemma. *Suppose* $\{d\}(x)\downarrow$ *or* $\{e\}(y)\downarrow$. *Then*

$$\min(|\{d\}(x)|, |\{e\}(y)|) \leq_E x, y \text{ (uniformly)}.$$

Proof. By effective transfinite recursion on min. The rough idea behind the recursion step is:

$$\min(|u|, |v|) = \max\{\min(|a|, |b|) \mid a <_U u \;\&\; b <_U v\},$$

where $<_U$ is the universal computation tree of Section 2. Some smoothing is needed to make this idea work, because if $u\uparrow$, then $\{a \mid a <_U u\rangle$ may not be E-recursive in u. Assume $d = 2^m \cdot 3^n$ and $e = 2^p \cdot 3^q$. (As in the proof of 2.5, only scheme T is considered.) Define:

$$\alpha = \min(|\{2^m \cdot 3^n\}(x)|, |\{2^p \cdot 3^q\}(y)|);$$
$$\alpha_1 = \min(|\{m\}(x)|, |\{p\}(y)|).$$

$\alpha_1 < \alpha$, so by recursion $\alpha_1 \leq_E x, y$. With the aid of Lemma 2.7, an $s \in \{\{m\}(x), \{p\}(y)\}$ can be selected so that $|s| = \alpha_1$.

Suppose s is $\{m\}(x)$. Define:

$$\beta_b = \min(|\{n\}(b)|, |\{p\}(y)|);$$
$$\beta = \sup^+\{\beta_b \mid b \in \{m\}(x)\}.$$

By recursion $\beta_b \leq_E x, y, b$ (uniformly in b), hence $\beta \leq_E x, y$. Now use 2.7 to see if

$$\alpha \leq \max(|\{m\}(x)|, \beta) + 1. \tag{1}$$

If (1) is false, then $\{n\}(b)\uparrow$ for some $b \in \{m\}(x)$, and so $\alpha = |\{2^p \cdot 3^q\}(y)|$.
If s is $\{p\}(y)$, then proceed as in the above paragraph with q, m in place of n, p. □

4.3 Proof of Gandy Selection. For simplicity drop x. By Lemmas 2.7 and 4.2 there exists a partial E-recursive ψ such that

$$\psi(r,e,k) \simeq \begin{cases} \{r\}(e,k+1)+1 & \text{if } \{r\}(e,k+1)\downarrow \,\&\, |\{r\}(e,k+1)| \leq |\{e\}(k)|, \\ 0 & \text{if } |\{e\}(k)| < |\{r\}(e,k+1)|. \end{cases}$$

The instructions for computing ψ are as follows. Use Lemma 4.2 to compute the min of $\{r\}(e,k+1)$ and $\{e\}(k)$. If the min is undefined, then $\psi(r,e,k)$ is undefined. If the min is defined, then use Lemma 2.7 to decide which case of the definition of ψ applies.

Kleene's fixed point theorem provides a c such that

$$\psi(c,e,k) \simeq \{c\}(e,k)$$

for all e and k. Observe that the computation of $\{c\}(e,0)$ compares $|\{c\}(e,1)|$ with $|\{e\}(0)|$; the computation of $\{c\}(e,1)$ compares $|\{c\}(e,2)|$ with $|\{e\}(1)|$; and so on. $\{c\}$ is designed to seek the least k such that $|\{e\}(k)|$ is the minimum of $\{|\{e\}(m)| \mid m < \infty\}$.

First note that

$$(k)[\{c\}(e,k+1)\downarrow \to \{c\}(e,k)\downarrow], \tag{1}$$

$$(k)[\{e\}(k)\downarrow \to \{c\}(e,k)\downarrow]. \tag{2}$$

Hence

$$(k)[\{e\}(k)\downarrow \to \{c\}(e,0)\downarrow]. \tag{3}$$

Now observe that

$$\{c\}(e,0)\downarrow \to (\mathrm{Ek})[|\{e\}(k)| < |\{c\}(e,k+1)|]. \tag{4}$$

Otherwise $\{c\}(e,k) > \{c\}(e,k+1)$ for all k, an absurdity.

It remains only to show

$$(\mathrm{Ek})[\{e\}(k)\downarrow] \to \{e\}(\{c\}(e,0))\downarrow. \tag{5}$$

Suppose the left side of (5) holds. By (3) and (4),

$$\{c\}(e,0)\downarrow, \quad \text{and} \quad |\{e\}(k)| < |\{c\}(e,k+1)|$$

for some k; let k_0 be the least such k. Then

$$\{c\}(e,k_0) = 0, \quad \text{and} \quad \{c\}(e,j) = \{c\}(e,j+1)+1$$

for all $j < k_0$. Hence $\{c\}(e,0) = k_0$. □

4.4 Corollary. *The class of E-recursively enumerable predicates is closed under finite disjunction and existential number quantification.*

Proof. Suppose n is a number variable and $P(n,x)$ is E-recursively enumerable. By Gandy selection there is a partial E-recursive $\phi(x)$ such that

$$(\text{En})\ P(n,x) \leftrightarrow \phi(x)\downarrow. \quad \square$$

4.5 Corollary. *Suppose $P(x,y)$ is E-recursively enumerable and*

$$(x)_{x\in z}(\text{Ey})[y \leq_E x \quad \& \quad P(x,y)].$$

Then there exists a partial E-recursive function f such that

$$(x)_{x\in z}[f(x)\downarrow \quad \& \quad P(x,f(x))].$$

Proof. The predicate, $\{e\}(x)\downarrow$ & $P(x,\{e\}(x))$, is E-recursively enumerable. By Gandy selection there is a partial E-recursive ϕ such that

$$(\text{Ee})[\{e\}(x)\downarrow \quad \& \quad P(x,\{e\}(x))] \leftrightarrow \phi(x)\downarrow, \text{ and}$$
$$\phi(x)\downarrow \rightarrow \{\phi(x)\}(x)\downarrow \quad \& \quad P(x,\{\phi(x)\}(x)).$$

Let $f(x) \simeq \{\phi(x)\}(x)$. □

The proof of Corollary 4.5 deserves a gloss. Suppose $R(x,y)$ is an E-recursively enumerable predicate and there is a y such that

$$(1) \qquad y \leq_E x \quad \text{and} \quad R(x,y).$$

Then Gandy selection legitimizes a search through the y's E-recursive in x until one is found that satisfies (1). In addition the search procedure is uniform in x and R. Other selection principles will be proved in later sections. They too can be viewed as legitimizing certain searches.

4.6 Corollary. *If $a \subseteq \omega$, then $E(a)$ is Σ_1 admissible.*

Proof. Let $D(x,y,z)$ be a Δ_0^{ZF} formula. Suppose

$$E(a)\ (x)_{x\in p}\ (\text{Ey})\, D(x,y,q)$$

for some $p, q \in E(a)$. Since $a \subseteq \omega$, every $y \in E(a)$ is E-recursive in a. It follows from Corollary 4.5 that there exists a partial E-recursive function f such that

$$(x)_{x\in p}\,[f(x,q)\downarrow \quad \& \quad D(x,f(x,q))].$$

Let $r = \{f(x,q) \mid x \in p\}$. Then $r \leq_E p, q$ and

$$E(a) \models (x)_{x\in p}(\mathrm{Ey})_{y\in r}\, D(x,y,q). \qquad \square$$

Let $\mathrm{Ad}_1(x)$ be the least Σ_1 admissible set with x as a member. $\mathrm{Ad}_1(x)$ is $L(\alpha, TC(\{x\}))$ for some Σ_1 admissible α determined by x. $\mathrm{Ad}_1(x)$ is E-closed, hence $E(x) \subseteq \mathrm{Ad}_1(x)$. It follows from Corollary 4.6 that $E(a)$, for $a \subseteq \omega$, equals $\mathrm{Ad}_1(a)$. According to Exercise 4.9, subsets of $E(a)$ are Σ_1 definable over $E(a)$ iff they are E-recursively enumerable in a.

The situation with respect to $E(2^\omega)$ is more complex. It will be seen that $E(2^\omega)$ is not Σ_1 admissible because it admits Moschovakis witnesses.

4.7 Corollary. *Suppose* $\kappa_r^{x,a} = \kappa^x$ for all $a \in TC(x)$. Then $E(x)$ is Σ_1 admissible.

Proof. Suppose $D(u,v,w)$ is a Δ_0^{ZF} formula, and

$$E(x) \models (u)_{u\in c}(\mathrm{Ev})\, D(u,v,p)$$

for some $c, p \in E(x)$. For the moment assume $c \subseteq TC(x)$ and $p = x$. By Proposition 2.10

$$E(x) = L(\kappa_r^{x,a},\ TC(\{x\}))$$

for all $a \in TC(x)$. By reflection, for each $a \in c$, there is a $\delta \leq_E x, a$ such that

$$L(\delta, TC(\{x\})) \models (\mathrm{Ev})\, D(a,v,p).$$

It follows from Corollary 4.5 that δ can be construed as a partial E-recursive function f of x and a, defined for all $a \in c$. Normann's bounding scheme, (4) of subsection 1.1, implies that $\{f(x,a) \mid a \in c\}$ is bounded above by some $\gamma \leq_E x, c$. Hence there is a $d \in E(x)$ such that

$$E(x)\ (u)_{u\in c}(\mathrm{Ev})_{v\in d}\, D(u,v,p);$$

d is $L(\gamma, TC(\{x\}))$. Thus $E(x)$ is Σ_1 admissible.

The assumptions, $c \subseteq TC(x)$ and $p = x$, have to be discharged. $TC(x)$ is infinite; otherwise $E(x)$ is HF, hence Σ_1 admissible. According to Exercise 4.11 there exists a $g \in E(x)$ such that g is a one-one map of c into $TC(x)$. Then

$$(u)_{u\in c}(\mathrm{Ev})\ D(u,v,p)$$

is equivalent to

$$(u)_{u\in g[c]}\ (\mathrm{Ev})\ D(g^{-1}(u),v,p).$$

Thus it is safe to assume $c \subseteq TC(x)$.

Recall that $p = \{e\}(x, b)$ for some $b \in TC(x)$. Then $D(u, v, p)$ is equivalent to $(E\beta)(\mathrm{Eb})_{b \in TC(x)}[\,|\{e\}(x, b)| = \beta \;\&\; D(u, v, \{e\}(x, b))]$. □

4.8–4.12 Exercises.

4.8. Prove the class of E-recursively enumerable predicates is closed under finite disjunction.

4.9. Assume $a \subseteq \omega$ and $x \subseteq E(a)$. Show x is E-recursively enumerable in a iff x is Σ_1 definable over $E(a)$.

4.10. Assume $a \subseteq \omega$. Show a is E-recursively enumerable in ω iff a is Π^1_1.

4.11. Assume x is infinite and $c \in E(x)$. Find a $g \in E(x)$ such that g is a one-one map of c into $TC(x)$.

4.12. Fix x. Show the predicate, γ is not x-reflecting, is E-recursive in x. Hint: make use of a Σ_1^{ZF} predicate $\mathscr{F}$, whose only free variable is x, such that

$$L(\kappa_r^x + 1,\ TC(\{x\})) \models \mathscr{F},$$

$$L(\kappa_r^x,\ TC(\{x\})) \models \sim\mathscr{F}.$$

Slaman has shown that $\mathscr{F}$ cannot be chosen effectively from x.

5. *Moschovakis Witnesses*

Suppose $\{e\}(x)$ diverges. A *Moschovakis witness* to the divergence of $\{e\}(x)$ is any infinite path in $>_U$, the universal computation tree, below $\langle e, x\rangle$. More precisely, a witness is a function $\lambda n\,|\,t(n)$ such that: $t(0) = \langle e, x\rangle$; and for each n, $t(n+1)$ is an immediate subcomputation instruction of $t(n)$. Note that "t is a witness to $\{e\}(x)\uparrow$" is E-recursively enumerable. This section is devoted to locating the least ordinal that suffices to enumerate a witness to the divergence of $\{e\}(x)$. In many cases that ordinal will belong to $E(x)$. Thus the process of forming an E-closure by closing with respect to convergent computations will often close with respect to divergence witnesses as well.

A structure $\mathscr{E}$ is said to admit divergence witnesses if for all $e < \omega$ and $x \in \mathscr{E}$: if $\{e\}(x)\uparrow$ then some witness to $\{e\}(x)\uparrow$ belongs to $\mathscr{E}$. It will be seen shortly that $L(\kappa)$ admits divergence witnesses if $L(\kappa)$ is E-closed but not Σ_1 admissible. The study of forcing and priority over E-closed structures in future chapters will depend heavily on the presence of divergence witnesses.

There is a connection between the current section and the Gandy basis theorem for Σ^1_1 predicates (cf. Part A). Suppose $x \in E(\omega)$ and $\{e\}(x)\uparrow$. Then there exists a witness t to $\{e\}(x)\uparrow$ first order definable over $L(\omega_1^{\mathrm{CK}})$. In addition t can be chosen so that $E(t) = L(\omega_1^{\mathrm{CK}}, t)$. Also $E(\omega)$ does not admit Moschovakis witnesses.

5.1 Theorem (Kechris). *Suppose $y \leq_E x$ and A is E-recursively enumerable in x. If $y - A$ is nonempty, then*

$$(\mathrm{Eb})[b \in y - A \quad \& \quad \kappa_r^{x,b} \leq \kappa_r^x].$$

Proof. First observe that

$$\kappa_0^{x,b} \leq \kappa_r^x \to \kappa_r^{x,b} \leq \kappa_r^x. \tag{1}$$

Let $\mathscr{F}$ be a Σ_1 sentence of ZF with x as its only parameter. $\mathscr{F}$ reflects from $\kappa_r^{x,b}$ down to $\kappa_0^{x,b}$. Suppose $\kappa_0^{x,b} \leq \kappa_r^x$. Then $\mathscr{F}$ reflects from $\kappa_r^{x,b}$ down to κ_0^x. Hence $\kappa_r^{x,b} \leq \kappa_r^x$, since κ_r^x is the greatest x-reflecting ordinal.

By (1) it suffices to find a $b \in y - A$ such that $\kappa_0^{x,b} \leq \kappa_r^x$. Suppose there is no such b with the intent of showing $y \subseteq A$. Then

$$y \subseteq A \cup \{b \mid \kappa_r^x < \kappa_0^{x,b}\}. \tag{2}$$

Note that the predicate, $\kappa_r^x < \kappa_0^{x,b}$, is E-recursively enumerable in x by Corollary 4.4 and Exercise 4.12 since it is equivalent to

$$(\mathrm{E}\delta)[\delta \leq_E x, b \quad \& \quad \delta \text{ is not } x\text{-reflecting}].$$

Thus y is contained in the union of two sets, as indicated in (2), each of which is E-recursively enumerable in x. For each $b \in y$, there is a $\delta_b \leq_E x, b$ such that either (i) or (ii) holds:

(i) δ_b is the length of a computation that puts y in A;
(ii) δ_b is not x-reflecting.

By Corollary 4.4 δ_b can be construed as a partial E-recursive function of x, b defined for all $b \in y$. Let

$$\delta^s = \sup\{\delta_b \mid b \in y\}.$$

Then the bounding scheme yields $\delta^s \leq_E x$ and so $\delta^s < \kappa_r^x$. If (ii) holds for some b, then $\delta_b > \kappa_r^x > \delta^s$. Hence (ii) never holds, and $y \subseteq A$. □

The Kechris basis theorem is analogous to the Gandy basis theorem for Σ_1^1 sets of reals (Part A). In both an element b of a "co-recursively enumerable" set is found so that some ordinal generated by b is minimized. This minimization is precisely what is needed to prove Lemma 5.3. The next result is a technicality required for the proof of 5.3.

5.2 Proposition. *Assume some wellordering of $TC(x)$ is E-recursive in x. Suppose $\langle c, y\rangle$ is an immediate subcomputation instruction of $\langle e, x\rangle$. Then some wellordering of $TC(y)$ is E-recursive in x, y. In addition if $\{c\}(y)\downarrow$, then some wellordering of $\{c\}(y)$ is E-recursive in x, y.*

Proof. Let e be $2^m \cdot 3^n$. Suppose $\{m\}(x)\downarrow$ and $z \in \{m\}(x)$. Let $v \leq_E x$ be a wellordering of $TC(x)$. By Proposition 2.8,

$$\{m\}(x),\ v \in L(\gamma, TC(\{x\}))$$

for some $\gamma \leq_E x$. A recursion on the ordinals less than γ extends v to a wellordering w of $L(\gamma, TC(\{x\}))$. $w \leq_E v, \gamma$, hence $w \leq_E x$. Since $L(\gamma, TC(\{x\}))$ is transitive, w induces a wellordering of $\{m\}(x)$ E-recursive in x, and a wellordering of $TC(z)$ E-recursive in x, z. If $\{n\}(z)\downarrow$, then another application of 2.8 yields a wellordering of $\{n\}(z)$ E-recursive in x, z. □

Lemma 5.3 occupies a central place in E-recursion theory. It was inspired by an early theorem of Harrington [1973] in the setting of Kleene recursion in objects of finite type. His result, in the language of E-recursion, is: if $a \in 2^\omega$ and $\{e\}(a, 2^\omega)\uparrow$, then some Moschovakis witness to $\{e\}(a, 2^\omega)\uparrow$ is seen to be such via a computation of height $\kappa_r^{2^\omega, a}$.

5.3 Lemma. *Assume some wellordering of $TC(x)$ is E-recursive in x. If $\{e\}(x)\uparrow$, then some Moschovakis witness to $\{e\}(x)\uparrow$ is first order definable over $L(\kappa_r^x, TC(\{x\}))$.*

Proof. The witness, $\lambda t\,|\,\langle e_t, x_t\rangle$, is defined by recursion on $t < \omega$. $\langle e_0, x_0\rangle$ is $\langle e, x\rangle$. Fix $t \geq 0$ and assume $\langle e_t, x_t\rangle$ has already been defined so that

(1) $$\{e_t\}(x_t)\uparrow,$$

(2) $$x_t \in L(\kappa_r^x, TC(\{x\})), \text{ and}$$

(3) $$\kappa_r^x \geq \kappa_r^{x_0, \ldots, x_t}.$$

Let $e_t = 2^m \cdot 3^n$. (As in the proof of 2.5, only scheme (T) is considered.) (3) implies

(4) $$\kappa_r^x \geq \kappa_0^{x_0, \ldots, x_t} \geq \kappa_0^{x_t}.$$

Hence anything E-recursive in x_t belongs to $L(\kappa_r^x, TC(\{x\}))$. In particular, if $\{m\}(x_t)\downarrow$, then its computation tree belongs to $L(\kappa_r^x, TC(\{x\}))$. Hence examination of $L(\kappa_r^x, TC(\{x\}))$ reveals whether or not $\{m\}(x_t)\downarrow$.

Case A: $\{m\}(x_t)\uparrow$.

Define $\langle e_{t+1}, x_{t+1}\rangle = \langle m, x_t\rangle$.

Case B: $\{m\}(x_t)\downarrow$.

By Proposition 5.2 there is a wellordering $\leq_w$ of $\{m\}(x_t)$ E-recursive in x_t. The choice of w can be made in a uniform way with the aid of Gandy selection (Theorem 4.1), as in the remarks following the proof of Corollary 4.5, because the predicate, $\{c\}(x_t)\downarrow$ & $\{c\}(x_t)$ is a wellordering of $\{m\}(x_t)$, with c as a free variable, is E-recursively enumerable. Define

$$e_{t+1} = n, \text{ and}$$

$$x_{t+1} = w\text{-least } u\,[u \in \{m\}(x_t)\ \&\ |\{n\}(u)| \geq \kappa_r^x].$$

Observe that $\{x_{t+1}\}$ is first order definable over $L(\kappa_r^x, TC(\{x\}))$. It remains to be seen that

$$\{n\}(x_{t+1})\uparrow \ \&\ \kappa_r^{x_0,\ldots,x_t} \geq \kappa_r^{x_0,\ldots,x_{t+1}}. \tag{5}$$

The Kechris basis theorem (Lemma 5.1) provides a z such that

$$z \in \{m\}(x_t) \ \&\ \{n\}(z)\uparrow \ \&\ \kappa_r^{x_0,\ldots,x_t} \geq \kappa_r^{x_0,\ldots,x_t,z}. \tag{6}$$

Let z_0 be the w-least z that satisfies (6). (5) is proved by showing $x_{t+1} = z_0$. Clearly $x_{t+1} \leq z_0$.

Every $z <_w z_0$ is such that either $\{n\}(z)\downarrow$ or

$$\kappa_r^{x_0,\ldots,x_t} < \kappa_r^{x_0,\ldots,x_t,z}. \tag{7}$$

As in the proof of (1) of Theorem 5.1, (7) is equivalent to

$$\kappa_r^{x_0,\ldots,x_t} < \kappa_0^{x_0,\ldots,x_t,z}. \tag{8}$$

Thus for all $z <_w z_0$, either $\{n\}(z)\downarrow$ or (8) holds. (8) is equivalent to

$$(\mathrm{E}e)[\{e\}(x_0,\ldots,x_t,z)\downarrow \ \&\ \kappa_r^{x_0,\ldots,x_t} < \{e\}(x_0,\ldots,x_t,z)].$$

It follows from Gandy selection that there is a partial E-recursive function $f(x_0,\ldots,x_t,z)$, defined for all $z <_w z_0$, such that either

$$|\{n\}(z)| = f(x_0,\ldots,x_t,z) \quad \text{or} \quad \kappa_r^{x_0,\ldots,x_t} < f(x_0,\ldots,x_t,z). \tag{9}$$

Let $\gamma = \sup\{f(x_0,\ldots,x_t,z) \mid z <_w z_0\}$. The bounding scheme gives $\gamma \leq_E x_0,\ldots, x_t, z_0$. Consequently $\gamma < \kappa_r^{x_0,\ldots,x_t}$ by (6), and so

$$\{n\}(z)\downarrow \quad \text{for all} \quad z < z_0$$

by (9). If $x_{t+1} <_w z_0$, then

$$|\{n\}(x_{t+1}| < \kappa_r^{x_0,\ldots,x_t} \leq \kappa_r^x \leq |\{n\}(x_{t+1})|,$$

an absurdity. □

A reformulation of Lemma 5.3 will prove helpful. Let $>_U^\gamma$ be the universal computation tree restricted to computations of height less than γ. To be precise, clause (v)(b) of the definition of $>_U$ in subsection 2.1 is modified as follows. Replace "$\{d_i\}(x_1,\ldots,x^n)$ converges" by "$|\{d_i\}(x_1,\ldots,x_n)| < \gamma$". In the arguments to come only scheme (T) of the proof of Lemma 2.5 will be considered. The precise definition of $>_U^\gamma$ requires that "$\{m\}(x)\downarrow$" of clause (*Ta*) be replaced by

"$|\{m\}(x)| < \gamma$". Let $T^{\gamma}_{\langle e, x\rangle}$ be the tree-like object consisting of $\langle e, x\rangle$ and that part of $>^{\gamma}_{U}$ below $\langle e, x\rangle$.

5.4 Reformulation of Lemma 5.3. Suppose some wellordering of $TC(x)$ is E-recursive in x. If $\{e\}(x)\uparrow$, then

$$T^{\kappa^x_r}_{\langle e, x\rangle} \text{ is not wellfounded,}$$

and both it and its left-most infinite path are first order definable over $L(\kappa^x_r, TC(\{x\}))$.

The term "left-most infinite path" in 5.4 refers to the witness $\lambda t\,|\,x_t$ defined in the proof of Lemma 5.3. "Left-most" is appropriate because x_{t+1} is the w-least z such that $z \in \{m\}(x_t)$ & $\{n\}(z)\uparrow$, when case B holds.

5.4 indicates the importance of another parameter. For all z define

$$\theta^z = \mu\gamma(e)[\{e\}(z)\uparrow \to T^{\gamma}_{\langle e, z\rangle} \text{ not wellfounded}].$$

Harrington's ground breaking results [1973] are:

$$\kappa^{2^\omega, a}_r = \theta^{2^\omega, a}$$

$$\kappa^{2^\omega, a, b}_r \geq \kappa^{2^\omega, a}_r \quad \text{for all} \quad a, b \in 2^\omega.$$

Warning: Slaman [1985a] has found an x and y such that

$$\kappa^x_r > \theta^x \quad \text{and} \quad \kappa^{x, y}_r < \kappa^x_r.$$

5.5 Lemma

(*i*) $\kappa^u_r \geq \theta^u$ *for all* u.
(*ii*) $\theta^{u, v} \geq \theta^u$ *for all* u *and* v.
(*iii*) $\kappa^x_r = \theta^x$ *if some wellordering of* x *is* E*-recursive in* x.
(*iv*) $\kappa^{x, u}_r \geq \kappa^x_r$ *for all* u, *if some wellordering of* x *is* E*-recursive in* x.

Proof. (i) Repeat the proof of Lemma 5.3 stripped of most definability considerations. Assume $\langle e_t, u_t\rangle$ has been defined so that

$$\{e_t\}(u_t)\uparrow \quad \text{and} \quad \kappa^u_r \geq \kappa^{u_0, \ldots, u_t}_r.$$

In case B choose u_{t+1} so that

$$u_{t+1} \in \{m\}(u_t),\ \{n\}(u_{t+1})\uparrow \quad \text{and} \quad \kappa^x_r \geq \kappa^{u_0, \ldots, u_{t+1}}_r.$$

The choice is made with the aid of Lemma 5.1. Hence $|\{m\}(u_t)| < \kappa^u_r$ when case B holds. Thus all the ordinals needed for checking convergence in the course of defining $\lambda t\,|\,u_t$ are less than κ^u_r. (Only scheme (T) is being considered.) So $\lambda t\,|\,u_t$ is a path through $>^{\kappa^x_r}_U$.

(ii) follows from the existence of an e^* such that the instructions for computing $\{e^*\}(u, v)$ are essentially the same as those for $\{e\}(u)$. Then $T_{\langle e^*, u, v\rangle}$ consists of finitely many trivial nodes above a copy of $T_{\langle e, u\rangle}$.

(iv) $\kappa_r^{x,u} \geq \theta^{x,u} \geq \theta^x = \kappa_r^x$ by (i), (ii) and (iii).

(iii) By (i) $\kappa_r^x \geq \theta^x$. In a moment it will be shown that there exists a recursive function h such that for all $e < \omega$:

(1) $$\{e\}(x)\downarrow \leftrightarrow \{h(e)\}(x)\downarrow;$$

(2) if $\{h(e)\}(x)\uparrow$, then $T_{\langle h(e), x\rangle}$ has a unique infinite path.

Intuitively, $T_{\langle h(e), x\rangle}$ consists of the leftmost, infinite path of $T_{\langle e, x\rangle}$ and everything to the left of that path.

Suppose $\{e\}(x)\uparrow$. By (1) and (2) $T_{\langle h(e), x\rangle}$ has a unique infinite path t. t is first order definable over $L(\theta^x, TC(\{x\}))$. Suppose for a contradiction that $\theta^x < \kappa_r^x$. Then by reflection there is a divergence witness for $\{h(e)\}(x)\uparrow$ E-recursive in x.

But then for any e the question, does $\{e\}(x)\downarrow$?, can be effectively decided by searching through the sets E-recursive in x until either a wellfounded $T_{\langle e, x\rangle}$ turns up or a divergence witness to $\{h(e)\}(x)\uparrow$ is found. The search is effective by Gandy selection, as in the remarks following the proof of Corollary 4.5.

h is defined by abridging the Normann schemes so as not to allow computations to continue past the leftmost path. Suppose $e = 2^m \cdot 3^n$. (Only scheme (T) is being considered.) The immediate subcomputation instructions of $\langle h(e), x\rangle$ will be as follows:

(a) $\langle h(m), x\rangle$ is an immed. subcomp. instruc. of $\langle h(e), x\rangle$;

(b) suppose $\{h(m)\}(x)\downarrow$; let $<_w$ be the wellordering of $\{h(m)\}(x)$ E-recursive in x supplied by Proposition 5.2. If $\{h(n)\}(z)\downarrow$ for all $z <_w y$, then $\langle h(n), y\rangle$ is an immed. subcomp. instruc. of $\langle h(e), x\rangle$.

Let $h \simeq \{d\}$, where $\{d\}$ is a fixed point of $\psi(c)$; that is $\{d\} \simeq \{\psi(d)\}$.

$\psi(c)$ is a partial recursive function such that the instructions for computing $\{\{\psi(c)\}(e)\}(x)$ are as follows.

First compute $\{\{c\}(m)\}(x)$. If it converges, then compute $<_w$, a wellordering of $\{\{c\}(m)\}(x)$, as in Proposition 5.2. Then compute $\{\{c\}(n)\}(y)$ for all $y \in \{\{c\}(m)\}(x)$ *in the order* imposed by $<_w$. If this procedure terminates, then the final value is

$$\{\{\{c\}(n)\}(y) \mid y \in \{\{c\}(m)\}(x)\}.$$

Otherwise the procedure is stopped by the failure of $\{\{c\}(m)\}(x)$ to converge, or by the failure of $\{\{c\}(n)\}(y_0)$ to converge, where y_0 is the w-least y such that $\{\{c\}(n)\}(y_0)\uparrow$. In the latter case the procedure does compute $\{\{c\}(n)\}(y)$ for all $y <_w y_0$, but not for any $y >_w y_0$. □

5.6 Lemma. *Let x be a set of ordinals. Suppose $L(\kappa, x)$ is E-closed but not Σ_1 admissible. Then $\kappa_r^{x,y} < \kappa$ for all $y \in L(\kappa, x)$.*

Proof. Suppose $\kappa_r^{x,y} \geq \kappa$ for some $y \in L(\kappa, x)$ with the intent of showing $L(\kappa, x)$ Σ_1 admissible. Assume

$$L(\kappa, x) \models (u)_{u \in d}\, (\mathrm{Ev})\, \mathscr{F}(u, v, p).$$

$d, p \in L(\kappa, x)$, and $\mathscr{F}$ is Δ_0^{ZF}. For the moment assume

$$\kappa_r^{x,y,p,b} \geq \kappa_r^{x,y} \quad \text{for all } b \in d. \tag{1}$$

Then by reflection,

$$(b)_{b \in d}\, (\mathrm{Ec})\, [c \in L(\kappa_0^{x,y,p,b}, TC(\{x, y, p, b\})) \quad \& \quad \mathscr{F}(b, c, p)].$$

In fact

$$(b)_{b \in d}\, (\mathrm{Ec})\, [c \leq_E x, y, p, b \quad \& \quad \mathscr{F}(b, c, p)].$$

It follows from Corollary 4.5 that c can be construed as a partial recursive function of x, y, p, b defined for all $b \in d$. Let

$$e = \{c(x, y, p, b) \mid b \in d\}.$$

Then $e \in L(\kappa)$, since $e \leq_E x, y, p, d$ and $L(\kappa)$ is E-closed. Consequently

$$L(\kappa, x) \models (u)_{u \in d}\, (\mathrm{Ev})_{v \in e}\, \mathscr{F}(u, v, p).$$

To prove (1) apply Lemma 5.5 (iv). All that is needed is a wellordering w of $TC(\{x, y\})$ E-recursive in x, y. Since $\kappa_r^{x,y} \geq \kappa$, there is a wellordering of $TC(\{x, y\})$ in $L(\kappa_r^{x,y}, TC(\{x, y\}))$. Then w exists by reflection.

5.7 Theorem. *Suppose x is a set of ordinals and $L(\kappa, x)$ is E-closed. If $L(\kappa, x)$ is not Σ_1 admissible, then $L(\kappa, x)$ admits Moschovakis witnesses.* □

Proof. Suppose $y \in L(\kappa, x)$ and $\{e\}(y)\uparrow$. Let $w \in L(\kappa, x)$ be a wellordering of $TC(\{x, y\})$. By Lemma 5.6, $\kappa_r^{x,y,w} < \kappa$. According to Lemma 5.3 there is a witness t to $\{e\}(y)\uparrow$ first order definable over

$$L(\kappa_r^{x,y,w}, TC(\{x, y, w\})).$$

$t \leq_E \kappa_r^{x,y,w}, x, y, w$; hence $t \in L(\kappa, x)$, since $L(\kappa, x)$ is E-closed. □

5.8 Theorem. *Let x be a set of ordinals. The following are equivalent.*

(*i*) *$E(x)$ is not Σ_1 admissible.*
(*ii*) *$\kappa_r^{x,y} \in E(x)$ for all $y \in E(x)$.*
(*iii*) *$E(x)$ admits Moschovakis witnesses.*

Proof. (i) → (ii) by Lemma 5.6. (ii) → (iii) as in the proof of Theorem 5.7. Assume (iii) to prove (i). For each $a \in TC(x)$, define $f(e, a, x)$ to be the least γ such that $|\{e\}(a, x)| < \gamma$ or some witness to $\{e\}(a, x)\uparrow$ is first order definable over $L(\gamma, TC(\{x\}))$.
By (iii) $f(e, a, x)$ is defined on $\omega \times TC(x) \times \{x\}$. f is Σ_1 over $E(x)$, and

$$E(x) \subseteq L(\text{sup range } f, TC(\{x\})).$$

Consequently sup range $f = \kappa^x$, and $L(\kappa^x, x)$ is not Σ_1 admissible. □

It is not known if Lemma 5.6 can be strengthened so that its conclusion reads: Then $\kappa_r^y < \kappa$ for all $y \in L(\kappa, x)$. Slaman [1985] has found a Σ_1 inadmissible E-closed set that does not admit Moschovakis witnesses. Thus the supposition, x is a set of ordinals, of Theorem 5.7 was not lightly made. Existing proofs of the existence of Moschovakis witnesses rely on some form of dependent choice (DC_ω) obtained by either effectively wellordering the underlying set x or closing it under the formation of ω-sequences. As an example of the latter, let $x = 2^\omega$. Then $(2^\omega)^\omega = 2^\omega$ and so $E(2^\omega)$ satisfies DC_ω, if it is agreed that V, the class of all sets, satisfies DC_ω. Otherwise DC_ω can be justified in $E(2^\omega)$ by assuming either (a) $V = L$, or (b) $V = L(2^\omega)$ and the axiom of determinateness holds in $L(2^\omega)$ (Kechris).

5.9 Recursive Enumerability on $\mathscr{E}$. Suppose $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$. A is said to be E-recursively enumerable on $\mathscr{E}$ if for some $p \in \mathscr{E}$ and $e < \omega$,

$$A = \{z \mid z \in \mathscr{E} \ \&\ \{e\}(p, z)\downarrow\}. \tag{1}$$

Equivalently, $A = \mathscr{E} \cap B$ for some B E-recursively enumerable in some $p \in \mathscr{E}$.

Warning: A can be E-recursively enumerable on $\mathscr{E}$ without being E-recursively enumerable in any $p \in \mathscr{E}$.

In the next chapter it will be shown by a forcing argument that there exists an E-closed $L(\kappa)$ such that $L(\kappa)$ is not E-recursively enumerable in any $p \in L(\kappa)$.

5.10 Theorem. *Let x be a set of ordinals. Then (i) ↔ (ii).*

(i) $E(x)$ is Σ_1 admissible.

(ii) For all $A \subseteq E(x)$: A is Σ_1 definable over $E(x)$ iff A is E-recursively enumerable on $E(x)$.

Proof. (i) → (ii). Let $E(x) = L(\kappa, x)$. Suppose $A \subseteq E(x)$ and

$$z \in A \leftrightarrow L(\kappa, x) \models (\text{Ev})\ \mathscr{F}(z, v, p).$$

for some Δ_0^{ZF} $\mathscr{F}$ and $p \in L(\kappa, x)$. By Theorem 5.8 there is a $y \in L(\kappa, x)$ such that $\kappa_r^{y,x} \geq \kappa$. As in the proof of Lemma 5.6, v can be construed as a partial E-recursive

function of x, y, p and z defined for all $z \in A$. Then

$$z \in A \leftrightarrow f(x, y, p, z)\downarrow \quad \& \quad L(\kappa, x) \models \mathscr{F}(z, f(x, y, p, z), z),$$

and so A is E-recursively enumerable on $E(x)$ via y, p.

Now assume (ii) holds and (i) fails in hope of a contradiction. By Theorem 5.8, $E(x)$ admits Moschovakis witnesses, and so the predicate,

$$z \in E(x) \ \& \ \{e\}(e, z)\uparrow$$

with z and e as free variables, is Σ_1 definable over $E(x)$. Hence it is E-recursively enumerable on $E(x)$ via some parameter $q \in E(x)$. Then the set

$$\kappa_q = \{e \mid \{e\}(e, q)\uparrow\}$$

is E-recursively enumerable on $E(x)$ via q, hence E-recursively enumerable in q, since $x \in \omega$ is an E-recursive predicate. Thus for some $c \in \omega$,

$$\kappa_q = \{e \mid \{c\}(e, q)\downarrow\}.$$

But then $\{c\}(c, q)\uparrow$ iff $\{c\}(c, q)\downarrow$. □

5.11 The Divergence-Admissibility Split. Theorems 5.8 and 5.10 yield a useful split of all E-closed $L(\kappa)$'s into two *disjoint* classes.

