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AND

THE FOUNDATIONS OF MATHEMATICS

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Dedicated to

THORALF SKOLEM (1887–1963)

THE THEORY OF MODELS

PROCEEDINGS OF THE 1963 INTERNATIONAL SYMPOSIUM AT BERKELEY

Edited by

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PREFACE

This volume constitutes the proceedings of the International Symposium on the Theory of Models held at the University of California, Berkeley, from June 25 to July 11, 1963. The Symposium was sponsored by the Association for Symbolic Logic in cooperation with the Division of Logic, Methodology and Philosophy of Science of the International Union of History and Philosophy of Science.

The primary aim of the conference was to bring together from all countries, for review and exchange of ideas, scholars who have done significant work in the area of research in the foundations of mathematics generally known as "the theory of models." There were 160 registered participants from 17 countries; their names are recorded in a list at the end of this volume. Thirty-five scholars gave invited hour addresses and 13 additional 20-minute talks were contributed.

These proceedings contain 30 of the invited hour addresses as well as the papers of E. W. Beth (written jointly with J. J. F. Nieland), Roland Fraïssé, and Wolfram Schwabhäuser, who were invited to speak but were unable to attend the Symposium. Manuscripts for publication were not received from five invited speakers – Leon Henkin (Models of type theory), Roger C. Lyndon (Model theory and group theory), Andrzej Mostowski (Status of some theorems of model theory in the non-elementary case), Czesław Ryll-Nardzewski (Probabilities on models), and Alfred Tarski (Definability of recursive sets and undecidability of theories). In addition 4 short papers and 7 abstracts are included which cover all contributors except R. O. Gandy (β-models for analysis) and John McCarthy (A model for causality). The invited papers of C. C. Chang and E. P. Specker were written jointly with H. Jerome Keisler and Simon Kochen, respectively; the contributed paper of Walter Felscher was written jointly with Gudrun Jarfe.

These proceedings also contain a foreword on terminology and an 841-entry bibliography of the theory of models which grew out of a preliminary version prepared by Karel de Bouvère and distributed to participants in the Symposium. An essay preceding the bibliography attempts to explain what has been included under the heading "theory of models."

For convenience in finding papers in the volume the invited addresses have been arranged in alphabetical order by author, followed by the short papers and abstracts. But as an aid to students the papers and abstracts have also been organized by subject matter into eight categories, and a list of papers thus grouped precedes the foreword on terminology. The eight categories suggest the range and scope of the contemporary theory of models.

The Organizing Committee of the Symposium consisted of J. W. Addison, Alonzo Church, William Craig (Treasurer), Leon Henkin (Co-Chairman), S. C. Kleene, Roger C. Lyndon, Abraham Robinson, J. Barkley Rosser, Dana S. Scott (Secretary), Patrick Suppes, Alfred Tarski (Co-Chairman), and Robert L. Vaught.

The bulk of the support for the Symposium was granted by the U.S. National Science Foundation. Additional aid came from the U.S. National Academy of Sciences – National Research Council, from the University of California, Berkeley, and from the sponsors.

The editors are grateful for the help of a large number of logicians in the Berkeley area who read the manuscripts as they were submitted and suggested corrections and clarifications. Among these readers were E. W. Adams, R. F. Barnes, Jr., G. M. Benson, A. Cobham, A. Daigneault, T. E. Frayne, P. G. Hinman, M. Jean, D. L. Kreider, M. R. Krom, J. J. LeTourneau, J. I. Malitz, J. D. Monk, D. C. Peterson, W. N. Reinhardt, G. E. Reyes, R. M. Robinson, W. H. Rupley, A. L. Selman, J. H. Silver, P. Suppes, and B. F. Wells III. The editors also take this opportunity of acknowledging the valuable contribution made by the numerous participants in the Symposium who made helpful suggestions regarding the bibliography, and of thanking P. G. Hinman, who read the entire second printer's proof with care, independently of the editors. Finally, the editors join with all of the Symposium participants in saluting the outstanding dedication and service of Dana Scott, Secretary of the Organizing Committee, and June Lewin, Secretary of the Symposium.

University of California, Berkeley THE EDITORS

EVERT WILLEM BETH

1908 – 1964

Because of illness Professor Beth was unable to accept the invitation to speak at the Symposium, but he did agree to submit a manuscript of the paper he would have presented. Several months after the receipt of this manuscript the editors received the sad news of his death on April 12, 1964. He was an influential and prolific writer on logic; in particular he often wrote on the theory of models, contributing both to its development and to the wider understanding of its concepts, methods, results, and significance. Among his many works he is especially remembered for the beautiful theorem in the theory of definition which bears his name. His personal charm, wide ranging interests in science and education, and warm-hearted friendliness endeared him to a very wide circle of colleagues in the field. For all of these reasons his collaboration in the continued development of the theory of models will be keenly missed.

THE EDITORS

LIST OF PAPERS
GROUPED BY SUBJECT MATTER

In a very rough way the papers presented at the Symposium can be classified into eight groups. Naturally many of the papers do not fall neatly into exactly one group — some have parts belonging to each of several groups and some, in a strict sense, do not belong to any of them. For example, Feferman's paper falls partly into Group V and partly into Group VI, while much of Julia Robinson's paper lies outside of the theory of models; the latter paper was invited because of its close interrelation with and importance for the theory of models. Actually the entire Group IV, which deals with the mathematical aspects of relational structures and categories, falls largely outside model theory. However, since the theory of models deals with the interrelationship of languages and *structures*, it was considered important to invite several papers dealing chiefly with structures themselves; and the papers on categories were solicited in view of the possibility that another general method of organizing mathematics might be revealed as closely connected with and important for the theory of models.

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FOREWORD ON TERMINOLOGY

In a relatively new field such as the theory of models it is not surprising that terminology has not yet reached a settled state. Especially for students and others beginning work in model theory this may prove a tiresome inconvenience. In an effort to make learning easier for them and to give this volume greater unity the editors decided to *suggest* to authors, on an experimental basis, some uniform terminology and notation.

By promising to summarize these terminological suggestions here in one place at the front of the volume, we have made it possible for the individual authors to omit repetitious explanations of much of their own terminology and notation. Authors were, of course, free to diverge from these suggestions and in a number of cases it proved convenient for them, for a variety of purposes, to do so. In such cases they were asked to include the necessary substitute terminological explanations in their individual papers as usual.

Following is a resume of the principal suggestions.

The basic mathematical objects considered in the theory of models have been variously called "relational systems", "relational structures", "realizations", "possible realizations", "models", "semi-models", "pseudo-models", "frames", etc. The term "*relational structure*" is suggested here.

Structures of a given similarity type, corresponding to some language L, may then be referred to as "structures *for* L" or "L-structures". This terminology preserves the word "system" for its other quite different uses in logic and preserves the word "model" for its basic use, as in "a model of a sentence" or "a model of a set of sentences". "*Universe*" is suggested for that set, associated with each structure, which has been alternatively called its "domain", "base", "set of individuals", "support", etc.

The word "language" has been used in a wide variety of ways in model theory and logic. Three chief uses have been: (i) in reference to a collection of primitive symbols and rules of formation; (ii) in reference to (i) together with a scheme of interpretation, i.e., a specification of meaning to the absolute constants and of allowable kinds of interpretation to variables (including parameters and arbitrary constants); and (iii) in

reference to (ii) together with a scheme of deduction (e.g., axioms and rules of inference). It is suggested that "*language*" be reserved for entities fitting roughly into category (ii). Other words, free of semantical or deductive connotations, should be used for entities fitting roughly into category (i); one such possibility is "*grammar*". For the well-formed formal expressions of a language the words "*formula*" and "*term*" are recommended, according as the expression is of the propositional or of the denotative type. "*Sentence*" is suggested for formulas without free occurrences of variables.

For use as logical symbols in formal expressions of an object language, or as names for such symbols, the following are suggested:

$$\mathsf{T} \quad \mathsf{F} \quad \neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad \forall \quad \exists \quad =$$

(where in particular T is the truth symbol, F is the falsity symbol, and $=$ is the identity symbol). These symbols may appear either as above or in a lightface variation of them.

For the fundamental notion of semantical (or model-theoretic) consequence "$\models$" is suggested (over, e.g., "$\models$" or "$\|\!-$"). The recommended usage is suggested by the following examples. Let Σ be a set of formulas, φ be a formula, $\mathfrak{A}$ be a structure, and $\mathscr{K}$ be a class of structures. Then "$\models \varphi$" means "φ is (logically) valid", "$\Sigma \models \varphi$" means "φ is a semantical consequence of Σ", "$\models_{\mathfrak{A}} \varphi$" means "$\varphi$ is valid in $\mathfrak{A}$", and "$\models_{\mathscr{K}} \varphi$" means "$\varphi$ is valid in all structures of $\mathscr{K}$". For deductive (or proof-theoretic) consequence "$\vdash$" is recommended.

As names for and as variables ranging over certain kinds of objects, letters from the following fonts of type are recommended:

(1) elements of the universe of a structure = *lower case italic letters*

(2) relations and operations of a structure = *capital italic letters*

(3) structures = *capital German letters*

(4) the universe of a structure = *the corresponding capital italic letter*

(5) classes of structures = *capital script letters*

(6) symbols in an object language for elements of the universe of a structure = *lower case Tamalpais (i.e. Gill lightface) letters*

(7) symbols in an object language for relations and operations of a structure = *capital Tamalpais (i.e. Gill lightface) letters*

(8) formulas of an object language = *lower case Greek letters*

(9) sets of formulas of an object language = *capital Greek letters*

It is not intended that these fonts be used exclusively for these purposes, of course. For example, lower case Greek letters might also be used for ordinal numbers. If a lower case italic letter denotes a certain object then the corresponding Tamalpais (Gill lightface) letter may sometimes be used without further explanation to denote a name for that object.

THE EDITORS

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Symposium on the Theory of Models; North-Holland Publ. Co., Amsterdam (1965)

THE METHOD OF ALTERNATING CHAINS *

J. W. ADDISON

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0. Introduction. A common construction occurring in contributions to the theory of models (see, for example, A. Robinson [56e], Chang [59], Keisler [60], Addison [62], Krom [63, Dissertation, Chapter V]) is the development of an *alternating chain* of structures, i.e. of a sequence of structures lying alternately in each of two given classes of structures. Although not always originally formulated in these terms, it has turned out that a substantial number of the applications of these chains can be formulated in terms of *separability* (or *interpolability*) *tests*. We begin by illustrating these ideas with a simple, concrete example.

Consider a pure first-order predicate language $\mathscr{L}$ with equality with exactly one parameter – the binary predicate parameter F. Suppose one wondered how "tight" the familiar implication $\exists a \forall b Fab \models \forall b \exists a Fab$ is, or equivalently, how "close together" the class $\mathscr{A}$ of models of $\exists a \forall b Fab$ and the class $\mathscr{B}$ of models of $\neg \forall b \exists a Fab$ are. One measure of this could be given in terms of how complicated a sentence φ of $\mathscr{L}$ must be in order to be interpolable between $\exists a \forall b Fab$ and $\forall b \exists a Fab$ (in the sense that $\exists a \forall b Fab \models \varphi \models \forall b \exists a Fab$), or how complicated a class $\mathscr{C}$ must be in order to separate $\mathscr{A}$ from $\mathscr{B}$ (in the sense that $\mathscr{A} \subseteq \mathscr{C}$ and $\mathscr{C} \cap \mathscr{B} = \emptyset$). For example, can such a $\mathscr{C}$ be both $\vee_2{}^0$ and $\wedge_2{}^0$ (in the terminology of Addison [62a])? By constructing (as is easily done) an alternating chain $\langle \mathfrak{S}_n : n \in \omega \rangle$ such that for any n in ω $\mathfrak{S}_{2n} \in \mathscr{A}$, $\mathfrak{S}_{2n+1} \in \mathscr{B}$, and $\mathfrak{S}_{n+1}$ is an extension of $\mathfrak{S}_n$, one shows that this is impossible. For assume the contrary, let $\mathfrak{T}$ be the common union of $\{\mathfrak{S}_{2n} : n \in \omega\}$ and of $\{\mathfrak{S}_{2n+1} : n \in \omega\}$, and recall that $\wedge_2{}^0$ classes are closed under unions of extension-chains. Then $\mathfrak{T} \in \mathscr{C}$ (since $\mathfrak{S}_{2n} \in \mathscr{A} \subseteq \mathscr{C} \in \wedge_2{}^0$) and $\mathfrak{T} \in \sim \mathscr{C}$ (since $\mathfrak{S}_{2n+1} \in \mathscr{B} \subseteq \sim \mathscr{C} \in \wedge_2{}^0$), a contradiction.

* This research was supported in part by the U.S. National Science Foundation under Grant GP-1842.

This example illustrates the fact that a useful test for the inseparability of two disjoint classes $\mathscr{A}$, $\mathscr{B}$ of structures by an $\vee_2^0 \cap \wedge_2^0$ class is the existence of an extension-chain of length ω alternating from $\mathscr{A}$ to $\mathscr{B}$.[1]

Does this *alternating chain test* constitute a *general* method for determining the $(\vee_2^0 \cap \wedge_2^0)$-inseparability of disjoint classes of structures? In the important case where the classes are elementary (or even $\vee_{1,1}^1$) it turns out that it does indeed.

The existence of an extension-chain of length ω alternating from $\mathscr{A}$ to $\mathscr{B}$ assures, of course, the existence of longer and longer extension-chains of finite length, or, equivalently (by reversing these chains), of longer and longer finite substructure-chains. For elementary $\mathscr{A}$, $\mathscr{B}$ a compactness argument can be used to show that the converse is also true. Therefore an alternate necessary and sufficient condition for the $(\vee_2^0 \cap \wedge_2^0)$-inseparability of elementary classes $\mathscr{A}$, $\mathscr{B}$ is the existence of longer and longer finite substructure-chains alternating from $\mathscr{A}$ to $\mathscr{B}$.

In the special case where the two disjoint classes are complementary, separability tests reduce to *classification tests*—in the above situation, for example, we see that a necessary and sufficient condition for an elementary class to be $\vee_2^0 \cap \wedge_2^0$ is the existence of a finite bound on the lengths of substructure chains alternating from the class to its complement. Thus, for example, a sequence of longer and longer finite substructure-chains alternating in and out of the class of models of $\exists a \forall b Fab$ assures us that this sentence is not logically equivalent to an $\wedge_2^0$ sentence of $\mathscr{L}$. And such simple facts as these, although often expected, are often not trivial to prove. Thus alternating chain tests are in part a contribution to one of the oldest and most central problems of logic—that of determining whether or not a given definition can be "simplified".

The alternate separability test mentioned above naturally suggests the question whether there is any simple grammatical significance to the existence or nonexistence of alternating substructure-chains of a given finite length. This question leads to the discovery of a new "hierarchy" which, rather unexpectedly, appears to share a number of properties with some of the well-known hierarchies studied in descriptive set theory, recursive function theory, and the theory of models. By the classification test this hierarchy fills up the class of $\vee_2^0 \cap \wedge_2^0$ sets and thereby sheds light on the structure of this class.

These findings have been accompanied by the discovery that as early

[1] For a still better, practical understanding of the method of alternating chains see Footnote 2.

as 1914 Hausdorff (see [14]) had studied a cousin of this hierarchy in descriptive set theory and had developed a method of residues that bears a close analogy to the method of alternating chains. Although his method of residues yields only classification tests, it can be generalized (cf. Addison [62]) to give separability tests as well.

In the present paper we initiate a general study of the class of *difference hierarchies*, which includes the hierarchy of $\vee_2^0 \cap \wedge_2^0$ sets mentioned above, Hausdorff's hierarchy, and a variety of other hierarchies in the theory of models, descriptive set theory, and recursive function theory. Included in the study is a *generalized method of alternating chains* which applies to a large number of the examples.

This method has been a powerful heuristic tool in the study of difference hierarchies. Not only has it provided the only known proofs of some results about particular hierarchies, but it has suggested general theorems about difference hierarchies which have subsequently been established by purely Boolean-algebraic methods.

For simplicity, in the present paper we restrict our study to hierarchies of length $\leqslant \omega$; however, many of the results do extend to the transfinite. We plan a separate publication on these extensions, which present special problems outside the algebraic spirit of the present paper.

In the present paper we focus attention on general methods useful for studying the structure of difference hierarchies, reserving for later the application of these methods to particular hierarchies. We begin by outlining in axiomatic form some of the important structural properties of hierarchies.

1. An axiomatic approach to hierarchies. The word "hierarchy" is usually used informally in mathematical discussions, so it no doubt suggests more or less structure to different people. We adopt here a reasonably broad definition and then introduce descriptive adjectives to cover additional structural properties that are often present.

By a *family of sets* we mean a function to a class of sets. "Hierarchy" is most often used in the literature in reference to a special kind of family of sets – namely a family of sets (or classes) of sets. The inclusion relation on the classes of sets induces a partial order on the domain (or index set) of the hierarchy, and in most cases this is a partial well order if not indeed a well order. Balancing brevity against generality we include a restriction to well order in our definition. Since the use of a family of classes in place of a class of classes is simply a matter of notational

convenience, we can without any real loss of generality restrict the domains of hierarchies to be ordinal numbers (i.e. to be initial segments of ordinal numbers).

Definition 1.0. *For any* $\mathfrak{H}$

$\mathfrak{H}$ *is a* prehierarchy *if and only if* $\mathfrak{H}$ *is a function, the domain of* $\mathfrak{H}$ *is an ordinal, the range of* $\mathfrak{H}$ *is a class of classes of sets, and:*

H1. $\mathfrak{H}$ *is* nondecreasing, *i.e. for any* μ, ν *in the domain of* $\mathfrak{H}$ *if* $\mu < \nu$, *then* $\mathfrak{H}(\mu) \subseteq \mathfrak{H}(\nu)$.

Let $\mathfrak{H}$ be a prehierarchy with domain μ. We call the union of the union of the range of $\mathfrak{H}$ its *universe* and denote it by "$|\mathfrak{H}|$", and we refer to $\mathfrak{H}$ as a *μ-prehierarchy over* $|\mathfrak{H}|$. We will usually use the subscript functional notation, writing "$\mathfrak{H}_\nu$" in place of "$\mathfrak{H}(\nu)$". We call the union of the range of $\mathfrak{H}$ its *scope* and denote it by "$\mathfrak{H}_{(\mu)}$". More generally, for any ordinal ν in $\mu+1$ we write "$\mathfrak{H}_{(\nu)}$" as an abbreviation for "$\bigcup\{\mathfrak{H}_\varrho : \varrho \in \nu\}$". For any ν in μ we say that $\mathfrak{H}$ is *stationary at* ν if and only if $\mathfrak{H}_\nu = \mathfrak{H}_{(\nu)}$.

In general a prehierarchy cannot be recaptured from its range, but if it is *one-to-one* this can be done. Into the definition of hierarchy we put a slightly stronger property.

Definition 1.1. *For any* $\mathfrak{H}$

$\mathfrak{H}$ *is a* hierarchy *if and only if* $\mathfrak{H}$ *is a prehierarchy and:*

H2. $\mathfrak{H}$ *is* never stationary (*i.e. for any* ν *in the domain of* $\mathfrak{H}$: $\mathfrak{H}_\nu \neq \mathfrak{H}_{(\nu)}$).

Sometimes in the literature prehierarchies which are stationary only at limit ordinals (i.e. one-to-one prehierarchies) or only beyond a certain ordinal have been called "hierarchies", but it seems slightly advantageous to adopt the terminology chosen here.

A fundamental operation on hierarchies is *dualization* (which we denote by "$^-$"). For any μ-prehierarchy $\mathfrak{H}$ by the *dual* $\mathfrak{H}^-$ of $\mathfrak{H}$ we mean $\langle\{|\mathfrak{H}| \sim A : A \in \mathfrak{H}_\nu\} : \nu \in \mu\rangle$.

Beyond the fundamental axioms H1, H2 there are a variety of special principles directly or indirectly involving dualization to which we wish to draw attention.

Definition 1.2. *For any ordinal* μ *and any* μ*-prehierarchy* $\mathfrak{H}$:

H3. $\mathfrak{H}$ *is* balanced *if and only if for any* ν *in* μ $\mathfrak{H}_{(\nu)} \cup \mathfrak{H}^-_{(\nu)} \subseteq \mathfrak{H}_\nu \cap \mathfrak{H}^-_\nu$;

H4. $\mathfrak{H}$ *is* selfdual *if and only if for any* ν *in* μ $\mathfrak{H}_\nu = \mathfrak{H}^-_\nu$;

H4′. $\mathfrak{H}$ *is* lateral *if and only if for any* ν *in* μ $\mathfrak{H}_\nu \neq \mathfrak{H}^-_\nu$;

H5. $\mathfrak{H}$ *is* separable *if and only if for any* ν *in* μ $\mathfrak{H}_\nu$ *has the first separation property with respect to* $|\mathfrak{H}|$ *(i.e. for any disjoint sets* A, B *in* $\mathfrak{H}_\nu$ *there exists a set* C *in* $\mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$ *such that* $A \subseteq C$ *and* $C \cap B = \emptyset$*);*

H5.′ $\mathfrak{H}$ *is* inseparable *if and only if for any* ν *in* μ $\mathfrak{H}_\nu$ *does not have the first separation property;*

H6. $\mathfrak{H}$ *is* reducible *if and only if for any* ν *in* μ $\mathfrak{H}_\nu$ *has the reduction property (i.e. for any sets* A, B *in* $\mathfrak{H}_\nu$ *there exist sets* A', B' *in* $\mathfrak{H}_\nu$ *such that* $A' \cap B' = \emptyset$, $A' \subseteq A$, $B' \subseteq B$, *and* $A' \cup B' = A \cup B$*);*

H6′. $\mathfrak{H}$ *is* irreducible *if and only if for any* ν *in* μ $\mathfrak{H}_\nu$ *does not have the reduction property;*

H7. $\mathfrak{H}$ *is* perfect *if and only if for any* ν *in* μ $\mathfrak{H}_{(\nu)} \cup \mathfrak{H}_{(\nu)}^- = \mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$;

H7′. $\mathfrak{H}$ *is* coarse *if and only if for any* ν *in* μ $\mathfrak{H}_{(\nu)} \cup \mathfrak{H}_{(\nu)}^- \neq \mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$.

Note that 'lateral', 'inseparable', and 'irreducible' are distinct from 'nonselfdual', 'nonseparable', and 'nonreducible', respectively.

We dualize each of these properties by use of the prefix "co". For example, we say $\mathfrak{H}$ is *coseparable* if and only if $\mathfrak{H}^-$ is separable.

We localize each of these properties by use of the preposition "at". For example, for any ν in μ we say $\mathfrak{H}$ is *separable at* ν if and only if $\mathfrak{H}_\nu$ has the first separation property with respect to $|\mathfrak{H}|$.

Certain obvious relations hold between these properties. For example, balanced = cobalanced, selfdual = coselfdual, lateral = colateral, and perfect = coperfect. Every selfdual or perfect prehierarchy is balanced and every reducible prehierarchy is coseparable.

2. Difference hierarchies. The hierarchy of $\vee_2^0 \cap \wedge_2^0$ sets, suggested by finite alternating chains and mentioned at the end of Section 0, turns out to belong to a wide class of hierarchies based on a common principle of generation. Since one of the fundamental operations involved in the generation of these hierarchies is set-theoretic difference, denoted by "$\sim$", we call such hierarchies "difference hierarchies".

Difference hierarchies are naturally defined in terms of the algebraic operation of "iterated difference", which we denote by "∇", regard as a function on the class of finite sequences of sets, and define recursively (according to the length of the sequence) as follows:

(0) $\nabla\langle A_i : i \in 0\rangle = \emptyset$;

(1) $\nabla\langle A_i : i \in n+1\rangle = A_0 \sim \nabla\langle A_{i+1} : i \in n\rangle$.

Definition 2.0. *For any class $\mathscr{R}$ of sets:*
$\mathfrak{D}(\mathscr{R}) = \langle \{\forall A : A : n \to \mathscr{R}\} : n \in \omega \rangle$.

Equivalently, $\mathfrak{D}(\mathscr{R})$ is the set family on ω defined recursively by: (i) $\mathfrak{D}(\mathscr{R})_0 = \{\emptyset\}$; (ii) $\mathfrak{D}(\mathscr{R})_{n+1} = \{A \sim B : A \in \mathscr{R},\ B \in \mathfrak{D}(\mathscr{R})_n\}$.

We usually write "$\mathfrak{D}_n(\mathscr{R})$" in place of "$\mathfrak{D}(\mathscr{R})_n$", and follow an analogous convention in similar situations later. For any set X and any function F we denote by "$\mathscr{P}X$" the class of all subsets of X and by "$\mathscr{D}F$" the domain of F. If $X \subseteq \mathscr{D}F$, we denote $\{F(x) : x \in X\}$ by "$F^*(X)$". For any n in ω a set sequence A on n is called *decreasing* (*nonincreasing*) if and only if for any $i+1$ in n $A_i \supset A_{i+1}$ ($A_i \supseteq A_{i+1}$).

Theorem 2.1. *For any set $\mathfrak{k}$ and any subclass $\mathscr{R}$ of $\mathscr{P}\mathfrak{k}$:*

(i) *if $\emptyset \in \mathscr{R}$, then $\mathfrak{D}(\mathscr{R})$ is an ω-prehierarchy;*

(ii) *if $\emptyset,\ \mathfrak{k} \in \mathscr{R}$, then $\mathfrak{D}(\mathscr{R})$ is a balanced ω-prehierarchy.*

Proof. (i) We prove $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$ by induction. Basis: $n = 0$ or $n = 1$. $\mathfrak{D}_{(0)}(\mathscr{R}) = \emptyset \subseteq \mathfrak{D}_0(\mathscr{R}) = \mathfrak{D}_{(1)}(\mathscr{R}) = \{\emptyset\} \subseteq \mathscr{R} = \mathfrak{D}_1(\mathscr{R})$. Induction step: $n = k+2$. Let $A \in \mathfrak{D}_{(k+2)}(\mathscr{R})$. By the hypothesis of the induction $\mathfrak{D}_{(k+2)}(\mathscr{R}) = \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}_{k+1}(\mathscr{R})$. Hence $A = B \sim C$ for some B in $\mathscr{R}$ and C in $\mathfrak{D}_k(\mathscr{R})$. So by the hypothesis of the induction $C \in \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}_{k+2}(\mathscr{R})$.

(ii) By (i) $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$. From this it also follows that $\mathfrak{D}^-_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$. From $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$ will follow $\mathfrak{D}^-_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$, so it suffices to prove $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$. We do this by induction on n. Basis: $n = 0$. $\mathfrak{D}_{(0)}(\mathscr{R}) = \emptyset \subseteq \mathfrak{D}^-_0(\mathscr{R})$. Induction step: $n = k+1$. Let $A \in \mathfrak{D}_{(k+1)}(\mathscr{R})$. By (i) $\mathfrak{D}_{(k+1)}(\mathscr{R}) = \mathfrak{D}_k(\mathscr{R})$, so $A \in \mathfrak{D}_k(\mathscr{R})$. Hence, since $\mathfrak{k} \in \mathscr{R}$, $\mathfrak{k} \sim A \in \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}^-_{k+1}(\mathscr{R})$.