Class I: $L(\kappa)$ admits Moschovakis witnesses.

Class II: $L(\kappa)$ is Σ_1 admissible; and for all $A \subseteq L(\kappa)$, A is Σ_1 definable over $L(\kappa)$ iff A is E-recursively enumerable on $L(\kappa)$.

In brief, either $L(\kappa)$ admits divergence witnesses or Σ_1 equals E-recursively enumerable on $L(\kappa)$, but not both. A typical application of the split, made in Chapter XIII occurs in the solution to Post's problem in the sense of E-recursion theory for every E-closed $L(\kappa)$. If $L(\kappa) \in$ Class II, then the admissibility methods of Part B suffice. If $L(\kappa) \in$ Class I, then Moschovakis witnesses shoulder most of the burden of proof.

For proof of the split first assume $L(\kappa) \neq E(x)$ for any x. Every $x \in L(\kappa)$ is wellordered in $L(\kappa)$, so Lemma 5.3 implies $L(\kappa) \in I$ and $\{\langle e, x\rangle \mid \{e\}(x)\uparrow\}$ is Σ_1 over $L(\kappa)$. If $L(\kappa)$ were in II, then $\{\langle e, x\rangle \mid \{e\}(x)\uparrow\}$ would be E-r.e.; hence E-recursive, an absurdity.

Now suppose $L(\kappa) = E(x)$ for some x. It is safe to assume x is a wellordering. Then by Theorems 5.8 and 5.10, $L(\kappa) \notin I$ iff $L(\kappa)$ is Σ_1 admissible iff "Σ_1 definability over" agrees with "E-recursively enumerable on" for $L(\kappa)$.

5.12–5.17 Exercises

5.12. Make precise and prove the reformulation of Lemma 5.3 given in subsection 5.4.

5.13. Supply the details of the proof of Lemma 5.5 (i).

5.14. Suppose $A \subseteq q \in \mathscr{E}$, $\mathscr{E}$ is E-closed, and A is E-recursively enumerable on $\mathscr{E}$ via p. Show A is E-recursively enumerable in p, q.

5.15. Show $E(\omega_1)$ is not Σ_1 admissible.

5.16. (Kleene) Show $2^\omega \cap E(\omega)$ is E-recursively enumerable in ω.

5.17. Suppose $P(\gamma)$ is E-co-recursively enumerable in a, and $(E\gamma)_{\gamma<c}\, P(\gamma)$. Let γ_∞ be $\mu\gamma_{\gamma<c}\, P(\gamma)$. Show

$$\kappa_r^{\gamma_\infty, a, c} \leq \kappa_r^{a,c}.$$

Chapter XI
Forcing Computations to Converge

In this chapter several notions of forcing over E-closed structures are studied. Only set forcing is examined. The term "set forcing" refers to forcing relations that are elements of the ground structure. Forcing over an E-closed structure poses questions peculiar to E-recursion theory. How can computations be forced to converge or diverge? There does not appear to be a uniform answer. Much depends on the nature of the forcing relation under scrutiny. The two major examples studied below are countably closed, and countable chain, forcing. Both preserve E-closure, yet do so in different ways. An example of set forcing that does not preserve E-closure is given in Section 1. Note that set forcing over a Σ_1 admissible set always preserves Σ_1 admissibility.

As an application of forcing it will be shown that certain naturally enumerable sets are not E-recursively enumerable. Recall that $2^\omega \cap E(\omega)$ is E-recursively enumerable in ω (cf. Exercise 5.16.X). It will be seen that $2^{2^\omega} \cap E(2^\omega)$ is not E-recursively enumerable in $2^\omega, b$ for any $b \in 2^\omega$ if $E(2^\omega) \models (\text{card } (2^\omega) \text{ is regular})$ (cf. Exercise 2.5).

1. Set Forcing over $L(\kappa)$

Assume $L(\kappa)$ is E-closed and not Σ_1 admissible. Thus there is a greatest cardinal in the sense of $L(\kappa)$ (cf. Part C) denoted by $\text{gc}(\kappa)$. It is convenient to think of the generic object G as a subset of $\text{gc}(\kappa)$. The primary question is: if G is generic, is $L(\kappa, G)$ E-closed? For a "no" answer, let $L(\kappa)$ be $E(\omega_1)$, and G a Levy-collapse of ω_1 to ω. Then $L(\kappa, G) = L(\kappa, b)$ for some $b \subseteq \omega$. If $L(\kappa, b)$ is E-closed, then $L(\kappa, b) = E(b)$ is Σ_1 admissible, and so $L(\kappa)$ is Σ_1 admissible, an absurdity by Exercise 5.15.X.

Thus set forcing over $E(\omega_1)$ must preserve ω_1 if it is to preserve E-closure. It turns out that preserving ω_1 does not guarantee that E-closure will be preserved, but something close to that is true. This point is discussed further at the end of the present chapter.

1.1 The Forcing Language. Suppose $G \subseteq \text{gc}(\kappa)$ as above. It follows from Proposition 2.8.X that the members of $L(\kappa, G)$ are just those sets of the form $\{e\}(a, G)$ such that $a \in L(\kappa)$ and $|\{e\}(a, G)| < \kappa$. The terms of the forcing language,

$\mathscr{L}(\kappa, G)$, are designed to name the elements of $L(\kappa, G)$. Unavoidably some do not; they refer to divergent computations. The primitive terms are $\mathscr{G}$ and $\underline{a}$ for each $a \in L(\kappa)$. A general term is of the form $\{e\}(t_1, \ldots, t_n)$, where t_i $(1 \leq i \leq n)$ is a term. By composition every term is equivalent to one of the form $\{e\}(\underline{a}, \mathscr{G})$.

The other primitives are: ε; $|\ |$; set variables $x, y, z, \ldots$ ranging over $L(\kappa, G)$; ordinal variables $\beta, \gamma, \delta, \ldots$ ranging over κ, quantifiers and propositional connectives.

The formulas of $\mathscr{L}(\kappa, \mathscr{G})$ are as follows.

(i) $|\{e\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$. (i) says that the computation indicated converges and has height $\sigma (< \kappa)$. (i) is a typical ranked formula.

(ii) $(E\delta)[|\{e\}(\underline{a}, \mathscr{G})| = \delta]$. This says that the computation indicated converges and has height less than κ. A typical unranked formula.

(iii) $x \in \{e\}(\underline{a}, \mathscr{G})$. Here is a formula whose meaning is elusive. If $\{e\}(a, G)$ converges and has height $\sigma < \kappa$, then $\{e\}(\underline{a}, \mathscr{G})$ names $\{e\}(a, G)$. Otherwise $\{e\}(\underline{a}, \mathscr{G})$ does not name any member of $L(\kappa, G)$.

1.2 The Forcing Relation. Let $\mathscr{P} = \langle P, \geq \rangle$ be a notion of forcing that belongs to $L(\kappa)$; in short $\mathscr{P} \in L(\kappa)$. The elements of P, denoted by $p, q, r, \ldots$, are called forcing conditions. If $p \geq q$, then p is said to be extended by q; intuitively, q says more about $\mathscr{G}$ than p does.

$\mathscr{P}$ determines a forcing relation $|\!\vdash$ defined by effective transfinite recursion on $\sigma < K$. Three formal entities are defined simultaneously:

(a) $p |\!\vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$.
(b) $\mathscr{T}(p, e, a, \mathscr{G}, \sigma)$.
(c) $q |\!\vdash s \in \{e\}(\underline{a}, \mathscr{G})$.

(b) is a set of terms adequate for naming all the elements of $\{e\}(a, G)$ if G is generic, $p \in G$ and (a) holds. In (c), s is a member of (b), and q extends the p mentioned in (b).

At the ground level ($\sigma = 0$) take $\{e\}(\underline{a}, \mathscr{G})$ to be $\mathscr{G}$ for simplicity. Then for all p and q,

$p |\!\vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{0}$,
$\mathscr{T}(p, e, a, \mathscr{G}, 0) = \{\underline{\delta} | \delta < \mathrm{gc}(\kappa)\}$, and
$q |\!\vdash \underline{\delta} \in \{e\}(\underline{a}, \mathscr{G})$ iff $(q |\!\vdash \underline{\delta} \in \mathscr{G})$ is a ground zero forcing fact.

(Assume $\mathscr{P}$ includes a list of all ground zero forcing facts of the form $q |\!\vdash \underline{\delta} \in \mathscr{G}$ or $r |\!\vdash \underline{\gamma} \notin \mathscr{G}$.)

Above ground level ($\sigma > 0$) take e to be $2^m \cdot 3^n$ (scheme T of 2.5.X). The definition of

$$p |\!\vdash |\{2^m \cdot 3^n\}(\underline{a}, \mathscr{G})| = \underline{\sigma} \tag{1}$$

has three clauses:

$$(E\gamma)_{\gamma < \underline{\sigma}}[p |\!\vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\gamma}]; \tag{2}$$

$$p |\!\vdash (x)(E\tau)_{\tau < \underline{\sigma}}[x \in \{m\}(\underline{a}, \mathscr{G}) \to |\{n\}(x)| = \tau]; \tag{3}$$

$$p |\!\vdash (\tau)_{\tau < \underline{\sigma}}[|\{m\}(\underline{a}, \mathscr{G})| = \tau \vee (\mathrm{Ex})[x \in \{m\}(\underline{a}, \mathscr{G}) \ \ \& \ \ |\{n\}(x)| \geq \tau]]. \tag{4}$$

To make sense out of (2)–(4), recall some fundamental aspects of forcing:

$p \Vdash (\mathrm{Ex})\, \mathscr{F}(x)$ iff $p \Vdash \mathscr{F}(c)$ for a suitable term c;
$p \Vdash \sim \mathscr{F}$ iff $(q)_{p \geq q} \sim [q \Vdash \mathscr{F}]$;
$p \Vdash (\mathrm{x})\, \mathscr{F}$ iff $p \Vdash \sim (\mathrm{Ex}) \sim \mathscr{F}$

Then note that the γ of (2) is less than σ, hence $\mathscr{T}(p, m, a, \mathscr{G}, \gamma)$ was defined prior to stage σ. The subformula, $x \in \{m\}(\underline{a}, \mathscr{G})$, gains a meaning by letting x range over the terms in $\mathscr{T}(p, m, a, \mathscr{G}, \gamma)$. If s is such a term, then

$$q \Vdash s \in \{m\}(\underline{a}, \mathscr{G})$$

was defined prior to stage σ for all $q \geq p$.

Next define

$$\mathscr{T}\{p, 2^m \cdot 3^n, \underline{a}, \mathscr{G}, \sigma\} = \{\{n\}(s) \mid s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)\}.$$

Finally, define $q \Vdash t \in \{2^m \cdot 3^n\}(\underline{a}, \mathscr{G})$ iff for some $s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)$,

$$t = \{n\}(s) \text{ and}$$
$$q \Vdash [s \in \{m\}(\underline{a}, \mathscr{G}) \quad \& \quad (E\tau)_{\tau < \sigma}(|\{n\}(s)| = \tau)].$$

Recall from subsection 1.1 that each general term t of $\mathscr{L}(\kappa, \mathscr{G})$ is equivalent to a term of the form $\{e\}(\underline{a}, \mathscr{G})$. It will often be convenient to replace $(\underline{a}, \mathscr{G})$ by t, as in Proposition 1.3.

1.3 Proposition. *The relations,*

$p \Vdash |\{e\}(t)| = \underline{\sigma}$,
$s \in \mathscr{T}(p, e, t, \sigma)$ *and*
$q \Vdash s \in \{e\}(t)$,
are E-recursive in σ, $\mathscr{P}$ *uniformly.*

Proof. By effective transfinite recursion on σ. The definition given in subsection 1.2 is such a recursion. At stage σ all quantifiers in the definition are bounded either by $\mathscr{P}$ or by sets defined at earlier stages. Recall the proofs of Lemmas 2.6 and 2.7 of Chapter X. □

1.4 The Tree of Possibilities. In subsection 1.2 a downward view of forcing computations to converge was adopted. This view parallels the downward view of convergence of Chapter X. It is possible to develop an upward approach based on hierarchies. The latter makes forcing over an E-closed $L(\kappa)$ look very much like forcing over L in conventional set-theoretic presentations. The downward view of computation seems closer than the upward to the nature of computation. An instruction is given and then pursued without any foreknowledge of convergence.

The upward approach deals only with convergent computations. It builds long ones by combining short ones. One advantage of proceeding downward is that it leads to the all important concept of divergence witness. Another is that most arguments for preserving E-closure in generic extensions turn on a related concept.

To study divergence witnesses in generic extensions, a forcing counterpart of $>_U$, the universal computation tree, is needed. It is denoted by $>_V$ and is called the tree of possibilities. A node on $>_V$ is a triple $\langle p, e, t\rangle$, where p is a forcing condition, $e < \omega$ and t is a term of $\mathscr{L}(\kappa, \mathscr{G})$. $>_V$ below $\langle p, e, t\rangle$ corresponds to $>_U$ below $\langle e, t\rangle$ for all generic G that satisfy p. A preliminary definition of $\langle p, e, t\rangle >_V \langle q, n, s\rangle$ is

$$(1) \qquad p \geq q \quad \text{and} \quad q \Vdash^* [\langle e, t\rangle >_U \langle n, s\rangle].$$

($\Vdash^*$ refers to weak forcing; $p \Vdash^* \mathscr{F}$ iff $p \Vdash \sim\sim \mathscr{F}$).
To make (1) precise, $>_U$ has to be expressed within the forcing language $\mathscr{L}(\kappa, \mathscr{G})$. Suppose $e = 2^m \cdot 3^n$.

$$(2)(a) \qquad p \Vdash [\langle 2^m \cdot 3^n, t\rangle >_U \langle m, t\rangle].$$

$$(2)(b) \qquad \text{If } p \Vdash |\{m\}(t)| = \underline{\gamma},\ s \in \mathscr{T}(p, m, t, \gamma) \text{ and}$$
$$p \Vdash s \in \{m\}(t), \quad \text{then} \quad p \Vdash [\langle 2^m \cdot 3^n, t\rangle >_U \langle n, s\rangle].$$

Formula (2) holds for all p, t and s, and indicates how to define $p \Vdash [\langle n, s\rangle$ is an immediate subcomputation instruction of $\langle e, t\rangle]$. The idea is to mimic the definition of immediate subcomputation instruction given in subsection 2.1.X with the help of the forcing definitions given in subsection 1.2. Then $p \Vdash a >_V b$ iff there exist $a_0, \ldots, a_n$ such that $a_0 = a$, $b = a_n$ and

$$p \Vdash [a_{i+1} \text{ is an immed. subcomp. instruc. of } a_i] \text{ for all } i < n.$$

The most pressing question about $>_V$ is: if $p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma|$, does it follow that $>_V$ below $\langle p, e, t\rangle$ is wellfounded?

An affirmative answer would be useful, since the key properties of $\Vdash$ could then be established by induction on the well-founded parts of $>_V$, as in the next lemma. The answer is yes and no. Yes for countably closed forcing, and no for countable chain condition forcing, both of which preserve E-closure.

1.5 Lemma. *Suppose* $p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma]$ *and* $>_V$ *is wellfounded below* $\langle p, e, t\rangle$. *Then*

$$p \Vdash^* |\{e\}(t)| \leq \underline{\gamma}$$

for some $\gamma \leq_E p, t, \mathscr{P}$ (*uniformly*).

Proof. γ is computed by effective transfinite recursion on $>_V$ below $\langle p, e, t\rangle$. Roughly, γ is the height of the wellfounded subtree of $>_V$ below $\langle p, e, t\rangle$. As usual

e is $2^m \cdot 3^n$ (scheme T of the proof of Lemma 2.5.X). By recursion

$$p \Vdash^* |\{m\}(t)| \leq \underline{\delta}$$

for $\delta \leq_E p, t, \mathscr{P}$, since $\langle p, e, t\rangle >_V \langle p, m, t\rangle$.

Define $\langle p', \sigma\rangle \in K$ by

$$p \geq p' \quad \& \quad \sigma \leq \delta \quad \& \quad p' \Vdash |\{m\}(t)| = \sigma.$$

K makes explicit all the values of $|\{m\}(t)|$ allowed by p. By Proposition 1.3, $K \leq_E p, t, \mathscr{P}$.

Fix $\langle p', \sigma\rangle \in K$ and $s \in T(p', m, t, \sigma)$. Then

$$p' \Vdash (E\beta)[s \in \{m\}(t) \to |\{n\}(s)| = \beta].$$

Define $q \in J(p', \sigma, s)$ by

$$p' \geq q \quad \& \quad q \Vdash s \in \{m\}(t).$$

If $q \in J(p', \sigma, s)$, then $q \Vdash^* (E\beta)[|\{n\}(s) = \beta]$. Again by 1.3,

$$J(p', \sigma, s) \leq_E p', \sigma, s, t, \mathscr{P}.$$

For each $q \in J(p', \sigma, s)$ there is by recursion some $\rho \leq_E q, s, \mathscr{P}$ such that

$$q \Vdash^* |\{n\}(s)| \leq \rho.$$

By Corollary 4.5.X to Gandy's selection principle, ρ can be construed as a partial E-recursive function of q, s and $\mathscr{P}$. As q varies over $J(p', \sigma, s)$, ρ is bounded by some

$$\rho_{J(p', \sigma, s)} \leq_E J(p', \sigma, s), s, \mathscr{P}.$$

As s varies over $\mathscr{T}(p', m, t, \sigma)$, $\rho_{J(p', \sigma, s)}$ is bounded by some

$$\rho_{\mathscr{T}(p', m, t, \sigma)} \leq_E p', \sigma, t, \mathscr{P}.$$

Finally, as $\langle p', \sigma\rangle$ varies over K, $\rho_{\mathscr{T}(p', m, t, \sigma)}$ is bounded by some $\rho_K \leq p, t, \mathscr{P}$. Let γ be the strict supremum of δ and ρ_K. □

1.6 Effective Bounding. Suppose $\mathscr{P} \in L(\kappa)$ is a notion of forcing such that: if

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma], \tag{1}$$

then $p \Vdash^* |\{e\}(t)| \leq \underline{\gamma}$ for some $\gamma \leq_E p, t, \mathscr{P}$ (uniformly). Then $\mathscr{P}$ is said to satisfy effective bounding. According to Lemma 1.5, if $>_V$ is wellfounded below $\langle p, e, t\rangle$ whenever (1) holds, then $\mathscr{P}$ satisfies effective bounding.

G is said to be $\mathscr{P}$-generic if for every sentence $\mathscr{F}$ of $\mathscr{L}(\kappa, \mathscr{G})$,

$$(\mathrm{Ep})_{p \in G}[p \Vdash \mathscr{F} \quad \text{or} \quad p \Vdash \sim \mathscr{F}].$$

G is viewed both as a subset of $\mathrm{gc}(K)$, and as a consistent set of forcing conditions. Thus $\delta \in G$ iff $q \Vdash \delta \in G$ for some $q \in G$. If G is $\mathscr{P}$-generic, then for every sentence $\mathscr{F}$,

$$L(\kappa, G) \vDash \mathscr{F} \quad \text{iff} \quad (\mathrm{Ep})_{p \in G}[p \Vdash F]. \tag{2}$$

(2) is proved in a standard fashion by induction on the rank and complexity of $\mathscr{F}$ (cf. Part A, Chapter IV).

1.7 Lemma. *If $\mathscr{P}$ satisfies effective bounding and G is $\mathscr{P}$-generic, then $L(\kappa, G)$ is E-closed.*

Proof. Suppose not. Then for some e and $a \in L(\kappa)$,

$$\{e\}(a, G)\downarrow \quad \text{and} \quad |\{\dot{e}\}(a, G)| \geq \kappa.$$

$T_{\langle e, \langle a, G\rangle\rangle}$ must have a node z such that $|T_z| = \kappa$. So it is safe to assume

$$|\{e\}(a, G)| = \kappa. \tag{1}$$

Since G is generic there is a $p \in G$ such that p forces (1). Let e be $2^m \cdot 3^n$. Then

$$p \Vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\sigma}$$

for some $\sigma \leq_E p, a, \mathscr{P}$ by effective bounding. p also forces

$$(x)(E\beta)[x \in \{m\}(\underline{a}, \mathscr{G}) \to |\{n\}(x)| = \beta]. \tag{2}$$

In (2) x ranges over $T(p, m, \underline{a}, \mathscr{G}, \sigma)$, which is E-recursive in $p, a, \mathscr{P}$ according to Proposition 1.3.

Now proceed as in the proof of Lemma 1.5. Obtain a bound $\rho_0 \leq_E p, a, \mathscr{P}$ on the possible values of $|\{n\}(s)|$ as s ranges over $\mathscr{T}(p, m, a, \mathscr{G}, \sigma)$. Then p forces $|\{e\}(\underline{a}, \mathscr{G})|$ to be at most $\sup^+(\rho_0, \sigma)$. The latter is less than κ, since it is E-recursive in elements of $L(\kappa)$. □

In the next Section it will be shown that every countably closed notion of forcing satisfies effective bounding. The approach taken consists of proving $>_\nu$ is well-founded below $\langle p, e, t\rangle$ whenever 1.6(1) holds, and then applying Lemma 1.5. This approach fails for countable chain condition notions of forcing, which nonetheless will be seen to satisfy effective bounding.

2. *Countably Closed Forcing*

Again assume $L(\kappa)$ is E-closed and not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a notion of forcing. $\mathscr{P}$ is said to be *countably closed* if for every $(\lambda_n | p_n) \in L(\kappa)$:

$$(n)[p_n \geq p_{n+1}] \to (\mathrm{Eq})(n)[p_n \geq q].$$

For an example, assume $L(\kappa) \models [\mathrm{cf}(\mathrm{gc}(\kappa)) > \omega]$. (Recall that "gc" denotes greatest cardinal and "cf" cofinality.) Let P be the set of all $p \in L(\kappa)$ such that p is a function from some $\delta < \mathrm{gc}(\kappa)$ into 2. Then $P \in L(\kappa)$, since $P \subseteq L(\mathrm{gc}(\kappa))$. Say $p \geq q$ if $\mathrm{dom}\, p \subseteq \mathrm{dom}\, q$ and $p(x) = q(x)$ for all $x < \mathrm{dom}\, p$. $\mathscr{P}$ is countably closed because $\mathrm{gc}(\kappa)$ has uncountable cofinality in $L(\kappa)$. For q take $\cup \{p_n | n < \omega\}$.

Countably closed forcing over an E-closed structure originated in Sacks [1980] in the setting of Kleene recursion in ${}^3E$. Slaman was the first to point out the role of countable closure in Sacks [1980]. A slightly abstract account of countably closed forcing over an E-closed structure can be found in Sacks–Slaman [1987], as well as an extension of the method to class forcing.

Warning: there is an assumption not made explicit in the statement of Theorem 2.1 and needed in its proof for the application of Lemma 5.3.X. $\mathscr{P}$, and all $p \in P$, are effectively equivalent to sets of ordinals.

Theorem 2.1. *Suppose $L(\kappa)$ is E-closed, but not Σ_1 admissible, and $\mathscr{P}$ is a countably closed notion of forcing in $L(\kappa)$. Then every $\mathscr{P}$-generic extension of $L(\kappa)$ is E-closed.*

Proof. By Lemmas 1.5 and 1.7 it is enough to show $>_V$ is wellfounded below $\langle p, e, t \rangle$ when

$$p \| \vdash^* (E\sigma)[|\{e\}(t)| = \sigma]. \tag{1}$$

The idea is to convert an infinite descending path, if there is one, below $\langle p, e, t \rangle$ into a $q \leq p$ that forces the existence of a Moschovakis witness to the divergence of $\{e\}(t)$.

The proof of Lemma 1.5 includes an effective procedure for computing γ from $p, e, t, \mathscr{P}$ by recursion on $>_V$ below $\langle p, e, t \rangle$. The procedure is welldefined whether or not $>_V$ is wellfounded below $\langle p, e, t \rangle$. Consequently Lemma 1.5 can be rephrased as follows.

There exists a $\{g\}$ such that if (1) holds and $>_V$ is well-founded below $\langle p, e, t \rangle$, then

$$\{g\}(p, e, t)\downarrow \quad \text{and} \quad p \| \vdash^* |\{e\}(t)| \leq \{g\}(p, e, t).$$

(For simplicity, argument $\mathscr{P}$ of $\{g\}$ is suppressed.)

Observe that if $\{g\}(p, e, t)\downarrow$, then $>_V$ is wellfounded below $\langle p, e, t \rangle$. This is so because the computation of the value of $\{g\}(p, e, t)$ assigns an ordinal rank to each node $\langle p', e', t' \rangle$ of $>_V$ below $\langle p, e, t \rangle$. That rank is $\{g\}(p', e', t')$. In short, $\{g\}$ cannot finish if there is an ∞ descending path below $\langle p, e, t \rangle$.

Suppose $\{g\}(p, e, t)\uparrow$. The plan is to compare $>_V$ below $\langle p, e, t\rangle$ with $>_U$ below $\langle g, \langle p, e, t\rangle\rangle$. There is a witness, $\lambda n|z_n$, to the divergence of $\{g\}(p, e, t)$ inside $L(\kappa)$ by Theorem 5.7.X. In addition

$$z_0 = \langle g, \langle p, e, t\rangle\rangle \quad \text{and} \quad z_n >_U z_{n+1}.$$

The proof of Lemma 1.5 defines $\{g\}$ in such a way that $\{g\}(p, e, t)\uparrow$ if and only if there exists some $\langle q', m', s\rangle$ such that

(2) $\{g\}(q', m', s)\uparrow$, and

(3) $\langle q', m', s\rangle$ lies immediately below $\langle p, e, t\rangle$ in $>_V$.

It follows that z must contain some information of the sort expressed by (2) and (3). Thus there is an n and a $\langle q', m', s\rangle$ such that

$$z_n = \langle g, \langle q', m', s\rangle\rangle$$

and $\langle q', m', s\rangle$ satisfies (2) and (3). Continuing in this fashion extracts a sequence $\lambda r|w_r$ from $\lambda n|z_n$ such that $w_0 = \langle p, e, t\rangle$ and for all r:

$$w_r >_V w_{r+1},$$
$$w_r \text{ is of the form } \langle p_r, e_r, t_r\rangle,$$
$$\{g\}(p_r, e_r, t_r)\uparrow,$$
$$(\text{En})[z_n = \langle g, \langle p_r, e_r, t_r\rangle\rangle].$$

Since $\mathscr{P}$ is countably closed, there is a q such that $p_r \geq q$ for all r. Then q weakly forces $\lambda r|\langle e_r, t_r\rangle$ to be a Moschovakis witness to $\{e\}(t)\uparrow$. □

2.2 Non-enumerability. A typical application of countably closed forcing is the proof that certain naturally enumerable sets are not E-recursively enumerable. For example, it follows from Theorem 2.3 that $E(\omega_1)$ is not E-recursively enumerable in any $b \in E(\omega_1)$. The case for regarding $E(\omega_1)$ as naturally enumerable is as follows. Begin with ω_1 and iterate first order definability. At a limit stage λ, look back and see if there is an x already enumerated and an e such that $|\{e\}(x)| = \lambda$. If so, collect everything already enumerated and continue. Otherwise stop. The procedure is predicative in nature, because $|\{e\}(x)| = \lambda$ only if all the immediate subcomputations of $\{e\}(x)$ converge and have been enumerated prior to stage λ. This way of laying out $E(\omega_1)$ is close in spirit to a standard way of laying out all finite computations in classical recursion theory.

Theorem 2.4, together with Corollary 4.6.X, explain why $E(\omega)$ is E-recursively enumerable. Recall that $\text{gc}(\kappa)$ is the greatest cardinal in the sense of $L(\kappa)$, if there is one.

2.3 Theorem (Sacks 1986). *Assume $L(\kappa)$ is not Σ_1 admissible and*

$$L(\kappa) \models [\mathrm{gc}(\kappa) \text{ is regular}].$$

Then $2^{\mathrm{gc}(\kappa)} \cap L(\kappa)$ is not E-recursively enumerable in any $b \in L(\kappa)$.

Proof. Recall the forcing conditions defined at the beginning of Section 2. Suppose (in hope of a contradiction) that for some $b \in L(\kappa)$,

$$2^{\mathrm{gc}(\kappa)} \cap L(\kappa) = \{x \mid \{e\}(b, x)\downarrow\}.$$

By Theorem 2.1 the null condition ϕ weakly forces $L(\kappa, \mathscr{G})$ to be E-closed. By Theorem 5.7.X, ϕ weakly forces $L(\kappa, \mathscr{G})$ to admit divergence witnesses. Thus ϕ weakly forces $(E\delta)[|\{e\}(t)| = \delta \vee (\mathrm{Ew})(w \in L(\delta)\ \&\ w$ is a divergence witness to $\{e\}(t)\uparrow)]$ for every term t. Hence *either* (1) or (2) holds:

(1) $$\phi \Vdash^* (E\sigma)[|\{e\}(\underline{b}, \mathscr{G})| = \sigma].$$

(2) There exist q, w and $\delta < \kappa$ such that

$$q \Vdash^* [w \in L(\underline{\delta}, \mathscr{G}) \quad \&\quad w \text{ witnesses } \{e\}(\underline{b}, \mathscr{G})\uparrow].$$

If (1) holds, then there will be a $G \subseteq \mathrm{gc}(\kappa)$ such that $\{e\}(b, G)\downarrow$ but $G \notin L(\kappa)$.
If (2) holds, then there will be a $G \subseteq \mathrm{gc}(\kappa)$ such that $G \in L(\kappa)$ but $\{e\}(b, G)\uparrow$.
Assume (1) holds. According to Theorem 2.1

(3) $$\phi \Vdash^* |\{e\}(\underline{b}, \mathscr{G})| < \underline{\lambda}$$

for some limit $\lambda < L(\kappa)$. The language $\mathscr{L}_\lambda(\mathscr{G})$ consists of sentences of rank at most λ and has enough expressive power to give a complete account of the computation of $\{e\}(b, G)$. A more detailed definition of $\mathscr{L}_\lambda(\mathscr{G})$ follows. As usual, only scheme T is considered. Let $e = 2^m \cdot 3^n$.

(4) (i) $(E\sigma)_{\sigma < \underline{\lambda}}[|\{e\}(\underline{b}, \mathscr{G})| = \sigma]$ belongs to $\mathscr{L}_\lambda(\mathscr{G})$.
(ii) If $\rho < \lambda$ and $p \Vdash |\{2^m \cdot 3^n\}(t)| = \rho$, then all of the following belong to $\mathscr{L}_\lambda(\mathscr{G})$:

$$(x)(E\tau)_{\tau < \underline{\rho}}[x \in \{m\}(t) \to |\{n\}(x)| = \tau];$$

$$(\tau)_{\tau < \underline{\rho}}(\mathrm{Ex})[x \in \{m\}(t) \to |\{n\}(x)| \geq \tau];$$

$$(E\sigma)_{\sigma < \underline{\rho}}[|\{m\}(t)| = \sigma].$$

(x ranges over the terms in $\mathscr{T}(p, 2^m \cdot 3^n, t, \rho)$.)

(iii) Every instance of every subformula of every sentence in $\mathscr{L}_\lambda(\mathscr{G})$ belongs to $\mathscr{L}_\lambda(\mathscr{G})$.

Proposition 1.3 implies $L_\lambda(G) \in L(\kappa)$. Let $f \in L(\kappa)$ be a map from $\mathrm{gc}(\kappa)$ onto $L_\lambda(\kappa)$.

There exists an $S \subseteq \mathrm{gc}(\kappa)$ such that $S \notin L(\kappa)$ but $(S \cap \delta) \in L(\kappa)$ for all $\delta < \mathrm{gc}(\kappa)$. S is needed in the construction of G to insure that $G \notin L(\kappa)$. If κ is countable, then S is any ω-sequence through $\mathrm{gc}(\kappa)$ not in $L(\kappa)$. If $\mathrm{gc}(\kappa)$ is a regular L-cardinal, then S exists because κ is not Σ_1 admissible, hence less than the next L-cardinal after $\mathrm{gc}(\kappa)$. Otherwise S is obtained by a Jensenian fine structure argument such as Theorem 4.1 of Sacks [1986].

The desired G is generic with respect to every sentence of $\mathscr{L}_\lambda(\mathscr{G})$ and encodes S. The characteristic function of G is a union of forcing conditions p_δ of $(\delta < \mathrm{gc}(\kappa))$ defined by recursion on δ.

$$p_0 = \phi. \quad p_\gamma = \cup \{p_\beta \mid \beta < \gamma\} \quad \text{if } \gamma \text{ is a limit.} \tag{5}$$

$$q_\delta = \text{least } q \geq p_\delta \quad \text{such that} \quad q \Vdash f_\delta \quad \text{or} \quad q \Vdash \sim f_\delta.$$

$$p_{\delta+1} = p_\delta{}^\frown\langle 0\rangle \quad \text{if} \quad \delta \in S, \quad p_\delta{}^\frown\langle 1\rangle \quad \text{if} \quad \delta \notin S.$$

$q^\frown\langle i\rangle = q \cup \langle \mathrm{dom}\, q, i\rangle$.) Observe that

$$p_\delta \text{ is a forcing condition (i.e. } p_\delta \in L(\kappa)), \text{ and} \tag{6}$$

$$\{p_\gamma \mid \gamma < \delta\} \leq_E S \cap \delta, \mathrm{gc}(\kappa), \lambda, f, \delta \quad \text{(uniformly in } \delta). \tag{7}$$

The proof of (6) and (7) is a simultaneous induction on δ. When δ is a limit, then (6) follows from (7) because $\mathrm{gc}(\kappa)$ is regular in $L(\kappa)$ and $(S \cap \delta) \in L(\kappa)$. When δ is a successor, then (7) follows from Proposition 1.3.

Hence G is well defined. By (3) and (4) $\{e\}(b, G)\downarrow$. $G \notin L(\kappa)$ because $S \notin L(\kappa)$ and

$$S \leq_E G, \mathrm{gc}(\kappa), \lambda, f. \tag{8}$$

(8) is a consequence of

$$\langle p_\delta, S \cap \delta\rangle \leq_E G, \mathrm{gc}(\kappa), \lambda, f, \delta \quad \text{(uniformly in } \delta), \tag{9}$$

and (9) is derived by recursion on δ. (In fact, S and G have the same E degree modulo $\mathrm{gc}(\kappa), \lambda, f$.)

Now assume (2) holds. Let λ be a limit greater than δ. Repeat the construction of G given in (5) with p_0 replaced by q and S by 0. Then G is generic with respect to all sentences of $\mathscr{L}_\lambda(\mathscr{G})$, and so $\{e\}(b, G)\uparrow$. According to (7), $G \leq_E \mathrm{gc}(\kappa), \lambda, f$, hence $G \in L(\kappa)$. □

The proof of Theorem 2.3 relied heavily on the Σ_1 inadmissibility of $L(\kappa)$. The next result assumes Σ_1 admissibility and generalizes the classic result of Kleene: $2^\omega \cap E(\omega)$ is E-recursively enumerable in ω.

2.4 Theorem. *If z is a set of ordinals and $E(z)$ is Σ_1 admissible, then $E(z)$ is E-recursively enumerable in some element of $E(z)$.*

Proof. $E(z) = L(\kappa, z)$ for some κ by Proposition 2.10.X. According to Theorem 5.8.X, there is a $y \in E(z)$ such that $\kappa_r^{z,y} \geq \kappa$. Any $u \in L(\kappa, z)$ can be encoded by some v, a set of ordinals, in $L(\kappa, z)$. Hence there is a $v \in L(\kappa, z)$ such that $v \subseteq \kappa$ and $z, y \leq_E v$. By reflection it is safe to assume $v \leq_E z, y$. Since $v \equiv_E z, y$ it follows that $\kappa_0^v = \kappa_0^{v,z}$ and $\kappa_r^v = \kappa_r^{z,y}$. Hence

$$\kappa_r^{v,x} \geq \kappa \quad \text{for all } x$$

by Lemma 5.5.X(iv).

Assume $x \in E(z)$ with the intention of computing $O(x)$ from v, x. $O(x)$ is the unique δ such that

$$\delta < \kappa \leq \kappa_r^{v,x} \quad \text{and} \quad x \in L(\delta + 1, z) - L(\delta, z).$$

By reflection $O(x) \leq_E v, x$. Gandy selection (Theorem 4.1.X) provides an e such that for all $x \in E(z)$,

$$\{e\}(v, x)\downarrow \quad \text{and} \quad \{e\}(v, x) = O(x).$$

$E(z)$ is enumerated as follows. Assume x is arbitrary. Try to compute $\{e\}(v, x)$. If $(e)(v, x)\downarrow$, then enumerate x if $E(z)$ is not a subset of $L(\{e\}(v, x), z)$. □

A partial converse to Theorem 2.4 exists (Sacks [1986]). If γ is an ordinal and $E(\gamma)$ is E-recursively enumerable in some $b \in E(\gamma)$, then $E(\gamma)$ is Σ_1 admissible. The argument does not relativize to a set z of ordinals.