Let $\mathscr{R}$ be a class of sets containing $\emptyset$. Let ν be the greatest ordinal in $\omega+1$ such that $\mathfrak{D}(\mathscr{R})|\nu$ is a hierarchy. We call ν the *dimension* of $\mathfrak{D}_{(\omega)}(\mathscr{R})$ over $\mathscr{R}$ and we call $\mathfrak{D}(\mathscr{R})|\nu$ the *difference hierarchy generated by* $\mathscr{R}$.

To develop an understanding of difference hierarchies it is useful to study some of the algebraic properties of ∇. As Hausdorff noted it is convenient for this purpose to introduce a closely related algebraic operation—"alternating series"—which we denote by "$\oplus$", also regard as a function on the class of finite sequences of sets, and define recursively as follows:

(2) $\oplus\langle A_i : i \in 0\rangle = \emptyset$;

(3) $\oplus\langle A_i : i \in 2n+1\rangle = \oplus\langle A_i : i \in 2n\rangle \cup A_{2n}$;

(4) $\oplus\langle A_i : i \in 2n+2\rangle = \oplus\langle A_i : i \in 2n+1\rangle \sim A_{2n+1}$.

In the following sequence of lemmas we develop (following in part the work of Hausdorff [14]) some of the basic algebraic facts about ∇ and $\oplus$.

Lemma 2.2. *For any set X, any n in ω, and any set sequence A on n:*

(i) $\nabla\langle X \cap A_i : i \in n\rangle = X \cap \nabla\langle A_i : i \in n\rangle$;

(ii) $\oplus\langle A_i \cup X : i \in n\rangle = \begin{cases} \oplus\langle A_i : i \in n\rangle \sim X & \text{if } n \text{ is even,} \\ \oplus\langle A_i : i \in n\rangle \cup X & \text{if } n \text{ is odd.} \end{cases}$

Proof. (i) By induction on n. Basis: $n=0$. $\emptyset = X \cap \emptyset$. Induction step: $n=k+1$. $X \cap \nabla\langle A_i : i \in k+1\rangle = X \cap (A_0 \sim \nabla\langle A_{i+1} : i \in k\rangle)$ $= (X \cap A_0) \sim (X \cap \nabla\langle A_{i+1} : i \in k\rangle) =$ [by the hypothesis of the induction] $(X \cap A_0) \sim \nabla\langle X \cap A_{i+1} : i \in k\rangle = \nabla\langle X \cap A_i : i \in k+1\rangle$.

(ii) By induction on n. Basis: $n=0$. $\emptyset = \emptyset \sim X$. Induction step: $n=k+1$. Case 1: k is even. $\oplus\langle A_i \cup X : i \in k+1\rangle = \oplus\langle A_i \cup X : i \in k\rangle \cup (A_k \cup X)$ $=$ [by the hypothesis of the induction] $(\oplus\langle A_i : i \in k\rangle \sim X) \cup (A_k \cup X)$ $= \oplus\langle A_i : i \in k\rangle \cup A_k \cup X = \oplus\langle A_i : i \in k+1\rangle \cup X$. Case 2: k is odd. $\oplus\langle A_i \cup X : i \in k+1\rangle = \oplus\langle A_i \cup X : i \in k\rangle \sim (A_k \cup X) =$ [by the hypothesis of the induction] $(\oplus\langle A_i : i \in k\rangle \cup X) \sim (A_k \cup X)$ $= (\oplus\langle A_i : i \in k\rangle \sim A_k) \sim X = \oplus\langle A_i : i \in k+1\rangle \sim X$.

Lemma 2.3. *For any positive n in ω and any set sequence A on n:*

(i) $\nabla\langle A_i : i \in n\rangle = \nabla\langle A_0 \cap A_i : i \in n\rangle$;

(ii) $\oplus\langle A_i : i \in n\rangle = \oplus\langle A_i \cup A_{n-1} : i \in n\rangle$.

Proof. (i) $\nabla\langle A_0 \cap A_i : i \in n\rangle =$ [by Lemma 2.2(i)] $A_0 \cap \nabla\langle A_i : i \in n\rangle$ $= A_0 \cap (A_0 \sim \nabla\langle A_{i+1} : i \in n-1\rangle) = A_0 \sim \nabla\langle A_{i+1} : i \in n-1\rangle = \nabla\langle A_i : i \in n\rangle$.

(ii) Case 1: n is even. $\oplus\langle A_i \cup A_{n-1} : i \in n\rangle =$ [by Lemma 2.2(ii)] $\oplus\langle A_i : i \in n\rangle \sim A_{n-1} = (\oplus\langle A_i : i \in n-1\rangle \sim A_{n-1}) \sim A_{n-1} = \oplus\langle A_i : i \in n\rangle$. Case 2: n is odd. $\oplus\langle A_i \cup A_{n-1} : i \in n\rangle =$ [by Lemma 2.2(ii)] $\oplus\langle A_i : i \in n\rangle \cup A_{n-1} = (\oplus\langle A_i : i \in n-1\rangle \cup A_{n-1}) \cup A_{n-1} = \oplus\langle A_i : i \in n\rangle$.

Lemma 2.4. *For any n in ω and any nonincreasing set sequence A on $n+1$:*

$\oplus\langle A_i : i \in n+1\rangle = A_0 \sim \oplus\langle A_{i+1} : i \in n\rangle$.

Proof. By induction on n. Basis: $n=0$. $\emptyset \cup A_0 = A_0 = A_0 \sim \emptyset$. Induction step: $n=k+1$. Case 1: k is even. $\oplus\langle A_i : i \in k+2\rangle$ $= \oplus\langle A_i : i \in k+1\rangle \sim A_{k+1} =$ [by the hypothesis of the induction]

$(A_0 \sim \oplus\langle A_{i+1} : i \in k\rangle) \sim A_{k+1} = A_0 \sim (\oplus\langle A_{i+1} : i \in k\rangle \cup A_{k+1})$
$= A_0 \sim \oplus\langle A_{i+1} : i \in k+1\rangle$. Case 2: k is odd. $\oplus\langle A_i : i \in k+2\rangle$
$= \oplus\langle A_i : i \in k+1\rangle \cup A_{k+1} =$ [by the hypothesis of the induction]
$(A_0 \sim \oplus\langle A_{i+1} : i \in k\rangle) \cup A_{k+1} =$ [since $A_0 \supseteq A_{k+1}$, which holds since A is nonincreasing]
$A_0 \sim (\oplus\langle A_{i+1} : i \in k\rangle \sim A_{k+1}) = A_0 \sim \oplus\langle A_{i+1} : i \in k+1\rangle$.

The close relationship between ∇ and $\oplus$ is given by the following result.

Theorem 2.5. *For any n in ω and any nonincreasing set sequence A on n:*

$\nabla A = \oplus A$.

Proof. By induction on n. Basis: $n = 0$. $\nabla A = \emptyset = \oplus A$. Induction step: $n = k+1$. $\nabla\langle A_i : i \in k+1\rangle = A_0 \sim \nabla\langle A_{i+1} : i \in k\rangle =$ [by the hypothesis of the induction] $A_0 \sim \oplus\langle A_{i+1} : i \in k\rangle =$ [by Lemma 2.4] $\oplus\langle A_i : i \in k+1\rangle$.

Lemma 2.6. *For any n in ω and any nonincreasing set sequence A on $n+2$:*

$\oplus\langle A_i : i \in n+2\rangle = (A_0 \sim A_1) \cup \oplus\langle A_{i+2} : i \in n\rangle$.

Proof. By induction on n. Basis: $n = 0$. $(\emptyset \cup A_0) \sim A_1 = (A_0 \sim A_1) \cup \emptyset$. Induction step: $n = k+1$. $\oplus\langle A_i : i \in k+3\rangle =$ [by Lemma 2.4]
$A_0 \sim (A_1 \sim \oplus\langle A_{i+2} : i \in k+1\rangle) =$ [since $A_0 \supseteq A_1 \supseteq A_2 \supseteq \oplus\langle A_{i+2} : i \in k+1\rangle$, which holds since A is nonincreasing] $(A_0 \sim A_1) \cup \oplus\langle A_{i+2} : i \in n\rangle$.

Lemma 2.7. *For any n in ω and any nonincreasing set sequence A on $2n$:*

$\oplus\langle A_i : i \in 2n\rangle = \bigcup\{A_{2i} \sim A_{2i+1} : i \in n\}$.

Proof. By induction on n. Basis: $n = 0$. $\oplus\emptyset = \emptyset = \bigcup\emptyset$. Induction step: $n = k+1$. $\oplus\langle A_i : i \in 2k+2\rangle =$ [by Lemma 2.6] $(A_0 \sim A_1) \cup \oplus\langle A_{i+2} : i \in 2k\rangle$
$=$ [by the hypothesis of the induction] $(A_0 \sim A_1) \cup \bigcup\{A_{2i+2} \sim A_{2i+3} : i \in k\}$
$= \bigcup\{A_{2i} \sim A_{2i+1} : i \in n\}$.

Lemma 2.8. *For any set $\mathfrak{l}$, any n in ω, and any nonincreasing sequence A on n to $\mathscr{P}\mathfrak{l}$:*

$$\oplus\langle \mathfrak{l} \sim A_{n-1-i} : i \in n\rangle = \begin{cases} \oplus\langle A_i : i \in n\rangle & \text{if } n \text{ is even,} \\ \mathfrak{l} \sim \oplus\langle A_i : i \in n\rangle & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We omit "ł" when it appears immediately in front of "$\sim$". Case 1: n is even. By induction on n. Basis: $n=0$. $\oplus\emptyset=\oplus\emptyset$. Induction step: $n=k+2$. Let $B=\langle\sim A_{k+1-i}: i\in k+2\rangle$. Then B is nonincreasing since A is. $\oplus\langle A_i: i\in k+2\rangle=$ [by (4), (3)] $(\oplus\langle A_i: i\in k\rangle\cup A_k)\sim A_{k+1}$ $=$ [since A is nonincreasing] $\oplus\langle A_i: i\in k\rangle\cup(A_k\sim A_{k+1})$ $=(\sim A_{k+1}\sim\sim A_k)\cup\oplus\langle A_i: i\in k\rangle=$ [by the hypothesis of the induction] $(\sim A_{k+1}\sim\sim A_k)\cup\oplus\langle\sim A_{k-1-i}: i\in k\rangle$ $=(B_0\sim B_1)\cup\oplus\langle B_{i+2}: i\in k\rangle=$ [by Lemma 2.6, since B is nonincreasing] $\oplus\langle B_i: i\in k+2\rangle=\oplus\langle\sim A_{k+1-i}: i\in k+2\rangle$. Case 2: n is odd. Let B be A "preceded" by ł, i.e. let $B=\langle 0,ł\rangle\cup\langle A_{i-1}: 0<i\in n+1\rangle$. Since A is nonincreasing so is B. $ł\sim\oplus\langle A_i: i\in n\rangle=B_0\sim\oplus\langle B_{i+1}: i\in n\rangle=$ [by Lemma 2.4, since B is nonincreasing] $\oplus\langle B_i: i\in n+1\rangle=$ [by Case 1, since B is nonincreasing] $\oplus\langle\sim B_{n-i}: i\in n+1\rangle=$ [by (4)] $\oplus\langle\sim B_{n-i}: i\in n\rangle\sim(\sim ł)$ $=\oplus\langle\sim B_{n-i}: i\in n\rangle=\oplus\langle\sim A_{n-1-i}: i\in n\rangle$.

The algebraic results established thus far now put us in a position to give (in Theorems 2.10 and 2.11) a variety of alternate characterizations of difference hierarchies. These characterizations are in turn useful in studying the structure of these hierarchies.

Definition 2.9. *For any class $\mathscr{R}$ of sets:*

(i) $\mathfrak{E}(\mathscr{R})=\langle\{\oplus A: A: n\to\mathscr{R}\}: n\in\omega\rangle$;

(ii) $\mathfrak{D}'(\mathscr{R})=\langle\{\forall A: A: n\to\mathscr{R}$ *and* A *is nonincreasing*$\}: n\in\omega\rangle$;

(iii) $\mathfrak{E}'(\mathscr{R})=\langle\{\oplus A: A: n\to\mathscr{R}$ *and* A *is nonincreasing*$\}: n\in\omega\rangle$.

Theorem 2.10. *For any class $\mathscr{R}$ of sets:*

(i) $\mathfrak{D}(\mathscr{R})=\mathfrak{D}'(\mathscr{R})$ *if* $\mathscr{R}$ *is closed under* $\cap$;

(ii) $\mathfrak{D}'(\mathscr{R})=\mathfrak{E}'(\mathscr{R})$;

(iii) $\mathfrak{E}'(\mathscr{R})=\mathfrak{E}(\mathscr{R})$ *if* $\mathscr{R}$ *is closed under* $\cup$.

Proof. (i) $\mathfrak{D}(\mathscr{R})=\mathfrak{D}'(\mathscr{R})$. It is immediate that $\mathfrak{D}'(\mathscr{R})\subseteq\mathfrak{D}(\mathscr{R})$. We show $\mathfrak{D}_n(\mathscr{R})\subseteq\mathfrak{D}_n'(\mathscr{R})$ by induction on n. Basis: $n=0$. $\mathfrak{D}_0(\mathscr{R})=\{\emptyset\}=\mathfrak{D}_0'(\mathscr{R})$. Induction step: $n=k+1$. Let $X\in\mathfrak{D}_{k+1}(\mathscr{R})$. Then for some sequence — say A — on $k+1$ to $\mathscr{R}$ $X=\forall\langle A_i: i\in k+1\rangle=$ [by Lemma 2.3(i)] $\forall\langle A_0\cap A_i: i\in k+1\rangle$. $\forall\langle A_{i+1}: i\in k\rangle\in\mathfrak{D}_k(\mathscr{R})$, so by the hypothesis of the induction we may assume $\langle A_{i+1}: i\in k\rangle$ is nonincreasing. Hence so too is $\langle A_0\cap A_{i+1}: i\in k\rangle$, and hence so is $\langle A_0\cap A_i: i\in k+1\rangle$. Since A is to $\mathscr{R}$ and $\mathscr{R}$ is closed under $\cap$, $A_0\cap A_i\in\mathscr{R}$ for any i in $k+1$. Hence $X\in\mathfrak{D}'_{k+1}(\mathscr{R})$.

(ii) $\mathfrak{D}'(\mathscr{R}) = \mathfrak{E}'(\mathscr{R})$. This is a consequence of Theorem 2.5.

(iii) $\mathfrak{E}'(\mathscr{R}) = \mathfrak{E}(\mathscr{R})$. It is immediate that $\mathfrak{E}'(\mathscr{R}) \subseteq \mathfrak{E}(\mathscr{R})$.

We show $\mathfrak{E}_n(\mathscr{R}) \subseteq \mathfrak{E}_n'(\mathscr{R})$ by induction on n. Basis: $n=0$. $\mathfrak{E}_0(\mathscr{R}) = \{\emptyset\} = \mathfrak{E}_0'(\mathscr{R})$. Induction step: $n = k+1$. Let $X \in \mathfrak{E}_{k+1}(\mathscr{R})$. Then for some sequence – say A – on $k+1$ to $\mathscr{R}$ $X = \oplus\langle A_i : i \in k+1\rangle =$ [by Lemma 2.3(ii)] $\oplus\langle A_i \cup A_k : i \in k+1\rangle$. $\oplus\langle A_i : i \in k\rangle \in \mathfrak{E}_k(\mathscr{R})$, so by the hypothesis of the induction we may assume $\langle A_i : i \in k\rangle$ is nonincreasing. Hence so too is $\langle A_i \cup A_k : i \in k\rangle$, and hence so is $\langle A_i \cup A_k : i \in k+1\rangle$. Since A is to $\mathscr{R}$ and $\mathscr{R}$ is closed under $\cup$, $A_i \cup A_k \in \mathscr{R}$ for any i in $k+1$. Hence $X \in \mathfrak{E}'_{k+1}(\mathscr{R})$.

Theorem 2.11. *Let $\mathscr{R}$ be a class of sets closed under $\cup$ and $\cap$, let $\mathfrak{F}(\mathscr{R}) = \langle\{X$: X is a union of n disjoint $\mathfrak{D}_2(\mathscr{R})$ sets$\}$: $n \in \omega\rangle$, and let $\mathfrak{F}'(\mathscr{R}) = \langle\{\bigcup\{A_{2i} \sim A_{2i+1} : i \in n\}$: $A : 2n \to \mathscr{R}$, A is nonincreasing$\}$: $n \in \omega\rangle$. Then for any n in ω:*

$\mathfrak{D}_{2n}(\mathscr{R}) = \mathfrak{F}_n(\mathscr{R}) = \mathfrak{F}_n'(\mathscr{R})$.

Proof. (i) $\mathfrak{D}_{2n}(\mathscr{R}) \subseteq \mathfrak{F}_n(\mathscr{R})$. Let $X \in \mathfrak{D}_{2n}(\mathscr{R})$. Then by Theorem 2.10(i)–(ii) there exists a nonincreasing sequence – say A – on $2n$ to $\mathscr{R}$ such that $X = \oplus A$. Hence by Lemma 2.7 $X = \bigcup\{A_{2i} \sim A_{2i+1} : i \in n\}$. But the $\mathfrak{D}_2(\mathscr{R})$ sets $A_{2i} \sim A_{2i+1}$ for i in n are disjoint, since A is nonincreasing. Hence $X \in \mathfrak{F}_n(\mathscr{R})$.

(ii) $\mathfrak{F}_n(\mathscr{R}) \subseteq \mathfrak{F}_n'(\mathscr{R})$. By induction on n. Basis: $n=0$. $\bigcup\emptyset = \bigcup\emptyset$. Induction step: $n = k+1$. Let $X \in \mathfrak{F}_n(\mathscr{R})$. Then for some sequence – say A – on k to $\mathfrak{D}_2(\mathscr{R})$, and some sets – say U, W – in $\mathscr{R}$, $X = (U \sim W) \cup \bigcup\{A_i : i \in k\}$ and $\{U \sim W\} \cup \{A_i : i \in k\}$ is a class of disjoint sets. By the hypothesis of the induction there is a nonincreasing sequence – say B – on $2k$ to $\mathscr{R}$ such that $\bigcup\{A_i : i \in k\} = \bigcup\{B_{2i} \sim B_{2i+1} : i \in k\}$. Let $V = U \cap W$. Then $U \sim W = U \sim V$, so $X = (U \sim V) \cup \bigcup\{B_{2i} \sim B_{2i+1} : i \in k\}$. Since U, $W \in \mathscr{R}$ and $\mathscr{R}$ is closed under $\cap$, $V \in \mathscr{R}$. Since $U \sim V$ is disjoint from A_i for any i in k, $U \sim V$ is disjoint from $\bigcup\{A_i : i \in k\}$, and so from $\bigcup\{B_{2i} \sim B_{2i+1} : i \in k\}$, and so from $B_{2i} \sim B_{2i+1}$ for any i in k. Let $C = B \cup \{\langle -2, \bigcup\mathscr{R}\rangle, \langle -1, \bigcup\mathscr{R}\rangle, \langle 2k, \emptyset\rangle, \langle 2k+1, \emptyset\rangle\}$. Then C is nonincreasing since B is. Let D be the sequence on $2k+2$ which takes even numbers to U and odd numbers to V. Let $E = \langle (C_{i-2} \cap D_i) \cup C_i : i \in 2k+2\rangle$. We complete the proof by showing: (a) $E : 2k+2 \to \mathscr{R}$; (b) E is nonincreasing; (c) $X = \bigcup\{E_{2i} \sim E_{2i+1} : i \in k+1\}$.

Proof of (a): E is on $2k+2$ by definition. E is to $\mathscr{R}$ since B is to $\mathscr{R}$, for any Z in $\mathscr{R}$ $\bigcup\mathscr{R} \cap Z = Z$ and $Z \cup \emptyset = Z$, U, $V \in \mathscr{R}$, and $\mathscr{R}$ is closed under $\cap$ and $\cup$.

Proof of (b): Let $0<j\in 2k+2$ and let $x\in E_j=(C_{j-2}\cap D_j)\cup C_j$. Assume $x\in C_j$. Then $x\in C_{j-1}$ since C is nonincreasing so $x\in(C_{j-3}\cap D_{j-1})\cup C_{j-1}=E_{j-1}$. Now assume $x\in C_{j-2}\cap D_j$. Case 1: j is even. Then $D_j=U$, so $x\in C_{j-2}\cap U$. Then $x\in C_{j-3}$ since C is nonincreasing. If $x\in C_{j-1}$ then $x\in(C_{j-3}\cap D_{j-1})\cup C_{j-1}=E_{j-1}$. If $x\notin C_{j-1}$, then $x\in C_{j-2}\sim C_{j-1}$. Hence, since $C_{j-2}\sim C_{j-1}$ is disjoint from $U\sim V$, $x\notin U\sim V$. But $x\in U$, so $x\in V$. Hence $x\in(C_{j-3}\cap V)=C_{j-3}\cap D_{j-1}\subseteq E_{j-1}$. Case 2. j is odd. Then $D_j=V$, so $x\in C_{j-2}\cap V$. Then $x\in C_{j-3}\cap U\subseteq(C_{j-3}\cap D_{j-1})\cup C_{j-1}=E_{j-1}$, since C is nonincreasing and $U\supseteq V$.

Proof of (c): Part I: $\subseteq$. Let $x\in X=(U\sim V)\cup\bigcup\{B_{2i}\sim B_{2i+1}:i\in k\}$. Assume on the one hand that $x\in U\sim V$. Since C is nonincreasing every element of $\bigcup\mathscr{R}$ is in exactly one set in $\{C_{i-1}\sim C_i:i\in 2k+1\}$. Assume $x\in C_{j-1}\sim C_j$ for j in $2k+1$. Then j must be even since for odd j $U\sim V$ is disjoint from $C_{j-1}\sim C_j$. Since $x\in C_{j-1}$ and C is nonincreasing, $x\in C_{j-2}$; hence, since $x\in U$, $x\in(C_{j-2}\cap U)\cup C_j=(C_{j-2}\cap D_j)\cup C_j=E_j$. Since $x\in U\sim V$, $x\notin C_{j-1}\cap V$; since $x\in C_{j-1}\sim C_j$ and C is nonincreasing, $x\notin C_{j+1}$. Hence $x\notin(C_{j-1}\cap V)\cup C_{j+1}=(C_{j-1}\cap D_{j+1})\cup C_{j+1}=E_{j+1}$. Therefore $x\in E_j\sim E_{j+1}\subseteq\bigcup\{E_{2i}\sim E_{2+i+1}:i\in k+1\}$. Now assume on the other hand that $x\in B_{2j}\sim B_{2j+1}$ for j in k. Case 1: $x\notin U$. Since $x\in B_{2j}$, $x\in(C_{2j-2}\cap D_{2j})\cup B_{2j}=E_{2j}$; since $x\notin B_{2j+1}$ and $x\notin U\supseteq V$, $x\notin(C_{2j-1}\cap V)\cup B_{2j+1}=(C_{2j-1}\cap D_{2j+1})\cup C_{2j+1}=E_{2j+1}$. Hence $x\in E_{2j}\sim E_{2j+1}$. Case 2: $x\in U$. Then $x\in B_{2j}\cap U$, so $x\in(C_{2j}\cap D_{2j+2})\cup C_{2j+2}=E_{2j+2}$. Since $x\notin C_{2j+1}$, $x\notin C_{2j+1}\cap D_{2j+3}$, and since further C is nonincreasing $x\notin C_{2j+3}$. Hence $x\notin(C_{2j+1}\cap D_{2j+3})\cup C_{2j+3}=E_{2j+3}$. Therefore $x\in E_{2j+2}\sim E_{2j+3}\subseteq\bigcup\{E_{2i}\sim E_{2i+1}:i\in k+1\}$. Part II: $\supseteq$. Let $x\in\bigcup\{E_{2i}\sim E_{2i+1}:i\in k+1\}$. Then for some number—say j— in $k+1$ $x\in E_{2j}\sim E_{2j+1}=[C_{2j-2}\cap U)\cup C_{2j}]\sim[(C_{2j-1}\cap V)\cup C_{2j+1}]$. Hence $x\in C_{2j-2}\cap U$ or $x\in C_{2j}$. Case 1: $x\in C_{2j-2}\cap U$. $x\notin C_{2j-1}$ or $x\notin V$. If $x\notin C_{2j-1}$ then $x\in C_{2j-2}\sim C_{2j-1}$. Hence, since $C_{-2}\sim C_{-1}=\emptyset$, $0<j$. Hence, since $0<j\in k+1$, $C_{2j-2}\sim C_{2j-1}=B_{2j-2}\sim B_{2j-1}$, so $x\in\bigcup\{B_{2i}\sim B_{2i+1}:i\in k\}\subseteq X$. If $x\notin V$, then $x\in U\sim V\subseteq X$. Case 2: $x\in C_{2j}$. Since $x\notin E_{2j+1}$, $x\notin C_{2j+1}$. Hence $x\in C_{2j}\sim C_{2j+1}$. $C_{2k}\sim C_{2k+1}=\emptyset\sim\emptyset=\emptyset$, so $j\neq k$. Hence $x\in\bigcup\{C_{2i}\sim C_{2i+1}:i\in k\}=\bigcup\{B_{2i}\sim B_{2i+1}:i\in k\}\subseteq X$.

(iii) $\mathfrak{F}_n'(\mathscr{R})\subseteq\mathfrak{D}_{2n}(\mathscr{R})$. Let $X\in\mathfrak{F}_n'(\mathscr{R})$. Then there is a nonincreasing sequence—say A—on $2n$ to $\mathscr{R}$ such that $X=\bigcup\{A_{2i}\sim A_{2i+1}:i\in n\}$. Hence by Lemma 2.7 $X=\oplus A$. Hence by Theorem 2.10 (i)–(ii) $X\in\mathfrak{D}_{2n}(\mathscr{R})$.

The following theorem, due to Hausdorff, characterizes the scope of difference hierarchies and suggests some of their usefulness.

Theorem 2.12. *For any set* $\mathfrak{k}$ *and any subclass* $\mathscr{R}$ *of* $\mathscr{P}\mathfrak{k}$:

(i) *if* $\mathscr{R}$ *is closed under* $\cup$ *and* $\cap$, *then* $\mathfrak{D}_{(\omega)}(\mathscr{R})$ *is the least class of sets including* $\mathscr{R}$ *and closed under* $\cup$, $\cap$, *and* $\sim$ (*difference*);

(ii) *if* $\mathfrak{k} \in \mathscr{R}$ *and* $\mathscr{R}$ *is closed under* $\cup$ *and* $\cap$, *then* $\mathfrak{D}_{(\omega)}(\mathscr{R})$ *is the universe of the Boolean algebra of subsets of* $\mathfrak{k}$ *generated by* $\mathscr{R}$.

Proof. (i) Cf. Hausdorff [14]. (ii) This follows from (i).

The relationship between the difference hierarchy generated by a class $\mathscr{R}$ of subsets of a set $\mathfrak{k}$ and the difference hierarchy generated by the class of complements (relative to $\mathfrak{k}$) of sets in $\mathscr{R}$ is explained by the next theorem.

Theorem 2.13. *For any classes* $\mathscr{R}$, $\mathscr{S}$ *of sets closed under* $\cap$ *such that* $\mathscr{R} = \{\bigcup \mathscr{S} \sim X\colon X \in \mathscr{S}\}$ *and any* n *in* ω:

(i) $\mathfrak{D}_{2n}(\mathscr{R}) = \mathfrak{D}_{2n}(\mathscr{S})$;

(ii) $\mathfrak{D}_{2n+1}(\mathscr{R}) = \mathfrak{D}^{-}_{2n+1}(\mathscr{S})$.