Further results on the enumerability of $L(\kappa)$, when $L(\kappa)$ is E-closed, are given in Sacks [1986]. A complete resolution of the problem is available when $L(\kappa)$ is not Σ_1 admissible. The admissible case is open.

Suppose $L(\kappa)$ is E-closed and not Σ_1 admissible. Then (i) and (ii) are equivalent.

(i) $2^{\mathrm{gc}(\kappa)} \cap L(\kappa)$ is E-recursively enumerable in some $b \in L(\kappa)$.

(ii) $L(\kappa) \models [\omega < \text{cofinality } (\mathrm{gc}(\kappa)) < \mathrm{gc}(\kappa)]$; or $L(\kappa) \models [\omega = \text{cofinality } \mathrm{gc}(\kappa)]$ and every amenable subset of $\mathrm{gc}(\kappa)$ belongs to $L(\kappa)$.

(A set $Z \subseteq \mathrm{gc}(\kappa)$ is *amenable* (Jensen) if $(Z \cap \delta) \in L(\mathrm{gc}(\kappa))$ for all $\delta < \mathrm{gc}(\kappa)$.)

The cofinality ω case is handled by a variant of Green's compactness theorem [1974]. The uncountable singular cofinality case is managed by fine structure results of S. Friedman [1981].

2.5–2.6 Exercises

2.5. Suppose

$$E(2^\omega) \models [\mathrm{card}(2^\omega) = \omega_1].$$

Show $E(2^\omega) \cap 2^{2^\omega}$ is not E-recursively enumerable in any $b \in 2^\omega$. Normann selection, Corollary 1.4.XII, makes it possible to replace [card $(2^\omega) = \omega_1$] by [card (2^ω) is regular].

2.6. Suppose $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P}$ be a countably closed, set notion of forcing in $L(\kappa)$. (Heed the warning just before Theorem 2.1.) Show for all $a \in L(\kappa)$

$$(p)(\mathrm{Eq})(p \geq q \quad \& \quad q \Vdash [\kappa_r^{\mathscr{G},a} \leq \kappa_r^{\mathscr{P},p,a}]).$$

3. *Enumerable Forcing Relations*

Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing as in Section 1. Recall the concept of effective bounding from subsection 1.6: if

$$p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma], \tag{1}$$

then $p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \leq \underline{\gamma}$ for some $\gamma \leq_E p, \underline{a}, \mathscr{P}$ (uniformly). According to Lemma 1.7, $L(\kappa, G)$ is E-closed if G is $\mathscr{P}$-generic and $\mathscr{P}$ satisfies effective bounding. Another consequence of effective bounding is: relation (1) is E-recursively enumerable on $L(\kappa)$. More intuitively, the forcing relation restricted to "r.e. sentences" is "r.e.". This last follows from effective bounding via Corollary 4.4.X. The purpose of the present section is to prove the converse. The result clarifies the nature of forcing over E-closed structures, but does not help to show any particular notion of forcing preserves E-closure.

The notion of recursive enumerability on $L(\kappa)$ is boldface; any member of $L(\kappa)$ can occur as a parameter. For simplicity it is assumed in Theorem 3.1 that the only parameter needed for the enumeration of relation (1) above is $\mathscr{P}$. This assumption is in agreement with the definition of effective bounding, which requires γ to be computed from p, a, $\mathscr{P}$.

Throughout the present section it is taken for granted that G, $\mathscr{P}$, and all $p \in P$, are equivalent to sets of ordinals in some uniform, effective fashion. Thus it is safe to apply 5.3–5.6 of Chapter X.

3.1 Theorem (Sacks & Slaman). *Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. Then* (*i*) $\equiv$ (*ii*).

(*i*) *$\mathscr{P}$ satisfies effective bounding.*

(*ii*) *The relation, $p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma]$, is E-recursively enumerable on $L(\kappa)$.*

Proof. (i) implies (ii) by Corollary 4.4.X.

Assume (ii) holds. For simplicity suppress $\mathscr{P}$. Suppose in hope of a contradiction that (1) holds

$$p \Vdash^* (E\sigma)[\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}}]. \tag{1}$$

Gandy selection implies there is an e^* such that:

$$\{e^*\}(p, a, G)\downarrow \quad \text{iff} \quad (\text{Ee})[\infty > |\{e\}(a, G)| \geq \kappa_r^{\underline{a}, \underline{p}}].$$

Gandy chooses the shortest computation, so (1) is equivalent to $p \Vdash^*(E\sigma)[|\{e^*\}(\underline{p}, \underline{a}, \mathscr{G})| = \sigma]$. Then (ii) implies

(2) (1) is E-recursively enumerable on $L(\kappa)$.

Let f be such that on $L(\kappa)$:

(3) $\{f\}(p, e, b)\downarrow$ iff

$$p \Vdash^*(E\sigma)[|\{e\}(\underline{b}, \mathscr{G})| = \sigma].$$

There exists a recursive function t such that

$$\{t(c)\}(\langle p, a\rangle, G) \simeq \begin{array}{l} 0 \text{ if some } w \leq_E a, p, G \text{ is a} \\ \quad \text{witness to } \{f\}(p, c, \langle p, a\rangle)\uparrow, \\ \text{undefined otherwise.} \end{array}$$

Let e_0 be a fixed point of t. Then

(4) $\{e_0\}(\langle p, a\rangle, G)\downarrow$ iff some $w \leq_E a, p, G$ is a witness to $\{f\}(p, e_0, \langle p, a\rangle)\uparrow$.

To see that

(5) $$\{f\}(p, e_0, \langle p, a\rangle)\uparrow,$$

assume otherwise. Then by (3),

$$p \Vdash^*(E\sigma)[|\{e_0\}(\langle \underline{p}, \underline{a}\rangle, \mathscr{G})| = \sigma].$$

Choose a generic $G_0 \in p$. Then $\{e_0\}(\langle p, a\rangle, G_0)\downarrow$, and by (4), $\{f\}(p, e_0, \langle p, a\rangle)\uparrow$. So (5) is true. It follows from Lemma 5.3.X that there is a witness w to (5) such that $w \leq_E \kappa_r^{a, p}$. Consider a generic $G_1 \in p$. By (1),

$$\kappa_r^{a, p} < \kappa_r^{a, G_1} \leq \kappa_r^{p, a, G_1}.$$

By reflection there is a witness w to (5) E-recursive in p, a, G_1. By (4),

(6) $$\{e_0\}(\langle p, a\rangle, G_1)\downarrow.$$

Note that (6) holds thanks to a computation of height $h \leq_E \kappa_r^{a, p} < \kappa$. Thus

(7) $$|\{e_0\}(\langle p, a\rangle, G_1)| < \kappa.$$

Since (7) holds for *all* generic $G_1 \in p$, it follows that

$$p \Vdash^* (E\sigma)[|\{e_0\}(\langle \underline{p}, \underline{a}\rangle, \mathscr{G})| = \sigma].$$

But then (3) contradicts (5) with $\langle p, a\rangle$ in place of b. Hence supposition (1) was false, and so

$$\text{not}\,[p \Vdash^* (E\sigma)(\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}})] \tag{8}$$

is proved for all e, p and a.

Now (i) can be proved. Assume

$$p \Vdash^* (E\sigma)[|\{e\}(\underline{a}, \mathscr{G})| = \sigma].$$

By Lemma 3.2 below, it suffices to show

$$p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| < \kappa_r^{\underline{a},\underline{p}}. \tag{9}$$

Suppose (9) fails in order to contradict (8). So there is a $q \leq p$ such that

$$q \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\underline{a},\underline{p}}. \tag{11}$$

For $q \leq p$, the negation of (11) is equivalent to

$$(E\gamma)[\gamma < \kappa_r^{a,p} \quad \& \quad (\mathrm{Er})_{q \geq r}(r \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\gamma})]. \tag{12}$$

By Proposition 1.3, "$r \Vdash |\{e\}(\underline{a}, \mathscr{G})| = \underline{\gamma}$" is E-recursive in γ. By reflection, (12) is E-recursively enumerable in a, p. (Recall that parameter $\mathscr{P}$ is being suppressed.) Hence (11), for $q \leq p$, is co-E-recursively enumerable in a, p. Kechris's basis theorem (5.1.X) supplies a $q_0 \leq p$ such that (11) holds and $\kappa_r^{a,p,q_0} \leq \kappa_r^{a,p}$. But then

$$q_0 \Vdash^* (E\sigma)(\sigma = |\{e\}(\underline{a}, \mathscr{G})| \geq \kappa_r^{\langle \underline{a}, \underline{p}\rangle, \underline{q_0}}),$$

which contradicts (8) with p replaced by q_0 and a by $\langle a, p\rangle$. □

The next lemma relates bounds on κ_r to bounds on κ_0 in generic extensions. It is a useful general fact about forcing over E-closed structures.

3.2 Lemma. *Assume $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. If*

$$p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \leq \kappa_r^{p,a,\mathscr{P}},$$

then

$$p \Vdash^* |\{e\}(\underline{a}, \mathscr{G})| \leq \gamma$$

for some $\gamma \leq_E p, a, \mathscr{P}$.

Proof. By Lemma 5.5.X,

$$\kappa_r^{p,a,\mathscr{P}} \leq \kappa_r^{p,a,q,\mathscr{P}} \tag{1}$$

for all $q \in P$.

Assume $e = 2^m \cdot 3^n$. Then

$$(q)_{p \geq q} (\mathrm{Er})_{q \geq r} (E\delta)[r \mid\!\vdash |\{m\}(\underline{a}, \mathscr{G})| = \delta]. \tag{2}$$

Clearly the δ of (2) is less than $\kappa_r^{p,a,\mathscr{P}}$. It follows from (1) and reflection that the r and δ of (2) can be taken to be E-recursive in $p, a, q, \mathscr{P}$. By Corollary 4.5.X to Gandy selection, r and δ can be taken to be partial E-recursive functions of $p, a, q, \mathscr{P}$. As q ranges over all possible extensions of p, δ is bounded by some $\delta_0 \leq_E p, a, \mathscr{P}$. Hence

$$p \mid\!\vdash^* |\{m\}(\underline{a}, \mathscr{G})| \leq \underline{\delta}_0.$$

Fix $q \leq p$ and suppose

$$q \mid\!\vdash |\{m\}(\underline{a}, \mathscr{G})| = \underline{\delta} \tag{3}$$

for some δ. Then $\delta \leq_E q, p, a, \mathscr{P}$ by Proposition 1.3, since $\delta \leq \delta_0$. Let K_q^t be the set of all $r \leq q$ such that

$$r \mid\!\vdash t \in \{m\}(\underline{a}, \mathscr{G}) \qquad (t \in \mathscr{T}(q, m, a, \mathscr{G}, \delta)).$$

For each $r \in K_q^t$, there is an $s \leq r$ such that

$$s \mid\!\vdash |\{n\}(t)| = \underline{\sigma} \quad \text{for some} \quad \sigma < \kappa_r^{p,a,\mathscr{P}}.$$

The argument given above for δ_0 is now repeated three times. As q ranges over K_q^t, σ is bounded by some $\sigma_0 \leq_E t, q, p, a, \mathscr{P}$. As t ranges over $\mathscr{T}(q, m, a, \mathscr{G}, \delta)$, σ_0 is bounded by some $\sigma_1 \leq_E q, p, a, \mathscr{P}$. As q ranges over all extensions of p that satisfy (3), σ_1 is bounded by some $\sigma_2 \leq_E p, a, \mathscr{P}$.

Let γ be $\max(\sigma_2, \delta_0) + 1$. □

3.3 Exercise. Formulate and prove a theorem similar to 3.1 for class forcing.

4. *Countable-Chain-Condition Forcing*

Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ be a set notion of forcing. If $p, q \in P$ and

$$\sim (\mathrm{Er})[p \geq r \quad \& \quad q \geq r],$$

then p and q are said to be *incompatible*. A set of mutually incompatible conditions is called an antichain. $\mathscr{P}$ is said to satisfy the *countable chain condition* (c.c.c.) in $L(\kappa)$ if every antichain in $L(\kappa)$ is countable in $L(\kappa)$. In a moment it will be shown that c.c.c. forcing preserves E-closure. It will be seen from the proof that c.c.c. forcing differs radically from countably closed forcing. The former preserves the κ_r spectrum, but the latter does not (cf. Corollary 4.4 and Exercise 2.6).

The standard argument from set theory that finitely supported, iterated c.c.c. forcing is c.c.c. works without change inside $L(\kappa)$. An application of this fact to a problem in E-recursion is given in Section 5.

The next theorem proves more than is promised above. Let ρ be a regular cardinal in the sense of $L(\kappa)$. $\mathscr{P}$ is said to satisfy the *ρ-chain condition* in $L(\kappa)$ if every antichain in $L(\kappa)$ has cardinality less than ρ. (Note that the ω_1-chain condition is equivalent to the countable chain condition.) Selection plays a part in the study of forcing relations that fulfill the ρ-chain condition. Let γ be a cardinal in the sense of $L(\kappa)$. $L(\kappa)$ is said to obey *less-than-γ* selection if there exists a partial E-recursive (in γ and possibly another parameter) function f such that for all $e < \omega$, $\delta < \gamma$ and $p \in L(\kappa)$:

$$(\mathrm{Ex})_{x<\delta}[\{e\}(p,x)\downarrow] \to [f(e,\delta,p)\downarrow \quad \& \quad \{e\}(p,f(e,\delta,p))\downarrow].$$

Thanks to Gandy selection, $E(\omega_1^L)$ obeys less-than-ω_1 selection (cf. Exercise 4.6). In contrast $E(\omega_{\omega_1}^L)$ does not obey less-than-ω_2 selection (cf. Exercise XII.1.5). A theorem of Normann implies $E(\gamma)$ obeys less-than-γ selection when γ is a regular cardinal of L (cf. Theorem XII.1.3).

To grasp more quickly the proof of Theorem 4.1, assume $L(\kappa)$ is $E(\omega_1)$ and $\rho = \omega$. Then only cases 0 and 2 apply.

4.1 Theorem (Sacks 1986). *Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ a set notion of forcing. Assume $\mathscr{P}$ satisfies the ρ-chain condition, and $L(\kappa)$ obeys less-than-ρ selection. If there exist r and δ such that*

$$p \geq r \quad \textit{and} \quad r \Vdash |\{e\}(t)| = \underline{\delta},$$

then there exist such r and δ E-recursive in p, t, $\mathscr{P}$, ρ (and background parameters).

Proof. Let $c \leq \kappa$ be such that $t, \mathscr{P}, \rho \in L(c)$, and gc exists, the greatest cardinal in the sense of $L(c)$. All cofinalities below are in the sense of $L(c)$. For simplicity $\mathscr{P}$ and all background parameters such as gc or those needed for less-than-ρ selection are suppressed. (As usual $\mathscr{P}$ and all $p \in P$ come effectively coded as relations on ordinals.) Define

$$\min(p,e,t) \simeq \min_{\delta}(\mathrm{Er})_{p \geq r}[r \Vdash |\{e\}(t)| = \underline{\delta}].$$

An effective transfinite recursion on $\min(p,e,t)$, henceforth called the main recursion, will show that $\min(p,e,t)$ is E-recursive in p, t, ρ uniformly. Proving the theorem is equivalent to computing $\min(p,e,t)$ by Proposition 1.3.

Assume $e = 2^m \cdot 3^n$ and $\min(p, e, t)$ is defined. Then $\min(p, m, t)$ is defined and, by the main recursion, equals some $\gamma_0 \leq_E p, t, \rho$. Let

$$\mathscr{T}(p, m, t, \gamma_0) = \{t_i \mid i < \theta\}$$

for some $\theta \leq_E p, t, \rho$ such that $\theta \leq \text{gc}$.

For each $i < \theta$, Q_i will be a set of conditions that settles the fate of t_i. For all $q \in Q_i$:

(1a) $$p \geq q;$$

(1b) $$q \mid\!\vdash t_i \in \{m\}(t) \quad \text{or} \quad q \mid\!\vdash t_i \notin \{m\}(t);$$

(1c) $$\text{if} \quad q \mid\!\vdash t_i \in \{m\}(t), \text{ then } q \mid\!\vdash (E\delta)[\,|\{n\}(t_i)| = \delta\,].$$

Initially all Q_i's are empty. The final Q_i's will be such that

(2) $$\bigcap_{i<\theta} Q_i \neq \varnothing.$$

Intuitively (2) means there is a condition r such that for all i, the conditions in Q_i are *dense* in r. More precisely, (2) means

$$(\text{Er})(r_1)_{r \geq r_1}(i)_{i<\theta}(\text{Eq})(q \in Q_i \;\&\; r_1, q \text{ are compatible}).$$

The Q_i's are built up simultaneously by recursion on $\beta \leq \theta$, henceforth called the beta-recursion. During stage β conditions are added to Q_i for various $i < \beta$. At the end of stage β

$$\cap\{Q_i \mid i < \beta\} \neq \varnothing.$$

Stage β. There are four cases.

Case 0: $\beta < \rho$. A subrecursion of length ρ adds conditions to Q_i for various $i < \beta$. At stage $\beta_\gamma (\gamma < \rho)$ at most one q is added to at most one Q_i.

Stage β_γ. Suppose $\cap\{Q_i \mid i < \beta\} = \varnothing$ after all the additions prior to stage β, and prior to stage β_γ; otherwise go to the next stage. It is safe to assume each Q_i already contains all $q \leq p$ such that

$$q \mid\!\vdash t_i \notin \{m\}(t),$$

because all such q's can be added to Q_i at stage β_0. Q_i can be treated as if it were a condition by taking

$$Q_i \geq r \text{ to mean } (q_1)_{r \geq q_1}\,(\text{Eq})(q \in Q_i \;\&\; q_1, q \text{ are compatible}).$$

Thus (2) means $(\text{Er})(i)\,(Q_i \geq r)$, and for all $i < \beta$ the steps taken at stage β_0 imply

$$(p - Q_i) \mid\!\vdash^* t_i \in \{m\}(t)$$

unless $p - Q_i$ is not a condition (that is, there is no $q \leq p - Q_i$). Since $\min(p, e, t)$ is defined, there is an $r \leq p$ such that for all i,

$$r \Vdash^* [t_i \in \{m\}(t) \to (E\delta)(|\{n\}(t_i)| = \delta)].$$

But $p = \cup\{p - Q_i | i < \beta\}$, so there must be an $i < \beta$ such that

$$\min(p - Q_i, n, t_i) \tag{3}$$

is defined. According to the main recursion, (3), if defined, is E-recursive in $p - Q_i$, t_i, ρ uniformly. Note that $p - Q_i$ and t_i are E-recursive in certain parameters and i uniformly. Consequently less-than-ρ selection makes it possible to compute an $i < \rho$ for which (3) is defined; call it i_0 and let δ_0 be the value of (3) when $i = i_0$. From δ_0 a q_0 can be computed via Proposition 1.3 such that

$$p - Q_{i_0} \geq q_0 \quad \& \quad q_0 \Vdash t_{i_0} \in \{m\}(t) \quad \& \quad q_{i_0} \Vdash \{n\}(t_{i_0})| \leq \delta_0.$$

Add q_0 to Q_{i_0}.

For each $i < \beta$, the conditions added to Q_i as γ increases from 0 to ρ, are mutually incompatible. Since $\beta < \rho$, it follows from the ρ-chain condition that for all sufficiently large γ, nothing is added to any $Q_i (i < \beta)$. Thus for all sufficiently large γ, $\cap\{Q_i | i < \beta\}$ is nonempty at stage β_γ, hence nonempty at the end of stage β.

Case 1: $\mathrm{gc} > \beta \geq \rho > \mathrm{cf}(\beta)$. Similar to Case 0. Let β^+ be the least cardinal greater than β in the sense of $E(c)$. A subrecursion of length β^+ adds conditions to Q_i for various $i < \beta$. Since $\rho > \mathrm{cf}(\beta)$, there is a $W \subseteq \beta$ such that $\sup W = \beta$ and the ordertype of W is less than ρ. Define

$$Q_i^- = \cap\{Q_j | j < i\}.$$

Then $\cap\{Q_i^- | i \in W\} = \cap\{Q_j | j < \beta\}$. Stage β_γ ($\gamma < \rho \cdot \beta^+$) acts directly on at most one Q_i^- for some $i \in W$. The restriction of i to W makes it possible to use less-than-ρ selection as in case 0.

The augmentation of some Q_i^- ($i \in W$) at stage β_γ is performed with the aid of procedure P_i developed at stage i of the beta-recursion. This makes sense because $i < \beta$. P_i, given an empty Q_i^-, delivered a non-empty Q_i^-. P_i did its work by adding conditions to various Q_j's ($j < i$). P_i is effective because it is based on the main recursion. P_i converged because: if each Q_j is extended to a maximal antichain $Q_j^m \leq p$, then $\cap\{Q_j^m | j < i\}$ must weakly force $(E\delta)(j)_{j<i}[|\{n\}(t_i)| = \delta]$. P_i slowly builds up the antichains and stops as soon as their intersection is nonempty.

P_i^+ is a slight extension of P_i. It proceeds in precisely the same manner as P_i but is allowed to begin with a nonempty Q_i^-, denoted by start (Q_i^-). P_i^+ stops when

$$\text{finish } (Q_i^-) - \text{start } (Q_i^-) \neq \varnothing.$$

Thus P_i^+ is said to *augment* Q_i^-. P_i^+ converges iff such an augmentation is possible, that is iff

$$\cap\{Q_j^m \mid j < i\} - \text{start}(Q_i^-) \neq \varnothing.$$

Stage β_γ (for case 1). Suppose $Q_\beta^- = \varnothing$ after all the additions and augmentations prior to stage β, and prior to stage β_γ; otherwise go to the next stage. The argument given in Case 0 that (3) is defined now shows $P_i^+(Q_i^-)$ is defined for some $i \in W$. Thanks to the main recursion, $P_i^+(Q_i^-)$, if defined, is E-recursive in certain parameters and i uniformly. Consequently less-than-ρ selection computes such an i; call it i_0. Augment $Q_{i_0}^-$ to $P_{i_0}^+(Q_{i_0}^-)$.

Now consider the action of stage β_γ as γ increases from 0 to β^+. $Q_j (j < \beta)$ is always an antichain, so additions are made to Q_j at a set of stages of cardinality less than ρ. Hence for all sufficiently large γ, nothing is added to any $Q_j (j < \beta)$, and no $Q_i^- (i \in W)$ is augmented. Thus for all sufficiently large γ, Q_β^- is nonempty at stage β_γ, hence nonempty at the end of stage β.

Case 2: $\text{cf}(\beta) \geq \rho$. According to the β-recursion, $Q_i^- \neq \varnothing$ for all $i < \beta$. It follows from the ρ-chain condition that $Q_\beta^- \neq \varnothing$.

Case 3: $\text{gc} = \beta$ & $\rho > \text{cf}(gc)$. By Exercise 2.11.XII, there is a uniform method for choosing an element of a nonempty E-recursively enumerable (in p) subset of β, for $p \in E(c)$. Proceed as in Case 0.

At the end of the β-recursion, the end of stage θ, $Q_\theta^- \neq \varnothing$. In other words there is an $r_0 \leq Q_\theta^-$ such that

$$r_0 \Vdash^* |\{e\}(t)| \leq \underline{\delta}_0,$$

where δ_0 is the $\sup^+$ of γ_0 and all the values of (3) computed during the β-recursion. The value of $\min(p, e, t)$, needed for the main recursion, is computable from δ_0 with the aid of Proposition 1.3. □

4.2 Countable Chain Selection. The proof of Theorem 4.1 (ρ-chain condition forcing) should be contrasted with that of theorem 2.1 (countably closed forcing). The latter relies on Moschovakis witnesses and reflection phenomena, while the former is a selection argument that holds for a narrow class of Σ_1 relations. Consider the special case of $L(K) = E(\omega_1)$ and $\rho = \omega_1$. Let $\mathscr{P}$ be a countable-chain-condition notion of forcing in $E(\omega_1)$. The relation on p, e and t given by

$$(\text{Er})(p \geq r \Vdash (E\sigma)[|\{e\}(t)| = \sigma]) \tag{1}$$

is Σ_1 over $E(\omega_1)$. The proof of Theorem 4.1 shows how to compute an r (if there is one) that satisfies the matrix of (1) from p, t and $\mathscr{P}$. It follows that relation (1) is E-recursively enumerable on $E(\omega_1)$. Thus Theorem 4.1 is a selection theorem that shows certain Σ_1 relations, derived recursively from a chain condition, are E-recursively enumerable.

The absence of κ_r in the proof of Theorem 4.1 accounts for the difference between Exercise 2.6 and Corollary 4.4.

$\mathscr{P}$-genericity was defined in subsection 1.6. The point of Corollary 4.3 is at its sharpest when

$$L(\kappa) \models [\mathrm{gc}(\kappa) \text{ is singular}].$$

4.3 Corollary. *Suppose $L(\kappa)$ is E-closed and $\mathscr{P} \in L(\kappa)$ is a countable-chain-condition, set notion of forcing. Then every $\mathscr{P}$-generic extension of $L(\kappa)$ is E-closed.*

Proof. According to Exercise 4.6, Gandy selection implies $L(\kappa)$ obeys less than ω_1 selection. By Theorem 4.1 $\mathscr{P}$ satisfies effective bounding. Now apply Lemma 1.7. □

4.4 Corollary (Slaman). *Let $L(\kappa)$ be E-closed and $\mathscr{P} \in L(\kappa)$ a set notion of forcing. Assume $L(\kappa) = E(\mathrm{gc}(\kappa))$, $\mathscr{P}$ satisfies the ρ-chain condition, and $L(\kappa)$ obeys less-than-ρ selection. Then*

$$\kappa_r^{G,a} \leq \kappa_r^a$$

for all $\mathscr{P}$-generic G and $a \in L(\kappa)$. (Parameters ρ, $\mathrm{gc}(\kappa)$ and $\mathscr{P}$ are suppressed.)

Proof. For simplicity suppress $\mathscr{P}$, ρ and various background parameters. It suffices to show

$$\phi \Vdash^* [\kappa_0^{\mathscr{G},a} < \kappa_r^a]. \tag{1}$$

To prove (1) a maximal antichain is constructed. r_γ and $\delta_\gamma (\gamma < \rho)$ are defined by recursion on γ. Suppose there are r and δ such that

(1a) r is incomparable with r_β for all $\beta < \gamma$, and

(1b) $(\mathrm{Ee})(r \Vdash [|\{e\}(\mathscr{G}, \underline{a})| = \underline{\delta}])$.

Note that (1a) is equivalent to $r \Vdash (\beta)_{\beta < \gamma}\, (\mathscr{G} \notin r_\beta)$. Let $\langle r_\gamma, \delta_\gamma \rangle$ be such an r and δ computed from

$$\{\langle r_\beta, \delta_\beta, \beta \rangle | \beta < \gamma\}, a$$

with aid of Theorem 4.1 and Gandy selection. If there are no such r and δ, then $\langle r_\gamma, \delta_\gamma \rangle$ is undefined.

The ρ-chain condition implies $\langle r_\gamma, \delta_\gamma \rangle$ is undefined for some $\gamma < \rho$; let γ_∞ be the least such. γ_∞ is the least solution of a predicate co-E-recursively enumerable in a. By Exercise 5.17.X, a corollary to Kechris's basis theorem (5.1.X), $\kappa_r^{a,\gamma_\infty} \leq \kappa_r^a$. Let

$$\delta_\infty = \sup \{\delta_\gamma | \gamma < \gamma_\infty\}.$$

Then $\delta_\infty \leq_E \gamma_\infty, a$, and so $\delta_\infty < \kappa_r^a$. Finally

$$\phi \Vdash^* \kappa_0^{\mathscr{G},a} \leq \delta_\infty. \qquad \square$$

4.5–4.6 Exercises

4.5. Let $\mathscr{P}$ be Cohen forcing with finite conditions on a real. Suppose $L(\kappa)$ is E-closed and $G \subseteq \omega$ is $\mathscr{P}$-generic over $L(\kappa)$. Show

$$\kappa_0^{G,a} = \kappa_0^a$$

for all $a \in L(\kappa)$. (Note: do not assume $a \subseteq \kappa$.)

4.6. Let ρ be ω_1 in the sense of some E-closed $L(\kappa)$. Show $L(\kappa)$ obeys less-than-ρ selection.

5. *Normann Selection and Singular Cardinals*

Several definitions are needed for the sake of an application of countable-chain-condition forcing over $E(\omega_{\omega_1})$. Let R be an arbitrary class of sets. The Normann schemes, (1)–(6) of Section 1.X, relativize readily to R. Simply replace $\{\ \}$ by $\{\ \}^R$ throughout (1)–(6), and add a new scheme,

(7) $$\{e\}^R(x_1, \ldots, x_n) = R \cap x_1 \quad \text{if} \quad e = \langle 7, n \rangle.$$

A partial function f from V into V is *partial E-recursive relative to R* if there exists an e such that $f(x) \simeq \{e\}^R(x)$ for all $x \in V$. Thus "partial E-recursive" is equivalent to "partial E-recursive relative to the empty set".

Let A be a transitive set. A is said to be E-closed relative to R if

$$\vec{y} \in A \quad \& \quad f(\vec{y})\downarrow \rightarrow f(\vec{y}) \in A$$

for every f partial E-recursive relative to R. The E-closure of an arbitrary set x relative to R, in symbols $E(x; R)$, is the least E-closed (relative to R) $y \subseteq TC(\{x\})$. Observe that

$$E(x; R) = E(x; R \cap E(x; R)).$$

Assume $R \subseteq E(2^\omega)$ and

$$E(2^\omega; R) \models [2^\omega \text{ is wellordered}].$$

It follows there is a greatest cardinal in the sense of $E(2^\omega; R)$; call it gc. Assume

(1) $$E(2^\omega; R) \models [\text{gc is regular}].$$

Statement (1) is Norman's regularity assumption. It is the principal hypothesis in Normann's selection theorem and its extensions discussed in the next chapter. One

such extension has as its conclusion:

(2) If $\delta < gc$ and A is E-recursively enumerable relative to R, then $(A \cap \delta) \in E(2^\omega; R)$.

(2) is informally put as: every *small R–RE* class belongs to $E(2^\omega; R)$. (2) figured prominently in Normann's study of Post's problem. These matters will be discussed in greater detail in the next two chapters. The matter at hand is: is (1) necessary for (2)? The answer is no according to the next result.

5.1 Theorem. *There exists a model M of ZFC in which for some $R \subseteq \omega_{\omega_1} \times 2^\omega$:*

$$E(2^\omega; R) \models [\mathrm{card}\,(2^\omega) = \omega_{\omega_1}],$$

and every small R–RE class belongs to $E(2^\omega; R)$.

Proof. Let $L(\alpha)$ be a model of *ZFC*, and $\mathscr{P}$ be Cohen forcing with finite conditions designed to produce a generic real. $\mathscr{P}_{\omega_{\omega_1}}$ is the result of iterating $\mathscr{P}$ with finite support ω_{ω_1} times. Let $L(\kappa)$ be $E(\omega_{\omega_1})$. $\mathscr{P}_{\omega_{\omega_1}} \in L(\kappa)$ and satisfies the countable chain condition in $L(\kappa)$. Let R be $\mathscr{P}$-generic over $L(\alpha)$, hence over $L(\kappa)$. Then $L(\kappa, R) = E(R)$ by Corollary 4.1. Standard set-theoretic arguments show $L(\alpha, R)$ is a model of *ZFC*, call it M, with the same cardinals as $L(\alpha)$. Note that in M, $L(\kappa, R) = E(2^\omega; R)$.

A typical small *R–RE* class is

$$A = \{\gamma \mid \{e\}^R(\gamma)\downarrow \quad \& \quad \gamma < \delta\}, \tag{1}$$

where $\delta < \omega_{\omega_1}$. To see that $A \in E(2^\omega; R)$, let

$$A_\gamma = \{p \mid p \Vdash (E\sigma)[|\{e\}^{\mathscr{R}}(\gamma)| = \sigma]\}$$

for each $\gamma < \delta$. In $L(\alpha)$: A_γ is equivalent to a countable antichain A^0_γ; $\{\langle \gamma, A^0_\gamma \rangle \mid \gamma < \delta\}$ has the same complexity as a bounded subset of ω_{ω_1}, hence belongs to $L(\omega_{\omega_1})$. Then

$$\gamma \in A \leftrightarrow \gamma < \delta \quad \& \quad (\mathrm{Ep})[p \in A^0_\gamma \quad \& \quad R \text{ satisfies } p],$$

and so $A \in E(2^\omega; R)$. □

An application of iterated, countable-chain-condition forcing, deeper than that of Theorem 5.1, has been made by Slaman [1983]. He shows: if ZFC is consistent, then ZFC and $2^\omega = \omega_{\omega_1}$ and the extended plus-one hypothesis is consistent. See subsection 3.7.XII for the missing definitions. Normann proved the extended plus-one hypothesis with the aid of his regularity assumption, formula (1) of Section 5. Slaman [1983] showed that regularity is not essential by an almost-disjoint forcing

argument based on Harrington [1973]. In Slaman's model,

$$E(2^{\omega}) \models [\mathrm{card}\,(2^{\omega}) = \omega_{\omega_1}]$$

and the extended plus-one hypothesis holds. It is not known if the hypothesis fails in some model of ZFC in which

$$E(2^{\omega}) \models [2^{\omega} \text{ is wellordered}].$$

6. *Further Forcing*

Countably closed and countable chain condition forcings are special cases of Axiom A forcing, an invention of Baumgartner. Sacks [199?] shows that Axiom A forcing preserves E-closure, as does Shelah's proper forcing. Also given is an example of set forcing that preserves cardinals but not E-closure.

Some results on class forcing over E-closed structures are available. One such proves Theorem 6.1; the forcing conditions are E-pointed, perfect trees.

6.1 Theorem (Sacks & Slaman [1987]). *Let $L(\kappa)$ be countable, E-closed and not Σ_1 admissible with greatest cardinal* $\mathrm{gc}(\kappa)$. *If*

$$L(\kappa) \models [\mathrm{cf}(\mathrm{gc}(\kappa)) > \omega],$$

then for some $G \subseteq \omega_1^{L(\kappa)}$, $L(\kappa, G) = E(G)$.

A partial converse to Theorem 6.1 is proved in the next chapter, Corollary 5.2.XII. It states: if x is a set of ordinals and

$$E(x) \models [\text{cofinality of the greatest cardinal is } \omega],$$

then $E(x)$ is Σ_1 admissible.

Another use of class forcing over an E-closed structure, inspired by Steel forcing, has been found by Slaman. At one swoop he settles many questions.

6.2 Theorem (Slaman [1985]). *Let $L(\kappa)$ be countable and E-closed. There exists an $x \subseteq 2^{\omega}$ such that:*

(*i*) $L(\kappa, x) = E(x)$;
(*ii*) $L(\kappa, x)$ *does not admit Moschovakis witnesses*;
(*iii*) *if $L(\kappa)$ is Σ_1 admissible but not the E-closure of any $y \in L(\kappa)$, then $L(\kappa, x)$ is Σ_1 admissible, but Σ_1 over $L(\kappa, x)$ differs from E-recursively enumerable on $L(\kappa, x)$*;
(*iv*) *if $\kappa_r^a < \kappa$ for all $a \in L(\kappa)$, then $\kappa_r^a < \kappa$ for all $a \in L(\kappa, x)$*;
(*v*) *there is no Σ_1 formula $\mathscr{F}(x)$ such that for all $a \in L(\kappa, x)$,*

$$L(\kappa_r^a + 1, x) \models \mathscr{F}(\underline{a}) \quad \textit{but} \quad L(\kappa_r^a, x) \models \sim \mathscr{F}(\underline{a}).$$

The content of 6.2 is considerable. (i) says each countable E-closed ordinal is the least such relative to some set of reals. (ii) says the E-closure of a set of reals can be inadmissible yet not admit Moschovakis witnesses. Thus the assumption that x is a set of ordinals, in Thoerem 5.7.X, was needed. (v) says there is no uniform (in a) failure of Σ_1 reflection at $\kappa_r^a + 1$ in $L(\kappa, x)$ for some $x \subseteq 2^\omega$. It is tempting to think there is a uniform failure having to do with existence of Moschovakis witnesses. That idea succeeds in $E(2^\omega)$, as pointed out by Harrington [1973]. For each $a \subseteq \omega$, $\kappa_r^{a, 2^\omega}$ is the least ordinal that suffices to construct witnesses for all e such that $\{e\}(a, 2^\omega)\uparrow$. The idea also succeeds in $E(z)$ when z is a set of ordinals and $E(z)$ is inadmissible.