Proof. (i) Let X be a set. Then $X \in \mathfrak{D}_{2n}(\mathscr{R})$ if and only if [by Theorem 2.10 (i)–(ii)] there is a nonincreasing sequence A on $2n$ to $\mathscr{R}$ such that $X = \oplus A$ if and only if [by Lemma 2.8, using $\bigcup \mathscr{S}$ for $\mathfrak{k}$] there is a nonincreasing sequence B on $2n$ to $\mathscr{S}$ such that $X = \oplus B$ if and only if [by Theorem 2.10 (i)–(ii)] $X \in \mathfrak{D}_{2n}(\mathscr{S})$.

(ii) Let X be a set. Then $X \in \mathfrak{D}_{2n+1}(\mathscr{R})$ if and only if [by Theorem 2.10 (i)–(ii)] there is a nonincreasing sequence A on $2n+1$ to $\mathscr{R}$ such that $X = \oplus A$ if and only if [by Lemma 2.8, using $\bigcup \mathscr{S}$ for $\mathfrak{k}$] there is a nonincreasing sequence B on $2n+1$ to $\mathscr{S}$ such that $\bigcup \mathscr{S} \sim X = \oplus B$ if and only if [by Theorem 2.10 (i)–(ii)] $X \in \mathfrak{D}^{-}_{2n+1}(\mathscr{S})$.

3. Alternating chains. The chains discussed in Section 0 were extension-chains (and substructure-chains) of relational structures of a given similarity type. In formulating the generalized method of alternating chains we generalize first to allow more general kinds of relations connecting adjacent values of the chains, second to allow nonempty sets of structures in place of single structures as values of the chains, and third to allow subsets of an arbitrary set in place of subsets of a structure space.

Definition 3.0. *For any set* $\mathfrak{k}$, *any unary partial operation* $\mathscr{C}$ *on* $\mathscr{P}\mathfrak{k}$, *any* A, B *in* $\mathscr{P}\mathfrak{k}$, *any* n *in* ω, *and any* S:

S is a $\mathscr{C}$-chain of length n alternating from A to B if and only if S is a sequence on n to $\mathscr{DC} \sim \{\emptyset\}$, for any k in n $S_k \subseteq A$ if k is even and $S_k \subseteq B$ if k is odd, and for any positive k in n $S_k \subseteq \mathscr{C}(S_{k-1})$.

The following theorem and its corollaries make clear the intimate relationship between alternating chains and difference hierarchies.

Theorem 3.1 (Separability test)[2]. *Let $\mathfrak{l}$ be a set, let $\mathscr{C}$ be a $\subseteq$-preserving unary partial operation on $\mathscr{P}\mathfrak{l}$, let $\mathscr{R}$ be a subclass of $\mathscr{DC}$ on which $\mathscr{C}$ is the identity, let $n \in \omega$, and let $A, B \in \mathscr{P}\mathfrak{l}$. Then:*

(i) *A is $\mathfrak{D}_n(\mathscr{R})$-separable from B only if there does not exist a $\mathscr{C}$-chain of length $n+1$ alternating from A to B.*

Let $\mathscr{S}$ be a subclass of $\mathscr{DC}$ closed under $\cap$ and assume further that $\emptyset \in \mathscr{R}$, $\mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{R} \subseteq \mathscr{S}$, for any Z in $\mathscr{S}$ $Z \subseteq \mathscr{C}(Z)$, and $A, B \in \mathscr{S}$. Then conversely:

(ii) *A is $\mathfrak{D}_n(\mathscr{R})$-separable from B if there does not exist a $\mathscr{C}$-chain of length $n+1$ alternating from A to B.*

Proof. (i) By induction. Basis: $n=0$. Assume A is $\mathfrak{D}_0(\mathscr{R})$-separable from B—say by X. Then $X=\emptyset$, since $\mathfrak{D}_0(\mathscr{R})=\{\emptyset\}$. Hence $A=\emptyset$, or equivalently, there is no $\mathscr{C}$-chain of length 1 alternating from A to B. Induction step: $n=k+1$. Assume A is $\mathfrak{D}_{k+1}(\mathscr{R})$-separable from B—say by $X \sim Y$, where $X \in \mathscr{R}$ and $Y \in \mathfrak{D}_k(\mathscr{R})$. Then $A \subseteq X$, $X \cap B \subseteq Y$, and $A \cap Y = \emptyset$. Assume, for reductio ad absurdum, that there is a $\mathscr{C}$-chain of length $k+2$, say S, alternating from A to B. Then for any even p in $\mathscr{D}S$ $S_p \subseteq A$. Hence, since $A \subseteq X$, $S_p \subseteq X$. Hence, since S is to $\mathscr{DC}$, $X \in \mathscr{R} \subseteq \mathscr{DC}$, and $\mathscr{C}$ is $\subseteq$-preserving, $\mathscr{C}(S_p) \subseteq \mathscr{C}(X)$. But since $X \in \mathscr{R}$ and $\mathscr{C}$ is the identity on $\mathscr{R}$, $\mathscr{C}(X)=X$. Hence $\mathscr{C}(S_p) \subseteq X$. Hence, since for any positive (and hence any odd) q in $\mathscr{D}S$ $S_q \subseteq \mathscr{C}(S_{q-1})$, it follows that $S_q \subseteq X$ for any odd q in $\mathscr{D}S$. But for any odd q in $\mathscr{D}S$ $S_q \subseteq B$, and hence $S_q \subseteq X \cap B$. Hence, since $X \cap B \subseteq Y$, $S_q \subseteq Y$. Therefore $\langle S_{i+1} : i \in k+1 \rangle$ is a $\mathscr{C}$-chain of length $k+1$ alternating from Y to A. Hence by the hypothesis of the induction Y is not $\mathscr{D}_k(\mathscr{R})$-separable from A. But $A \cap Y = \emptyset$, so Y itself separates Y from A, contradicting $Y \in \mathfrak{D}_k(\mathscr{R})$. (ii) By induction. Basis: $n=0$. Assume there is no $\mathscr{C}$-chain of length 1 alternating from A to B. Then, since $A \in \mathscr{S} \subseteq \mathscr{DC}$, $A=\emptyset$. Hence $\emptyset$ separates A from B. But $\mathfrak{D}_0(\mathscr{R})=\{\emptyset\}$, so A is $\mathfrak{D}_0(\mathscr{R})$-separable from B.

[2] For a still better, practical understanding of the method of alternating chains see Footnote 1.

Induction step: $n=k+1$. Assume there is no $\mathscr{C}$-chain of length $k+2$ alternating from A to B. Since $A \in \mathscr{S}$, $A \in \mathscr{DC}$. Let $X=\mathscr{C}(A)$. Then $X \in \mathscr{S}$, since $A \in \mathscr{S}$ and $\mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{S}$. Hence $X \cap B \in \mathscr{S}$, since $B \in \mathscr{S}$ and $\mathscr{S}$ is closed under $\cap$. If $A=\emptyset$, then A is $\mathfrak{D}_0(\mathscr{R})$-separable from B and hence by Theorem 2.1(i), since $\emptyset \in \mathscr{R}$, A is $\mathfrak{D}_{k+1}(\mathscr{R})$-separable from B. If $A \neq \emptyset$, there is no $\mathscr{C}$-chain of length $k+1$ alternating from $X \cap B$ to A; for if there were we could "insert" the nonempty subset A of A in front of it to get, since $A \in \mathscr{S} \subseteq \mathscr{DC}$ and $X \cap B \subseteq X=\mathscr{C}(A)$, a $\mathscr{C}$-chain of length $k+2$ alternating from A to $X \cap B$, and hence from A to B. Hence, since $A, X \cap B \in \mathscr{S}$, by the hypothesis of the induction $X \cap B$ is $\mathfrak{D}_k(\mathscr{R})$-separable from A – say by Y. But $X \in \mathscr{R}$, since $A \in \mathscr{S}$ and $\mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{R}$; hence $X \sim Y \in \mathfrak{D}_{k+1}(\mathscr{R})$. Furthermore $A \subseteq \mathscr{C}(A)=X$, since for any Z in $\mathscr{S}$ $Z \subseteq \mathscr{C}(Z)$; hence $X \sim Y$ separates A from B. Therefore A is $\mathfrak{D}_{k+1}(\mathscr{R})$-separable from B.

Example 3.2. We return for illustration to the situation discussed in Section 0. Let $\mathscr{L}$ be a pure first-order predicate language with equality, let $\mathfrak{k}$ be the class of all $\mathscr{L}$-structures, let $\mathscr{C}$ be the operation on $\mathscr{P}\mathfrak{k}$ of taking closure under substructures, let $\mathscr{R}$ be $\wedge^0_{1\Delta}(\mathscr{L})$, and let $\mathscr{S}$ be $EC_\Delta(\mathscr{L})$. Then it follows from Theorem 3.1 [and a compactness argument] that for any n in ω and any disjoint sets A, B in $EC_\Delta(\mathscr{L})$ A is $\mathfrak{D}_n(\wedge^0_{1\Delta}(\mathscr{L}))$-separable from B [if and only if A is $\mathfrak{D}_n(\wedge^0_1(\mathscr{L}))$-separable from B] if and only if there is no substructure-chain of length $n+1$ alternating from A to B.

Corollary 3.3. *Let $\mathfrak{k}$ be a set, let $\mathscr{C}$ be a $\subseteq$-preserving unary partial operation on $\mathscr{P}\mathfrak{k}$, let $\mathscr{R}$ be a subclass of $\mathscr{DC}$ on which $\mathscr{C}$ is the identity, let $n \in \omega$, and let $A \in \mathscr{P}\mathfrak{k}$. Then:*

(i) *$A \in \mathfrak{D}_n(\mathscr{R})$ only if there does not exist a $\mathscr{C}$-chain of length $n+1$ alternating from A to $\mathfrak{k} \sim A$.*

Let $\mathscr{S}$ be a subclass of $\mathscr{DC}$ closed under $\cap$ and assume further that $\emptyset \in \mathscr{R}$, $\mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{R} \subseteq \mathscr{S}$, for any Z in $\mathscr{S}$ $Z \subseteq \mathscr{C}(Z)$, and A, $\mathfrak{k} \sim A \in \mathscr{S}$. Then conversely:

(ii) *$A \in \mathfrak{D}_n(\mathscr{R})$ if there does not exist a $\mathscr{C}$-chain of length $n+1$ alternating from A to $\mathfrak{k} \sim A$.*

Definition 3.4. *For any set $\mathfrak{k}$, any subclass $\mathscr{S}$ of $\mathscr{P}\mathfrak{k}$, and any unary partial operation $\mathscr{C}$ on $\mathscr{P}\mathfrak{k}$:*

$\mathfrak{A}(\mathscr{S}, \mathscr{C}) = \langle \{X\colon X, \mathfrak{l} \sim X \in \mathscr{S}$ *and there does not exist a* $\mathscr{C}$*-chain of length* $n+1$ *alternating from* X *to* $\mathfrak{l} \sim X\} \colon n \in \omega \rangle$.

Let $\mathfrak{l}$ be a set, $\mathscr{S}$ be a subclass of $\mathscr{P}\mathfrak{l}$, and $\mathscr{C}$ be a unary operation on $\mathscr{P}\mathfrak{l}$. Then it follows easily from Definitions 3.0 and 3.4 that $\mathfrak{A}(\mathscr{S}, \mathscr{C})$ is a balanced ω-prehierarchy if $\emptyset, \mathfrak{l} \in \mathscr{S}$. Let ν be the greatest ordinal in $\omega+1$ such that $\mathfrak{A}(\mathscr{S}, \mathscr{C})|\nu$ is a hierarchy. Then we call $\mathfrak{A}(\mathscr{S}, \mathscr{C})|\nu$ the *alternating chain hierarchy generated by* $\langle \mathscr{S}, \mathscr{C} \rangle$.

An interesting and important special case of Corollary 3.3 arises in the theory of closure set rings. By a *closure ring* we mean a structure $\langle X, +, \cdot, 0, 1, C \rangle$, where $\langle X, +, \cdot, 0, 1 \rangle$ is a Boolean ring with unit and C is a Kuratowski closure operation on X, i.e. C is a unary operation on X such that: (K1) for any x, y in X $C(x \circ y) = C(x) \circ C(y)$; (K2) for any x in X $x \leqslant C(x)$; (K3) $C(0) = 0$; (K4) for any x in X $C(x) = C(C(x))$. (Here we write "$u \circ v$" for "$(u+v)+(u \cdot v)$" and "$u \leqslant v$" for "$u \circ v = v$".)

Corollary 3.5. *For any set* $\mathfrak{l}$, *subclass* $\mathscr{S}$ *of* $\mathscr{P}\mathfrak{l}$, *and unary operation* $\mathscr{C}$ *on* $\mathscr{S}$ *such that* $\langle \mathscr{S}, \Delta, \cap, \emptyset, \mathfrak{l}, \mathscr{C} \rangle$ *is a closure ring, and any* n *in* ω:

$$\mathfrak{D}_n(\mathscr{C}^*(\mathscr{S})) \cap \mathscr{S} \cap \{X\colon \mathfrak{l} \sim X \in \mathscr{S}\} = \mathfrak{A}_n(\mathscr{S}, \mathscr{C}).$$

Proof. We will apply Corollary 3.3. Let $\mathscr{R} = \mathscr{C}^*(\mathscr{S})$. Since $\mathscr{DC} = \mathscr{S} \subseteq \mathscr{P}\mathfrak{l}$, $\mathscr{C}$ is a unary partial operation on $\mathscr{P}\mathfrak{l}$. By (K1) $\mathscr{C}$ is $\subseteq$-preserving. Since $\mathscr{C}\colon \mathscr{S} \to \mathscr{S}$, $\mathscr{R} = \mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{S} = \mathscr{DC}$. By (K4) $\mathscr{C}$ is the identity on $\mathscr{R}$. Since $\langle \mathscr{S}, \Delta, \cap, \emptyset, \mathfrak{l} \rangle$ is a Boolean ring with unit, $\mathscr{S}$ is closed under $\cap$ and $\emptyset \in \mathscr{S}$. Also by (K3) $\mathscr{C}\emptyset = \emptyset$, so $\emptyset \in \mathscr{C}^*(\mathscr{S}) = \mathscr{R}$. Since $\mathscr{R} = \mathscr{C}^*(\mathscr{S})$, $\mathscr{C}^*(\mathscr{S}) \subseteq \mathscr{R} \subseteq \mathscr{S}$. By (K2) for any Z in $\mathscr{S}$ $Z \subseteq \mathscr{C}(Z)$.

(i) Let $A \in \mathfrak{D}_n(\mathscr{R})$ and assume $A, \mathfrak{l} \sim A \in \mathscr{S}$. Then by Corollary 3.3(i) there is no $\mathscr{C}$-chain of length $n+1$ alternating from A to $\mathfrak{l} \sim A$. Hence, since $A, \mathfrak{l} \sim A \in \mathscr{S}$, $A \in \mathfrak{A}(\mathscr{S}, \mathscr{C})$.

(ii) Let $A \in \mathfrak{A}_n(\mathscr{S}, \mathscr{C})$. Then $A, \mathfrak{l} \sim A \in \mathscr{DC} = \mathscr{S}$ and there is no chain of length $n+1$ alternating from A to $\mathfrak{l} \sim A$. Hence by Corollary 3.3(ii) $A \in \mathscr{D}_n(\mathscr{R})$.

As Corollary 3.5 is intended to suggest, all of the results proved in this paper could, with slight modifications, be developed in the algebra of abstract closure rings.

4. Applications. In a separate publication we use the general tools developed here to study the structure of a number of particular difference hierarchies arising in the theory of models, descriptive set theory, and recursive function theory. As an illustration we cite here two samples.

Example 4.0. Let $\mathscr{L}$ be a pure propositional language with infinitely many propositional parameters. Let $\mathscr{R}$ be the class of classes of models of positive sentences of $\mathscr{L}$. Then $\mathfrak{D}(\mathscr{R})$ is a balanced, lateral, coseparable, perfect ω-hierarchy separable at odd ordinals and inseparable at even ordinals. The scope of $\mathfrak{D}(\mathscr{R})$ is the class of classes of models of sentences of $\mathscr{L}$.

Example 4.1. Let $\mathscr{R}$ be the set of recursively enumerable sets of natural numbers. Then $\mathfrak{D}(\mathscr{R})$ is a balanced, lateral, reducible, coseparable, inseparable, coarse ω-hierarchy. The scope of $\mathfrak{D}(\mathscr{R})$ is a small proper subclass of the class of Δ_2^0 sets of natural numbers.

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SEMANTIC CONSTRUCTION OF LEWIS'S SYSTEMS S4 AND S5

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1. Introduction. Even though in recent years modal logic has been treated on a semantic basis by Guillaume [58], K. J. J. Hintikka [1], Kanger [57], and Kripke [59, 63, 63a], the present report [2] on a similar approach is still felt to serve a certain purpose. We shall restrict ourselves to explaining the main lines of our investigation [3].

2. Formulas. Formulas are built from atoms $A, B, C, \ldots$ by the construction, from any given U and V, of $\overline{U}$ (negation), NU (necessity), $U \rightarrow V$ (material implication), $U \prec V$ (strict implication), $U \wedge V$ (conjunction), and $U \vee V$ (disjunction). Actually, except for a few remarks, negation, strict implication, conjunction, and disjunction will play no rôle in our exposition.

3. Semantics. An S4-*valuation* is an ordered quadruple $\langle P, \leqslant, p^0, f\rangle$; here P is a set, $\leqslant$ a partial ordering of P, p^0 the first element of P in accordance with that ordering, and f a function which associates with each *node* p in the *tree* P a *pseudo-valuation* $w=f(p)$; such a pseudo-valuation w connects with every formula U a truth-value $w(U) \in \{0, 2\}$ in accordance with the following conditions:

(S1) If $w(U)=2$ and $w(V)=0$, then $w(U \rightarrow V)=0$; otherwise, $w(U \rightarrow V)=2$.

(S2) If $w=f(p)$ and if for every p' and w' such that $p' \leqslant p$, $w'=f(p')$ we have $w'(U)=2$, then $w(NU)=2$; otherwise, $w(NU)=0$.

Throughout the following discussion we shall, as far as possible, avoid referring to the set P and its elements p. With this purpose in mind, we shall write $w \leqslant w'$ whenever, for certain nodes p, $p' \in P$, we have

† See the note following the preface.

[1] K. J. J. Hintikka, private communication, dated May 1959.

[2] Based on work done under research contract No 010-60-12 between Euratom and the University of Amsterdam; cf. Beth [61], nos. 2, 6, 7, and 13.

[3] Further details on semantic tableaux and related notions are found in Beth [62].

$p \leqslant p'$, $w = f(p)$, $w' = f(p')$; we let $w^0 = f(p^0)$. We use this convention immediately in stating the following remark:

(S3) If $w(NU) = 2$ then, for every $w' \leqslant w$, we have $w'(NU) = 2$.

4. Sequents. For any (finite) sets K and L of formulas, we may consider the *sequent* K/L, or:

True	False
K	L

The *valuation problem* connected with the sequent K/L will be concerned with the construction of an S4-valuation $\langle P, \leqslant, p^0, f \rangle$ such that $w^0(U) = 2$ for every formula $U \in K$ and $w^0(V) = 0$ for every formula $V \in L$.

If no such S4-valuation can be found, then we shall say that the sequent K/L is S4-*valid*. If the sequent $\emptyset/\{V\}$, written $\emptyset/V$, is S4-valid, then the formula V will be called an S4-*identity*.

5. Closure and reduction schemata; semantic tableaux. Valuation problems connected with sequents can be systematically treated by applying certain operations that can be conveniently represented in the form of closure and reduction schemata, as follows.

(i)

True	False
K'	L'
U	U

(ij^a)

True	False	
K'	L	
$U \to V$		
1 | 2	1 | 2	
V	U	

(ij^b)

True	False
K	L'
	$U \to V$
U	V

(iij^a)

True	False	
K	L	
NU		
U		
U		

(iij^b)

True		False	
K		L_{at}	
		L'	
		NV	
1	2	1	2
		V	L'

	True	False
	K'	L
	NU	
	.....	
	.....	
(iij^{aa})	NU	

	True	False
		NV
(iij^{bb})		V

For an explanation of the schemata (i)–(ij^{b}) we refer to Beth [62].

In an application of schema (iij^{b}), the formulas in the succedent are grouped into three classes: the class L_{at} of all atoms, the formula NV that is to be treated, and the class L' of all remaining formulas. The formulas A in L_{at} are given the truth-value $w(A)=0$. Furthermore, we look for pseudo-valuations w', $w''<w$ such that $w'(V)=0$ and $w''(Y)=0$ for every formula Y in L'. Accordingly, the splitting of the semantic tableau is *disjunctive*: the closure of one subtableau suffices for the closure of the original tableau.

Schema (iij^{bb}) corresponds to a situation in which the operation can be simplified.

Schema (iij^{a}) is based upon the consideration that, if $w(NU)=2$, then $w(U)=2$ and, moreover, $w'(U)=2$ for every $w'<w$. Schema (iij^{aa}) corresponds to (S3).

The results of successive applications of the above schemata can be represented in a semantic tableau, as described in Beth [62]; a concrete example is given below.

	True		False	
			(1) $(NA \to NB) \to N(NA \to B)$	
	(2) $NA \to NB$		(3) $N(NA \to B)$	
	1	2	1	2
		(5) NB	(4) NA	
(iij^{bb})	.....		.....	(6) $NA \to B$
		(7) NA		(8) B
(iij^{a})	.....	(9) B		
(iij^{b})	.....			
	11	12	11	12
			(6) $NA \to B$	(10) A
	(7) NA		(8) B	
(iij^{a})	(11) A			

Theorem 1. *A sequent K/L is* S4*-valid if and only if the corresponding semantic tableau is closed and, if the semantic tableau is not closed, then it provides a solution to the valuation problem connected with the sequent K/L.*

Example. Let us consider the sequent:

$$\emptyset/(NA \to NB) \to N(NA \to B);$$

we obtain the foregoing semantic tableau.

This semantic tableau is not closed and it produces the following S4-valuation which clearly provides a solution to the valuation problem connected with the given sequent.

Nodes		A	B	NA	NB	$NA \to NB$	$NA \to B$	$N(NA \to B)$	$(\ ) \to N(\ )$
0	w^0	0	0	0	0	2	2	0	0
0 0	$w\ w'$	2 0	0 0	2 0	0 0	0 2	0 2	0 2	2 2

If in the construction of a (non-closed) semantic tableau we meet with a conjunctive splitting, then in constructing a corresponding S4-valuation we select one of the two resulting subtableaux (if one of these two subtableaux happens to be closed we must, of course, select the other one). But, if we meet with a disjunctive splitting, then both of the subtableaux have to be taken into account and, accordingly, there will be a fork in the tree structure formed by the nodes.

6. Axiomatization. Let us now consider the following axiom-schemata:

(I) $U \to (V \to U)$,

(II) $[U \to (V \to W)] \to [(U \to V) \to (U \to W)]$,

(III) $[(U \to V) \to U] \to U$,

(IV) $NU \to U$,

(V) $N(NU \to V) \to (NU \to NV)$,

and the following schemata of deduction:

$$(\mathrm{i}^0)\ \frac{\begin{matrix} U \to V \\ U \end{matrix}}{V} \qquad\qquad (\mathrm{ij}^0)\ \frac{U}{NU}$$

Theorem 2. *The axiom-schemata* (I)–(V), *together with the deduction schemata* (i^0) *and* (ij^0), *provide an axiomatization for the set of all* S4-*identities.*

Proof. It is not difficult to see that every theorem that can be obtained from the axiom-schemata by applying the deduction schemata is an S4-identity.

Furthermore, it is known that the closure and reduction schemata (i)–(ij^b) can be replaced by the axiom-schemata (I)–(III) combined with deduction schema (i^0).

To complete our proof, we first note that deduction schema (ij^0) corresponds to reduction schema (iij^{bb}). Furthermore, an application of reduction schema (iij^a) can always be replaced by an application of reduction schema (iij^{aa}) and then eliminated by virtue of axiom-schema (IV).

Finally, we observe that a typical application of reduction schema (iij^b) can be replaced by an application of reduction schema (iij^{bb}), provided a suitable instance of axiom-schema (V) be inserted into the left column of the tableau.

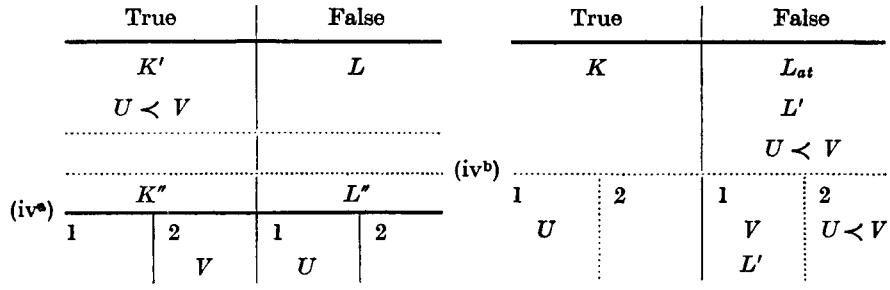
	True	False
	K'	L_{at}
	NU	L'
		NV
(iij^b)	U	V

	True		False	
	K'		$[L_{at}]$	
	NU		$[L']$	
	$N(NU \to V) \to (NU \to NV)$		NV	
		$NU \to NV$	$N(NU \to V)$	
(iij^{bb})	.....	..... | NV	$NU \to V$	NU |
	NU		V	
	U			

This completes our proof.

7. S4 in terms of strict implication. Since $U \prec V$ can be defined as $N(U \to V)$, we obtain for strict implication the following semantical rule:

(S4) If, for every $w' \leqslant w$, we have either $w'(U)=0$ or $w'(V)=2$, then $w(U \prec V)=2$; otherwise $w(U \prec V)=0$. Accordingly, we have the following reduction schemata:



The method applied in the proof of Theorem 2 can be applied again in proving:

Theorem 3. *The axiom-schemata*:

(VI) $U \prec U$,

(VII) $(U \prec V) \prec [W \prec (U \prec V)]$,

(VIII) $[U \prec (V \prec W)] \prec [(U \prec V) \prec (U \prec W)]$,

together with the deduction schema:

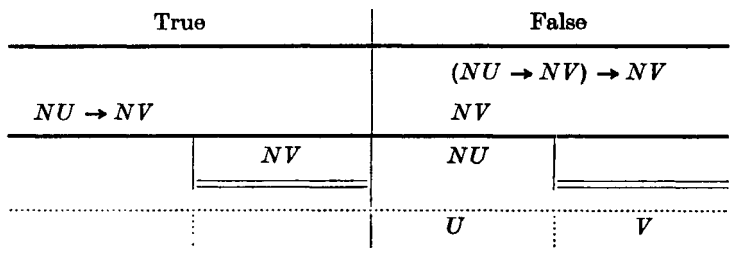
$$(\text{iij}^0)\ \frac{U \prec V \qquad U}{V}$$

provide an axiomatization for the set of all those S4*-identities which contain only* $\prec$.