A question not answered by Theorem 6.2 is: is there a set x of ordinals and a $y \in E(x)$ such that $E(x)$ is inadmissible and $\kappa_r^y \notin E(x)$?

Chapter XII
Selection and k-Sections

Selection theorems, unlike so many results in recursion theory, have the virtue of being positive. They reveal uniformities not immediately apparent to the most discerning recursion theorist. They are often based on farfetched computations that converge slowly. A typical selection theorem addresses the following question. Suppose A is E-recursively enumerable in b, and $A \cap x$ is nonempty. Is there a uniform method for computing an element of $A \cap x$ from b, x? More generally, since x may not be effectively wellorderable, is there a uniform procedure for computing a nonempty $y \subseteq A \cap x$ from b, x? As a rule y is obtained by computing an ordinal θ such that the set of all computations from b of length at most θ suffices to enumerate some elements of $A \cap x$.

Let Selection (x) denote the following principle: there exists a partial E-recursive function f such that for all $e \leq \omega$ and all b,

$$(1) \qquad (\mathrm{Ez})_{z \in x}[\{e\}(z, b)\downarrow] \leftrightarrow f(e, b)\downarrow \quad \& \quad (\mathrm{Ez})_{z \in x}[|\{e\}(z, b)| \leq f(e, b)].$$

Gandy selection, Theorem 4.1.X, is equivalent to Selection (ω).

Selection (2^ω) and Selection (ω_1) are false. If the first were true, then $E(2^\omega)$ would be Σ_1 admissible, but the existence of Moschovakis witnesses in $E(2^\omega)$ gives rise to a $\Sigma_1^{E(2^\omega)}$ map from $\omega \times 2^\omega$ onto $E(2^\omega)$. Connections between selection and admissibility are made in Section 2 below.

Restricting the enumeration parameter b of (1) can make a difference, and even more so the introduction of a *special* parameter p into f. For example, Grilliot selection, established in Section 1, implies (1) when

$$x = 2^\omega, \quad b \in 2^{2^\omega} \quad \text{and} \quad p = 2^{2^\omega}.$$

(The $f(e, b)$ of (1) is replaced by $f(e, b, p)$.)

To sum up prematurely, a selection theorem involves a fixed set x and a collection C of procedures for enumerating elements of x. C is defined in terms of Gödel numbers e, enumeration parameters b, and additional predicates R. A typical member of C is $\{e\}^R(z, b)$, where z ranges over x. The selection theorem asserts the existence of a uniform method of computing a nonempty $y \subseteq x$ such that $\{c\}^R(z, b)\downarrow$ for all $z \in y$. y is completed from x, b, p; R, where p is a special parameter independent of b and R.

1. Grilliot Selection

Let f and g be functions with domain x and range $\subseteq \omega \times (2^x \times \{2^x\})$. For simplicity assume x is transitive, closed under pairing and $\omega \subseteq x$. Define $g \leq f$ by

$$(z)_{z\in x}[g(z) \text{ is an immediate subcomputation instruction of } f(z)].$$

Let $\min f = \min\{|f(z)||z \in x\}$.

Assume $\min f < \infty$. It was Grilliot's idea to compute $\min f$ from f, 2^x by recursion on the ordinals less than $\min f$. The recursion equation he had in mind was

$$\min f = \max_{g<f}{}^{+}(\min g). \tag{1}$$

An immediate obstacle to the recursion is:

$$\{g|g<f\} \quad \text{is not } E\text{-recursive in } f, 2^x.$$

It is overcome by approximating $\{g|g<f\}$ E-recursively in $f, 2^x$. For each ordinal β, $g <^\beta f$ is the β-th approximation of $g < f$. $g <^\beta f$ means: for all $z \in x$, $g(z)$ is seen to be an immediate subcomputation instruction of $f(z)$ by referring to the set of all computations (from $f, 2^x$) of length at most β. $f(z)$ is of the form $\langle e, \langle u, 2^x\rangle\rangle$, but from now on 2^x will often be suppressed. For example, suppose $f(z)$ is $\langle 2^m \cdot 3^n, u\rangle$. Then $\langle 2^m, u\rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ without any reference to computations. If $|\langle 2^m, u\rangle| < \infty$ and $y \in \{m\}(u)$, then $\langle n, y\rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ by referring to computations of length $\leq |\langle 2^m, u\rangle|$. Note that Exercise 1.6 makes it safe to regard every subcomputation instruction as being of the form $\langle d, u\rangle$ for some $u \subseteq x$. It follows from Lemma 2.7.X that

$$\{g|g <^\beta f\} \leq_E f, 2^x \quad \text{uniformly in } \beta, f, 2^x.$$

If β is too small, then the use of $<^\beta$ in place of $<$ in (1) may produce a false minimum, a value less than $\min f$.

Harrington and MacQueen [1976] completed Grilliot's idea. They introduced a notion of iteration that has proved valuable in the study of selection. Their idea is to apply (1) to generate a positive contribution to $\min f$. A sufficiently long sequence of such contributions adds up to $\min f$. An ordinal computed at a given stage can serve as β at the next stage.

The proof of Theorem 1.1 makes use of a set-theoretic choice principle, AC_x:

$$(f)[(z)_{z\in x}(f(z) \neq \phi) \to (\mathrm{E}g)(z)_{z\in x}(g(z) \in f(z))].$$

AC_{2^ω} holds if V, the class of all sets, satisfies ZFC. AC_{2^ω} need not hold in $L(2^\omega)$, the

natural setting for the axiom of determinateness. Nonetheless, ZF implies Theorem 1.1 holds for $x = 2^\omega$ in $L(2^\omega)$ (cf. Exercise 1.2).

1.1 Theorem (Grilliot 1969, Harrington & MacQueen 1976). *Let* $f: x \to \omega \times (2^x \times \{2^x\})$ *and suppose* $|f(z)| < \infty$ *for some* $z \in x$. *Then*

$$\left(\min_{z \in x} |f(z)|\right) \leq_E f, 2^x \quad \textit{uniformly.}$$

Proof. Note that $x \leq_E f$. Let θ be the supremum of all ordinals β such that there exists a map of x onto β. Observe that $\theta \leq_E 2^x$ thanks to the assumptions on x made at the beginning of Section 1.

$t(\gamma)$ is defined by recursion on $\gamma < \theta$ with the intention that $t(\gamma)$ equal $\min f$ for all sufficiently large γ. Thus $\min f$ will be $\sup\{t(\gamma) | \gamma < \theta\}$.

Stage γ. Let $t^-(\gamma) = \sup\{t(\delta) | \delta < \gamma\}$.
Case I: $t^-(\gamma) \geq \min f$. Set $t(\gamma) = \min f$.
Case II: $t^-(\gamma) < \min f$. Set

$$t(\gamma) = \max^+\{\min g | g <^{t^-(\gamma)} f \quad \& \quad t^-(\gamma) \leq \min g\}. \tag{1}$$

If case II holds and there is a g that satisfies the rightside condition of (1), then $t^-(\gamma) < t(\gamma)$. To find such a g, AC_x is invoked. For each $z \in x$, the sought-after g has the following properties: $g(z)$ is seen to be an immediate subcomputation instruction of $f(z)$ by examining computations of length $\leq t^-(\gamma)$; $t^-(\gamma) \leq |g(z)|$. To check there exists a suitable choice for $g(z)$, let $f(z)$ be $\langle 2^m \cdot 3^n, u\rangle$. $\langle m, u\rangle$ is seen to be an immediate subcomputation instruction of $f(z)$ without any examination of computations. If $t^-(\gamma) \leq |\langle m, u\rangle|$, then $\langle m, u\rangle$ can serve as $g(z)$. Otherwise there is a $y \in \{m\}(u)$ such that $t^-(\gamma) \leq |\langle n, y\rangle|$. If there were no such y, then $|f(z)|$ would be less than $\min f$.

Suppose case II holds for every $\gamma < \theta$. Then a partial map of x onto θ exists, contrary to the definition of θ. For each $z \in x$, let $f(z)$ be $\langle 2^{m_z} \cdot 3^{n_z}, u_z\rangle$. Call z troublesome if $f(z)\uparrow$ but $|\langle m_z, u_z\rangle| < \infty$. Fix $\gamma < \theta$. If $t^-(\gamma) \geq |\langle m_z, u_z\rangle|$ for every troublesome z, then $t(\gamma + 1) \geq \min f$ by reasoning as in the previous paragraph. Let z be troublesome and such that $t^-(\gamma) < |\langle m_z, u_z\rangle|$ and $|\langle m_z, u_z\rangle|$ has the least possible value. Then $t(\gamma) \geq |\langle m_z, u_z\rangle|$. Map z to γ. □

1.2 Exercise. Show Theorem 1.1 holds for $x = 2^\omega$ in ZF plus $V = L(2^\omega)$. (Replace AC_x by a "blurred" choice principle.)

The proof of Theorem 1.1 is unaffected by the presence of an additional class R (cf. Section 5.XI for $\{e\}^R(u)$). If the values of f are $\langle e, u; R\rangle$, then $\min f$ can be computed as above from $f, 2^x; R$.

A more sage observation concerns the role of 2^x in the proof of 1.1. The true power set of x is not needed. For example, let x be an ordinal less than some

cardinal in the sense of $L(\kappa)$. Thus $TC(x) = x$ and $(2^x \cap L(\kappa)) \in L(\kappa)$. The proof of 1.1 can now be repeated with the understanding that $f, g, \ldots$ are functions in $L(\kappa)$. θ becomes x^+, the next cardinal after x in $L(\kappa)$. If an additional class R is present, then some caution is necessary. R might blow up 2^x to a "proper class"; that is, $2^x \cap L(\kappa; R)$ might not be an element of $L(\kappa; R)$ (cf. Section 5.XI for the definition of $L(\kappa; R)$). Of course $L(\kappa; R)$ had better be E-closed relative to R. There can be trouble even if $R \subseteq x$. All goes well if $L(\kappa; R) \cap 2^x$ is an element of $L(\kappa; R)$. (These matters are discussed further in Section 5.)

The next theorem is an excellent example of the power of Grilliot's idea despite the absence of the true power set operation.

1.3 Theorem (Normann). *Suppose x is an unbounded subset of ρ and ρ is a regular cardinal in the sense of $E(x)$. If $\delta < \rho$ and $C \subseteq \delta$ is nonempty and E-recursively enumerable in x, then some element of C is E-recursive in δ, x uniformly.*

Proof. Suppose C is

$$\{z | z < \delta \quad \& \quad \{e\}(z, x)\downarrow\}.$$

Let $f(z)$ be $\langle e, \langle z, x\rangle\rangle$ for all $z < \delta$. To find an element of C, it suffices to compute $\min f$ as in the proof of Theorem 1.1 with some small changes. All functions such as f and g now map δ into $\omega \times (\rho \times \{x\})$. θ becomes ρ. Since ρ is regular, Hajnal's theorem implies each bounded subset of ρ in $E(x)$ belongs to $L(\rho, x)$. Hence $\{g | g <^{\beta} f\} \in E(x)$ for all $\beta \in E(x)$, and is in fact E-recursive in β, x, f uniformly. □

1.4 Corollary (Normann). *Suppose $W \in E(2^\omega)$ is a wellordering of 2^ω whose height is a regular cardinal in the sense of $E(2^\omega)$. If $a, c \in 2^\omega$ and $C \subseteq \{b | b <_W a\}$ is nonempty and E-recursively enumerable in c, 2^ω, then some $b \in C$ is E-recursive in a, c, W uniformly.*

Corollary 1.4 was devised by Normann for the sake of his work on Post's problem for $E(2^\omega)$, and has also proved useful in forcing arguments over $E(2^\omega)$. Not much is known about the role of regularity in 1.4. There is a model of ZFC, due to Slaman 1983, in which there is a wellordering W of 2^ω in $E(2^\omega)$ of singular height in the sense of $E(2^\omega)$, and in which the conclusion of 1.4 holds. It is not known if there is a model of ZFC in which there is a singular W in $E(2^\omega)$ as in Slaman's model, but in which the conclusion of 1.4 fails.

In Section 5 of this Chapter, Theorem 1.3 and Corollary 1.4 will be strengthened so as to allow an additional predicate. The Hajnal argument fails in the presence of an additional predicate, and consequently the Grilliot approach also fails.

1.5–1.6 Exercises.

1.5. Show $E(\omega^L_{\omega_1})$ does not obey less-than-ω_2 selection.

1.6. Assume z is transitive, closed under pairing, and $\omega \subseteq z$. Suppose $v \in z$. Show each immediate subcomputation instruction of $\langle e, \langle v, z\rangle\rangle$ is equivalent to one of the form $\langle d, \langle u, z\rangle\rangle$ for some $u \in z$. (cf. 2.8.X, 4.11.X and 2.7.XI.

2. *Moschovakis Selection*

Recall that Gandy selection implies $E(\omega)$ is Σ_1 admissible, or restated, $E(R(\omega))$ is Σ_1 admissible. ($R(0) = \phi$, $R(\alpha+1) = 2^{R(\alpha)}$, and $R(\lambda)$ is $\cup\{R(\alpha)|\alpha < \lambda\}$ for limit λ.) Moschovakis combined Gandy and Grilliot selection to show $E(R(\alpha))$ is Σ_1 admissible if α has cofinality ω inside $E(R(\alpha))$.

$k \in E(R(\alpha))$ is said to be an ω-sequence through α if dom $k = \omega$ and sup range $k = \alpha$.

2.1 Theorem (Moschovakis). *Suppose*

$$E(R(\alpha)) \models [\textit{cofinality } \alpha = \omega].$$

Let $k \in E(R(\alpha))$ be an ω-sequence through a. Assume f: $R(\alpha) \to \omega \times R(\alpha) \times \{R(\alpha)\}$. If $|f(x)| < \infty$ for some $x \in R(\alpha)$, then

$$\left(\min_x |f(x)|\right) \leq_E f, R(\alpha), k \quad (\textit{uniformly}).$$

Proof. A typical value of f is $\langle e, u, R(\alpha)\rangle$, where $u \in R(\alpha)$. For simplicity suppress the parameter $R(\alpha)$. The immediate subcomputation instructions of $\langle e, u\rangle$ are of the form $\langle c, v\rangle$, where $c < \omega$ and $v \in E(R(\alpha))$. The "Grilliot aspect" of the proof below requires that v be construed effectively as an element of $R(\alpha)$ and that can be managed by a one-one map of $\{e\}(u)$ into $R(\alpha)$. Exercise 2.7 supplies such a map by an effective transfinite recursion on $|\{e\}(u)|$. Thus the set of immediate subcomputation instructions of $f(x)$ can be written as

$$\{f(x)(y)|y \in S_x \subseteq R(\alpha)\}.$$

For example, consider $f(x) = \langle 2^m \cdot 3^n, u\rangle$. Take 0 to be in S_x and $f(x)(0)$ to be $\langle 2^m, u\rangle$. If $\{m\}(u)\downarrow$, then $f(x)(y)$ ranges over $\{\langle n, y\rangle | y \in \{m\}(u)\}$ as y ranges over $S_x - \{0\}$.

The procedure for computing min$f$ has countably many cases. If min$f < \infty$, then at least one case yields min$f$. Gandy selection weaves the cases into a single procedure. The intention is to compute min f by a recursion on the ordinals less than min f, referred to below as the main recursion.

Case $\langle i, j\rangle$. Assume: $i < j$,

$$\min f = |f(x_0)| \quad \text{for some } x_0 \in R(k(i)), \tag{1}$$

$$|f(x_0)| = \sup{}^+ \{|f(x_0)(y)| \, | \, y \in S_{x_0} \cap R(k(j))\}.$$

The method of Theorem 1.1 is used to build up min f from min g ($g < f$) via the main recursion. Now $g < f$ means:

$$\text{dom } g = R(k(i));$$

$$(x)\,[x \in \text{dom } g \to g(x) \in S_x \cap R(k(j))].$$

For each $x \in R(k(i))$, let S_x^β be the set of all $y \in S_x \cap R(k(j))$ such that $f(x)(y)$ is seen to be an immediate subcomputation instruction of $f(x)$ by examining computations of height at most β. $g <^\beta f$ now means:

$$(x)\,[x \in R(k(i)) \to g(x) \in S_x^\beta].$$

$\Pi\{S_x^\beta | x \in R(k(i))\}$ can be viewed as a subset of $R(k(j))^{R(k(i))}$, hence as a small subset of $R(\alpha)$. It follows that $\{g | g <^\beta f\}$ is E-recursive in f, β, $R(\alpha)$ uniformly. Thus the Grilliot idea, as described in Section 1, applies to this case because $R(\alpha)$ is closed under the power set operation. Let θ be the supremum of $|W|$ for all wellfounded W in $R(\alpha)$. Then $R(k(i))$ cannot be mapped onto θ, and so the argument of Theorem 1.1 can be repeated with $R(k(i))$ in place of x.

Case $\langle i, 0\rangle$. Assume (1) holds. Let

$$|f(x)|_j = \sup^+ \{|f(x)(y)| \,|\, y \in S_x \cap R(k(j))\}.$$

Assume $|f(x_0)|_j < |f(x_0)|$ for all $j < \omega$. For each $j < \omega$, let $f(x)_j$ be a computation instruction such that

$$|f(x)_j| = |f(x)|_j.$$

Define $g < f$ by:

$$\text{dom } g = R(k(i)) \quad \text{and}$$
$$(x)(x \in R(k(i)) \to g(x) \in \{f(x)_j \,|\, j < \omega\}.$$

Now proceed as in case $\langle i, j\rangle$. Observe that $\omega^{R(k(i))}$ can be construed as a small subset of $R(\alpha)$; hence $\{g | g <^\beta f\}$ is E-recursive in f, β, $R(\alpha)$ uniformly.

The above cases give rise to a function t with domain ω^2 such that: t is partial E-recursive in f, $R(\alpha)$, k; if case (i, j) holds, then $t(i, j)\downarrow$ and equals $\min f$. If $\min f < \infty$, then Gandy selection computes an i and j such that $t(i, j)\downarrow$. □

2.2 Grilliot Selection Functions. For each $\langle e, b\rangle \in E(x)$, let

$$K_{e,b}^x = \{z | \{e\}(z, x, b)\downarrow\} \cap TC(x).$$

f is said to be a Grilliot selection function for $E(x)$ if for all $\langle e, b\rangle$ in $E(x)$:

$$K_{e,b}^x \neq \phi \to \phi \neq f(e, b) \in E(x) \cap 2^{K_{e,b}^x}.$$

Note that f chooses a nonempty subset, rather than an element of, $K_{e,b}^x$. If some wellordering of $TC(x)$ is E-recursive in x, then f makes it possible to choose an element of $K_{e,b}^x$.

2.3 Lemma. *Let f be a Grilliot selection function for $E(x)$. If f is partial E-recursive in some $b \in E(x)$, then $E(x)$ is Σ_1 admissible.*

Proof. Let g be a many-valued $\Sigma_1^{E(x)}$ map with domain d. The task at hand is to find an $r \in E(x)$ so that $g(v)$ has a value in r for all $v \in d$. For simplicity assume $TC(x) = (TC(x))^{<\omega}$. Let t be a partial E-recursive (in x) map of $\omega \times TC(x)$ onto $E(x)$. t is a restriction of a universal partial E-recursive function. With the help of t, the range of g can be thought of as a subset of $TC(x)$. Let D be a Δ_0^{ZF} formula such that

$$g(v) = w \quad \text{iff} \quad E(x) \models (\text{Ey})D(v, w, y).$$

D may have parameters from $E(x)$. Every member of $E(x)$ is E-recursive in x, a for some $a \in TC(x)$, so it is safe to assume that the only parameters are x and a for some $a \in TC(x)$. For each $v \in d$, the set

$$K_v = \{z \mid t(z)\downarrow \quad \& \quad D(v, (t(z))_0, (t(z))_1)\}$$

is nonempty and E-recursively enumerable in v, x, a. f computes a nonempty subset of K_v from v. Then the desired r is

$$\{(t(z))_0 \mid (\text{Ev})(z \in f(v) \quad \& \quad v \in d)\}. \qquad \square$$

The next lemma is a partial converse of Lemma 2.3.

Lemma 2.4. *Suppose x is a set of ordinals and $E(x)$ is Σ_1 admissible. Then there exists a Grilliot selection function for $E(x)$ partial E-recursive via some $b \in E(x)$.*

Proof. According to Theorem 5.8.X there is a $y \in E(x)$ such that $\kappa_r^{x,y} \notin E(x)$. Since x is a set of ordinals, there is a $w \in E(x)$ such that w is a relation on ordinals and $\langle x, y\rangle \leq_E w$. ($w$ encodes the ε-relation restricted to $TC(\{\langle x, y\rangle\})$.) By reflection there is such a w E-recursive in x, y. Hence $\kappa_r^{x,y} = \kappa_r^w$.

Suppose $C \subseteq TC(x)$ is nonempty and E-recursively enumerable in x, a for some $a \in TC(x)$. $\kappa_r^{w,a} \geq \kappa_r^w$ by Lemma 5.5(iv).X. Since C has an element via a computation in $E(x)$ it has one via a computation of height less than $\kappa_r^{w,a}$, hence one recursive in w, a. Thus a nonempty subset of C can be computed from w, a. $\square$

Corollary 2.5 (Moschovakis). *Suppose*

$$E(R(\alpha)) \models [\textit{cofinality } \alpha = \omega].$$

Then $E(R(\alpha))$ is Σ_1 admissible.

Proof. Combine Theorem 2.1 and Lemma 2.3. $\square$

2.6 Admissibility of $E(R(\alpha))$. Assume $\alpha \geq \omega$. When is $E(R(\alpha))$ Σ_1 admissible? A complete answer to this question is presently unavailable. If α is a successor, then $R(\alpha)^\omega \subseteq R(\alpha)$, and so $R(\alpha)$ admits Moschovakis witnesses, and hence is not Σ_1 admissible (cf. exercise 2.8).

Assume α is a limit. If

(1) $$E(R(\alpha)) \models [\text{cofinality } \alpha = \omega],$$

then $E(R(\alpha))$ is Σ_1 admissible by Corollary 2.5. This result can be stretched a bit. Let g: $\lambda \to \alpha$ be a cofinality sequence for α in $E(R(\alpha))$. The proof of Theorem 2.1 will work with ω replaced by λ if Gandy selection is replaceable by less-than-λ^+ selection. (cf. Exercise 2.11.) The latter principle implies: there exists a uniform method of computing an element of a subset of λ, if that subset is nonempty and E-recursively enumerable in some $b \in E(R(\alpha))$.

If cofinality $\alpha > \omega$ in V, then $E(R(\alpha))$ admits Moschovakis witnesses and hence is not Σ_1 admissible. All that is actually needed here is: the $\Sigma_2^{E(R(\alpha))}$ cofinality of α is greater than ω.

Suppose $\Sigma_2^{E(R(\alpha))}$ cofinality of α equals ω and (1) is false. This is a dead zone as far as the question of Σ_1 admissibility of $E(R(\alpha))$ is concerned.

2.7–2.11 Exercises

2.7. Find a recursive function h such that for all x and u the following holds. If $u \in TC(x)$ and $\{e\}(TC(x), u)\downarrow$, then $\{h(e)\}(TC(x), u)\downarrow$ and its value is a one-one map of $\{e\}(TC(x), u)$ into $TC(x)$.

2.8. Show $E(R(\beta + 1))$ is not Σ_1 admissible for all $\beta \geq \omega$. Show $E(R(\omega_{\omega_1}))$ is not Σ_1 admissible.

2.9 (Grilliot). Assume α is a successor and $P(x, y)$ is an E-recursively enumerable (in $R(\alpha)$) relation on $R(\alpha)$. Show

$$(\mathrm{E}y)[y \in R(\alpha - 1) \quad \& \quad P(x, y)]$$

is an E-recursively enumerable (in $R(\alpha)$) relation on $R(\alpha)$.

2.10. Let α be $\omega_{\omega_1}^L$. Show there is no uniform method for selecting an element of a nonempty E-recursively enumerable (in α) subset of ω_1^L.

2.11. Suppose γ is the greatest cardinal in the sense of $E(\gamma)$, and is singular in $E(\gamma)$. Assume less-than-(cofinality γ)$^+$-selection holds as defined at the beginning of Section 4.XI. Show there is a uniform method for selecting an element of a nonempty E-recursively enumerable (in p) subset of γ, where $p \in E(\gamma)$.

3. *Plus-One Theorems*

In the manner of Kleene 1959 define

$$\begin{aligned} tp(0) &= \omega, \\ tp(\alpha + 1) &= 2^{tp(\alpha)}, \quad \text{and} \\ tp(\lambda) &= \cup \{tp(\alpha) \mid \alpha < \lambda\} \quad (\text{limit } \lambda). \end{aligned}$$

Note that $tp(\alpha)$ is transitive and $tp(\alpha) \subseteq tp(\beta)$ if $\alpha \leq \beta$. The elements of $tp(\alpha)$ are referred to as the *objects* of type α. $tp(\alpha)$ is equivalent to $R(\alpha + \omega)$. Suppose $0 < \delta < \alpha + 2$; the *δ-section* of a type $\alpha + 2$ F is:

$$\delta - sc(F) = \{z | z \in tp(\delta) \quad \& \quad z \leq_E tp(\alpha); F\}. \tag{1}$$

Note the semicolon (;) preceding F in (1). It indicates that F is present as an additional class. Thus "$z \leq_E tp(\alpha); F$" means $z = \{e\}^F(tp(\alpha))$ for some e (cf. Section 5.XI for further details of "; F".)

For all α, ${}^{\alpha+2}E$ is the function from $tp(\alpha + 1)$ into 2 defined by:

$${}^{\alpha+2}E(x) = \begin{matrix} 1 & \text{if } x \neq \varnothing \\ 0 & \text{if } x = \varnothing. \end{matrix}$$

${}^{\alpha+2}E$ is equivalent to a type $\alpha + 2$ object. In the language of Kleene 1959 an object F of type $\alpha + 2$ is said to be *normal* if ${}^{\alpha+2}E$ is recursive (in the sense of Kleene) in F. E-recursion theory has normality built in; equality is an E-recursive predicate. An important result of Kleene 1959 is:

$$1 - sc({}^2E) = \text{HYP}. \tag{2}$$

It is reasonable to think of $1 - sc({}^3E)$ as a continuation of the hyperarithmetic hierarchy of reals, since $E(tp(\delta))$ has a hierarchic structure imparted to it by Proposition 2.10.X. There are difficulties with this point of view, because some of the reals in $1 - sc({}^3E)$ result from computations of uncountable length. It is natural to ask, as Hinman and Shoenfield did, if there exists a type 2 object F whose 1-section is the same as that of ${}^3E$. An affirmative answer means that type 3 objects are not essential to the definition of $1 - sc({}^3E)$. In addition it follows that the members of $1 - sc({}^3E)$ can be arranged in a hierarchy that resembles HYP, since $E(\omega; {}^2E)$ is similar in form to $E(\omega; F)$. F can be regarded as a jump operator, and then $1 - sc({}^3E)$ is the result of iterating the F jump through the ordinals E-recursive in F.

Moschovakis has asked if it is possible to conclude anything about the type of F from the constitution of its δ-sections ($\delta < \alpha + 2 =$ type of F). The answer appears to be: "very little".

3.1 Theorem (Sacks 1974, 1977). *Let F be an object of type $\alpha + 2$. Assume: $\delta < \alpha + 2$; δ is a successor; and α is a successor if $\delta > 1$. Then there exists a G of type $\delta + 1$ such that*

$$\delta - sc(F) = \delta - sc(G).$$

The proof of Theorem 3.1 splits in two: $\delta > 1$ and $\delta = 1$. Proposition 3.5 and Theorem 3.6 cover $\delta = 1$.

3.2 Plus-One for $\delta > 1$. The proof is an application of Grilliot selection to an otherwise straightforward forcing construction. G is going to be a function from $tp(\delta) \times tp(\delta - 1)$ into 2. Strictly speaking G is not of type $\delta + 1$, but G is certainly equivalent to an object of type $\delta + 1$. For each such G and $x \in tp(\delta)$, let

$$g(x) = \{y \mid y \in tp(\delta - 1) \quad \& \quad G(x, y) = 0\}. \tag{1}$$

Thus g maps $tp(\delta)$ into $tp(\delta)$. For any set b and class C the structure $L(\beta, b; C)$ is defined by recursion on β.

$L(0, b; C) = TC(b)$.

$L(\beta + 1, b; C) =$ set of all sets first order definable over $L(\beta, b; C)$; "$x \in C$" is added to the atomic formulas of ZF.

$L(\lambda, b; C) = \cup \{L(\beta, b; C) \mid \beta < \lambda\}$.

For example, if $x \in L(\beta, tp(\delta - 1); G)$, then $g(x) \in L(\beta + 1, tp(\delta - 1); G)$.

Let γ be the ordertype of a wellfounded relation whose field is $tp(\delta - 1)$, and which is E-recursive in $tp(\alpha)$; F. Let $\lambda_{\delta-1}^{tp(\alpha); F}$ be the sup of all such γ's. $\kappa_0^{tp(\delta-1); G}$ is going to be $\lambda_{\delta-1}^{tp(\alpha); F}$. The plan is to construct $L(\kappa_0^{tp(\delta-1); G}, tp(\delta - 1); G)$ in countably many steps.

$p, q, r, \ldots$ are forcing conditions. p specifies an object of the form $L(\beta, t(\delta - 1); H)$, where H maps $tp(\delta) \times tp(\delta - 1)$ into 2. The specification includes the diagram of $L(\beta, tp(\delta - 1); H)$, the set of all sentences true in $L(\beta, tp(\delta - 1); H)$ in the language of ZF with names for H and all the members of $L(\beta, tp(\delta - 1); H)$. p of course does not specify all of H, but only the partial object

$$L(\beta, tp(\delta - 1); H) \cap H. \tag{2}$$

Denote (2) by H_p. If p specifies $L(\beta, tp(\delta - 1); H)$, then

$$L(\beta, tp(\delta - 1); H) = L(\beta, tp(\delta - 1); H_p).$$

A forcing condition p must have the following properties.

(3) (i) p specifies $L(\beta_p, tp(\delta - 1); H_p)$.

(ii) p is first order definable over $L(\delta, tp(a); F)$ for some $\delta \leq_E tp(\alpha)$; F. The first order definition takes only $tp(\alpha)$ and F as parameters; thus it is specifiable by a Gödel number $< \omega$. (Hence $p \leq_E tp(\alpha)$; F.)

(iii) p has a *top*; namely, for some e

$$|\{e\}^{H_p}(tp(\delta - 1))| = \beta_p. \tag{4}$$

The presence of a partial, rather than total, object in (4) is legitimate. The computation of $\{e\}^{H_p}(tp(\delta - 1))$ calls values of H_p only for arguments in $L(\beta_p, tp(\delta - 1); H_p)$.

Clause (3)(iii) insures that $\beta_p \leq_E tp(\delta-1); H_p$.

Define $p \geq q$ to be: $\beta_p \leq \beta_q$ and $H_q{\restriction} p = H_p$.

The forcing relation $\Vdash$ treats the logical connectives in standard fashion. In addition:

$$p \Vdash \{e\}^{\mathcal{G}}(\underline{z})\downarrow \quad \text{iff} \quad L(\beta_p, tp(\delta-1); H_p) \models \{e\}^{\mathcal{G}}(\underline{z})\downarrow.$$

($z \in L(\beta_p, tp(\delta-1); H_p)$, $\mathcal{G}$ is interpreted as H_p and $\underline{z}$ as z.)

$$p \Vdash \{e\}^{\mathcal{G}}(\underline{z})\uparrow \quad \text{iff} \quad (q)_{p \geq q} \sim [q \Vdash \{e\}^{\mathcal{G}}(\underline{z})\downarrow].$$

A generic sequence $\{p_n | n < \omega\}$ is constructed by stages. $\cup\{p_n | n < \omega\}$ will specify $L(\kappa_0^{tp(\delta-1);G}, tp(\delta-1); G)$.

Stage 0. $p_0 = \varnothing$.

Stage $n+1$. Let $e = (n)_1$.

Case 1: n is even. The purpose of this case is to settle whether or not $\{e\}^{\mathcal{G}}(tp(\delta-1))$ converges.

Subcase 1a: there is a $p \leq p_n$ such that

$$p \Vdash \{e\}^{\mathcal{G}}(tp(\delta-1))\downarrow. \tag{4b}$$

Such a p can be chosen to be p_{n+1} via Gandy selection, since (4b) is E-recursively enumerable.

Subcase 1b: otherwise. Then p_{n+1} is p_n.

Case 2: n is odd. Let z_e be the e-th member of the δ-section of F. The purpose of this case is to add z_e to the δ-section of G. According to (3)(iii) there is an m such that

$$|\{m\}^{H_{p_n}}(tp(\delta-1))| = \beta_{p_n}. \tag{5}$$

Let x_n denote $L(\beta_{p_n}, tp(\delta-1); H_{p_n})$. (5) implies x_n is E-recursive in $tp(\delta-1); H_{p_n}$. As in Exercise 3.8, there is a $y_n \subseteq tp(\delta-1)$ such that $y_n \notin x_n$ and y_n is first order definable over x_n. The definition in question is lightface; its only parameter is $tp(\delta-1)$ and it treats H_{p_n} as an additional predicate. Extend p_n to p_{n+1} so that

$$\beta_{p_{n+1}} = \beta_{p_n} + 1, \quad \text{and} \tag{6}$$

$$H_{p_{n+1}}(y_n, y) = 0 \leftrightarrow y \in z_e.$$

Then z_e can be computed from $tp(\delta-1); H_{p_{n+1}}$ with the aid of (5) and (6).

Let $p_\infty = \lim_n p_n$, $\beta_\infty = \sup \beta_{p_n}$ and $H_\infty = \cup\{H_{p_n} | n < \omega\}$. Thus p_∞ specifies $L(\beta_\infty, tp(\delta-1); H_\infty)$. Clause (ii) of (3) implies $\beta_\infty \leq \lambda_{\delta-1}^{tp(\alpha);F}$. To see that $\beta_\infty \geq \lambda_{\delta-1}^{tp(\alpha);F}$, let z_e be a wellfounded relation on $tp(\delta-1)$ E-recursive in $tp(\alpha); F$. Case 2 insures that $z_e \leq_E tp(\delta-1); H_{p_n}$ for some n. Case 1 guarantees that $|z_e|$, the ordertype of z_e, is computed from $tp(\delta-1); H_{p_n}$ for some n. Hence $|z_e| \leq \beta_n$ for some n.

Let $G = H_\infty$ on domain H_∞, and 0 elsewhere. Then $\kappa_0^{t(\delta-1);G} \geq \beta_\infty$, and by case 2, the δ-section of F is the set of all $z \in tp(\delta)$ such that $z \leq_E tp(\delta-1)$; G via a computation of height $< \beta_\infty$.

It remains only to show $\kappa_0^{t(\delta-1);G} \leq \beta_\infty$. Suppose not. Then for some d,

$$\infty > |\{d\}^G(tp(\delta-1))| \geq \beta_\infty.$$

Let

(7a) $\quad \gamma_0 = |\{d\}^G(tp(\delta-1))|, \quad$ and $\quad$ let r specify $L(\gamma_0, tp(\delta-1); G)$.

Observe that r satisfies clauses (i) and (iii) of (3) (with r in place of p), and also (7b) and (8).

(7b) $\quad r$ is first order definable over $L(\delta, tp(\alpha); F)$ for some δ.

(8) $\quad r \leq_E b, tp(\alpha); F \quad$ for some $b \in tp(1)$.