8. Further metamathematical results. The above approach provides a simple decision method for S4 as well as simple proofs for various metamathematical results; in particular, we mention:

Theorem 4 (*Gödel* [33]). *A formula* $NU \vee NV$, *or* $(NU \rightarrow NV) \rightarrow NV$, *is an* S4*-identity, if and only if either* U *or* V *is an* S4*-identity.*

Proof. It is sufficient to construct the semantic tableau:



9. S4 and topology. Somewhat more attention will be devoted to the connections between S4 and topology. It will now be convenient to state the semantic rule for negation:

(S5) If $w(U)=2$, then $w(\overline{U})=0$; and if $w(U)=0$, then $w(\overline{U})=2$.

Conjunction, disjunction, and equivalence can then be introduced in the same way as in classical logic.

A *topological space* (in the larger sense) is an ordered couple $\langle M, Int\rangle$ consisting of a non-empty set M and an operation Int that associates with any set $X \subseteq M$ a set $IntX \subseteq M$ in accordance with the following conditions:

(T1) $IntX \subseteq X$,

(T2) $Int(IntX)=IntX$,

(T3) $Int(X \cap Y)=IntX \cap IntY$,

(T4) $IntM=M$.

A (topological) *valuation* over $\langle M, Int\rangle$ is a function G that associates with every formula U a set $G(U) \subseteq M$ in accordance with the following conditions:

(T5) $G(U \rightarrow V)=[M-G(U)] \cup G(V)$,

(T6) $G(NU)=Int(G(U))$,

(T7) $G(\overline{U})=M-G(U)$.

Theorem 5. *If U is an* S4*-identity, $\langle M, Int\rangle$ is a topological space, and G is a valuation over $\langle M, Int\rangle$, then $G(U)=M$.*

The proof is straightforward.

Theorem 6 (*McKinsey* [41]). *If U is not an* S4*-identity, then we can find a topological space $\langle M, Int\rangle$ and a valuation H over $\langle M, Int\rangle$ such that $H(U) \neq M$.*

Proof. Since U is not an S4-identity, we can find an S4-valuation $\langle P, \leqslant, p^0, f\rangle$ such that $w^0(U)=0$. Let any S4-valuation of this kind be given.

Then as our set M we take the set P of all nodes. For every $p \in P$, the set of all $p' \in P$ such that $p' \leqslant p$ will be denoted as a *neighborhood* $M^{(p)}$. And, if $X \subseteq M$, then $IntX$ will be the union of all those neighborhoods $M^{(p)}$ that are included in X.

If F is any formula, then $H(F)$ will consist of all nodes p such that for $w=f(p)$ we have $w(F)=2$.

It is now easy to see that $\langle M, Int\rangle$ is a topological space (in the wider sense) and that H is a valuation over $\langle M, Int\rangle$. Moreover, since $w^0(U)=0$, we have $w^0 \in M-H(U)$ and hence $H(U) \neq M$, as required.

10. The System S5. A similar treatment of the System S5 will now be described much more briefly.

An *S5-valuation* is an ordered couple $\langle W, w^0\rangle$, where $w^0 \in W$ and where W is a set of pseudo-valuations which are now required to satisfy the following condition:

(S2+) If for every $w' \in W$ we have $w'(U)=2$, then $w(NU)=2$; otherwise, $w(NU)=0$.

Conditions (S1) and (S5) remain unchanged, whereas (S3) and (S4) must be slightly modified.

The whole above discussion can be repeated for the System S5, but for the introduction of an additional reduction schema:

	True	False
	K	L
(iij^{a+})	U	
	K'	L'
	NU	

Accordingly, the above axiomatization of the System S4 is extended into an axiomatization of the System S5 by adding the axiom-schema:

(IX) $(NU \rightarrow NV) \rightarrow N(NU \rightarrow V)$.

An axiomatization of the set of all S5-identities which contain only $\prec$ is obtained by adding to the above axiom-schemata (VI)–(VIII) the following version of Peirce's law:

(X) $[\{(U \prec V) \prec W\} \prec (U \prec V)] \prec (U \prec V)$.

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CONTINUOUS MODEL THEORY

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Introduction. This paper is a brief report on the joint work of the authors in a new area of research in model theory. As such the paper only serves to indicate the kind of research the authors have done in this subject and it is not to be regarded as a complete and final rendering of their findings. A complete account of their joint work will shortly appear as a volume in the Annals of Mathematics Studies series.

In what follows we shall give only the statements of definitions and theorems. When we refer to standard or well-known results of 2-valued model theory, we shall not always give complete bibliographical or historical references. All such precise references will be found in the forthcoming monograph. The style of this paper will be informal; however, we hope that it will convey the spirit of our work.

The symbols $\in$, $\subseteq$, $\cup$, $\cap$, $-$, $\bigcup$, $\bigcap$, and $\{x : \ldots x \ldots\}$ shall have their usual set-theoretical meanings. We shall work in naive set theory with the axiom of choice. We identify each ordinal with the set of all smaller ordinals and we define a cardinal as an initial ordinal. 0 is used for the empty set as well as the smallest ordinal. The letters i, j, k, m, n shall denote finite ordinals and cardinals. In general, small Greek letters μ, ν denote ordinals and the letters α, β are reserved for cardinals. α^+ denotes the cardinal successor of α, and 2^α denotes the cardinal 2 to the power α. The cardinal sum of two cardinals α, β is given by $\alpha + \beta$. As usual, ω denotes the smallest infinite cardinal.

Let X be a set. The cardinal of X shall be denoted by $|X|$. We let $X^* = \{Y : Y \subseteq X \text{ and } Y \neq 0\}$. Binary relations over a set X are defined as sets of ordered pairs $\langle x, y \rangle$ of elements x, y belonging to X. The converse of a relation H is denoted by $\breve{H}$. If Y is a set and H a relation we let

$$H[Y] = \{y : \langle x, y \rangle \in H \text{ for some } x \in Y\}.$$

We sometimes write xHy for $\langle x, y \rangle \in H$. Functions are defined as special binary relations in the usual manner. A function f with domain

X and range included in Y is sometimes graphically presented as $f : X \to Y$. The cartesian power of X to the Y is denoted by X^Y. For a finite cartesian power, we identify X^n with $X \times X \times \ldots \times X$ n-times, so that elements of X^n may be considered as ordered n-tuples. The special notation X^∞ is used to denote the set of functions $f \in X^\omega$ such that f is eventually constant. That is,

$$X^\infty = \{f \in X^\omega : \text{for some } n,\ f(n) = f(m) \text{ for all } m \geqslant n\}.$$

Let $(X, \mathscr{S})$ be a topological space. Thus $\mathscr{S}$ is the collection of all open sets of X. An open basis for $\mathscr{S}$ is denoted by $\mathscr{S}_0$. Elements of $\mathscr{S}_0$ are referred to as the basic open sets, and complements of elements of $\mathscr{S}_0$ are called basic closed sets. The closure of a subset Y of X is denoted by $\overline{Y}$. The topology $\mathscr{S}^*$ on the set X^* is defined as follows: $\mathscr{S}^*$ is the least topology on the set X^* such that

(i) whenever V is open in X, the set $\{Y \in X^* : \overline{Y} \subseteq V\}$ is open in $\mathscr{S}^*$, and

(ii) whenever U is open in X, the set $\{Y \in X^* : Y \cap U \neq 0\}$ is open in $\mathscr{S}^*$.

When we speak of the topological space X^*, we mean the space $(X^*, \mathscr{S}^*)$. The topology $\mathscr{S}^*$ was defined by Vietoris [23] and he proved that if X is compact, then X^* is also compact.

Let us now consider, for any non-empty set I, the product space X^I with the usual product topology, which we shall denote by $\mathscr{S}^I$. By Tychonoff's theorem the space $(X^I, \mathscr{S}^I)$ is compact Hausdorff if $(X, \mathscr{S})$ is compact Hausdorff. An open set V in X^I will be called *singular* if, for some $i \in I$ and $U \in \mathscr{S}_0$,

$$V = \{h \in X^I : h(i) \in U\}.$$

An open set V in X^I is called *basic* if it is a finite intersection of singular open sets. The set of basic open sets in X^I is the usual open base for the space X^I, and the set of singular open sets in X^I is the usual open subbase for X^I. A closed set Y in X^I is called *singular*, or *basic*, if its complement is a singular, or basic, open set, respectively. Each closed set of X^I is then an intersection of basic closed sets.

1. 2-valued model theory. To put our study in the proper perspective, we begin by rephrasing the basic definitions and ideas of 2-valued model theory in a form more suitable for our purposes. These basic

definitions and ideas in 2-valued model theory can be found in Tarski [52b] and [54].

Consider the set $2=\{0, 1\}$. A first-order language with identity is constructed in the usual manner from

the set of predicates: $\equiv$, $S(x, y, z)$;

the set of individual variables: $v_0, v_1, \ldots$;

the set of connectives: $\vee$, $\wedge$, $\neg$;

the set of quantifiers: $\exists$, $\forall$.

The sets of formulas and sentences are defined in the usual manner. Connectives are interpreted as functions mapping the cartesian powers 2^n into 2. Thus the functions $\vee$, $\wedge$ map 2^2 into 2 and the function $\neg$ maps 2 into 2 in the usual manner. Quantifiers are interpreted as functions mapping 2^* into 2. Thus the functions $\exists$, $\forall$ map 2^* into 2 in the usual manner, that is, for each $Y \in 2^*$; $\exists(Y)=\sup Y$ and $\forall(Y)=\inf Y$. We let the small Roman letter l denote the triple $\langle 2, \{\vee, \wedge, \neg\}, \{\exists, \forall\}\rangle$ and we speak of l as the *2-valued logic*. We let Φ_l and Σ_l denote the sets of formulas and sentences of l, respectively.

The class of models A for l, denoted by $\mathscr{M}_l$, consists of all pairs $A=\langle R, S\rangle$ where R is a non-empty set and S is a mapping of R^3 into 2. For each formula $\varphi \in \Phi_l$, model $A=\langle R, S\rangle \in \mathscr{M}_l$, and function $r \in R^\infty$ an element $\varphi[A, r] \in 2$ can be defined in the usual manner. Basically, $\varphi[A, r]$ has the value 1 or 0 depending on whether the formula φ holds or fails to hold in the model A under the interpretation of the individual variables given by r. If $\varphi \in \Sigma_l$ then $\varphi[A, r]$ is independent of r and is denoted by $\varphi[A]$.

To introduce the usual notions of elementary classes and elementary equivalence, we first introduce the discrete topology on 2. Thus 2 becomes a compact Hausdorff space. Consider the product space 2^{Σ_l}. Any non-empty subset of 2^{Σ_l} is called a *theory*. (Our usage of the term theory is somewhat different from the usual usage.) The *theory of a model* $A \in \mathscr{M}_l$, in symbols $\mathrm{Th}(A)$, is defined to be that function in 2^{Σ_l} such that

$$\mathrm{Th}(A)(\varphi)=\varphi[A] \text{ for each } \varphi \in \Sigma_l.$$

For a class $\mathscr{K} \subseteq \mathscr{M}_l$, the *theory of* $\mathscr{K}$ is defined as

$$\mathrm{Th}(\mathscr{K})=\{\mathrm{Th}(A) : A \in \mathscr{K}\}.$$

Thus $\mathrm{Th}(\mathscr{K}) \subseteq 2^{\Sigma_l}$ and $\mathrm{Th}(\mathscr{K})$ is a theory if $\mathscr{K} \neq 0$. Given a theory $\theta \subseteq 2^{\Sigma_l}$, the *class of all models of* θ, in symbols $\mathrm{Mod}(\theta)$, is defined by

$$\mathrm{Mod}(\theta) = \{A \in \mathscr{M}_l : \mathrm{Th}(A) \in \theta\}.$$

A is a *model of the theory* θ if $\mathrm{Th}(A) \in \theta$, or equivalently, $A \in \mathrm{Mod}(\theta)$. A class $\mathscr{K} \subseteq \mathscr{M}_l$ is an *elementary class*, in symbols $\mathscr{K} \in \mathbf{EC}_\Delta$, if $\mathscr{K} = \mathrm{Mod}(\theta)$ for some closed theory θ in 2^{Σ_l}. If θ is closed and singular, then $\mathscr{K}$ is called a *singular elementary class* ($\mathscr{K} \in \mathbf{EC}_s$), and if θ is basic, then $\mathscr{K}$ is called a *basic elementary class* ($\mathscr{K} \in \mathbf{EC}$). Notice that due to the properties of the connectives $\vee$, $\wedge$, and $\neg$, every basic elementary class is a singular elementary class and **EC** is closed under union, intersection, and complementation. Finally, two models A, B are *elementarily equivalent* if $\mathrm{Th}(A) = \mathrm{Th}(B)$.

Defining $\mathbf{EC}_\Delta$ as the collection of all closed subsets of $\mathscr{M}_l$, we obtain a topology on the class $\mathscr{M}_l$, called the *elementary topology on* $\mathscr{M}_l$. The elementary topology is not Hausdorff. However, under the identification of elementarily equivalent models it becomes Hausdorff.

With the terminology introduced above, one could state a number of theorems in 2-valued model theory in purely topological terms. For instance, it is easily seen that the compactness theorem is equivalent to the statement that the elementary topology is compact or that $\mathrm{Th}(\mathscr{M}_l)$ is closed in 2^{Σ_l}.

2. X-valued model theory. We now repeat the basic definitions of Section 1 with 2 replaced by an arbitrary topological space X containing 2 as a subset. Our purpose is to obtain theorems in X-valued model theory which, when X is taken to be the space 2, immediately specialize to well-known results of 2-valued model theory. While such generalizations may not lead to new results in 2-valued model theory, we believe that such a study will clarify our understanding of 2-valued model theory. These generalizations in general have much more subtle proofs than their 2-valued analogs. Hence they give us new insights into why the theorems hold in the simple case $X = 2$.