$\kappa_0^{tp(\alpha);F}$ can serve as δ in (7), partly because any object of the form

$$L(\gamma, tp(\delta-1); G) \quad (\gamma < \kappa_0^{tp(\delta-1);G})$$

can be construed as an object of type α. H_∞ was defined over $\kappa_0^{tp(\alpha);F}$. The b of (8) is a map from ω into ω such that for all n, $p_n = \{b(n)\}^F(tp(\alpha))$. r is not quite a forcing condition; nonetheless it makes sense to say

(9) $$r \Vdash \{\underline{d}\}^{\mathcal{G}}(t(\delta-1))\downarrow.$$

Fix n so that $(n)_1 = d$ and n is even. Let Q be the set of all $q \leq p_n$ that satisfy (3)(i) and (3)(iii) (with q in place of p), and (7) (with q in place of r). Then

$$L(\kappa_0^{b,tp(\alpha);F}, b, tp(\alpha); F) \cap Q \neq \varnothing$$

It follows from Lemma 3.3 (simple reflection), a consequence of Grilliot selection, that

$$\kappa_r^{tp(\alpha);F} \geq \kappa_0^{b,tp(\alpha);F}.$$

Consequently there is a $q \in Q$ such that $q \leq_E tp(a); F$. One such q can be found by first minimizing the δ of (7b) and then minimizing the Gödel number of the required first order definition. But then subcase (1a) of stage $n+1$ guarantees that

$$|\{d\}^G(tp(\delta-1))| < \beta_\infty. \quad \square$$

Lemma 3.3 (Sacks 1974). *Assume α is a successor and F is an object of type $\alpha+2$. Then*

$$\kappa_r^{tp(\alpha);F} \geq \kappa_0^{a,tp(\alpha);F}$$

for all $a \in tp(\alpha-1)$. (*Simple reflection*)

Proof. Theorem 1.1 (Grilliot selection) is easily revised to show: let f map $tp(\alpha-1)$ to values of the form $\{e\}^F(y, tp(\alpha))$ $(y \in tp(\alpha))$ and suppose $|f(z)| < \infty$ for some $z \in tp(\alpha-1)$; then

$$\left(\min_z |f(z)|\right) \leq_E f, tp(\alpha); F.$$

Let $\mathscr{S}$ be a Σ_1^{ZF} sentence whose only parameters are $tp(\alpha)$ and F. Suppose

$$(\mathrm{Ea})\,[a \in tp(\alpha-1) \quad \& \quad L(\kappa_0^{a,tp(\alpha);F}, tp(\alpha); F) \models \mathscr{S}\,].$$

Thus there is an $a \in tp(\alpha-1)$ and a $\delta \leq_E a, tp(\alpha); F$ such that

$$L(\delta, tp(a); F) \models \mathscr{S}.$$

Grilliot selection implies the least such δ is E-recursive in $tp(\alpha); F$. □

3.4 Plus-One for $\delta = 1$. Suppose Z is a countable subset of $tp(1)(=2^\omega)$. Z is said to be an *abstract 1-section* if there is a Σ_1 admissible set A satisfying Σ_1 dependent choice and such that $Z = 2^\omega \cap A$. Note that Z is an abstract 1-section iff Z_A, the set of sets coded by reals in Z, satisfies Σ_1 dependent choice and hence is Σ_1 admissible (cf. Exercise 3.9). Proposition 3.5 and Theorem 3.6 imply the plus-one theorem for $\delta = 1$.

3.5 Proposition. *If F is an object of type $\alpha+2$, then $1-sc(F)$ is an abstract 1-section.*

Proof. Let w be a real and y a set. Let "w codes y" mean: w codes the diagram of $\langle TC(\{y\}), \varepsilon\rangle$. Then the relation, w codes y, is E-recursively enumerable.

Let A be the set of sets coded by reals in $1-sc(F)$. Suppose

$$\langle A, \varepsilon\rangle \models (\mathrm{Ey})D(\underline{a}, y)$$

for some $a \in A$ and $\Sigma_1^{ZF} D$. The set

$$(1) \qquad \{w \mid w \in 1-sc(F) \quad \& \quad w \text{ codes } y \quad \& \quad D(\underline{a}, y)\}$$

is E-recursively enumerable in $a, tp(\alpha); F$ uniformly. By Gandy selection there is a method for computing an element of (1) from $a, tp(\alpha); F$ uniformly. The method, iterated ω times, establishes Σ_1 DC in A. □

3.6 Theorem (Sacks 1970). *If Z is an abstract 1-section, then there exists a type 2 G such that $1-sc(G) = Z$.*

Proof. G is going to be a function from $2^\omega \times \omega$ into 2. The construction of G roughly resembles that of Section 3.2. The reflection argument that concludes 3.2 will be replaced by an application of Σ_1 dependent choice within Z_A, the set of all sets coded by reals in Z. Consequently the details of the forcing argument now matter more than they did in 3.2. As before a forcing condition p has the following properties.

(1) (i) p specifies $L(\beta_p, \omega; H_p)$, and $L(\beta_p, \omega; H_p) \models (x)$ (x is countable).
(ii) $p \in Z_A$.
(iii) p has a *top*; namely, for some m

$$|\{m\}^{H_p}(\omega)| = \beta_p.$$

The second half of (i) can be deleted (cf. Exercise 3.11). The forcing relation $\Vdash$ is defined as in 3.2 with some extra clauses for bounding scheme situations.

(2) $$p \Vdash \{2c \cdot 3d\}^{\mathcal{G}}(\omega)\downarrow \quad \text{iff}$$

(3) $$p \Vdash \{c\}^{\mathcal{G}}(\omega)\downarrow \quad \text{and}$$

(4) $$p \Vdash (x)[x \in \{c\}^{\mathcal{G}}(\omega) \to \{d\}^{\mathcal{G}}(\omega, x)\downarrow].$$

By Exercise 2.7 it is safe to regard $\{c\}^G(\omega)$ as a subset of ω. Thus if (3) holds, then (4) means:

(5) $$(q)_{p \geq q}(i)_{i \in \omega}(\mathrm{Er})_{q \geq r}[r \Vdash i \notin \{c\}^{\mathcal{G}}(\omega) \quad \text{or} \quad r \Vdash \{d\}^{\mathcal{G}}(\omega, i)\downarrow].$$

The generic sequence $\{p_n | n < \omega\}$ is constructed by stages as in 3.2.
Stage 0. $p_0 = \varnothing$.
Stage $n+1$. let $e = (n)_1$.
Case 1: n is even.
Subcase 1a: there is a $p \leq p_n$ such that

$$p \Vdash \{e\}^{\mathcal{G}}(\omega)\downarrow.$$

Any such p will do for p_{n+1}.
Subcase 1b: otherwise. If e is not of the form $2^c \cdot 3^d$ let p_{n+1} be p_n. Otherwise there is a q such that either

(6) $$q \Vdash \{c\}^{\mathcal{G}}(\omega)\uparrow$$

or for some $i < \omega$,

(7) $$q \Vdash \{c\}^{\mathcal{G}}(\omega)\downarrow \quad \text{and}$$
$$q \Vdash i \in \{c\}^{\mathcal{G}}(\omega) \quad \text{and} \quad q \Vdash \{d\}^{\mathcal{G}}(\omega, i)\uparrow.$$

Any such q will do for p_{n+1}.

Case 2: n is odd. Let z_e be the e-th element of Z. Let x_n denote $L(\beta_{p_n}, \omega; H_{p_n})$. Since p_n has a top, β_{p_n} is the length of a computation from $\omega; H_{p_n}$. As in Exercise 3.8, there is a $y_n \subseteq \omega$ such that $y_n \notin x_n$ and y_n is first order definable over x_n. The definition in question is lightface; its only parameter is ω and it treats H_{p_n} as an additional atomic predicate. Extend p_n to p_{n+1} so that

$$\beta_{p_{n+1}} = \beta_{p_n} + 1, \quad \text{and} \quad H_{p_{n+1}}(y_n, y) = 0 \leftrightarrow y \in z_e. \tag{8}$$

Then z_e can be computed from $\omega; H_{p_{n+1}}$.

Let $p_\infty = \lim_n p_n$, $\beta_\infty = \sup \beta_{p_n}$ and $H_\infty = \cup \{H_{p_n} | n < \infty\}$. Thus p_∞ specifies $L(\beta_\infty, \omega; H_\infty)$. Let λ_0^Z be the least ordinal not in Z_A. Clause (ii) of (1) implies $\beta_\infty \leq \lambda_0^Z$. To check that $\beta_\infty \geq \lambda_0^Z$, let $z_e \in Z_A$ be a wellfounded relation on ω. Case 2 insures $z_e \leq_E \omega; H_{p^n}$ for some n. Case 1 guarantees that $|z_e|$, the ordertype of z_e, is computed from $\omega; H_{p_n}$ from some n, hence $|z_e| \leq \beta_{p^n}$ for some n.

Let $G = H_\infty$ on the domain of H_∞, and 0 elsewhere. Then $\kappa_0^{\omega; G} \geq \beta_\infty$, and by case 2,

$$2^\omega \cap L(\beta_\infty, \omega; G) = Z_A.$$

It need only be shown that $\kappa_0^{\omega; G} \leq \beta_\infty$. Suppose not. Then there is an e such that

$$|\{e\}^{\mathcal{G}}(\omega)| = \beta_\infty. \tag{9}$$

Since β_∞ is a limit, e can be presumed to be $2^c \cdot 3^d$. Go to stage $n+1$ for n even and $(n)_1 = e$. If subcase (1b) holds, then either (6) or (7) holds (with p_{n+1} in place of q), contrary to (9). Hence subcase (1a) holds. Thus

$$p_{n+1} \Vdash |\{c\}^{\mathcal{G}}(\omega)| \leq \beta_{p_{n+1}},$$

and (5) holds with p_{n+1} in place of q. The matrix of (5) is Σ_1 over Z_A. Since Σ_1 DC holds in Z_A, there is a function $\lambda_i | r_i$ in Z_A such that $p_{n+1} = r_0$ and

$$(i)_{i \in \omega} [r_i \geq r_{i+1} \quad \& \quad (r_{i+1} \Vdash i \notin \{c\}^{\mathcal{G}} \quad \text{or} \quad r_{i+1} \Vdash \{d\}^{\mathcal{G}}(\omega, i)\downarrow)]. \tag{10}$$

Let $r_\infty = \cup \{r_i | i < \omega\}$. r_∞ specifies $L(\beta_{r_\infty}, \omega; H_{r_\infty})$. r_∞ is an excellent candidate for a condition that forces

$$|\{e\}^{\mathcal{G}}(\omega)| \leq \max^+(\beta_{p_{n+1}}, \beta_{r_\infty}) < \beta_\infty,$$

thereby contradicting (9). The only difficulty is the possibility that r_∞ does not have a top. It is handled by "minimizing" β_{r_i}. For each i, require that no proper initial segment of r_{i+1} satisfy the matrix of (10). A *proper initial segment* of r_{i+1} is a condition that specifies a proper initial segment of $L(\beta_{r_{i+1}}, \omega; H_{r_{i+1}})$. It follows that $\lambda_i | r_i$ is Σ_1 definable over $L(\beta_{r_\infty}, \omega; H_{r_\infty})$ but is not a member of it. Therefore

$L(\beta_{r_\infty}, \omega; H_{r_\infty})$ is not Σ_1 admissible, hence not E-closed by Gandy selection (cf. Exercise 3.10). In other words r_∞ has a top. □

3.7 Extended Plus-One. Suppose $1 \leq \delta < \alpha + 2$ and F is a type $\alpha + 2$ object. The extended δ-section of F is denoted by $ex - \delta - sc(F)$, and equals

$$\{z \mid z \in tp(\delta) \quad \& \quad (\mathrm{Eb})_{b \in tp(\delta - 1)} \quad (z \leq_E b, tp(\alpha); F)\}.$$

If $\delta = 1$, then $ex - \delta - sc = \delta - sc$. The extended 2-section of ${}^3E$ is simply $2^{2^\omega} \cap E(2^\omega)$ and conveys the same information as $E(2^\omega)$. It is hard not to ask: does there exist a type 3 G such that

$$ex - 2 - sc(G) = ex - 2 - sc({}^4E)?$$

Harrington has obtained a negative answer by assuming AD. On the other hand a positive answer follows from $2^\omega = \omega_1$. Normann has obtained a positive answer if (i) $2^\omega = \omega_2$ and Martin's axiom holds, or if (ii) there is a wellordering W of 2^ω in $E(2^{2^\omega})$ such that $|W|$ is regular inside $E(2^{2^\omega})$. Slaman has found a generic extension of L in which: (a) the answer is yes, and (b) there is a wellordering V of 2^ω in $E(2^{2^\omega})$ such that $|V|$ is a singular cardinal in $E(2^{2^\omega})$.

At this writing it is not known if there is a generic extension of L in which the answer is no and some wellordering of 2^ω belongs to $E(2^{2^\omega})$.

Hoole (1982) defines an abstract extended 2-section and shows every such is the extended 2-section of a type 3 object.

3.8–3.12 Exercises

3.8. Suppose $\beta = |\{e\}(x)|$. Show there is a $y \subseteq TC(\{x\})$ such that $y \notin L(\beta, TC(\{x\}))$ and y is first order definable over $L(\beta, TC(\{x\}))$ via a definition whose only parameter is x.

3.9. Let w be a real and y a set. w is said to code y if w is a binary relation isomorphic to $\langle TC(\{y\}), \varepsilon\rangle$. Suppose Z is an abstract 1-section. Show Z_A, the set of all sets coded by reals in Z, satisfies Σ_1 DC:

$$(x)(\mathrm{Ey})R(x, y) \to (\mathrm{Ef})(n)_{n < \omega} R(f(n), f(n+1)),$$

where R is Σ_1 over Z_A.

3.10. Suppose the structure $L(\beta, \omega; H)$ is $E(\omega; H)$; show it is Σ_1 admissible.

3.11. Show that the second half of 3.6(1)(i) follows from the first half and 3.6(1) (ii)–(iii). (Forcing condition simplification.)

3.12. Let D^1_2 be the set of all lightface Δ^1_2 subsets of ω. Show that D^1_2 is the 1-section of some type 2 object. (Also true for D^1_n when $n > 2$.)

4. *Harrington's Plus-Two Theorem*

Assume $\delta < \alpha+2$, δ is a successor, and F is a type $\alpha+2$ object. The *δ-envelope* of F is denoted by $\delta-en(F)$, and is defined (following Moschovakis) to be

(1) $$\{z \mid z \in tp(\delta) \quad \& \quad z \text{ is } E\text{-r.e. in } tp(\delta-1),\ tp(\alpha);\ F\}.$$

Note the occurrence of "$tp(\delta-1)$" on the extreme right of (1); it is superfluous if $\delta \leq_E tp(\alpha);\ F$; thus it is omitted if $\delta < \omega$. The δ-envelope carries the same information about F as the complete E-recursively enumerable (in $tp(\delta-1)$, $tp(\alpha);\ F$) subset of $tp(\delta-1)$. Early on Moschovakis observed that the 1-envelope of ${}^3E$ is not the 1-envelope of any type 2 object (cf. Exercise 4.5). It follows from Harrington's plus-two theorem that the 1-envelope of ${}^4E$ is the 1-envelope of a type 3 object. His argument requires Theorem 4.1, a reflection result analogous to Corollary 8.2.III.

Some of the notions occurring in 4.1 and its applications need elaboration. $P(X)$ is a formula of class-set theory whose only class variable is the free variable X. $P(X)$ is built up from the atomic formulas of ZF, and additional formulas of the form $y \in X$, by means of set quantifiers and logical connectives.

Suppose $\gamma < \kappa^{tp(\alpha);F}$ and $a \in tp(\alpha)$. γ is said to be E-constructive in a, $tp(\alpha)$; F if

$$\gamma = |\{e\}^F(a, tp(\alpha))|$$

for some e. Constructive in $a, tp(\alpha);\ F$ implies recursive in $a, tp(\alpha);\ F$, but not conversely (cf. Exercise 4.6). Of course each recursive ordinal is less than some constructive.

Suppose $A \in \delta-en(F)$. Thus

(1) $$A = \{b \mid \{e\}^F(b, tp(\delta-1), tp(\alpha);\ F)\downarrow\}$$

for some e. A can be thought of as being enumerated by stages. At stage σ enumerate all elements of $tp(\delta-1)$ that belong to A according to (1) by virtue of computations of length σ; the set of all such elements is A_σ. Thus A is enumerated without repetitions at stages E-constructive in b, $tp(\delta-1)$, $tp(\alpha);\ F$ as b ranges over A. For all γ define

$$A_{<\gamma} = \cup\{A_\sigma \mid \sigma < \gamma\}.$$

Let $\kappa_{\delta-1}^{tp(\delta-1),\,tp(\alpha);\,F}$ be the supremum of all ordinals E-constructive in b, $tp(\delta-1)$, $tp(\alpha);\ F$ as b ranges over $tp(\delta-1)$. Thus

$$A = A_{\kappa_{\delta-1}^{tp(\delta-1),\,tp(\alpha);\,F}}.$$

Theorem 4.1 (Harrington 1973). *Assume $\delta \leq \alpha$, δ and α are successors, and F is a type $\alpha+2$ object. Suppose $A \in \delta-en(F)$ and $b \in tp(\delta-1)$.* If

$$\{e\}^F(A, b, tp(\delta-1), tp(\alpha);\ F)\downarrow$$

then

$$\{e\}^F(A_{<\gamma}, b, tp(\delta-1), tp(\alpha); F)\downarrow$$

for some γ *E-constructive in* $(c, tp(\delta-1), tp(\alpha); F)$ *for some* $c \in tp(\delta-1)$ (*further reflection*).

If Theorem 4.1 seems too farfetched, let α and δ be 1, F be ${}^3E$, and A the complete, E-recursively enumerable (in 2^ω) subset of ω. Then 4.1 implies

$$\kappa_r^{2^\omega} \geq \kappa_0^{2^\omega, A} > \kappa_0^{2^\omega}.$$

Suppose $P(X)$ is a Σ_1 formula of class-set theory whose only class variable is the free variable X, and whose only parameters are X and 2^ω. If

$$L(\kappa_0^{2^\omega}, 2^\omega) \models P(A),$$

then 4.1 implies

$$L(\gamma, 2^\omega) \models P(A_{<\gamma})$$

for some γ E-constructive in 2^ω (cf. Corollary 8.2.III) A functions as an additional atomic predicate above. The integers in A can be regarded as notations for ordinals cofinal in $\kappa_0^{2^\omega}$, hence $P(A)$ can be rewritten as a Δ_0 formula. Then the truth-value of $P(A_{<\gamma})$ in $L(\kappa_0^{2^\omega}, 2^\omega)$ can be checked in $L(\gamma, 2^\omega)$.

4.2 Proof of Theorem 4.1. Let W be the set of all x such that x is a wellfounded relation whose field is a type δ object. Since $\alpha > 0$, $W \leq_E tp(\alpha)$. For each $x \in W$, $|x|$ is the ordinal height of x. If $\beta < |x|$, then x_β is the set of elements of the field of x of height β.

Let $\mathcal{O}_{\delta-1}$ be the set of ordinals E-constructive in d, $tp(\delta-1)$, $tp(\alpha)$; F as d ranges over $tp(\delta-1)$, and let t be an order preserving map of $\mathcal{O}_{\delta-1}$ onto an initial segment of ordinals.

A function f with domain W is computed from b, $tp(\delta-1)$, $tp(\alpha)$; F as follows. Three possibilities occur.

(i) $(Ec)[c \in (A - (\text{field of } x)]$.
(ii) $(E\beta)[x_\beta \neq A_{t^{-1}(\beta)}]$.
(iii) There is a $\sigma \leq |x|$ such that: $x_\beta = A_{t^{-1}(\beta)}$ for all $\beta < \sigma$; and

(1) $$\{e\}(A_{<\sigma}, b, tp(\delta-1), tp(\alpha); F)\downarrow.$$

As in Exercise 2.9, Grilliot selection implies (i), (ii) and (iii) are predicates (of x) E-recursively enumerable in b, $tp(\delta-1)$, $tp(\alpha)$; F. At least one of the three is true, and one such is provided by Gandy selection. Define $f(x)$ to be the length of the computation of the one selected.

Let $\gamma_0 = \sup\{f(x) | x \in W\}$. Let x_0 be the wellfounded relation on A induced by the enumeration of A by stages. Then $f(x_0)$ was defined by (iii). Hence (1) holds for some $\sigma \leq \gamma_0$. Since $\gamma_0 \leq_E b$, $tp(\delta-1)$, $tp(\alpha)$; F, there is a $\sigma_0 \geq \sigma$ such that σ_0 is E-constructive in c, $tp(\delta-1)$, $tp(\alpha)$; F for some $c \in tp(\delta-1)$, and $A_{<\sigma} = A_{<\sigma_0}$. □

4.3 Theorem (Harrington 1973). *Suppose $\delta < \alpha$, δ and α are successors, and F is a type $\alpha+2$ object. Then there exists a type $\delta+2$ object G such that*

$$\delta - en(G) = \delta - en(F).$$

Proof. G will map $tp(\delta+1) \times tp(\delta-1)$ into 2. G is defined by stages that correspond to the stages of the enumeration of $C^{\delta-1;F}$, the complete E-recursively enumerable (in $tp(\delta-1)$, $tp(\alpha)$; F) subset of $tp(\delta-1)$. At stage γ all elements of $tp(\delta-1)$ that belong to $C^{\delta-1;F}$ by virtue of computations of length γ are enumerated in $C^{\delta-1;F}_{\gamma}$. Stage γ is said to *exist* if there is a computation of length γ that puts an element in $C^{\delta-1;F}$. As $C^{\delta-1;F}$ is enumerated, the graph of G, restricted to

$$L(\kappa_{\delta-1}^{tp(\delta);G}, tp(\delta); G), \tag{1}$$

is enumerated in $tp(\delta-1)$, $tp(\alpha)$; F. ($\kappa_{\delta-1}^{tp(\delta);G}$ is the supremum of all ordinals E-constructive in b, $tp(\delta)$; G for all $b \in tp(\delta-1)$.) Thus the external aspects of the construction insure that $C^{\delta-1;G}$, the complete E-recursively enumerable (in $tp(\delta)$; G) subset of $tp(\delta-1)$, belongs to the δ-envelope of F, and hence that

$$\delta - en(G) \subseteq \delta - en(F). \tag{2}$$

Along the way steps are taken to insure that $C^{\delta-1;F}$ belongs to the δ-envelope of G. Thus the internal aspects of the construction guarantee the converse of (2).

Two difficulties arise: undershoot and overshoot. Undershoot means there is a γ such that stage γ exists but G is already defined on (1), that is, $C^{\delta-1;G}$ is already defined. Undershoot is possible because G is of lower type than F. Overshoot means the enumeration of $C^{\delta-1;F}$ comes to end before G is defined on (1), that is, before $C^{\delta-1;G}$ is defined. Both of these difficulties are overcome with the help of Harrington's reflection theorem (4.1).

Suppose stage γ exists. Let

$$\gamma^- = \sup\{\sigma \mid \sigma < \gamma \quad \& \quad \text{stage } \sigma \text{ exists}\}.$$

The construction so far has produced

$$L(\beta_{\gamma^-}, tp(\delta); G) = \bigcup_{\sigma<\gamma} L(\beta_\sigma, tp(\delta); G) \tag{3}$$

and G restricted to (3). As in Exercise 3.8, if (5) holds, there is a $y(\beta_{\gamma^-}) \subseteq tp(\delta)$ such that $y(\beta_{\gamma^-})$ is first order definable over, but not a member of, (3). The definition in question has $tp(\delta)$ as its only parameter and treats G as an additional atomic predicate; it is "complete Σ_1", hence uniform in every sense. $G(y(\beta_{\gamma^-}), b)$ is not yet defined for any $b \in tp(\delta-1)$.

Let

$$C^{\delta-1;F}_{\gamma} = \{b \mid b \in C^{\delta-1;F} \quad \& \quad |b| = \gamma\}. \tag{4}$$

($|b|$ is the length of the computation that puts b in $C^{\delta-1;F}$.)

Case 1: $\gamma^- < \gamma$ and stage γ^- exists. Assume (to be checked at end of each stage)

(5) $\beta_{\gamma-}$ is E-constructive in b, $tp(\delta-1)$; G for some $b \in tp(\delta-1)$.

Define

(6a) $$G(y(\beta_{\gamma-}), b) = \begin{cases} 1 & \text{if } b \in C_\gamma^{\delta-1;F}, \\ 0 & \text{if } b \notin C_\gamma^{\delta-1;F}. \end{cases}$$

Let $\beta_\gamma = \beta_{\gamma-} + \omega$, and extend G trivially to $L(\beta_\gamma, tp(\delta); G)$ by setting G equal to 0 whenever it needs to be defined. It follows from (5) and (6a) that the members of $C_\gamma^{\delta-1;F}$ can be enumerated at stage β_γ of the enumeration of $C^{\delta-1;G}$.

Case 2: case 1 does not hold. Let G_0 be the trivial extension of G from (3) to all of $tp(\delta-1) \times tp(\delta-1)$.

Subcase 2a: $\beta_{\gamma-} = \kappa_{\delta-1}^{tp(\delta);G_0}$ (undershoot). Whether or not subcase 2a holds can be computed from $\beta_{\gamma-}$, G_0, $tp(\alpha)$. The hierarchy

(6b) $$L(\kappa_{\delta-1}^{tp(\delta);G_0}, tp(\delta); G_0)$$

can be viewed as a $tp(\delta)$ object. There is a partial E-recursive (in G_0) map of $tp(\delta)$ onto (6b). Hence (6b) is encodable by an element of $tp(\delta+1)$. Thus Subcase 2a holds iff there is an element of $tp(\delta+1)$ that encodes (6b) and 2a holds within (6b). In short $tp(\delta+1)$ has the power to compare $\beta_{\gamma-}$ with $\kappa_{\delta-1}^{tp(\delta);G_0}$. Since $\delta < \alpha$, $tp(\alpha)$ also has the power. Define $G(y(\beta_{\gamma-}), b)$ as in (6a), and then trivially extend G to G_1 defined everywhere. Now Theorem 4.1 can be used to show

(7) $$\beta_{\gamma-} < \kappa_{\delta-1}^{tp(\delta);G_1}.$$

Fix $b_0 \in C_\gamma^{\delta-1;F}$. The construction assigns non-zero values to G only when required by (6a). $C^{\delta-1;F}$ is enumerated without repetitions. Thus

(8) $$\beta_{\gamma-} = \mu\tau[G(y(\tau), b_0) = 1].$$

Let A denote $C^{\delta-1;G_1}$. e is such that the computation of

(9) $$\{e\}(A, b_0, tp(\delta-1), tp(\alpha); G_1)$$

proceeds as follows. Enumerate A E-recursively in $tp(\delta)$; G. Let ρ be the supremum of all computations that put elements in A. The value of (9), if it converges, is

$$\mu\tau_{\tau \le \rho}[G(y(\tau), b_0)] = 1.$$

It follows from (8) that (9) converges to $\beta_{\gamma-}$. Theorem 4.1 implies (9) yields the same result when A is replaced by $A_{<\sigma}$ for some σ E-constructive in c, $tp(\delta)$; G_1 for some $c \in tp(\delta-1)$; G_1. Hence (7) holds. Let $\beta_\gamma = \sigma + \omega$, and let G be the restriction

of G_1 to $L(\beta_\gamma, tp(\delta-1); G_1)$. Then (7) is true when G_1 is replaced by G; moreover β_γ is E-constructive in c, $tp(\delta)$; G.

Subcase 2b: $\beta_{\gamma^-} < \kappa_{\delta-1}^{tp(\delta);\, G_0}$. Hence there is a $\tau \geq \beta_\gamma^-$ such that τ is E-constructive in b, $tp(\delta)$; G_0 for some $b \in tp(\delta-1)$. Let τ_γ be the least such τ. Define G to agree with G_0 on $L(\tau_\gamma, tp(\delta); G_0)$, and define $G(y(\tau_\gamma), b)$ as in (6a). Let $\beta_\gamma = \tau_\gamma + \omega$ and extend G trivially to $L(\beta_\gamma, tp(\delta); G)$.

To finish the construction, define

$$\beta_\infty = \sup\{\beta_\gamma | \gamma \text{ exists}\}$$

and extend G trivially to all of $tp(\delta+1) \times tp(\delta-1)$.

Let $C_{<\beta_\infty}^{\delta-1;\, G}$ be that part of $C^{\delta-1;\, G}$ enumerated via computations of height less than β_∞. The outward form of the construction implies $C_{<\beta_\infty}^{\delta-1;\, G}$ is E-recursively enumerable in $tp(\delta-1)$, $tp(\alpha)$; F. The only remaining problem is to show $C_{<\beta_\infty}^{\delta-1;\, G}$ is $C^{\delta-1;\, G}$ (overshoot). Suppose not. Thus there is a $\tau \geq \beta_\infty$ such that $C_\tau^{\delta-1;\, G} \neq \varnothing$. Let τ_∞ be the least such τ and choose $b_1 \in C_{\tau_\infty}^{\delta-1;\, G}$. Theorem 4.1 shows $b_1 \in C_{<\beta_\gamma}^{\delta-1;\, G}$ for some γ that exists. There is an e such that

$$\{e\}(C^{\delta-1;\, F}, b_1, tp(\delta-1), tp(\alpha); F) \tag{10}$$

is computed as follows: enumerate all of $C^{\delta-1;\, F}$; enumerate $C_{<\beta_\infty}^{\delta-1;\, G}$ simultaneously; look for the least τ such that $b_1 \in C_\tau^{\delta-1;\, G}$. According to Theorem 4.1, (10) converges when $C^{\delta-1;\, F}$ is replaced by $C_{<\gamma}^{\delta-1;\, F}$ for some γ that exists. Furthermore $\{e\}$ can require $C_{<\gamma}^{\delta-1;\, F}$ to be a fairly good imitation of $C^{\delta-1;\, F}$; in particular $\gamma^- = \gamma$. Consequently case 2b holds at stage γ and $b_1 \in C_{<\beta_\gamma}^{\delta-1;\, G}$. □

Harrington noted, and Moldstad 1977 proved, that the plus-one and plus-two theorems can be combined. For example, there is a type 3 G such that $1-en(G) = 1-en(^4E)$ and $2-sc(G) = 2-sc(^4E)$.

No satisfactory plus-three theorem is known.

4.4–4.6 Exercises

4.4 (Harrington). Define

$$\tau_0 = \kappa_0^{2^\omega}$$

$$\tau_{n+1} = \kappa_0^{\tau_n, 2^\omega} \quad (n < \omega).$$

Show $\kappa_r^{2^\omega} > \lim_n \tau_n$.

4.5 (Moschovakis). Show the 1-envelope of ^3E is not the 1-envelope of any type 2 object.

4.6 Show that an ordinal E-recursive in a, $tp(\alpha)$; F need not be E-constructive in a, $tp(\alpha)$; F. ($a \in tp(\alpha)$ & α a successor & F of type $\alpha+2$.)

5. *Selection with Additional Predicates*

The proofs of Normann selection (Theorem 1.3) and Moschovakis selection (Theorem 2.1) made use of a "power set" hypothesis. In the Normann case, Hajnal's theorem was invoked to show $2^\rho \cap E(\{x, \rho\})$ was an element of $E(\{x, \rho\})$. In the Moschovakis case strong use was made of the fact that

$$R(k(j))^{R(k(i))} \in R(\alpha)$$

if $k(j) \leq k(i) < \alpha$. In both cases the need for a "power set" hypothesis stemmed from Grilliot's approach to selection: compute $\min f$ by computing $\min g$ for all g "below" f. In this section $\min A$ is computed by considering subcomputation instructions of computation instructions in A. Thus functions (or subsets) are replaced by elements, and so there is no need to know some class of subsets of a set is a set.

The elemental approach to Moschovakis selection is well exemplified by the proof of Theorem 5.1. Let x be a set of ordinals. Then $E(x)$ has a greatest cardinal (in the sense of $E(x)$) denoted by $\mathrm{gc}(E(x))$. In general $\mathrm{gc}(E(x)) \leq \sup x$. For simplicity assume $\mathrm{gc}(E(x)) = \sup x$. Let $A \subseteq \sup x$ be E-recursively enumerable in x. Thus

$$A = \{a \mid a < \sup x \quad \& \quad \{e\}(a, x)\downarrow\}. \tag{1}$$

If $A \neq \phi$, then define

$$\min A \text{ to be } \min\{|\{e\}(a, x)| \mid a \in A\}.$$

Assume

$$E(x) \vDash [\text{cofinality } \mathrm{gc}(E(x)) = \omega].$$

The proof of 5.1 uses the elemental approach to compute $\min A$ from x and an ω-sequence through $\mathrm{gc}(E(x))$. Note one last time that there might be a cardinal ρ (in the sense of $E(x)$) such that $2^\rho \cap E(x)$ is not a member of $E(x)$. In that event there is little hope for Grilliot's approach.

Some *notational conventions* will be helpful in the proof of 5.1. Let $a, b, c, \ldots$ be ordinals less than $\sup x$. $a, b, c, \ldots$ will be used to denote (or encode) computation instructions, that is, nodes on the universal computation tree $<_U$. If the node is of the form $\langle e, \langle a, x\rangle\rangle$, then it can be readily encoded as an ordinal less than $\sup x$. An immediate subcomputation instruction of $\langle e, \langle a, x\rangle\rangle$ is of the form $\langle c, v\rangle$ for some $c < \omega$ and $v \in E(x)$. As in the beginning of the proof of Theorem 2.1, v can be construed effectively as an element of $\sup x$ (cf. Exercise 2.7), and so $\langle c, v\rangle$ can be encoded as an element of $\sup x$.

As an example of the above usage,

$$|a| = \sup^+\{|b| \mid b <_U a\}.$$

5.1 Theorem (Sacks & Slaman 1987). *Let x be a set of ordinals. Suppose in $E(x)$ there is a strictly ascending sequence $\{k_i | i < \omega\}$ of cardinals (in the sense of $E(x)$) such that*

$$\sup x = \sup\{k_i | i < \omega\}.$$

If $A \subseteq x$ is nonempty and E-recursively enumerable in x, then

$$\min A \leq_E x, \{k_i | i < \omega\} \quad (\textit{uniformly}).$$

Proof. The predicate, b is an immediate subcomputation instruction of a, is not E-recursive in x. A slight modification, b is an immediate subcomputation instruction of a via β, is. The essential clauses are:

$\langle m, u\rangle$ is an immediate subcomputation instruction of $\langle 2^m \cdot 3^n, u\rangle$ via β;
if $|\{m\}(u)| \leqslant \beta$ and $v \in \{m\}(u)$, then $\langle n, v\rangle$ is immediate subcomputation of $\langle 2^m \cdot 3^n, u\rangle$ via β.

The theorem is proved by effective transfinite recursion on min A, henceforth called the *main recursion.* There are countably many cases, at least one of which yields min A. They are woven into one procedure by Gandy selection.
Case $\langle i, j\rangle$: $i < j$ and there is an $a_0 \in A \cap k_i$ such that

$\min A = \min(A \cap k_i) = |a_0|$, and

$\min A = \sup^+ \{|b| \, | \, b < k_j \,\&\, b$ is an immediate subcomputation instruction of $a_0\}$.

In this case min A is computed by a recursion of length k_{j+1}. Fix $\alpha < k_{j+1}$ and assume $\beta(\gamma)$ has been computed for $\gamma < \alpha$. Let

$$\beta^-(\alpha) = \sup\{\beta(\gamma) | \gamma < \alpha\}.$$

Subcase 1: $\beta^-(\alpha) + 1 \geq \min(A \cap k_i)$. Define $\beta(\alpha)$ to be $\min(A \cap k_i)$.
Subcase 2: $\beta^-(\alpha) + 1 < \min(A \cap k_i)$. Let Z_α be the set of all b such that $b < k_j$ & $b\downarrow$ & $\beta^-(\alpha) < |b|$ & b is an immediate subcomputation instruction of some $a < k_i$ via $\beta^-(\alpha)$.

To see Z_α is nonempty consider the a_0 mentioned in the case hypothesis. Suppose a_0 is $\langle 2^m \cdot 3^n, u\rangle$. If $\beta^-(\alpha) < |\langle m, u\rangle|$, then $\langle m, u\rangle \in Z_\alpha$. This last assertion assumes that the encoding of nodes on $<_U$ by elements of $\sup x$ has the property that $\langle m, u\rangle \in k_j$ if $\langle 2^m \cdot 3^n, u\rangle \in k_j$. Suppose $\beta^-(\alpha) \geq |\langle m, u\rangle|$. Then for every $v \in \{m\}(u)$, $\langle n, v\rangle$ is an immediate subcomputation instruction of a_0 via $\beta^-(\alpha)$. Hence the case, and subcase, hypotheses provide a b in Z_α.

Z_α is E-recursively enumerable in $\beta^-(\alpha)$, x, k_i, k_j; and $\min Z_\alpha < \min A$, hence the main recursion can be applied to compute $\min Z_\alpha$. Define $\beta(\alpha)$ to be $\min Z_\alpha$.

$\sup\{\beta(\alpha) | \alpha < k_{j+1}\} \leq_E x, k_i, k_j$, so it need only be shown that $\beta(\alpha) = \min(A \cap k_i)$ for all sufficiently large α. Suppose not. Then subcase 2 holds for all

α. For each α, there is a $b_\alpha \in Z_\alpha \cap k_j$ such that

$$|b_\alpha| = \beta(\alpha) \quad \text{and} \quad (\gamma)_{\gamma<\alpha}\,[\beta(\gamma) < |b_\alpha|].$$

The map $\alpha \mapsto b_\alpha$ is an injection of k_{j+1} into k_j.

For the sake of case $\langle i, 0\rangle$, define

$$\gamma(j,a) = \sup^+\{|b| \mid b < k_j \;\&\; b \text{ is an immediate subcomputation instruction of } a\} \tag{1}$$

with the understanding that $\gamma(j,a)$ has a value iff $b\downarrow$ for all $b < k_j$ an immediate subcomputation instruction of a.

Define $\gamma(j,a,\beta)$ as in (1) with b restricted to immediate subcomputation instructions of a via β. $\gamma(j,a,\beta)$ is a partial E-recursive approximation of $\gamma(j,a)$.

Case $\langle i, 0\rangle$: $\min A = \min(A \cap k_i)$ and

(*a*) $[a \in A \cap k_i \;\&\; |a| = \min A \to (j)_{j<\omega}\,(\gamma(j,a) < \min A)]$.

In this case $\min(A \cap k_i)$ is computed by a recursion of length k_{i+1}. Fix $\alpha < k_{i+1}$ and suppose $\beta(\gamma)$ has been computed for all $\gamma < \alpha$.

Subcase 1: $\beta^-(\alpha)+1 \geq \min(A \cap k_i)$. Define $\beta(\alpha)$ to be $\min(A \cap k_i)$.

Subcase 2: $\beta^-(\alpha)+1 < \min(A \cap k_i)$. Let Y_α be the set of $\langle j, a\rangle$ such that

$$j < \omega \quad \&\quad a < k_i \quad \&\quad \beta^-(\alpha) < \gamma(j,a,\beta^-(\alpha)).$$

Y_α is nonempty by the same sort of argument used in subcase 2 of case $\langle i, j\rangle$ to show Z_α is nonempty. Y_α is E-recursively enumerable in $\beta^-(\alpha), x, \{k_j \mid j < \omega\}$. Hence the main recursion can compute

$$\beta(\alpha) = \min\{\gamma(j,a,\beta^-(\gamma)) \mid \langle j, a\rangle \in Y_\alpha\}.$$

For all sufficiently large α, $\beta(\alpha)$ is $\min(A \cap k_i)$. Suppose not. Then subcase 2 holds for all α. For each α there is a $\langle j, a\rangle \in \omega \times k_i$ such that

$$\beta(\alpha) = \gamma(j,a,\beta^-(\alpha)) \quad \&\quad (\gamma)_{\gamma<\alpha}\,(\beta(\gamma) < \beta(\alpha)).$$

Each such $\langle j, a\rangle$ is associated with at most two α's. But there are k_{i+1} α's associated with only $\omega \times k_i$ $\langle j, a\rangle$'s, an impossibility. The two α's, α_0 and α_1, occur as follows. Suppose a is $\langle 2^m \cdot 3^n, u\rangle$. $\beta^-(\alpha_0)$ is less than $|\langle m, u\rangle|$. Later, when $\beta^-(\alpha_1) \geq |\langle m, u\rangle|$, $\langle n, v\rangle$ is an immediate subcomputation instruction of a via $\beta^-(\alpha_1)$ for all $v \in \{m\}(n)$.

An effective selection procedure is defined above for case q uniformly in q. If $\min A < \infty$, then case q converges for some q. For all q, if q converges, then it converges to $\min A$. If some q converges, then Gandy selection computes one. □

The proof of Theorem 5.1 establishes more than has been stated. $\{k_i \mid i < \omega\}$ can be replaced by a singular sequence of length $\rho > \omega$, if there is an effective method M

for selecting elements of nonempty subsets of ρ E-recursively enumerable in x. M replaces Gandy selection at the end of the proof of 5.1.

$E(x)$ can be replaced by $E(x; R)$ in the statement of Theorem 5.1 without any noteworthy changes in the proof of 5.1. Easy relativization to R is typical of the elemental approach to selection. The assumption that x is a set of ordinals can be relaxed somewhat. The essential point to remember about the proof of 5.1 is the non-injectibility of k_{i+1} into k_i.

5.2 Corollary. *If x is a set of ordinals and*

$$E(x) \models [\textit{cofinality of greatest cardinal} = \omega],$$

then $E(x)$ is Σ_1 admissible.

Proof. Theorem 5.1 and Lemma 2.3. □

Now the elemental method is used to relativize Normann selection (Theorem 1.3) to an additional predicate.

5.3 Theorem (Griffor & Normann 1982). *Let ρ be a regular cardinal in the sense of $E(\rho; R)$. If $\delta < \rho$ and $C \subseteq \delta$ is nonempty and E-recursively enumerable in ρ; R, then some element of C is E-recursive in δ, ρ; R (uniformly).*

Proof. Let $a, b, c, \ldots$ denote ordinals less than ρ. Recall the remarks about notational conventions made immediately prior to Theorem 5.1. They apply to the present situation. Thus $a, b, c, \ldots$ will be used to denote (or encode) computation instructions, that is, nodes on $<_{U;R}$. If the node is of the form $\langle e, \langle a, x\rangle\rangle$ then it can be encoded by a slight modification of x. If b is a node of the form $\langle 2^m \cdot 3^n, u\rangle$, then its immediate predecessors are:

$$\langle m, u\rangle;$$

$$\langle n, v\rangle \text{ for all } v \in \{m\}(u) \quad \text{if} \quad \{m\}(u)\downarrow.$$

$\langle m, u\rangle$ will be denoted by b_0, and the $\langle n, v\rangle$'s, if they exist, by $b_1, b_2, \ldots, b_\alpha, \ldots$ $(\alpha < \rho)$. (If $\{m\}(u)$ has cardinality less than ρ, then some of the b_α's are dummies; the listing, b_α $(\alpha < \rho)$, is derived from the one-one map mentioned in Exercise 2.7).

Let $C = \{a \mid a < \delta \ \&\ \{e\}^R(a)\downarrow\}$. The intent is to compute

$$\min C = \min\{|\{e\}^R(a)| \mid a < \delta\}$$

by a recursion on $\min C$ henceforth called the main recursion. For simplicity let a denote node $\langle e, a\rangle$ of $<_{U;R}$. Thus $\min C$ is $\min\{|a| \mid a \in C\}$.

$\beta(\alpha)$ is computed by recursion on $\alpha < \rho$. Let $\beta^-(\alpha)$ be $\sup\{\beta(\gamma) \mid \gamma < \alpha\}$.
Case 1: $\min C \leq \beta^-(\alpha) + 1$. Define $\beta(\alpha) = \min C$.

Case 2: $\min C > \beta^-(\alpha)+1$. Consider an arbitrary $a < \delta$. If $|a_0| > \beta^-(\alpha)$, then define $t(a, \alpha) = 0$. Suppose $|a_0| \leq \beta^-(\alpha)$. Then for all $\beta < \rho$, a_β is seen to be an immediate subcomputation instruction of a via a computation of length at most $\beta^-(\alpha)$. Let

$$t(a, \alpha) \simeq \mu\beta[|a_\beta| > \beta^-(\alpha)].$$

It is intended that $t(a, \alpha)$ be defined and equal to β iff

$$(\gamma)_{\gamma<\beta}(a_\gamma\downarrow \;\&\; |a_\gamma| \leq \beta^-(\alpha)),\quad a_\beta\downarrow \text{ and } |a_\beta| > \beta^-(\alpha).$$

$t(a, \alpha)$ is defined for some $a\in C$, since $\min C = |a|$ for some $a\in C$. Use the main recursion to compute

$$\beta(\alpha) = \min_{a<\delta} |a_{t(a,\alpha)}|. \tag{1}$$

If case 1 holds for some α, then $\min C = \sup\{\beta(\alpha)|\ \alpha < \rho\}$.

Suppose case 2 holds for all α. Let a^α be the least a that satisfies (1). $\{\langle\alpha, a^\alpha\rangle|\ \alpha < \rho\}$ is E-recursive in x, ρ; R thanks to the main recursion. The regularity of ρ in $E(x; R)$ implies a^α equals some fixed a^∞ for all $\alpha\in Z$, an unbounded subset of ρ. Hence

$$(\alpha)_{\alpha\in Z}(\gamma)[\gamma < t(a^\infty, \alpha) \to |a_\gamma| < a_{t(a^\infty,\alpha)}],$$
$$|a^\infty| = \sup^+_{\gamma<\rho}|a^\infty_\gamma| = \sup^+_{\gamma\in Z}|a^\infty_{t(a^\infty,\gamma)}|$$
$$= \sup^+_{\gamma\in Z}\beta(\gamma) \quad = \sup^+_{\gamma<\rho}\beta(\gamma),$$

and so $\min C \leq \sup^+_{\gamma<\rho}\beta(\alpha)$. □

Theorem 5.3 extends Normann's result on the extended plus-one hypothesis ((ii) of subsection 3.7) from ${}^4E$ to an arbitrary type 4 object. It also extends his work on Post's problem for $E(2^\omega)$ to $E(2^\omega; R)$ (cf. Chapter XIII).

There appears to be little left to discover of a general nature about selection. Suppose ρ is a cardinal in the sense of $E(\rho)$, and $E(\rho)$ is not Σ_1 admissible. If ρ is regular in $E(\rho)$, then selection is false for unbounded E-recursively enumerable subsets of ρ, but true for bounded ones. If ρ is singular in $E(\rho)$, then selection holds for E-recursively enumerable subsets of ρ iff it holds for subsets of ρ_0, the cofinality of ρ in $E(\rho)$.

According to Exercise 2.10, selection fails for subsets of ω_1^L E-recursively enumerable in $\omega^L_{\omega_1}$.

The study of forcing over E-closed structures leads to new, highly specialized selection results, e.g. Theorem 4.1.XI.

Chapter XIII
E-Recursively Enumerable Degrees

Degree theory for subsets of E-closed structures differs markedly from the Σ_1 admissible case. On the surface the results are similar, but the modes of argument differ considerably. Post's problem once again has a positive solution, but this time without injuries and without repeated attempts to satisfy a given requirement. The presence of Moschovakis witnesses makes all the difference. Injuries do occur in the proof of Slaman's splitting theorem.

1. Regular Sets

Let $\mathscr{E}$ be transitive and $A, B \subseteq \mathscr{E}$. The relativization of E-recursiveness to B was introduced in Section 5.XI. In essence a new scheme,

$$\{c\}^B(x_1, \ldots, x_n) = B \cap x_i \quad (c = \langle 7, n, i \rangle)$$

is added to the original six. f is partial E-recursive relative to B if $f \simeq \{e\}^B$ for some $e < \omega$. D is E-recursively enumerable in p relative to B if

$$D = \{x \mid \{e\}^B(x, p)\downarrow\}$$

for some e. $\mathscr{E}$ is E-closed relative to B if

$$\{e\}^B(x)\downarrow \rightarrow \{e\}^B(x) \in \mathscr{E}$$

for all $e < \omega$ and $x \in \mathscr{E}$.

Assume $\mathscr{E}$ is E-closed. A is *E-reducible* to B on $\mathscr{E}$ (in symbols $A \leq_{\mathscr{E}} B$) if there exist $e < \omega$ and $p \in \mathscr{E}$ such that

(i) $\{e\}^B(x, p)\downarrow$ for all $x \in \mathscr{E}$,

(ii) $T_{\langle e, x, p; B\rangle} \in \mathscr{E}$ for all $x \in \mathscr{E}$, and

(iii) $A = \hat{x}[x \in \mathscr{E} \;\&\; \{e\}^B(x, p) = 0]$.

Note that $\{e\}^B(y)$ is not required to converge outside $\mathscr{E}$. (ii) says that the computation trees associated with the convergent computations of (i) belong to $\mathscr{E}$. The joint effect of (i) and (ii) is weaker than that of E-closure of $\mathscr{E}$ relative to B. (ii) is included for the sake of parity (cf. Part B): computations, arguments and values all have the same complexity as elements of $\mathscr{E}$. (ii) is costly; because of it, $\leq_{\mathscr{E}}$ need not be transitive.

If $\mathscr{E}$ is HF and $A, B \subseteq \omega$, then $A \leq_{\mathscr{E}} B$ iff A is Turing reducible to B.

If $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$, then A is said to be *subgeneric* if for all $e < \omega$ and $x \in \mathscr{E}$

$$\{e\}^A(x)\downarrow \rightarrow \{e\}^A(x) \in \mathscr{E}$$

In short $\mathscr{E}$ is E-closed relative to A, or shorter still, $\langle \mathscr{E}, A \rangle$ is E-closed.

If $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$, then A is said to be *E-recursively enumerable on $\mathscr{E}$* if there exist $e < \omega$ & $p \in \mathscr{E}$ such that

$$A = \hat{x}[x \in \mathscr{E} \ \& \ \{e\}(x, p)\downarrow].$$

If both A and $\mathscr{E} - A$ are E-recursively enumerable on $\mathscr{E}$, then A is *E-recursive on $\mathscr{E}$*.

Lemma 1.1. *Assume $L(\kappa)$ is E-closed. Suppose $A, B, C \subseteq L(\kappa)$ are E-recursively enumerable on $L(\kappa)$. If $A \leq_{L(\kappa)} B$ and $B \leq_{L(\kappa)} C$, then $A \leq_{L(\kappa)} C$.*

Proof. If C is subgeneric, then $A \leq_{L(\kappa)} C$ by composition.

Assume C is not subgeneric. Suppose $L(\kappa) = E(x)$ for some set x of ordinals. If $E(x)$ is Σ_1 admissible, then Proposition 1.2 (ii) implies $A \leq_{L(\kappa)} C$, since $\leq_\kappa$ (as defined in 3.2.VII) is transitive. If $E(x)$ is not Σ_1 admissible, then $E(x)$ admits Moschovakis witnesses by Theorem 5.8.X. If $L(\kappa) \neq E(x)$ for any set x of ordinals, then $L(\kappa) \neq E(x)$ for any $x \in L(\kappa)$. (Each $x \in L(\kappa)$ is coded by a relation (on ordinals) $\in L(\kappa)$.) Thus it is safe to assume $L(\kappa)$ admits Moschovakis witnesses.

Assume C is regular. Then any computation relative to C of height less than κ belongs to $L(\kappa)$. Since C is not subgeneric, there is a computation relative to C of height κ. Thus for some $e < \omega$ and $y \in L(\kappa) \cap 2^\kappa$:

$$(x)[x \in y \rightarrow T_{\langle e, x, C\rangle} \in L(\kappa)], \quad \text{and}$$
$$\kappa = \sup\{|\{e\}^C(x)| \ | x \in y\}.$$

Let $d < \omega$ and $p \in L(\kappa)$ be such that

$$A = \hat{z}\{z \in L(\kappa) \quad \& \quad \{d\}(z, p)\downarrow\}.$$

Then $z \notin A$ iff

(1) $\quad (\mathrm{Ex})(\mathrm{Ew})[x \in y \quad \& \quad w \in L(|\{e\}^C(x)|) \quad \& \quad w \text{ witnesses } \{d\}(z, p)\uparrow].$

It follows from (1) and the regularity of C that $A \leq_{L(\kappa)} C$. The least x satisfying (1) is computable from C by effective transfinite recursion on $y \subseteq \kappa$.

Assume C is not regular. Suppose

$$C = \{x \mid \{f\}(x, q)\downarrow \quad \& \quad x \in L(\kappa)\}$$

for some $q \in L(\kappa)$. Choose $y \in L(\kappa)$ so that $y \cap C \notin L(\kappa)$. Then

$$\sup\{|\{f\}(x, q)| \mid x \in y \cap C\} = \kappa, \quad \text{and}$$

$$(2) \qquad z \notin A \to (\mathrm{Ex})(\mathrm{Ew})[x \in (y \cap C) \ \& \ w \in L(|\{f\}(x, q)|) \ \& \ w \text{ witnesses } \{d\}(z, p)\uparrow].$$

(2) implies $A \leq_{L(\kappa)} C$ as (1) did. □

1.2 Proposition. *Suppose x is a set of ordinals, $E(x)$ is Σ_1 admissible, and $A, B \subseteq E(x)$.*

(i) A is E-recursively enumerable on $E(x)$ iff A is $\Sigma_1^{E(x)}$.

(ii) $A \leq_{E(x)} B$ iff A is $E(x)$-recursive in B (in the sense of admissibility theory).

Proof. Same as that in Section 5.10.X. □

Suppose $L(\kappa)$ is E-closed and $A, B \subseteq L(\kappa)$ are E-recursively enumerable on $L(\kappa)$. A and B have the same *degree* (in symbols $A \equiv_{L(\kappa)} B$) if each is E-reducible to the other on $L(\kappa)$. By Lemma 1.1, $\equiv_{L(\kappa)}$ is an equivalence relation, and so the degrees are well defined. The degree of A (in symbols $\underline{A}$) is

$$\{C \mid C \equiv_{L(\kappa)} A \ \& \ C \text{ is } E\text{-recursively enumerable on } L(\kappa)\}.$$

Proposition 1.2 solves Post's problem for $L(\kappa)$ when $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$ for some x a set of ordinals. Otherwise $L(\kappa)$ admits Moschovakis witnesses, and they carry the burden of proof in the solution of Post's problem.

A set $A \subseteq \mathscr{E}$, $\mathscr{E}$ E-closed, is said to be *regular* if $(A \cap x) \in \mathscr{E}$ for all $x \in \mathscr{E}$. Clearly subgenericity implies regularity. The converse is far from true (Exercise 1.4). If A is E-recursively enumerable on $\mathscr{E}$, then A is regular iff the "enumeration" of $A \cap x$ finishes inside $\mathscr{E}$. More precisely,

$$\{|\{e\}(y, p)| \mid y \in A \cap x\} \in \mathscr{E},$$

where $A = \{y \mid \{e\}(y, p)\downarrow\} \cap \mathscr{E}$ for some $p \in \mathscr{E}$. Consequently the next theorem is essential for dynamic arguments about $\mathscr{E}$ recursively enumerable degrees.

1.3 Theorem (Sacks 1985). *Assume $L(\kappa)$ is E-closed. Suppose $C \subseteq L(\kappa)$ is E-recursively enumerable on $L(\kappa)$. Then there exists a regular $B \subseteq L(\kappa)$, E-recursively enumerable on $L(\kappa)$, such that $B \equiv_{L(\kappa)} C$.*

Proof. Suppose $L(\kappa)$ is not of the form $E(x)$. Then every E-recursively enumerable-on-$L(\kappa)$ subset of $L(\kappa)$ is regular. Suppose A is E-recursively enumerable in p on $L(\kappa)$ and $x \in L(\kappa)$. Then $A \cap x$ is first order definable over $E(\{x, p\}) \in L(\kappa)$.

Now suppose $L(\kappa)$ is of the form $E(x)$. It is safe to assume x is a set of ordinals. If $E(x)$ is Σ_1 admissible, then apply Proposition 1.2 and Theorem 4.2.VII. Suppose $E(x)$ is not Σ_1 admissible and C is not regular. Then $A \leq_{L(\kappa)} C$ for every A E-recursively enumerable on $L(\kappa)$, as in the last part of the proof of Lemma 1.1. In other words, C is *complete*. It suffices to find a regular, complete B E-recursively enumerable on $L(\kappa)$. Recall that the completeness of C in the proof of 1.1 follows from the existence of a computation relative to C of height κ. B will be defined so that $\kappa \leq_E \rho_0, x; B$, where

$$(1) \qquad \rho_0 = \mu\rho(\mathrm{Eg})[g\colon \rho \to \kappa \quad \& \quad g \in \Sigma_1^{L(\kappa)} \quad \& \quad \text{sup range } g = \kappa].$$

g_0 of Exercise 1.5 is a strictly increasing $\Delta_0^{L(\kappa)}$ map on ρ_0 and

$$\text{sup range } g_0 = \kappa.$$

It follows that range g_0 is E-recursive on $L(\kappa)$. Intuitively κ is computable from ρ_0 and range g_0. That will in fact be the case after a slight modification of g_0 via f.

Since $L(\kappa) = E(x)$ there is, for each $\delta < \kappa$, an $\langle e, u \rangle \in \omega \times \sup x$ such that

$$(2) \qquad |\{e\}(u, x)| = \delta.$$

Let $f(\delta)$ be the "least" member of $\omega \times \sup x$ that satisfies (2). Thus $f(\delta) \leq_E \delta, x$, and $\delta \leq_E f(\delta), x$. A pairing device allows the range of f to be construed as a subset of x.

Assume g_0 has been altered so that it is continuous and $g_0(i) + \rho_0 < g_0(i+1)$ for all $i < \rho_0$. Define

$$B = \{g_0(i) + f(g_0(i+1)) \mid i < \rho_0\}.$$

B is E-recursively enumerable on $L(\kappa)$. Enumerate γ in B as follows. Compute i, the greatest j such that $g_0(j) < \gamma$. Let $v = \gamma - g_0(i)$. Compute $f^{-1}(v)$ from v, x as above. If $f^{-1}(v) = g_0(i+1)$, then enumerate γ. (1) implies B is regular.

Observe that $\{g_0(i) \mid i < \rho_0\}$ can be defined by an effective transfinite recursion relative to B.

$$g_0(\lambda) = \sup\{g_0(i) \mid i < \lambda\}.$$

$$\{g_0(i) + f(g_0(i+1))\} = B \cap (g_0(i), g_0(i+1)).$$

$$g_0(i+1) \leq_E f(g_0(i+1)), x.$$

But then κ is E-recursive in $\rho_0, x; B$ via a computation of height κ. Hence B is complete.

The main trick above is the use of f to provide enough power to pass from $g_0(i)$ to $g_0(i+1)$. This trick is due to Slaman, who greatly simplified the original proof of 1.3. □

1.4–1.5 Exercises

1.4. Find an $A \subseteq E(\omega_1)$ such that A is regular but not subgeneric.

1.5. Suppose $L(\kappa) = E(x)$ for some set x of ordinals. Define ρ_0 as in (1) of the proof of Theorem 1.3. Show there exists a continuous $\Delta_0^{L(\kappa)}$ map g_0 such that $g_0(i) + \sup x < g_0(i+1)$ for all i and sup range $g_0 = \kappa$.

2. *Projecta and Cofinalities*

Some structural properties of $L(\kappa)$ are needed to establish suitable indexings of Friedberg–Muchnik requirements in the next two sections.

ρ, the greater, and η, the lesser, E-recursively enumerable projectum, are defined as follows.

$$\rho^\kappa = \mu\gamma_{\gamma \leq \kappa}(\mathrm{Ef})[f \text{ is a partial } E\text{-recursive-on-}L(\kappa) \text{ map of } \gamma \text{ onto } L(\kappa)].$$

$$\eta^\kappa = \mu\gamma_{\gamma \leq \kappa}(\mathrm{EA})[A \in 2^\gamma - L(\kappa) \text{ and } A \text{ is } E\text{-recursively enumerable on } L(\kappa)].$$

As any student of Jensen's fine structure theory might expect, the next result says $\eta^\kappa = \rho^\kappa$. For simplicity it is assumed that $L(\kappa)$ is E-closed. Only small changes in the proof of Lemma 2.1 are needed when $L(\kappa)$ is rudimentarily closed rather than E-closed.

2.1. Lemma (Sacks 1985). $\eta^\kappa = \rho^\kappa$.

Proof. Let f be a partial E-recursive-on-$L(\kappa)$ map of ρ onto $L(\kappa)$.

If $\rho < \eta$, then $f^{-1}[\kappa] \in L(\kappa)$, and consequently $\kappa \in L(\kappa)$.

To see $\eta \geq \rho$, fix $\gamma < \rho$ and let $A \subseteq \gamma$ be E-recursively enumerable in $\rho \in L(\kappa)$ via Gödel number e with the intention of showing $A \in L(\kappa)$. It is safe to assume p is an ordinal, since each member of $L(\kappa)$ is E-recursive in some ordinal less than κ. Let g be a universal partial E-recursive function. The essential property of g is: for all x,

$$g[x] = \{\{e\}(z_0, \ldots, z_{n-1}) \mid \{e\}(z_0, \ldots, z_{n-1})\downarrow, e, n < \omega, z_i \in x\}.$$

Thus $E(x) = g[TC(\{x\})]$. Let

$$H = g[\gamma \cup \{p\}]. \tag{1}$$

H is said to be the *partial E-recursive hull* of $\gamma \cup \{p\}$. Observe that 2.8.X implies

$$z \in H \to O(z) \in H. \tag{1a}$$

$O(z)$ is the least δ such that $z \in L(\delta + 1) - L(\delta)$.

(1) implies $H \prec_0 L(\kappa)$, that is, an Δ_0^{ZF} sentence with parameters true in H, is also true in $L(\kappa)$. The argument proceeds by induction on length of sentences. Suppose $\mathscr{F}(x)$ is a Δ_0^{ZF} formula with parameters in H such that

$$L(\kappa) \models (\mathrm{Ex})_{x \in b} \mathscr{F}(x), \tag{2}$$

where $b \in H$. The problem is to find some $x \in b \cap H$. that satisfies $\mathscr{F}(x)$. Each z put in H by g is accompanied by a wellordering of z. z is the value of some $\{e\}(z_0, \ldots, z_{n-1})$, where every z_i is an ordinal. Hence the convergent computation of $\{e\}(z_0, \ldots, z_{n-1})$ virtually includes a computation of a wellordering of z, as in Proposition 5.2.X. Let $w \in H$ be a wellordering of b, and x_0 the w-least element of b that satisfies $\mathscr{F}(x)$. Then x_0 is E-recursive in b, w and the parameters of $\mathscr{F}(x)$, and so belongs to H.

Since $H \prec_0 L(\kappa)$, H is extensional, hence isomorphic to a transitive set H_0 via the collapsing map t (cf. 2.6.VII). It follows from (1a) that $H_0 = L(\beta)$ for some $\beta \leq \kappa$ (cf. 2.5.VII). By (1)

$$L(\beta) = g[\gamma \cup \{t(p)\}]. \tag{3}$$

Since ρ is an $L(\kappa)$-cardinal, it follows from (3) that g, slightly altered, partially maps a bounded initial segment of ρ onto $L(\beta)$. Consequently $\beta < \rho$. Next

$$A = \{x \mid x < \gamma \quad \& \quad \{e\}(x, p)\downarrow\},$$

hence $A \in L(\beta + 1) \in L(\kappa)$, if t preserves convergence and divergence facts about elements of H. This last follows from $H_0 \prec_0 L(\kappa)$. Suppose $x < \gamma$. Then

$$L(\beta) \models |\{e\}(x, t(p))| = \tau$$
$$\text{iff} \quad L(\kappa) \models |\{e\}(x, p)| = t^{-1}(\tau). \qquad \square$$

Let γ be an ordinal. A slight revision of the proof of Lemma 2.1 shows: if H is the partial E-recursive hull of a subset of γ, then the transitive collapse of H is isomorphic to some $L(\beta)$. The latter is an effective variation on the Jensen–Karp condensation lemma: if $H \prec_1 L(\kappa)$, then the transitive collapse of H is isomorphic to some $L(\beta)$. It is possible to develop a fine structure theory for L based on E-recursion. The idea is to replace Σ_1 by E-recursively enumerable systematically. It is not clear if any worthwhile applications result.

A relativization of Lemma 2.1 is needed for Slaman's density and splitting theorems in the next chapter. Lemma 2.2 relativizes 2.1 to B when $L(\kappa)$ is E-closed and B is regular and E-recursively enumerable on $L(\kappa)$. The following definitions assume nothing about κ or B.

ρ^B is the least $\gamma \leq \kappa$ such that for some $p \in L(\kappa; B)$, there exists a partial map

$$\lambda x |\{e\}^B(x, p)$$

of γ onto $L(\kappa; B)$ via computations in $L(\kappa; B)$.

η^B is the least $\gamma \leq \kappa$ such that there exists an $R \in 2^\gamma - L(\kappa; B)$ such that R is E-recursively enumerable on $L(\kappa; B)$ relative to B. This last means there are $e < \omega$ and $q \in L(\kappa; B)$ such that

$$R \cap \gamma = \{x \mid \{e\}^B(q, x)\downarrow \text{ via a computation in } L(\kappa; B)\}.$$

The proof of Lemma 2.2 makes strong use of the assumption that $L(\kappa)$ is E-closed. By contrast the proof of Lemma 2.1 succeeds when $L(\kappa)$ is rudimentarily closed.

2.2 Lemma (Slaman 1985). *Assume $L(\kappa)$ is E-closed. If $B \subseteq \kappa$ is regular and E-recursively enumerable on $L(\kappa)$, then $\eta^B = \rho^B$.*

Proof. Modeled on the proof of 2.1. The only difference is the presence of B, which causes no great conceptual difference save for the last part of the argument.

As in 2.1 $\eta^B \leq \rho^B$.

Fix $\gamma < \rho^B$ and let $A \subseteq \gamma$ be E-recursively enumerable in $p \in L(\kappa; B)$ relative to B via computations in $L(\kappa; B)$. Since B is regular, $L(\kappa; B) = L(\kappa)$. $\eta^B \geq \rho^B$ is proved by showing $A \in L(\kappa)$. Let g^B be a universal, partial E-recursive-in-B function such that for all x:

$$g^B[x] = \{\{e\}^B(z_0, \ldots, z_{n-1}) \mid \{e\}^B(z_0, \ldots, z_{n-1})\downarrow, e, n < \omega, z_i \in x\}.$$

Thus $E(x; B) = g^B[TC(\{x\})]$. Suppose B is E-recursively enumerable in q on $L(\kappa)$. Let

$$H = \{z \mid z \in g^B[\gamma \cup \{p, q\}] \text{ via a computation in } L(\kappa)\}. \tag{1}$$

Define $O^B(z) = \mu\alpha[z \in L(\alpha+1; B) - L(\alpha; B)]$. As in the proof of 2.1,

$$z \in H \to O^B(z) \in H. \tag{2}$$

It is safe to assume p and q are ordinals, since each element of $L(\kappa)$ is E-recursive in some ordinal below κ. B is a set of ordinals. Hence each z put in H by g^B is accompanied by a wellordering of z, as in Proposition 5.2.X. It follows, as in 2.1, that

$$\langle H; H \cap B\rangle \prec_0 \langle L(\kappa; B)\rangle. \tag{3}$$

According to (3) each Δ_0^{ZF} sentence with parameters in H, and with $y \in B$ as an additional atomic formula, is true in H iff it is true in $L(\kappa)$. (3) implies H is extensional, hence isomorphic to a transitive set via a collapsing map t. By (2)

$$t\colon H \xrightarrow{\approx} L(\beta), \quad \text{and}$$

$$\langle H, H \cap B\rangle \approx \langle L(\beta); \bar{B}\rangle,$$

where $\beta \leq \kappa$ and $\bar{B} = t[H \cap B]$.

Suppose $\beta < \kappa$. A is definable over $\langle L(\beta); \bar{B} \rangle$. $\bar{B}$ is E-recursively enumerable in $t(q)$ on $L(\beta)$ thanks to (3) and the E-closedness of $L(\kappa)$. Consequently A is definable over $L(\beta)$ and so belongs to $L(\kappa)$.

Suppose $A \notin L(\kappa)$. By the previous paragraph $\beta = \kappa$. (1) implies g^B maps some $\gamma_0 < \rho^B$ onto H via computations in H. Hence $g^{\bar{B}}$ maps γ_0 onto $L(\kappa)$ via computations in $L(\kappa)$. The existence of $g^{\bar{B}}$ almost contradicts the definition of η^B; it fails to do so because it is computed relative to $\bar{B}$ rather than B. The defect is remedied by using A to approximate $g^{\bar{B}}$. Suppose

$$A = \{v \mid v < \gamma \quad \& \quad \{c\}^B(v, p)\downarrow\}$$

with the understanding that only computations in $L(\kappa)$ are acceptable. Define $|v| = |\{c\}^B(v, p)|$.

Consider the definition of $g^{\bar{B}}$: $\bar{B}$ is E-recursively enumerable in $t(q)$ on $L(\kappa)$; $g^{\bar{B}}$ is computed relative to $\bar{B}$ via computations in $L(\kappa)$. The definition of $g_v^{\bar{B}}$ is obtained by substituting $L(|v|)$ for $L(\kappa)$ in the definition of $g^{\bar{B}}$. If $t(p)$ or $t(q) \notin L(|v|)$, then $g_v^{\bar{B}}(u)$ is undefined. Set $f(u, v)$ equal to $g_v^{\bar{B}}(u)$. f is partial E-recursive in p, $t(p)$, $t(q)$ relative to A via computations in $L(\kappa)$.

It remains only to show f is a partial map of $\gamma_0 \times \gamma$ onto $L(\kappa)$. Fix $u < \gamma_0$ and suppose $g^{\bar{B}}(u)$ is defined by some computation d in $L(\kappa)$. d uses $\bar{B} \cap \sigma$ for some $\sigma < \kappa$. ($\bar{B}$ is regular because B is.) Since $L(\kappa)$ is E-closed, $\bar{B} \cap \sigma$ is enumerated via computations of height less than τ for some $\tau < \kappa$. Thus $g^{\bar{B}}(u) = g_v^{\bar{B}}(u)$ for all sufficiently large $|v|$. Note that

$$\sup\{|v| \mid v \in A\} = \kappa,$$

since $A \notin L(\kappa)$. □

2.3 Proposition. *Let p be an ordinal and A a set of ordinals. If $\lambda x \mid \kappa_r^{p, x}$ does not attain a maximum on A, then*

$$\sup\{\kappa_r^{p, x} \mid x \in A\} \leq \sup\{\kappa_0^{p, y, z} \mid y, z \in A\}.$$

Proof. Suppose not. Then there are $c, d \in A$ such that

$$\kappa_r^{p, c} > \kappa_r^{p, d} \geq \sup\{\kappa_0^{p, y, z} \mid y, z \in A\}. \tag{1}$$

By Lemma 5.5.X, $\kappa_r^{p, c, d} > \kappa_r^{p, d}$. The intended contradiction is: $\kappa_r^{p, c, d}$ is a $\langle p, d \rangle$-reflecting ordinal. Suppose some Σ_1 sentence $\mathscr{F}$ about $\langle p, d \rangle$ is true below $\kappa_r^{p, c, d}$. Then $\mathscr{F}$ is true below $\kappa_0^{p, c, d}$. But then (1) implies $\mathscr{F}$ is true below $\kappa_r^{p, d}$. □

2.4 Proposition. *Assume $L(\kappa)$ is E-closed, and admits Moschovakis witnesses. If $A \subseteq \kappa$ is E-recursively enumerable on $L(\kappa)$ and incomplete, then A is subgeneric.*

Proof. The incompleteness of A implies A is regular as in the first half of the proof of Theorem 1.3. Hence $L(\kappa; A) = L(\kappa)$. If $x \in L(\kappa)$ and $\{e\}^A(x)\downarrow$, then $T_{\langle e, x; A\rangle} \in L(\kappa)$ by recursion on $|\{e\}^A(x)|$. Suppose e is $2^m \cdot 3^n$. Then by recursion $T_{\langle m, x; A\rangle}$ and $T_{\langle n, y; A\rangle}$ (for all $y \in \{m\}^A(x)$) belong to $L(\kappa)$. If $|\{m\}^A(x)| < \beta$ and

$$\sup^+ \{|\{n\}^A(y)| \mid y \in \{m\}^A(x)\} \leq \beta < \kappa, \tag{1}$$

then $T_{\langle e, x; A\rangle}$ is first order definable over $L(\beta, x; A)$, and so belongs to $L(\kappa)$ by the regularity of A.

Suppose there is no β that satisfies (1). Then for all $d < \omega$ and all $z \in L(\kappa)$,

$$\{d\}(z)\uparrow \leftrightarrow (\mathrm{E}y)[y \in \{m\}^A(x) \quad \& \quad \text{some } w \text{ first order definable over} \tag{2}$$

$$L(|\{n\}^A(y)|) \text{ witnesses } \{d\}(z)\uparrow].$$

The right side of (2) can be verified by a computation from A in $L(\kappa)$, because A is regular. But then A is complete. (cf. proof of Lemma 1.1.) □

The next theorem is needed for priority constructions in which requirements are indexed by ordinals less than ρ^κ. It will imply a block of requirements bounded below ρ^κ can be satisfied by some stage prior to the end of the construction.

2.5 Theorem (Sacks 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. If $p \in \kappa$ and $\gamma < \rho^\kappa$, then*

$$\sup \{\kappa_r^{p, \delta} \mid \delta < \gamma\} < \kappa.$$

Proof. Since ρ is either κ or an $L(\kappa)$-cardinal, it can be assumed that γ is closed under pairing in the sense that

$$x, y < \gamma \rightarrow (\mathrm{E}w)[w < \gamma \quad \& \quad x, y \leq_E w]. \tag{1}$$

To verify (1) let z be the standard 1-1 onto map from pairs of ordinals to ordinals. z is defined by recursion; at stage σ, $z(\sigma, \gamma)$ is defined for all $\gamma \leq \sigma$. Both z and z^{-1} are partial E-recursive. The w of (1) is $z(x, y)$.

(1) permits

$$\{\langle e, \delta\rangle \mid \{e\}(p, \delta)\downarrow \quad \& \quad e < \omega \quad \& \quad \delta < \gamma\} \tag{2}$$

to be treated as a subset of γ. It follows from Lemma 2.1 that $(2) \in L(\kappa)$. Consequently

$$\sup \{\kappa_0^{p, \delta} \mid \delta < \gamma\} < \kappa. \tag{3}$$

It follows from Theorem 5.8.X that $\kappa_r^{p, \delta} < \kappa$ for all $\delta < \kappa$. Suppose $\lambda w \mid \kappa_r^{p, w}$ does

not attain a maximum on $\{\delta \mid \delta < \gamma\}$. Then Proposition 2.3 yields

$$\sup\{\kappa_r^{p,\delta} \mid \delta < \gamma\} \leq \sup\{\kappa_0^{p,x,y} \mid x, y < \gamma\}. \tag{4}$$

The theorem follows from (1), (3) and (4). □

2.6 RE Cofinality. As in α-recursion theory, cofinality considerations figure prominently in E-recursively enumerable degree constructions. Lemma 2.7 is needed for the adaptation of Shore's blocking method (3.2.VIII) to E-recursion. For B a set of ordinals, let

$$|B| = \text{ordertype of } B.$$

Assume $L(\kappa)$ is E-closed. Define r.e. $\mathrm{cf}(\lambda)$, the *E-recursively enumerable-on-$L(\kappa)$ cofinality* of λ, to be

$$\mu\gamma(\mathrm{EB})[\sup B = \lambda \quad \& \quad |B| = \gamma \quad \& \quad B \text{ is } E\text{-recursively enumerable on } L(\kappa)] \tag{1}$$

The last clause of (1) means that B is enumerated via a parameter and computations in $L(\kappa)$. Note that r.e. $\mathrm{cf}(\kappa)$ is the same as $\sigma 1\mathrm{cf}(\kappa)$.

Suppose B satisfies (1). B can be transformed into a B^* that is enumerated in increasing fashion. By "increasing" is meant:

$$B^* = \{x \mid \{e^*\}(p^*, x)\downarrow\};$$
$$|x| = |\{e^*\}(p^*, x)| \quad \text{if} \quad \{e^*\}(p^*, x)\downarrow;$$
$$x, y \in B^* \quad \& \quad x < y \to |x| < |y|.$$

Define:

$$x \in B_0^* \leftrightarrow x \in B \quad \& \quad (y)[y \in B \quad \& \quad |y| < |x| \to y < x];$$
$$x \in B^* \leftrightarrow x \in B_0^* \quad \& \quad (y)[y \in B_0^* \quad \& \quad |y| = |x| \to y \geq x].$$

If no unbounded in λ subset of B belongs to $L(\kappa)$, then the computations necessary for the enumeration of B are unbounded in height below κ, and so the above definition of B^* succeeds. Otherwise, B can be trivially altered to the desired B^*.

2.7 Lemma. *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Then*

$$\text{r.e. } \mathrm{cf}(\rho^\kappa) = \text{r.e. } \mathrm{cf}(\kappa).$$

Proof. To show r.e. $\mathrm{cf}(\rho^\kappa) \geq$ r.e. $\mathrm{cf}(\kappa)$, let B satisfy the matrix of 2.6(1) with ρ^κ in place of λ. Suppose $p \in L(\kappa)$ and $\{e\}(x, p)$ is a partial map of ρ^κ onto $L(\kappa)$.

Define:

$$x^r = \sup\{\kappa_r^{p,y} \mid y \leq x\} \ (x < \rho^\kappa);$$
$$B^r = \{x^r \mid x \in B\}.$$

By Theorem 2.5, $B^r \subseteq \kappa$. Let $f(y)$ be the value of $\{e\}(y, p)$ when the latter converges. In that event

$$\kappa_r^{f(y)} \leq \kappa_r^{p,y} \tag{1}$$

follows from Lemma 5.3.X. p can be effectively encoded by a set of ordinals, so it is safe to assume p is such a set. According to 5.3.X, divergence witnesses for computations from p, y can defined over $L(\kappa_r^{p,y})$. If (1) is false, these witnesses can be found below $\kappa_r^{f(y)}$, and so can be computed from p, y. But then the standard complete E-recursively enumerable in p, y subset of κ would be E-recursive in p, y.

It follows from (1) that $\sup B^r = \kappa$ and that the ordertype of B^r is at most that of B.

Lemma 5.5.X implies that $\kappa_r^{p,y}$ is the least γ such that every computation from p, y is seen to converge or to have a divergence witness by level γ of $L(\kappa)$. It follows that B^r is E-recursive on $L(\kappa)$.

To show r.e. $\mathrm{cf}(\rho^\kappa) \leq$ r.e. $\mathrm{cf}(\kappa)$, let B be an unbounded subset of κ, E-recursively enumerable on $L(\kappa)$ and of ordertype r.e. $\mathrm{cf}(\kappa)$. As shown in subsection 2.6, B can be assumed to be enumerated in increasing fashion. Replace B by

$$B' = \{\delta \mid (\mathrm{Ex})(x \in B \ \ \& \ \ |x| = \delta)\}.$$

($|\delta|$ is the length of the computation that puts x in B.) B' and B have the same ordertype, and B' is E-recursive on $L(\kappa)$.

Let f be a partial map, E-recursive on $L(\kappa)$, from ρ^κ onto κ. Let

$$t\colon \text{r.e. } \mathrm{cf}(\kappa) \to \kappa$$

enumerate B' in increasing fashion. Assume $t(0) = 0$. For $x \in \mathrm{dom}\, f$, let $|x|$ be the length of the computation that puts x in $\mathrm{dom}\, f$. Define:

$$(\mathrm{dom}\, f)_\gamma = \{x \mid x \in \mathrm{dom}\, f \ \ \& \ \ t(\gamma) \leq |x| < t(\gamma) + 1\};$$

$$s(\gamma) = \mu x[x \in (\mathrm{dom}\, f)_\gamma].$$

Since $(\mathrm{dom}\, f) \notin L(\kappa)$, $s(\gamma)$ is defined on unboundedly many γ's below ρ^κ. Let C be the range of s. The ordertype of C is at most that of B'. Note that s^{-1} is E-recursive on $L(\kappa)$, because t is. Consequently C is E-recursively enumerable on $L(\kappa)$.

The idea behind the above argument is to map B into ρ^κ by a computable function whose inverse is computable. In general the inverse of a one-one partial E-recursive function is not E-recursive, so some care had to be exercised above. □

Theorem 2.5 and Lemma 2.7 are the only structural facts needed for the solution of Post's problem in the next section. Some further facts are needed for Slaman's splitting and density theorems. For Post's problem ordinals less than ρ^κ, the greater E-recursively enumerable projectum (of κ), suffice to index requirements. For splitting, ρ^κ tends to be too large and has to be replaced by $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ. The notion of tame Σ_2 projectum figured prominently in α-recursion theory (cf. Section 2.VIII).

2.8. The Tame Σ_1 Projectum. Assume κ is a limit ordinal. $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ, is

$$\mu\gamma_{\gamma \leq \kappa}(\mathrm{Ef})\,[f\colon \gamma \xrightarrow{\text{onto}} \kappa \quad \& \quad f \in \Sigma_1 \quad \& \quad f \text{ is tame}].$$

To say f is *tame* means:

$$(x)_{x < \mathrm{dom} f}(E\beta)_{\beta < \kappa}(z)_{z < x}\,[L(\beta) \models (Ey)\,[f(z) = y]].$$

In short, for a proper initial segment x of f, there is a bound below κ for the range of $f \restriction x$ and the associated existential witnesses.

If $L(\kappa)$ is E-closed and admits Moschovakis witnesses, then according to Exercise 2.16

$$t\sigma 1p(\kappa) \leq \rho^\kappa.$$

Let C be an E-recursively-enumerable-on-$L(\kappa)$, non-E-recursive, regular subset of κ. C ambiguously denotes the characteristic function of C. Define:

(1) $$\{v\}(y, p) \simeq \{e\}(y, p, q) \quad \text{if} \quad v = \langle e, q \rangle;$$

(2) $$\ell_p^C(v) = \sup\{z \mid (y)_{y < z}(C(y) = \{v\}(y, p))\};$$

(3) $$\delta_C = \mu\gamma_{\gamma \leq \kappa}(\mathrm{Ep})_{p < \kappa}\,[\kappa = \sup\{\ell_p^C(v) \mid v < \gamma\}].$$

$\ell_p^C(v)$ measures the length of agreement between $\lambda y\,|\,C(y)$ and $\lambda y\,|\,\{v\}(y, p)$. It is commonplace in splitting and density arguments for recursively enumerable sets (cf. Section 1.IX). δ_C is called the *C-recovery parameter*. It plays a central role in the proof of Slaman's splitting theorem. The larger δ_C is, the better, since difficulties occur as $\ell_p^C(x)$ approaches κ. More precisely, δ_C should be larger than the length of any proper initial segment of preservation-type requirements.

The remainder of this section is devoted to showing $\delta_C \geq t\sigma 1p(\kappa)$.

Suppose $A \subseteq \kappa$. A is said to be *tame* $\Sigma_1^{L(\kappa)}$ if A is Σ_1 definable (boldface) over $L(\kappa)$, and if for each $\gamma < \sup A$, the existential witnesses need to establish $A \cap \gamma$ all lie below some $\delta < \kappa$.

2.9 Proposition. *If A is tame $\Sigma_1^{L(\kappa)}$ and $\sup A < t\sigma 1p(\kappa)$, then $A \in L(\kappa)$.*

Proof. The idea is to form a tame Σ_1 hull H over which A is Σ_1 definable. To say H is tame means H is the range of a tame Σ_1 function applied to an ordinal. The

collapse of H is isomorphic to some $L(\beta)$. If $\beta < \kappa$, then $A \in L(\beta+1) \in L(\kappa)$. If $\beta = \kappa$, then there exists a tame Σ_1 map from sup A onto $L(\kappa)$.

Form H as follows. Put each $\delta <$ sup A into H. For each $\delta \in A$, put the L-least existential witness that establishes $\delta \in A$ into H. Let H_0 be H so far. If H_0 is bounded below κ, then $A \in L(\kappa)$.

Suppose H_0 is unbounded. Some "bounded" existential witnesses must now be added to H_0. Suppose

$$b_0, b_1, \ldots, b_{n-1} < b < \kappa,$$

$$\mathscr{F}_e(x, y_0, \ldots, y_{n-1}) \text{ is } \Delta_0, \quad \text{and}$$

$$L(\kappa) \models (\mathrm{Ex})_{x \in \underline{b}} \mathscr{F}(x, \underline{b}_0, \ldots, \underline{b}_{n-1}). \tag{1}$$

Let $g(e, b_0, \ldots, b_{n-1}, b)$ be the L-least $x \in b$ that satisfies $\mathscr{F}_e(x, \underline{b}_0, \ldots, \underline{b}_{n-1})$. If (1) is false, give g the value 0. g is Σ_1, and tame in the sense that any bound on the arguments of g gives rise to a bound on the values of g. Now close H_0 under g. Let

$$H = H_0 \cup g[H_0] \cup gg[H_0] \cup \ldots$$

A slight alteration of g, denoted by g below, produces a tame $\Sigma_1^{L(\kappa)}$ map from $(\text{sup } A)^{<\omega}$ onto H. The remainder of the proof requires a tame $\Sigma_1^{L(\kappa)}$ function f from sup A, or from some ordinal less than $tp\sigma 1(\kappa)$, onto H. Obtaining such an f from g is troublesome because $tp\sigma 1(\kappa)$ can be less than κ without being an $L(\kappa)$-cardinal. Let μ be the $L(\kappa)$-cardinality of sup A. Then

$$\text{sup } A = \mu \cdot \eta + \beta$$

for some $\beta < \mu$. The proof of the proposition proceeds by induction on sup A. If $\beta > 0$, then

$$A = (A \cap \mu \cdot \eta) \cup (A \cap [\mu \cdot \eta, \mu \cdot \eta + \beta))$$

and so $A \in L(k)$ by induction. Thus $\beta = 0$. On similar grounds η is a limit or 1.

Since μ is a cardinal, there is a one-one map r of μ onto $\mu^{<\omega}$. For each $\delta < \eta$, let t_δ be a one-one map of μ onto $\mu \cdot \delta$. t_δ induces a one-one map v_δ of $\mu^{<\omega}$ onto $(\mu \cdot \delta)^{<\omega}$. Define

$$f(x) = g \circ v_{\delta+1} \circ r \circ t_{\delta+1}^{-1}(x) \quad \text{when} \quad x \in [\mu \cdot \delta, \mu \cdot (\delta+1)).$$

f is a tame $\Sigma_1^{L(\kappa)}$ map from sup A onto H.

Since H is closed under g, H is extensional, hence isomorphic to a transitive set S via the collapsing map. The unboundedness of H, and its closedness under g, imply

$$z \in H \to O(z) \in H.$$

Consequently $S = L(\beta)$ for some $\beta \leq \kappa$. If $\beta < \kappa$, then $A \in L(\beta+1) \subseteq L(\kappa)$.

Suppose $\beta = \kappa$. Then $\bar{f}$, the collapse of f, is a tame $\Sigma_1^{L(\kappa)}$ map from sup A onto $L(\kappa)$, an impossibility since sup $A < t\sigma 1p(\kappa)$. □

The next result is the principal structural fact needed in Slaman's splitting and density theorems.

2.10 Lemma (Slaman 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Let $C \subseteq \kappa$ be regular, not E-recursive on $L(\kappa)$, and E-recursively enumerable on $L(\kappa)$. Then*

$$\delta_C \geq t\sigma 1p(\kappa).$$

Proof. Suppose $\delta_C < t\sigma 1p(\kappa)$ with the hope of showing C is E-recursive on $L(\kappa)$.

Recall the definition of δ_C, the recovery parameter, from subsection 2.8. Let p be an ordinal that serves in 2.8(3) to define δ_C. Thus

$$\kappa = \sup\{\ell_p^C(x) \mid x < \delta_C\}. \tag{1}$$

Define "γ establishes (Ey)$[C(y) \neq \{x\}(y, p)]$" to mean: the computations and Moschovakis witnesses in $L(\gamma + 1)$ show there is a y such that

$$\{x\}(y, p)\uparrow \quad \text{or}$$
$$[\{x\}(y, p)\downarrow \quad \& \quad C(y) \neq \{x\}(y, p)].$$

For each $x < \delta_C$, define

$$f(x) = \mu\gamma[\gamma \text{ establishes (Ey)}(C(y) \neq \{x\}(y, p))]. \tag{2}$$

f will help compute C. A reflection argument will supply bounds on $f(x)$. Fix $v < \delta_C$, and let

$$w = \sup^+\{\ell_p^C(x) \mid x < v\}.$$

For each $x < v$, $f(x)$ establishes an inequality in at least one of three ways.
Case 1: $\{x\}(y, p)\uparrow$ for some $y < w$. Let y_0 be the least such y. y_0 is the least element of a set co-E-recursively enumerable in x, p, w. Exercise 5.17.X, a corollary of Kechris's basis theorem (5.1.X), implies

$$\kappa_r^{x, p, w, y_0} \leq \kappa_r^{x, p, w}.$$

Hence $f(x) \leq \kappa_r^{x, p, w}$ by Lemmas 5.3.X and 5.6.X.
Case 2: case 1 does not hold, and there is a $y < w$ such that $C(y) = 0 \neq \{x\}(y, p)$. (Recall that "$C(y) = 0$" means $y \notin C$.) Let q be an ordinal parameter needed to E-recursively enumerate C on $L(\kappa)$. Let

$$S_2 = \{y \mid y < w \quad \& \quad \{x\}(y, p) = 1\}.$$

Then $S_2 \leq_E x, p, w$, and, as in case 1, $f(x) \leq \kappa_r^{x, p, q, w}$. In this case $f(x)$ establishes a Moschovakis witness for some y not in C.

Case 3: case 1 does not hold, and there is a $y < w$ such that $C(y) = 1 \neq \{x\}(y, p)$. Let

$$S_3 = \{y \mid y < w \quad \& \quad \{x\}(y, p) = 0\}.$$

Since C is regular, there is an ordinal $\theta < \kappa$ such that the computations in $L(\theta)$ suffice to enumerate $C \cap w$. Hence $f(x) \leq \kappa_0^{x, p, w} + \theta$.

The bounds provided in the above cases imply $\{f(x) \mid x < v\}$ is bounded below κ by Theorem 2.5.

Consequently f is a total, tame $\Sigma_1^{L(\kappa)}$ function from δ_C into κ. Note that the range of f is unbounded by (1). Since f is tame, it is safe to assume f is strictly increasing.

Define E to be

$$\{\langle x, z\rangle \mid x, z < \delta_C \quad \& \quad (\mathrm{E}y)_{y < f(z)}[C(y) \neq \{x\}(y, p)]\}.$$

E is tame $\Sigma_1^{L(\kappa)}$ because f is, or more precisely, because of the above proof that f is tame $\Sigma_1^{L(\kappa)}$. By Proposition 2.9, $E \in L(\kappa)$.

With the aid of E, C is E-recursive on $L(\kappa)$ as follows. Fix $y < \kappa$. There is a unique z_y such that

$$y \in [f(z_y), f(z_y + 1)).$$

The definition of f in (2) via the least ordinal operator implies

$$\{\langle x, f(x)\rangle \mid f(x) \leq y\} \leq_E y, p$$

uniformly in y. Thus z_{y+1} can be computed from y, and

$$(y')_{y' < f(z_{y+1})}[C(y') = \{x\}(y', p)]$$

for some x. Such an x can be extracted from E. □

2.11 Tame Σ_2 Cofinality. Tame Σ_2 functions are needed for the adaptation of Shore's blocking method to the proof of Slaman's splitting and density theorems. They arise in the course of approximating regular, E-recursively-enumerable-on-$L(\kappa)$ sets, when $L(\kappa)$ is E-closed. Suppose C is such a set. Let

$$g(x) = C \cap x \quad \text{and} \quad g_0(\sigma, x) = C^{<\sigma} \cap x$$

for all x, $\sigma < \kappa$. ($C^{<\sigma}$ is that part of C enumerated via computations of height less than σ.) g_0 is E-recursive on $L(\kappa)$, and the regularity of C implies

(1) $$\lim_{\sigma} g_0(\sigma, x) = g(x).$$

In fact

(2) $$(a)_{a < \kappa}(E\tau)(x)_{x < a}(\sigma)_{\sigma > \tau}[g_0(\sigma, x) = g(x)].$$

Let h: dom $h \to \kappa$, where dom $h \leq \kappa$. h is said to be E_2 if there exists an E-recursive-on-$L(\kappa)$ function h_0 such that

$$(x)_{x \in \text{dom } h}(E\tau)(\sigma)_{\sigma \geq \tau}[h_0(\sigma, x) = h(x)].$$

Thus g is E_2 by (1). h is said to be *tame* E_2 if there exists an E-recursive-on-$L(\kappa)$ function h_0 such that

$$(a)_{a < \text{dom } h}(E\tau)(x)_{x < a}(\sigma)_{\sigma > \tau}[h_0(\sigma, x) = h(x)].$$

Thus g is tame E_2 by (2).

Suppose $\lambda \leq \kappa$. The *tame* E_2 *cofinality* of λ (in symbols, $te2cf(\lambda)$) is

$$\mu\gamma_{\gamma \leq \lambda}(\text{Ef})\,[f\colon \gamma \to \lambda \quad \& \quad \text{sup range } f = \lambda \quad \& \quad f \text{ is tame } E_2].$$

According to Exercise 2.15,

$$te2cf(t\sigma 1p(\kappa)) = te2cf(\kappa).$$

This last fact will be quoted in the proof of Slaman's splitting theorem (4.1), where requirements are indexed by ordinals less than $t\sigma 1p(\kappa)$ and the number of blocks of requirements is $te2\text{cf}(\kappa)$.

2.12–2.16 Exercises

2.12. Define rec cf(λ), the E-recursive-on-$L(\kappa)$ cofinality of λ, to be

$$\mu\gamma(\text{EB})[\sup B = \lambda \quad \& \quad \text{ordertype of } B = \gamma \quad \& \quad B \text{ is } E\text{-recursive on } L(\kappa)].$$

Assume $L(\kappa)$ is E-closed and show r.e. cf(κ) = rec cf(κ).

2.13. Assume $L(\kappa)$ is E-closed. Show that there exists a function g from r.e. cf(ρ^κ) into ρ^κ such that:

(i) graph of g is E-recursively enumerable on $L(\kappa)$;
(ii) sup range $g = \rho^\kappa$;
(iii) g is strictly increasing;
(iv) $\tau_1 < \tau_2 <$ r.e. cf$(\rho^\kappa) \to |\langle \tau_1, g(\tau_1)\rangle| < |\langle \tau_2, g(\tau_2)\rangle|$.

($|\langle x, g(x)\rangle|$ is the length of the computation that enumerates $\langle x, g(x)\rangle$ in the graph of g.)

2.14. Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Show $\delta_C \leqslant \rho^\kappa$.

2.15. Assume $L(\kappa)$ is E-closed. Show $te2\text{cf}(t\sigma 1p(\kappa)) = te2\text{cf}(\kappa)$.

2.16. Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Show $t\sigma 1p(\kappa) \leq \rho^{\kappa}$.

3. *van de Wiele's Theorem*

van de Wiele [1982] found an important link between Σ_1 admissibility and E-recursiveness. His theorem explains why some Σ_1 functions are E-recursive and others are not. Intuitively, if f is Σ_1 and the search for existential witnesses needed to evaluate f does not extend too far, then f is E-recursive.

More precisely, let f be a total function from V into V. f is said to be *uniformly* Σ_1 *definable on every* Σ_1 *admissible set* if there exists a lightface Σ_1^{ZF} formula $\mathscr{F}(x, y)$ such that for every Σ_1 admissible set A:

$$f[A] \subseteq A;$$

$$f\restriction A = \{\langle a, b\rangle \mid \langle A, \varepsilon\rangle \models \mathscr{F}(\underline{a}, \underline{b})\}.$$

"lightface" means that all parameters in $\mathscr{F}(x, y)$ are finite ordinals.

van de Wiele's proof is an application of proof-theoretic methods originated by Girard. Subsequently S. Simpson found a proof based on the compactness theorem for first order logic. The argument below is in the spirit of E-recursion and is extracted from Slaman [1981]. The latter approach appears to give more information than any of the others.

Theorem 3.1 (van de Wiele 1982). *Let f be a total function from V into V. Then (i) and (ii) are equivalent.*

(i) f is E-recursive.

(ii) f is uniformly Σ_1 definable on every Σ_1 admissible set A.

Proof. (i) implies (ii), since the relation, c is a computation of $\{e\}$ (b), is Δ_0^{ZF}, and so Δ_0 bounding implies A is closed under forming computations.

Assume (ii) holds. Fix x. The argument takes place within

$$L(\kappa_r^x, TC(\{x\})). \tag{1}$$

If $f(x) \in (1)$, then $f(x) \in L(\kappa_0^x, TC(x))$ by reflection, and so $f(x) \leq_E x$ by Gandy selection. $f(x)$ will be located in (1) by means of a hull $Z \subseteq (1)$ that collapses to a Σ_1 admissible set.

A curious technical step precedes the definition of Z. Choose $p \in (1)$ so that

$$\kappa_r^{x,p} = \min\{\kappa_r^{x,q} \mid q \in (1)\}. \tag{2}$$

For the moment assume $TC(x)$ is countable. Z will be of the form $\{z_i \mid i < \omega\}$. The z_i's are chosen one at a time from

$$L(\kappa_r^{x,p}, TC(\{x, p\})) \tag{3}$$

so that for all $i < \omega$:

(4) $$z_0 = \langle x, p \rangle;$$

(5) $$\kappa_r^{z_0, \ldots, z_i} = \kappa_r^{z_0};$$

(6) $$Z \text{ is } E\text{-closed; and}$$

(7) $$(\mathrm{Ew})[w \in (3) \quad \& \quad \{e\}(z_i, w)\uparrow] \to (\mathrm{Ej})[\{e\}(z_i, z_j)\uparrow].$$

In order to see that (6) and (7) can be honored without violating (5), fix i and assume $z_0, \ldots, z_i$ have been chosen in accord with (5). If $\{e\}(z_0, \ldots, z_i) = w$, then $\langle z_0, \ldots, z_i, w \rangle \equiv_E \langle z_0, \ldots, z_i \rangle$, and so

$$\kappa_r^{z_0, \ldots, z_i, w} = \kappa_r^{z_0, \ldots, z_i}.$$

If the antecedent of (7) holds, then Kechris's basis theorem (5.1.X) supplies a w such that

$$\kappa_r^{z_0, \ldots, z_i, w} \leq \kappa_r^{z_0, \ldots, z_i}.$$

It follows from (2) and (4) that $\kappa_r^{z_0, \ldots, z_i, w} = \kappa_r^{z_0}$.

By (7), Z is extensional, hence isomorphic to a transitive set $\bar{Z}$ via a collapsing map t. For each $z \in Z$ let $\bar{z}$ be $t(z) \in \bar{Z}$. To see that $\bar{Z}$ is Σ_1 admissible, let $\mathscr{G}(u, v, y)$ be a Δ_0^{ZF} formula such that

(8) $$\bar{Z} \models (u)_{u \in \bar{a}}(\mathrm{Ev})\mathscr{G}(u, v, \bar{c})$$

for some $\bar{a}, \bar{c} \in \bar{Z}$. To obtain a bound on v in (8) it has to be shown that:

(9) $$(3) \models (u)_{u \in a}(\mathrm{Ev})\mathscr{G}(u, v, c).$$

The derivation of (9) from (8) is based on the following technical fact.

(10) $$[u \in (3) \quad \& \quad (3) \models (\mathrm{Ev})\mathscr{G}(u, v, c)] \to (\mathrm{Ev})(E\gamma)[\gamma \leq_E x, p, c, u$$
$$\& \quad v \in L(\gamma, TC(\{x, p, c, u\})) \quad \& \quad (3) \models \mathscr{G}(u, v, c)].$$

Assume the antecedent of (10) holds with v_0 as the existential witness. Then

$$v_0 \in L(\gamma, TC(\{x, p\})) \quad \text{for some} \quad \gamma < \kappa_r^{x, p, c, u}$$

by (2), and so the consequent of (10) follows by reflection.

Suppose (9) is false. Then U, the set of all u such that

$$u \in a \quad \& \quad (3) \models (v) \sim \mathscr{G}(u, v, c),$$

is nonempty. It follows from (10) that U is co-E-recursively enumerable in x, p, c, a. Then (7) implies some $u \in U$ belongs to Z. Hence

$$Z \models (\mathrm{Eu})_{u \in a}(v) \sim \mathscr{G}(u, v, c),$$

contrary to (8). Thus (9) follows from (8).

Let (10r) be the result of replacing "$u \in (3)$" by "$u \in a$", and "x, p, c, u" by "x, p, c, a, u", in (10). The proof of (10r) is the same as that of (10). Gandy selection makes it possible to construe the γ of (10r) as an E-recursive function of u on a. Let

$$\delta = \sup\{\gamma(u) | u \in a\}.$$

Then $\delta \leq_E x, p, c, a$, and $\delta \in Z$ by (6). Thus (9) implies

$$Z \models (u)_{u \in a}(\mathrm{Ev})_{v \in b}\mathscr{G}(u, v, c),$$

where b is $L(\delta, TC(\{x, p, c, a\}))$. But then $\bar{b}$ bounds v in (8) and the proof of the Σ_1 admissibility of $\bar{Z}$ is complete.

Let $\mathscr{F}(u, v)$ be the Σ_1^{ZF} formula that defines f in every Σ_1 admissible set according to (ii). Then

$$\bar{Z} \models (\mathrm{Ev})\mathscr{F}(\bar{x}, v). \tag{11}$$

In (11) $\bar{Z}$ can be replaced first by Z and then by (1). Finally by reflection

$$L(\beta, TC(x)) \models (\mathrm{Ev})\mathscr{F}(x, v) \tag{13}$$

for some $\beta \leq_E x$, and the unique v $(=f(x))$ of (13) can be computed from β.

Thus $f(x)$ is computed by using Gandy selection to extract it from $L(\kappa_0^x, TC(x))$. The presence of $f(x)$ in $L(\kappa_0^x, TC(x))$ was established above by assuming $TC(x)$ was countable. That assumption is eliminable by the Levy–Shoenfield absoluteness theorem. Let $\{e\}$ be the procedure established above for computing $f(x)$ from x when $TC(x)$ is countable. Suppose there is an x_0 such that

$$f(x_0) \neq \{e\}(x_0). \tag{14}$$

By virtue of $\mathscr{F}$, computations and Moschovakis witnesses, (14) is a Σ_1 statement about x_0 true in V. Let H be a countable Σ_1 substructure of V with x_0 as an element. Let $\bar{H}$ be the transitive collapse of H and $\overline{x_0}$ the image of x_0 in $\bar{H}$. Then

$$f(\overline{x_0}) \neq \{e\}(\overline{x_0})$$

is a Σ_1 statement true in $\bar{H}$. But $TC(\overline{x_0})$ is countable. □

3.2 Exercise. What is the role of the curious parameter p in the proof of Theorem 3.1?

4. *Post's Problem for E-Recursion*

The theorem to be proved is: assume $L(\kappa)$ E-closed; then there exist two subsets of $L(\kappa)$, each E-recursively enumerable on $L(\kappa)$, such that neither is E-reducible to the other. In short, there exist incomparable E-recursively enumerable degrees. A weird feature of the argument is the absence of injuries. Priorities are assigned to requirements, but only to insure that requirements are met before time runs out. Conflicts between requirements are avoided with the help of Moschovakis witnesses. Injuries occur in the next section in the proof of Slaman's splitting theorem.

The conceptual differences between the solutions to Post's problem for E-recursion and α-recursion arise from the differences between the notion of recursively enumerable (r.e.) set as domain of partial recursive function and as Σ_1 definable set. Classical approaches to Post's problem rely on a dynamic view of r.e. sets: an r.e. set is enumerated by stages. An α-recursively enumerable set A is defined by a Σ_1 formula $\mathscr{F}$. A is enumerated by limiting the existential quantifier in $\mathscr{F}$. For each $\sigma < \alpha$, $A^{<\sigma}$ is the subset of $L(\sigma)$ definable by $\mathscr{F}$. Then $A = \cup\{A^{<\sigma} | \sigma < \alpha\}$, and $\lambda^{\sigma} | A^{<\sigma}$ is α-recursive. Suppose B, the domain of a partial E-recursive function, is to be defined dynamically by sifting through all computations in $L(\kappa)$. At stage σ, elements are to be added to B based on computations of height at most σ. If x is added, then the computation that puts x in B must be E-recursive in x, p, where p is a fixed parameter independent of x, e.g. the parameter needed to enumerate B. It follows that x cannot wait around forever. Either x is put in B prior to stage $\kappa_0^{x,p}$ or it never gets into B. In contrast, in α-recursion it is never to late to add x. Normally $\kappa_0^{x,p} < \kappa$, so it will eventually be too late to add x. To make matters worse (or more interesting) there is no effective way of recognizing $\kappa_0^{x,p}$. Thus x's time may have expired without any awareness of the expiration affecting the enumeration of B.

It may now seem immensely difficult to satisfy positive requirements in the setting of E-recursion. It turns out to be simpler than first supposed, because the presence of Moschovakis witnesses makes it easier than usual to satisfy negative requirements.

Suppose $L(\kappa)$ admits Moschovakis witnesses. A Friedberg–Muchnik incompatibility requirement is handled by waiting for a computation to converge. If it converges, then measures are taken to create and preserve an inequality. If it diverges, then by some stage the divergence is evident. At that stage a Moschovakis witness is chosen, and its role as a divergence witness is preserved forever.

If $L(\kappa) = E(\omega_1)$, then the requirements are indexed by countable ordinals, and each can be satisfied in its turn. For an arbitrary Σ_1 inadmissible $L(\kappa)$ the requirements are indexed by ordinals less than ρ^{κ}. By Theorem 2.5

(1) $\qquad$ if $x < \rho^{\kappa}$ and $p \in \kappa$, then $\sup\{\kappa_r^{p,y} | y < x\} < \kappa$.

It will follow from (1) that any set of requirements indexed by ordinals less than $x < \rho$ can be satisfied by stage σ for some $\sigma < \kappa$.

4.1 Theorem (Sacks 1985). *Suppose $L(\kappa)$ is E-closed. Then there exist two subsets of $L(\kappa)$, each E-recursively enumerable on $L(\kappa)$, such that neither is E-reducible to each other on $L(\kappa)$ (Post's problem).*

Proof. First suppose $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$. According to Proposition 1.2 the solution to Post's problem for $L(\kappa)$ provided by α-recursion theory, Theorem 2.6.VIII, is also a solution in the sense of E-recursion theory. So assume $L(\kappa)$ is either Σ_1 inadmissible or the limit of E-closed structures. In either case $L(\kappa)$ admits Moschovakis witnesses.

For $y \in L(\kappa)$, let

$$\{y\}^A(x) \quad \text{be} \quad \{(y)_0\}^A(x, (y)_1)$$

with the proviso that $\{(y)_0\}$ is null when $(y)_0 \geq \omega$. Let f be a partial E-recursive-on-$L(\kappa)$ map of ρ^κ onto $L(\kappa)$. The requirements are:

$$\begin{array}{lll} \text{req. } 2x: & A \neq \{f(x)\}^B, \text{ and} & \\ \text{req. } 2x+1: & B \neq \{f(x)\}^A. & (x < \rho^\kappa) \end{array}$$

Each requirement belongs to a block labeled by an ordinal less than

$$\gamma_0, \text{ the } E\text{-recursively enumerable-on-}L(\kappa) \text{ cofinality of } \rho^\kappa.$$

By Exercise 2.13 there exists a map g of γ_0 into ρ^κ such that:

(1a) the graph of g is E-recursively enumerable on $L(\kappa)$;
(1b) sup (range g) $= \rho^\kappa$;
(1c) g is strictly increasing;
(1d) $\tau_1 < \tau_2 < \gamma_0 \to |\langle \tau_1, g(\tau_1)\rangle| < |\langle \tau_2, g(\tau_2)\rangle|$,

where $|\langle x, g(x)\rangle|$ is the length of the computation that enumerates $\langle x, g(x)\rangle$ in the graph of g.

In addition $g(0) = 0$. Define

$$(\text{req. } 2x) \in \text{block } 2\tau, \quad \text{and} \quad (\text{req. } 2x+1) \in \text{block } 2\tau+1,$$

$$\text{if } g(\tau) \leq x < g(\tau+1).$$

The construction of A and B has the form of an effective transfinite recursion on κ. At stage $\sigma < \kappa$ decisions are made by examining all computations in $L(\sigma+1)$. For convenience it is assumed that the length of a computation c, denoted as usual by

$|c|$, has been redefined so that $|c|$ exceeds all ordinals mentioned in c. Let

$$A^{<\sigma} = \{x | x \text{ put in } A \text{ prior to stage } \sigma\},$$
$$\text{and} \quad A^{\sigma} = A^{<\sigma+1} - A^{<\sigma}.$$

Each requirement is acted on at most once!

Req. $2x$ is *acted on* at stage σ if (2a)–(2d) hold. Assume (req. $2x$) $\in$ block 2τ.

(2a) Req. $2x$ has not been acted on prior to stage σ.

(2b) $|\langle \delta, g(\delta)\rangle| < \sigma$ for all $\delta \leq \tau$, and $|f(x)| < \sigma$.

(2c) Every requirement in block y, for all $y < 2\tau$, has been acted on prior to stage σ.

(2d) Let $\sigma(2\tau)$ be the first stage after all the activity of (2b) and (2c) has been completed. Then either (2da) or (2db) holds:

(2da) $\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle \sigma(2\tau), f(x)\rangle)\downarrow$ via a computation in $L(\sigma+1)$. ($\langle \sigma(2\tau), f(x)\rangle$ is an ordinal that encodes the ordered pair $(\sigma(2\tau), f(x))$ and exceeds both of its components.)

(2db) A Moschovakis witness to

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle \sigma(2\tau), f(x)\rangle)\uparrow$$

belongs to $L(\sigma+1)$ via a computation in $L(\sigma+1)$. (Recall that the relation, x witnesses $\{e\}(y)\uparrow$, is E-recursively enumerable.)

Suppose req. $2x$ is acted on at stage σ. If (2da) holds and

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle \sigma(2\tau), f(x)\rangle) = 0,$$

then $\langle \sigma(2\tau), f(x)\rangle$ is put in A at stage σ. Thus

$$A^{\sigma}(\langle \sigma(2\tau), f(x)\rangle) = 1.$$

Req. $2x+1$ is handled by swapping A and B, odd and even, above.

End of construction.

4.2 Proposition. *Suppose every requirement in block z is acted on. Then all such acts occur prior to stage σ for some $\sigma < \kappa$.*

Proof. Suppose $z = 2\tau$ and (req. $2x$) $\in$ block 2τ. All activity with respect to block y, for all $y < 2\tau$, occurs prior to stage $\sigma(2\tau)$. Let $p \in \kappa$ encode $B^{<\sigma(2\tau)}$, $\sigma(2\tau)$, q, where q is the parameter needed to compute f. Then req. $2x$ is acted on by stage $\kappa_r^{p,x}$; by that stage either convergence (2da) is evident or a Moschovakis witness (2db) has appeared. Theorem 2.5 yields

$$\sup\{\kappa_r^{p,v} | 2v \in \text{block } 2\tau\} < \kappa,$$

since $\sup(\text{block } 2\tau) < \rho^{\kappa}$. $\square$

4.3 Proposition. *Assume* $z < \text{r.e. cf}(\rho^\kappa)$. *Suppose every requirement in block* y, *for all* $y < z$, *is acted on. Then all such acts occur prior to stage* σ *for some* $\sigma < \kappa$.

Proof. If $u < v$, then no requirement in block v is acted on until every requirement in block u has been acted on. Hence if z is a successor, then the desired σ exists by Proposition 4.2.

Assume z is a limit. Recall $\sigma(y)$ from the definition of "acted on". By Proposition 4.2

$$\sigma(y) < \kappa \quad \text{for all} \quad y < z.$$

Let S be $\{\sigma(y) | y < z\}$. S is E-recursively enumerable on $L(\kappa)$, because the computations and Moschovakis witnesses that put $\sigma(y)$ in S all lie inside $L(\sigma(y)+1)$. The ordertype of S is at most z, since $\lambda y | \sigma(y)$ is non-decreasing. It follows from Lemma 2.7 that $z < \text{r.e. cf}(\kappa)$. Hence $\sup(S) < \kappa$. □

4.4 End of Proof of 4.1. An induction on block numbers shows every requirement is acted on. Suppose (req. $2x$) $\in$ block 2τ. Assume every requirement in block y, for all $y < 2\tau$ has been acted on. By Proposition 4.3, $\sigma(2\tau) < \kappa$. ($\sigma(2\tau)$ was defined in clause (2d) of the definition of "acted on".) As in the proof of Proposition 3.2, req. $2x$ is acted on by stage $\kappa_r^{p,x}$, where p encodes $B^{<\sigma(2\tau)}$, $\sigma(2\tau)$, q. End of induction.

Suppose req. $2x$ is acted on at stage σ, and

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle \sigma(2\tau), f(x)\rangle) \tag{1}$$

converges. Then clause (2da) holds at stage σ,

$$(1) \neq A^\sigma(\langle \sigma(2\tau), f(x)\rangle) = A(\langle \sigma(2\tau), f(x)\rangle), \quad \text{and}$$

$$B^{<\sigma(2\tau)} = B^{<\sigma} = B \cap \sigma \tag{2}$$

Consequently $A \neq \{f(x)\}^B$.

Suppose (1) diverges. Then clause (2db) holds at stage σ, and some Moschovakis witness w to the divergence of (1) belongs to $\mathrm{L}(\sigma+1)$. It follows from (2) that w also witnesses the divergence of

$$\{f(x)\}^B(\langle \sigma(2\tau), f(x)\rangle).$$

Thus again $A = \{f(x)\}^B$.

The most curious feature of the above version of Post's problem is the preservation of divergence witnesses in addition to the usual preservation of inequalities.

A final word on the E-recursive enumerability on $L(\kappa)$ of A and B. Suppose z is put in A at stage σ. Then z is of the form $\langle \sigma(2\tau), f(x)\rangle$ and (1) converges. σ is computable from $B^{<\sigma}$, z and some parameters r independent of z and σ. $B^{<\sigma}$ is computable from $L(\sigma(2\tau)), r$. Thus $\sigma \leq_E z, r$ (uniformly in z), and so A is E-recursively enumerable on $L(\kappa)$. □

4.5 Other Results. The proof of Theorem 4.1 shows:

(1) Assume $\langle L(\kappa; A), A\rangle$ is E-closed. If $\eta^{\kappa; A} = \rho^{\kappa; A}$, and $\kappa_r^{x; A} < \kappa$ for all $x \in L(\kappa; A)$, then $\langle L(\kappa; A), A\rangle$ admits a positive solution to Post's problem.

It is unlikely that $\eta^{\kappa; A} = \rho^{\kappa; A}$ in general. The selection theorem of Griffor and Normann, 5.3.XII, implies:

(2) If $L(\kappa; A) \models$ [there is a greatest cardinal and it is regular], then $\eta^{\kappa; A} = \rho^{\kappa; A}$.

The assumption in (1) on κ_r is related to inadmissibility, as in Lemma 5.6.X. To be precise:

If $\langle L(\kappa; A), A\rangle$ is E-closed, but not Σ_1 admissible, then $\kappa_r^{x; A} < \kappa$ for all $x \in L(\kappa; A)$.

The proof of Theorem 4.1 is applicable to some other transitive E-closed structures $\mathscr{E}$. Let κ be the least ordinal not in $\mathscr{E}$. $\mathscr{E}$ is said to be *effectively wellorderable* if there exists a partial E-recursive on $\mathscr{E}$ map f from κ onto $\mathscr{E}$ such that the only parameter needed to compute f is an ordinal.

(3) Assume $\mathscr{E}$ is E-closed and effectively wellorderable. If $\eta = \rho$ for $\mathscr{E}$, and $\mathscr{E}$ admists Moschovakis witnesses, then $\mathscr{E}$ admits a positive solution to Post's problem.

A consequence of (3) is:

(4) Suppose x is a set of ordinals and $L(\kappa, x)$ is not Σ_1 admissible. If $\eta = \rho$ for $L(\kappa, x)$, then $L(\kappa, x)$ admits a positive solution to Post's problem.

Normann's ground-breaking result [1975] on Post's problem for recursion in ${}^3E$ can be phrased as follows:

(5) If $E(2^\omega) \models$ [2^ω is wellorderable and of regular cardinality], then $E(2^\omega)$ admits a positive solution to Post's problem.

The quest for Theorem 4.1 was initiated by a desire to circumvent the assumption of regularity in (5). Normann needed it primarily to show $\eta = \rho$ for $E(2^\omega)$; his argument made use of a selection result, Theorem 1.3.XII. Slaman [1983] has shown:

There exists a model M of ZFC in which $E(2^\omega) \models$ [2^ω is wellorderable and of singular cardinality] and $\eta = \rho$ for $E(2^\omega)$.

It follows from (4) that Slaman's M admits a positive solution to Post's problem for $E(2^\omega)$.

According to 2.4 the A and B supplied by the proof of Theorem 4.1 are subgeneric. The proof yields a stronger property than subgenericity when

r.e. $\mathrm{cf}(\rho^\kappa)=\rho^\kappa$. The simplest example is:

(6) $E(\omega_1)$ admits a positive solution (A, B) to Post's problem such that for all $\delta < \omega_1$,

(7) $$\kappa_0^{\omega_1,\delta;A}=\kappa_0^{\omega_1,\delta;B}=\kappa_0^{\omega_1,\delta}.$$

A set A with property (7) is said to preserve the κ_0-spectrum of $E(\omega_1)$. It is not known if the κ_0-spectrum can be preserved when ρ fails to be r.e. regular.

At this writing nothing is known about situations where Post's problem has a negative solution, or where $\eta \neq \rho$.

(8) Does there exist a set x of ordinals such that $L(\kappa, x)$ is E-closed, is not Σ_1 admissible, and does not admit a positive solution to Post's problem for E-recursion theory?

(9) Does there exist a set x of ordinals such that $L(\kappa, x)$ is E-closed, is not Σ_1 admissible, and $\eta \neq \rho$ for $L(\kappa, x)$?

A negative answer for (8) implies one for (9) by (4). It might be simpler, however, to deal with (9) directly. Both questions are good candidates for forcing arguments, but none of the techniques of Chapter XI appear to be of any use. Perhaps the most pertinent open question about Post's problem is:

(10) Does there exist a model of ZFC in which

$$E(2^\omega) \models [2^\omega \text{ is wellorderable}]$$

and $E(2^\omega)$ does not admit a positive solution to Post's problem?

Exercises 4.6–4.8

4.6. Verify 4.5(1).

4.7. Show that the A and B defined in the proof of Theorem 4.1 are subgeneric.

4.8. Verify 4.5(6)–(7).

5. *Slaman's Splitting and Density Theorems*

Unlike the solution to Post's problem given in Section 4, the proof of Slaman's splitting theorem is a fullblown "finite" injury argument. A typical requirement is injured repeatedly, and the set of stages at which it is injured is a member of $L(\kappa)$. Splitting in E-recursion bears a superficial resemblance to splitting in α-recursion. Both utilize Shore blocking and preservation of initial segments of equalities, but in vastly different ways. In α-recursion the blocks were defined by a Σ_2 cofinality

function, tame by reason of Σ_1 admissibility. Since $L(\kappa)$ may not be Σ_1 admissible, the blocks in E-recursion are defined by a Σ_2 cofinality function chosen to be tame.

The presence of Moschovakis witnesses drives away some of the uncertainties that led to injuries in α-recursion. As in the solution to Post's problem for E-recursion, some preservations are needed to develop Moschovakis witnesses. Otherwise there might be relevant, convergent computations beyond $L(\kappa)$.

Assume C is E-recursively enumerable, but not E-recursive, on $L(\kappa)$. C is to be split into A and B. The requirements are indexed by ordinals less than $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ. Let f be a tame Σ_1 function from $t\sigma 1p(\kappa)$ onto κ. For each $x < t\sigma 1p(\kappa)$,

$$\text{req. } 2x \text{ concerns preservation of } \{f(x)\}^{A^{<\sigma}}, \text{ and}$$

$$\text{req. } 2x+1 \text{ concerns preservation of } \{f(x)\}^{B^{<\sigma}}.$$

Let λ_0 be the tame E_2 cofinality of $t\sigma 1p(\kappa)$, and g a tame E_2 function with domain λ_0 and range an unbounded subset of $t\sigma 1p(\kappa)$. In other words, there exists an recursive-on-$L(\kappa)$ function $g_0(\sigma, r)$ such that

$$(a)_{a<\lambda_0}(E\tau)(r)_{r<a}(\sigma)_{\sigma>\tau}[g_0(\sigma, r) = g(r)].$$

Define

$$\begin{aligned} &(\text{req. } 2x) \in \text{block } 2r && \text{and} \\ &(\text{req. } 2x+1) \in \text{block } 2r+1, && \text{if } \ g(r) \leq x < g(r+1). \end{aligned}$$

As usual let $C^{<\sigma}$ be the subset of C enumerated via computations of height less than σ. For convenience accept a computation of height σ only if all ordinals it mentions are less than σ. C^{σ} is $C^{<\sigma+1} - C^{<\sigma}$. Assume C^{σ} has at most one element; if it exists, denote it by v. At stage σ, v is put in A, or in B, but not in both. Suppose adding v to A would injure a requirement in block $2p$ but none in block $2r$ for any $r < p$. Further suppose adding v to B would injure a requirement in block $2q+1$ but none in block $2r+1$ for any $r < q$. Then v is put in A if $2q+1 < 2p$, and put in B if $2p < 2q+1$.

All events at stage σ are determined by computations and witnesses in $L(\sigma+1)$.

5.1 Theorem (Slaman 1985). *Assume $L(\kappa)$ is E-closed. Suppose $C \subseteq L(\kappa)$ is regular, E-recursively enumerable, and not E-recursive, on $L(\kappa)$. Then there exist A and B, each E-recursively enumerable on $L(\kappa)$, such that*

$$C = A \cup B,\ A \cap B = \varnothing,$$

and C is not E-reducible to either A or B on $L(\kappa)$.

Proof. First suppose $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$. According to Proposition 1.2, Shore's splitting theorem for α-recursion theory, Theorem 1.1.IX, is also a splitting theorem in the sense of E-recursion theory. Now assume $L(\kappa)$ is either Σ_1 inadmissible or the limit of E-closed structures. In either event $L(\kappa)$ admits Moschovakis witnesses. A preservation requirement on A at stage σ is a

promise to keep some elements of $\kappa - A^{<\sigma}$ out of A. Each such requirement is attached to (indexed by) an even ordinal. Req. $2x$ is *injured* at stage σ if v is added to A at stage σ contrary to some promise made on behalf of req. $2x$ at stage τ for some $\tau \leq \sigma$.

Req. $2x$ is *active* at stage σ if a preservation requirement is attached to $2x$ at stage σ, or if req. $2x$ is injured at stage σ, or if the block assigned to req. $2x$ changes at stage σ. A change in block assignment at stage σ means: a change from stage $\sigma - 1$ if σ is a successor; or the assignment was not constant on some upper segment of ordinals less than σ if σ is a limit.

It will be shown that req. $2x$ is inactive for all sufficiently large σ.

Preservation requirements on A at stage σ arise as follows.

Let $\{x\}_\sigma$ be $\{x\}$ restricted to computations of height at most σ. Define

$$\ell_\sigma^A(x) = \mu y_{y<\sigma}[C^\sigma(y) \neq \{f(x)\}_\sigma^{A^{<\sigma}}(y)].$$

For each $y \leq \ell_\sigma^A(x)$, promise to keep z out of A if

$$\{f(x)\}_\sigma^{A^{<\sigma}}(y) \tag{1}$$

converges and "$z \notin A^{<\sigma}$" is used in the computation of (1). If

$$\{f(x)\}_\sigma^{A^{<\sigma}}(\ell_\sigma^A(x)) \tag{2}$$

diverges, and if no Moschovakis witness to the divergence exists in $L(\sigma+1)$, then promise to keep all elements of $\sigma - A^{<\sigma}$ out of A.

Certain conventions must be kept in mind. $C^{<\sigma}$, $A^{<\sigma}$, and $B^{<\sigma}$ are subsets of σ and are defined by computations in $L(\sigma)$. Thus $A^{<\sigma} \in L(\sigma+1)$. The relation,

$$w \text{ is a witness to the divergence of (2),} \tag{3}$$

is E-recursive with x and σ as parameters. To say w exists in $L(\sigma+1)$ means: there is a sequence c (of computations) in $L(\sigma+1)$ that makes (3) true.

Suppose (2) diverges, and some w witnesses the divergence via $c \in L(\sigma+1)$. Fix w and c, and promise to keep z out of A if "$z \in A^{<\sigma}$" is needed in c. Once w and c are chosen, they remain fixed until some promise associated with them is broken.

B and the odd-numbered requirements are handled similarly.

Adding v (from C) to A or B is described above. The block assignments at stage σ are supplied by g_0, the tame E-recursive approximation of g.

The next lemma is the major part of the proof of Theorem 5.1.

5.2 Lemma. *Fix $b <$ dom g $(= \lambda_0)$. Suppose block z is inactive after stage σ_0 for all $z < b$. Then there exists a stage σ after which block b is inactive.*

Proof. Since C is regular, there is a $\sigma_1 > \sigma_0$ such that

$$C^{<\sigma_1} \cap \sigma_0 = C \cap \sigma_0.$$

Assume $g_0(\tau, z) = g(z)$ for all $\tau \geq \sigma_1$ and $z \leq b$. Suppose b is $2r$. Further assume $f(x)$ is defined prior to stage σ_1 for all $x < g(r+1)$. At stage $\sigma \geq \sigma_1$ all requirements in $\cup\{\text{block } z \mid z < b\}$ are safe from injury, since every relevant negative requirement is bounded above by σ_0. A typical requirement in block b $(= 2r)$ is req. $2x$: $C \neq \{f(x)\}^A$. It follows that req. $2x$ is not injured at stage σ for any $\sigma \geq \sigma_1$.

Consider the behavior of

$$\{f(x)\}_\sigma^{A^{<\sigma}}(y) \tag{1}$$

for $\sigma \geq \sigma_1$. If $y < \ell_\sigma^A(x)$ for some $\sigma \geq \sigma_1$, then (1) is preserved forever after. Define

$$t(x, y) \simeq \mu\sigma_{\sigma \geq \sigma_1}(y < \ell_\sigma^A(x)),$$
$$\{h(x)\}(y) \simeq \{f(x)\}_{t(x,y)}^{A^{<t(x,y)}}(y).$$

The length of block $2r$ is less than $t\sigma 1p(\kappa)$. Consequently Lemma 2.10 implies there is an $\ell < \kappa$ such that

$$(x)_{g(r) \leq x < g(r+1)}(\mathrm{E}y)_{y<\ell}[\{h(x)\}(y) \neq C(y)].$$

Define

$$y_x = \begin{cases} \ell \text{ if } \{h(x)\}(y)\downarrow, & \text{for all } y < \ell, \\ \mu y[\{h(x)\}(y)\uparrow] & \text{otherwise.} \end{cases}$$

Choose $\sigma_2 > \sigma_1$ so that $C \cap \ell = C^{<\sigma_2} \cap \ell$.

Suppose $y_x = \ell$. Then this fact is seen to be true by some ordinal less than $\kappa_0^{x,\ell,p}$ (that is, witnessed in $L(\kappa_0^{x,\ell,p})$. p is an ordinal independent of x that encodes several parameters such as b, σ_0, σ_1, σ_2 and others occurring in the definitions of g_0, f and C. Let.

$$w_x = \mu y[\{h(x)\}(y)\downarrow \text{ and is } \neq C(y)].$$

If $w_x \in C$, then this fact is evident by some ordinal less than $\kappa_0^{x,\ell,p}$. If $w_x \notin C$, then this fact is established by a Moschovakis witness first order definable over $\kappa_r^{w_x,x,\ell,p} < \kappa$ by Lemmas 5.3.X and 5.6.X. Exercise 5.17.X, a corollary of Kechris's basis theorem, implies

$$\kappa_r^{w_x,x,\ell,p} \leq \kappa_r^{x,\ell,p},$$

since w_x is the least member of a subset of ℓ E-co-recursively enumerable in x, ℓ, p.

In short: if $y_x = \ell$, then req. $2x$ is inactive after stage $\kappa_r^{x,\ell,p}$.

Now suppose $y_x < \ell$. Then, as above,

$$\kappa_r^{y_x,x,\ell,p} \leq \kappa_r^{x,\ell,p} \tag{2}$$

by 5.3, 5.6 and 5.17 of Chapter X. Let

$$w_x^* = \mu y_{y<y_x}[\{f(x)\}(y)\uparrow \text{ and is } \neq C(y)].$$

Assume $w_x^* < y_x$. If $w_x^* \in C$, then this fact is evident by stage $\kappa_0^{y_x,x,\ell,p}$, hence before $\kappa_r^{x,\ell,p}$ by (2). If $w_x^* \notin C$, then this fact is established by a Moschovakis witness definable over

$$\kappa_r^{w_x^*,y_x,x,\ell,p} \leq \kappa_r^{y_x,x,\ell,p}$$

by 5.17.X.

Thus req. $2x$ is inactive after stage $\kappa_r^{x,\ell,p}$. Hence block b is inactive after stage $\sup\{\kappa_r^{x,\ell,p} | x < g(b)\}$. The latter is less than κ by Theorem 2.5, since $b < t\sigma 1p(\kappa) \leq \rho^\kappa$. □

5.3 Lemma. *Fix $c <$ dom $g = \lambda_0$. Suppose for each $b < c$ there is a σ such that block b is inactive after stage σ. Then there is a σ such that for all $b < c$, block b is inactive after stage σ.*

Proof. A tame E_2 function i: $c \to \kappa$ is defined so that block b is inactive after stage $i(b)$. According to Exercise 2.15, $\lambda_0 = te2\text{cf}(\kappa)$, hence sup (range i) $< \kappa$.

Fix $b < c$. The procedure for E-recursively approximating $i(b)$ is derived from the proof of Lemma 5.2. Notation from 5.2 is used below.

Keep checking on activity in block z for all $z < b$. If any such activity occurs at stage σ, then the current guess at $i(b)$ has to be increased to σ. Eventually stage σ_0 is reached. And after that, stage σ_1 where

$$C^{<\sigma_1} \cap \sigma_0 = C \cap \sigma_0.$$

Meanwhile computations and Moschovakis witnesses are sought to establish inequalities for requirements in block b. Some additional waiting for C to settle down may be necessary as in the definition of σ_2. □

Note that the function i of the proof of Lemma 5.2 is tame E_2, rather than tame Σ_1, because it was necessary to wait for C to settle down on some proper initial segment of κ. This is why tame Σ_1 blocking was adequate for Post's problem but not for splitting.

The proof of Theorem 5.1 is readily completed. By construction $C = A \cup B$ and $A \cap B = \varnothing$. A and B are E-recursively enumerable on $L(\kappa)$, because the decision to put v in A or B at stage σ is based on effective consideration of $L(\sigma)$. Lemma 5.2 and 5.3 imply the desired inequalities are realized.

5.4 Theorem (Slaman 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Let $C, D \subseteq \kappa$ be regular and E-recursively enumerable on $L(\kappa)$. Suppose $D <_{L(\kappa)} C$. Then there exist A and B, each E-recursively enumerable on $L(\kappa)$, such that $A \cap B = \varnothing$, $C = A \cup B$, $C \not\leq_\kappa D$, A and $C \not\leq_\kappa D$, B.*

It follows from Theorem 5.4 that the E-recursively enumerable degrees are dense for every E-closed $L(\kappa)$. If $L(\kappa)$ does not admit Moschovakis witnesses, then $L(\kappa)$ is of the form $E(x)$ for some set x of ordinals, and is Σ_1 admissible, by Theorem 5.8.X. Consequently Shore's density Theorem (5.1.IX) for α-recursion theory applies to $L(\kappa)$ by Proposition 1.2.

The assumption in 5.4 that $L(\kappa)$ admit Moschovakis witnesses is necessary, because Lachlan (197?) has shown that the conclusion of 5.4 fails when $\kappa = \omega$.

The proof of Theorem 5.4 is left as the final exercise. Most, but not all, of the difficulties that arise, as the proof of Theorem 5.1 is extended to cover 5.4, were anticipated in Section 2. Note 2.2.

A final methodological point and question. Post's problem for inadmissible $L(\kappa)$'s was solved without injuries with the aid of Moschovakis witnesses. Priorities were necessary, but at most one attempt was made to satisfy each requirement. Splitting an E-recursively enumerable set C produced a κ-finite set of injuries for each requirement, only because of the need to guess repeatedly at proper initial segments of C; each requirement had to have true knowledge of some such segment before it could be satisfied. Since 5.1 is so close in nature to 5.4, the proof of density for E-closed structures that admit Moschovakis witnesses is a "finite" injury argument. At this writing there are no "infinite" injury arguments in E-recursion, possibly because the search for an inequality can always be resolved below κ by a computation or a Moschovakis witness.

Does there exist an "infinite" injury argument in the setting of E-recursion theory?

5.5 Exercise. Complete the proof of Theorem 5.4.

Bibliography

Aczel P, Richter W (1972) Inductive definitions and analogues of large cardinals. In: Conference in Math. Log., London 1970. Springer, Berlin Heidelberg New York, p 1–9

Bailey C (1984) Beta-degrees for weakly inadmissible sets. Ph.D. Thesis, Harvard University, Cambridge, MA

Barwise J (1979) Admissible sets and structures. Springer, Berlin Heidelberg New York

Cenzer D (1976) Monotonic inductive definitions over the continuum J. Symb. Log. 41: 188–198

Chong CT, Lerman M (1976) Hyperhypersimple α-r.e. sets. Ann. Math. Log. 9: 1–42

Chong CT (1979) Generic sets and minimal α-degrees, Trans. Amer. Math. Soc. 254: 157–169

Chong CT (1984) Techniques of Admissible Recursion Theory. Springer, Berlin Heidelberg New York

Church A, Kleene SC (1937) Formal definitions in the theory of ordinal numbers. Fund. Math. 28: 11–21

Dekker JCE (1954) A theorem on hypersimple sets. Proc. Amer. Math. Soc. 5: 791–796

Devlin K (1984) Constructibility. Springer, Berlin Heidelberg New York

Driscoll GC Jr. (1968) Metarecursively enumerable sets and their metadegrees, J. Symb. Log. 33: 389–411

Feferman S, Spector C (1962) Incompleteness along paths in progressions of theories. J. Symb. Log. 27: 383–390

Feferman S (1965) Some applications of the notions of forcing and generic sets. Fund. Math. 56: 325–345

Fenstad JE (1980) General recursion theory. Springer, Berlin Heidelberg New York

Friedberg R (1957a) A criterion for completeness of degrees of unsolvability, J. Symb. Log. 22: 159–160

Friedberg R (1957b) Two recursively enumerable sets of incomparable degrees of unsolvability. Proc. Nat. Acad. Sci. USA 43: 236–238

Friedman SD (1976) Recursion on Inadmissible Ordinals. Ph.D. Thesis, Massachusetts Institute of Technology

Friedman SD, Sacks GE (1977) Inadmissible recursion theory, Bull. Math. Soc. 83: 255–256

Friedman SD (1979) β-recursion theory. Trans. Amer. Math. Soc. 255: 173–200

Friedman SD (1981a) Uncountable admissibles II: compactness. Israel J. Math. 40: 129–149

Friedman SD (1981b) Negative solutions to Post's problem. II. Ann. Math. 113: 25–43

Gandy RO (1960) Proof of Mostowski's conjecture. Bull. Acad. Polon. Sci. Math. 8: 571–575

Gandy RO (1967) General recursive functionals of finite type and hierarchies of functionals. Ann. Fac. Sci. Univ. Clermont-Ferrand 35: 202–242

Gandy RO, Sacks GE (1967) A minimal hyperdegree, Fund. Math. 61: 215–223

Green J (1974) Σ_1 compactness for next admissible sets. J. Symb. Log. 39: 105–116

Griffor ER (1980) *E*-Recursively Enumerable Degrees. Ph.D. Thesis, Massachusetts Institute of Technology

Griffor ER, Normann D (1982) Effective confinalities and admissibility in *E*-recursion. Preprint, University of Oslo

Grilliot T (1969) Selection functions for recursive functionals. Notre Dame Jour. Formal Log. p. 225–234

Harrington L (1973) Contributions to recursion theory. in higher types. Ph.D. Thesis, (Massachusetts Institute of Technology, Cambridge MA)

Harrington L, MacQueen D (1976) Selection in abstract recursion theory, J. Symb. Log. 41: 153–158

Hinman P (1978) Recursion-Theoretic Hierarchies. Springer, Berlin Heidelberg New York

Homer S, Sacks GE (1983) Inverting the half-jump. Trans. Amer. Math. Soc. 278: 317–331

Hoole T (1982) Abstract extended 2-sections. Ph.D. thesis, Oxford University (Note: the date and title of this reference are approximate.)

Jech T (1978) Set Theory. Academic Press, New York

Jockusch C, Simpson S (1976) A degree-theoretic definition of the ramified analytical hierarchy, Ann. Math. Log. 10: 1–32
Kleene SC (1943) Recursive predicates and quantifiers, Trans. Amer. Math. Soc. 53: 41–73
Kleene SC (1952) Introduction to Metamathematics. Van Nostrand, New York.
Kleene SC (1955a) Arithmetical predicates and function quantifies, Trans. Amer. Math. Soc. 79: 312–340
Kleene SC (1955b) Hierarchies of number-theoretic predicates, Bull. Amer. Math. Soc. 61: 193–213
Kleene SC (1955c) On the forms of the predicates in the theory of constructive ordinals II, Amer. J. Math. 77: 405–428
Kleene SC (1959) Recursive functionals and quantifies of finite types I. Trans. Amer. Math. Soc. 91: 1–52
Kleene SC (1963) Recursive functionals and quantifies of finite types II, Trans. Amer. Math. Soc. 108: 106–142
Kreisel G (1961) Set theoretic methods suggested by the notion of potential totality. In: Infinitistic Methods. Pergamon, Oxford, 325–369
Kreisel G (1962) The axiom of choice and the class of hyperarithmetic functions, Indag. Math. 24: 307–319
Kreisel G, Sacks GE (1963) Metarecursive sets I, II (abstracts), J. Symb. Log. 28: 304–305
Kreisel G (1965) Model-theoretic invariants: applications to recursive and hyperarithmetic operations. In: Theory of models. North-Holland, Amsterdam, p 190–205
Kreisel G, Sacks GE (1965) Metarecursive sets, J. Symb. Log. 30 p 318–338
Kreisel G (1971) Some reasons for generalizing recursion theory. In: Logic Colloquium '69. North-Holland Amsterdam, p 139–198
Kripke S (1964) Transfinite recursion on admissible ordinals I, II (abstracts), J. Symb. Log. 29: 161–162
Lachlan AH (1966) Lower bounds for pairs of recursively enumerable degrees. Proc. London Math. Soc. 16: 537–569
Lachlan AH (1975) A recursively enumerable degree which will not split over all lesser ones. Ann. Math. Log. 9: 307–365
Levy A (1963) Transfinite computability (abstract), Notices Amer. Math. Soc. 10: 286
Lerman M, Sacks GE (1972) Some minimal pairs of α-recursively enumerable degrees. Ann. Math. Log. 4: 415–442
Lerman M (1974) Maximal α-r.e. sets. Trans. Amer. Math. Soc. 188: 341–386
Lerman M (1983) Degrees of unsolvability. Springer, Berlin Heidelberg New York
Louveau A (1980) A separation theorem for Σ^1_1 sets, Trans. Amer. Math. Soc. 260: 363–378
Maass W (1977a) Minimal pairs and minimal degrees in higher recursion theory. Zeit. Math. Logik 18: 169–186
Maass W (1977b) Contributions to α- and β-recursion theory. Habilitationsschrift, Universität München
Maass W (1978a) The uniform regular set theorem in α-recursion theory. J. Symb. Log. 43: 270–279
Maass W (1978b) Inadmissibility, tame r.e. sets and the admissible collapse. Ann. Math. Log. 13: 149–170
Machover M (1961) The theory of transfinite recursion. Bull. Amer. Math. Soc. 67: 575–578
Machtey M (1970) Admissible ordinals and intrinsic consistency. J. Symb. Log. 35: 389–400
Macintyre JM (1968) Contributions to metarecursion theory. Ph.D. Thesis, M.I.T., Cambridge MA
Macintyre JM (1973) Minimal α-recursion theoretic degrees. J. Symb. Log. 38: 18–28
MacQueen D (1972) Recursion in finite types. Ph.D. Thesis. M.I.T., Cambridge MA
Moldstad J (1977) Computations in higher types. Lecture Notes in Math. 574. Springer, Berlin Heidelberg New York
Moschovakis YN (1966) Many-one degrees of the $H_a(x)$ predicates. Parif. Jour. Math. 18: 329–342
Moschovakis YN (1980) Descriptive set theory. North-Holland, Amsterdam
Muchnik AA (1956) On the unsolvability of the problem of reducibility in the theory of algorithms. Dokl. Akad. Nauk SSSR, N.S. 108: 194–197
Normann D (1974) Imbedding of higher type theories. Preprint Series in Math. 16: Oslo
Normann D (1975) Degrees of functionals. Preprint Series in Math. 22: Oslo
Normann D (1978a) Set recursion. In: Generalized recursion theory II. North-Holland, Amsterdam, 303–320
Normann D (1978b) Recursion in ^3E and a splitting theorem. In: Essays on mathematical and philosophical logic. D. Reidel, Dordrecht, p 275–285
Ohashi K (1970) On a question of G.E. Sacks. J. Symb. Log. 35: 46–50
Owings JC Jr. (1969) Π^1_1-sets, ω-sets and metacompleteness. J. Symb. Log. 34: 194–204
Platek R (1966) Foundations of recursion theory. Ph.D. Thesis. Stanford University, Stanford CA
Post EL (1944) Recursively enumerable sets of positive integers and their decision problems. Bull. Amer. Math. Soc. 50: 284–316

Richter W (1967) Constructive transfinite number classes. Bull. Amer. Math. Soc. 73: 261–265
Rogers H Jr. (1967) Theory of recursive functions and effective computability. McGraw-Hill, New York
Sacks GE (1963a) Recursive enumerability and the jump operator. Trans. Amer. Math. Soc. 108: 223–239
Sacks GE (1963b) On the degrees less than $0'$. Annals of Math. 77: 211–231
Sacks GE (1964) The recursively enumerable degrees are dense. Ann. of Math. 80: 193–205
Sacks GE (1966) Post's problem, admissible ordinals and regularity. Trans. Amer. Math. Soc. 124: 1–23
Sacks GE (1969) Measure theoretic uniformity in recursion theory and set theory. Trans. Amer. Math. Soc. 142: 381–420
Sacks GE (1970) Recursion in objects of finite type. Proc. Internat. Cong. Math. 1: 251–254
Sacks GE (1971) Forcing with perfect closed sets. In Axiomatic set theory, Proc. Symposia in Pure Math. Amer. Math. Soc. 13: 331–355
Sacks GE, Simpson S (1972) The α-finite injury method. Ann. Math. Log. 4: 343–367
Sacks GE (1974) The 1-section of a type n-object. In: Generalized recursion theory. North-Holland, Amsterdam, p 81–96
Sacks GE (1976) Countable admissible ordinals and hyperdegrees. Advances in Math. 19: 213–262
Sacks GE (1977) The k-section of a type n-object. Amer. J. Math. 99: 901–917
Sacks GE (1980) Post's problem, absoluteness and recursion in finite types. In The Kleene Symposium, North-Holland, Amsterdam, p 201–222
Sacks GE (1985) Post's problem in E-recursion. In Proceedings of symposia in pure mathematics. Amer. Math. Soc. 42: 177–193
Sacks GE (1986) On the limits of E-recursive enumerability. Ann. of Pure and Applied Logic 31: 87–120
Sacks GE, Slaman TA (1987) Inadmissible forcing. Advances in Math. 66: 1–30
Sacks GE (199?) Set forcing over E-closed structures (to appear)
Shore RA (1974) Σ_n sets which are Δ_n incomparable (uniformly). J. Symb. Log. 39: 295–304
Shore RA (1975a) The irregular and non-hyperregular α-r.e. degrees, Israel J. Math. 22: 28–41
Shore RA (1975b) Splitting an α-recursively enumerable set. Trans. Amer. Math. Soc. 204: 65–78
Shore RA (1975c) Some more minimal pairs of α-recursively enumerable degrees (abstract). Notices Amer. Math. Soc. 22: A524–525
Shore RA (1976a) The recursively enumerable α-degrees are dense. Ann. Math. Log. 9: 123–155
Shore RA (1976b) Combining the density and splitting theorem for α-r.e. degrees (abstract). Notices Amer. Math. Soc. 23: A598
Simpson SG (1971) Admissible Ordinals and Recursion Theory. Ph.D. Thesis, M.I.T., Cambridge MA
Simpson SG (1974a) Degree theory on admissible ordinals. In Generalized recursion theory, Proceedings of the 1972 Oslo Symposium. North-Holland, Amsterdam, p 165–194
Simpson SG (1974b) Post's problem for admissible sets. In Generalized recursion theory, Proceedings of the 1972 Oslo Symposium, North-Holland, Amsterdam, p 437–441
Slaman TA (1981) Aspects of E-Recursion. Ph.D. Thesis. Harvard University, Cambridge MA
Slaman TA (1983) The extended plus-one hypothesis – a relative consistency result. Nagoya Math. J. 92: 107–120
Slaman TA (1985a) Reflection and forcing in E-recursion theory. Ann. Pure Appl. Log. 29: 79–106
Slaman TA (1985b) The E-recursively enumerable degrees are dense. In Proceedings of symposia in pure mathematics. Amer. Math. Soc. 42, 195–213
Soare RI (1987) Recursively enumerable sets and degrees. Springer, Berlin Heidelberg New York
Spector C (1955) Recursive well-orderings, J. Symb. Log. 20: 151–163
Spector C (1956) On degrees of recursive unsolvability, Ann. of Math. 64: 581–592
Spector C (1959) Hyperarithmetic quantifiers, Fund. Math. 48: 313–320
Stoltenberg-Hansen V (1977) Finite injury argument in infinite computation theories. Preprint Series in Math. 12: Oslo
Suzuki Y (1964) A complete classification of the Δ_2^1 functions, Bull. Amer. Math. Soc. 70: 246–253
Takeuti G (1960) On the recursive functions of ordinal numbers. J. Math. Soc. Japan 12: 119–128
Tanaka H (1968) A basis result for Π_1^1-sets of positive measure. Comment. Math. Univ. St. Paul 16: 115–127
Tugué T (1964) On the partial recursive functions of ordinal numbers. J. Math. Soc. Japan 16: 1–31
van de Wiele J (1982) Recursive dilators and generalized recursion. In Proceedings of the Herbrand Symposium. North-Holland, Amsterdam, p 325–332
Yang DP (1984) On the embedding of α-recursive presentable lattices in the α-recursive degrees below $0'$. J. Symb. Log. 49: 488–502
Yates CEM (1966) A minimal pair of recursively enumerable degrees. J. Symb. Log. 31: 159–168

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