Let $(X, \mathscr{S})$ be a topological space such that $2 \subseteq X$. An *X-valued logic* $\mathscr{L}$ is a triple $\langle X, \mathscr{C}, \mathscr{Q} \rangle$ where $\mathscr{C}$ is a set of finitary functions mapping X^n into X and $\mathscr{Q}$ is a set of functions of one argument mapping X^* into X. Formulas of $\mathscr{L}$ are built up in the usual way from

the set of predicates: $\equiv$, $\mathsf{S}(x, y, z)$;
the set of individual variables: $v_0, v_1, \ldots$;
the set of connectives: $\mathscr{C}$;
the set of quantifiers: $\mathscr{Q}$.

Atomic formulas are $v_i \equiv v_j$ and $\mathsf{S}(v_i, v_j, v_k)$. If $\varphi_1, \ldots, \varphi_n$ are formulas and $c \in \mathscr{C}$ is an n-ary connective, then $c(\varphi_1, \ldots, \varphi_n)$ is a formula. If φ is a formula and $q \in \mathscr{Q}$, then $(qv_i)\varphi$ is a formula. The notions of free and bound variables are defined as usual, and a sentence is a formula with no free variables. The set of formulas of $\mathscr{L}$ is denoted by $\Phi_{\mathscr{L}}$ and the set of sentences by $\Sigma_{\mathscr{L}}$. We shall drop the subscript $\mathscr{L}$ from $\Phi_{\mathscr{L}}$ and $\Sigma_{\mathscr{L}}$ (as well as from any other notation we shall introduce) whenever $\mathscr{L}$ is understood.

The class of models for $\mathscr{L}$, denoted by $\mathscr{M}_{\mathscr{L}}$, consists of all pairs $A = \langle R, S\rangle$ where $R \neq 0$ and $S : R^3 \to X$. For each formula $\varphi \in \Phi$, model $A \in \mathscr{M}$, and function $r \in R^\infty$, an element $\varphi[A, r] \in X$ is defined by induction as follows:

$$(v_i \equiv v_j)[A, r] = 1 \text{ if } r_i = r_j;$$
$$(v_i \equiv v_j)[A, r] = 0 \text{ if } r_i \neq r_j;$$
$$\mathsf{S}(v_i, v_j, v_k)[A, r] = S(r_i, r_j, r_k);$$
$$c(\varphi_1, \ldots, \varphi_n)[A, r] = c(\varphi_1[A, r], \ldots, \varphi_n[A, r]);$$
$$(qv_n)\varphi[A, r] = q(\{\varphi[A, r'] : r' \in R^\infty \text{ and } r'(m) = r(m) \text{ for all } m \neq n\}).$$

If $\varphi \in \Sigma$, then $\varphi[A, r]$ is independent of r and is denoted by $\varphi[A]$.

Consider the product space $(X^\Sigma, \mathscr{S}^\Sigma)$. In an exactly analogous manner, we define

$$\mathrm{Th}(A)(\varphi) = \varphi[A] \text{ for } \varphi \in \Sigma,\ A \in \mathscr{M},$$
$$\mathrm{Th}(\mathscr{K}) = \{\mathrm{Th}(A) : A \in \mathscr{K}\} \text{ for } \mathscr{K} \subseteq \mathscr{M}, \text{ and}$$
$$\mathrm{Mod}(\theta) = \{A \in \mathscr{M} : \mathrm{Th}(A) \in \theta\} \text{ for } \theta \subseteq \mathrm{X}^\Sigma.$$

The terms *theory, model of a theory* are defined as before, and the notions of $\mathbf{EC}_s$, $\mathbf{EC}$, and $\mathbf{EC}_\Delta$ are introduced. In each case, we replace notions of 2^{Σ_1} by notions of 2^Σ. Notice that in general $\mathbf{EC} \neq \mathbf{EC}_s$ and $\mathbf{EC}$ need not be closed under $\cup$, $\cap$, and complementation. We also define in a similar manner the *elementary topology* on $\mathscr{M}$ and the notion of *elementary equivalence.*

We are now ready to present a sample of the theorems that may

be proved in X-valued model theory. Before we do that we need some standard notions defined on the models of $\mathscr{M}$.

A model $B=\langle R', S'\rangle$ is a *submodel* of $A=\langle R, S\rangle$ if $R'\subseteq R$ and S' is obtained from S by restricting S to $(R')^3$. We write $B\subseteq A$ to mean that B is a submodel of A. B is an *elementary submodel* of A, in symbols $B\prec A$, if $B\subseteq A$ and

$$\varphi[A, r]=\varphi[B, r] \text{ for every } \varphi\in\Phi \text{ and } r\in(R')^\infty.$$

Let μ be an ordinal. A is a *union of chain of models* $A_\nu=\langle R_\nu, S_\nu\rangle$, $\nu<\mu$, in symbols $A=\bigcup_{\nu<\mu} A_\nu$, if

$$A_\nu\subseteq A_{\nu'} \text{ whenever } \nu\leqslant\nu'<\mu,$$

$$R=\bigcup_{\nu<\mu} R_\nu, \text{ and}$$

$$S=\bigcup_{\nu<\mu} S_\nu.$$

Clearly $A=\langle R, S\rangle$ is again a model of $\mathscr{M}$, and each $A_\nu\subseteq A$. A is a *union of an elementary chain* A_ν, $\nu<\mu$, if

$$A_\nu\prec A_{\nu'} \text{ whenever } \nu\leqslant\nu'<\mu,$$

and $A=\bigcup_{\nu<\mu} A_\nu$. Two models $A=\langle R, S\rangle$ and $B=\langle R', S'\rangle$ are *isomorphic*, in symbols $A\cong B$, if there is a one-to-one function f mapping R onto R' such that

$$S(x, y, z)=S'(f(x), f(y), f(z)) \text{ for all } x, y, z\in R.$$

A is *elementarily embeddable* in B if for some C, $A\cong C$ and $C\prec B$. For a model $A=\langle R, S\rangle$, we let $\|A\|=|R|$. A is said to be *finite* (*infinite*) if $\|A\|$ is finite (infinite).

3. Continuous model theory. The following theorem of X-valued model theory can easily be proved by methods similar to those found in Tarski–Vaught [57].

Theorem 1 (*Downward Löwenheim–Skolem*). *Suppose that* $|X|+|\mathscr{C}|+|\mathscr{Q}|+\omega\leqslant\alpha\leqslant\beta$, $A=\langle R, S\rangle$, $R''\subseteq R$, $|R''|\leqslant\alpha$, *and* $|R|=\beta$. *Then there exists an elementary submodel* $B=\langle R', S'\rangle$ *of* A *such that* $R''\subseteq R'\subseteq R$ *and* $|R'|=\alpha$.

In this most general form, Theorem 1 was also known to Mostowski

(around 1960). It follows from Theorem 1 that if a theory θ has a model of power β and

$$|X|+|\mathscr{C}|+|\mathscr{Q}|+\omega\leqslant\alpha\leqslant\beta,$$

then θ has a model of power α.

Theorem 2 (*Upward Löwenheim–Skolem*). *Suppose that* $\omega\leqslant\alpha$, $|X|+|\mathscr{C}|+|\mathscr{Q}|+\alpha\leqslant\beta$, *and* $\|A\|=\alpha$. *Then there exists a model* B *such that* $A\prec B$ *and* $\|B\|=\beta$.

Theorem 2 is in general not true. For instance, it is false if X is a countable discrete space. However, following an idea of Hanf, it is possible to establish the following.

Theorem 3 (*Hanf*). *There exists an infinite cardinal* β (*depending on* $|X|$, $|\mathscr{C}|$, *and* $|\mathscr{Q}|$) *such that for every theory* θ *if* θ *has a model of power no less than* β *then* θ *has a model in each power greater than or equal to* β.

As we have mentioned in Section 1, the compactness theorem can be stated in the following form.

Theorem 4 (*Compactness*). $\mathrm{Th}(\mathscr{M})$ *is closed in* X^{Σ}.

This theorem is, not surprisingly, false in general. It becomes a true theorem under the following assumptions on $\mathscr{L}$:

(1) $(X, \mathscr{S})$ is a compact Hausdorff space.

(2) Each n-ary connective $c\in\mathscr{C}$ is a continuous function on X^n to X.

(3) Each $q\in\mathscr{Q}$ is a continuous function on X^* to X.

We say $\mathscr{L}$ is a *continuous logic* if (1), (2), (3) are satisfied. Under these conditions both Theorems 2 and 4 are true. In fact, Theorems 1 and 2 can be stated in a sharper form with $|\mathscr{S}_0|$ replacing $|X|$. Using the same hypotheses on X as given in Chang–Keisler [62a], Mostowski [63] has also found a proof of the compactness theorem.

We hardly need to point out that since the space 2 has the discrete topology, conditions (1), (2), and (3) obviously hold for the logic l. Thus l is a continuous logic. Another example of a continuous logic is given by the following. Let X be the closed real unit interval $[0, 1]$ with the usual topology. Let $\mathscr{C}$ be the set of all functions generated from the functions $(1-x)$ and $\min(1, 1-x+y)$ by composition. Let $\mathscr{Q}=\{\sup, \inf\}$. Let $l=\langle X, \mathscr{C}, \mathscr{Q}\rangle$. The logic l is often referred to as the classical infinite-valued logic of Łukasiewicz. One can easily prove that l is a continuous

logic. The authors' work summarized in this paper was preceded by a study of [0, 1]-valued model theory by Chang [61].

To consider problems of a deeper nature, we need the following notions.

Let Σ^+ denote the set of all formulas of Φ which contains at most the variable v_0 free. Thus $\Sigma \subseteq \Sigma^+ \subseteq \Phi$. We shall identify constant functions $r \in R^\infty$ with elements $a \in R$. We write $\varphi[A, a]$ in case $\varphi \in \Sigma^+$ and r is the constant function a. Consider the product space X^{Σ^+} with the usual topology. The *extended theory* $\mathrm{Th}^+(A)$ of a model $A = \langle R, S\rangle$ is defined by $\mathrm{Th}^+(A) = \{f \in X^{\Sigma^+} : \text{for some } a \in R,\ f(\varphi) = \varphi[A, a] \text{ for every } \varphi \in \Sigma^+\}$.

Thus $\mathrm{Th}^+(A) \subseteq X^{\Sigma^+}$. We write $\overline{\mathrm{Th}^+(A)}$ to denote the closure of $\mathrm{Th}^+(A)$ in X^{Σ^+}. In 2-valued model theory, one can easily check that for $A, B \in \mathscr{M}_l$, $\mathrm{Th}(A) = \mathrm{Th}(B)$ if and only if $\overline{\mathrm{Th}^+(A)} = \overline{\mathrm{Th}^+(B)}$.

Suppose that for each ordinal μ, c_μ is an individual constant symbol. By the logic $\mathscr{L}(\mu)$ we mean $\mathscr{L}$ augmented by the set of individual constant symbols $\{c_\nu;\ \nu < \mu\}$. Thus $\mathscr{L}(0)$ is identical with $\mathscr{L}$. We extend the notions of a formula, a sentence, and a formula with at most the variable v_0 free in a natural way to the logic $\mathscr{L}(\mu)$ and obtain the sets $\Phi(\mu)$, $\Sigma(\mu)$, and $\Sigma^+(\mu)$. The class of models $\mathscr{M}(\mu)$ for $\mathscr{L}(\mu)$ consists of all triples $\langle R, S, a\rangle$ where $A = \langle R, S\rangle \in \mathscr{M}$ and $a \in R^\mu$. We write (A, a) for the triple $\langle R, S, a\rangle$. Thus

$$\mathscr{M}(\mu) = \{(A, a) : A \in \mathscr{M} \text{ and } a \in R^\mu\}.$$

As before, for each formula $\varphi \in \Phi(\mu)$, model $(A, a) \in \mathscr{M}(\mu)$, and $r \in R^\infty$, we can define an element $\varphi[(A, a), r] \in X$ in the natural way. Similarly, we define

$$\mathrm{Th}((A, a)) \in X^{\Sigma(\mu)} \text{ and } \mathrm{Th}^+((A, a)) \subseteq X^{\Sigma^+}(\mu).$$

A model $A = \langle R, S\rangle$ of $\mathscr{M}$ is said to be *α-saturated* if for every ordinal $\mu < \alpha$ and every sequence $a \in R^\mu$, $\mathrm{Th}^+((A, a))$ is a closed subset of $X^{\Sigma^+(\mu)}$. A model $A \in \mathscr{M}$ is *special* if it is a union of an elementary chain A_α, $\alpha < \|A\|$, where each A_α is α^+-saturated. (Notice that α ranges over cardinals.) In the case of 2-valued model theory, the notion of special is introduced in a somewhat different, though essentially the same, form in Morley–Vaught [62], and the notion of α-saturated can be found in Keisler [61e] and Vaught [63b].

The following are theorems of 2-valued model theory. (They can all be found in Morley–Vaught [62].)

Theorem 5. *Every finite model is special.*

Theorem 6. *Every infinite model A has a special elementary extension of power α for every infinite α such that $\alpha = \sum_{\beta<\alpha} 2^{\beta}$ and $\omega + \|A\| + |\mathscr{S}_0| + |\mathscr{C}| + |\mathscr{Q}| < \alpha$.*

Theorem 7. *If $\mathrm{Th}(A) = \mathrm{Th}(B)$, $\|A\| \leqslant \|B\|$, and B is special, then A is elementarily embeddable in B.*

Theorem 8. *If $\mathrm{Th}(A) = \mathrm{Th}(B)$, $\|A\| = \|B\|$, and A, B are both special, then $A \cong B$.*

Theorems 5 and 6 are also true in continuous model theory. In general, Theorems 7 and 8 are false in continuous model theory. For instance, one can find an example of a continuous logic $\mathscr{L}$ and two finite, elementarily equivalent models of $\mathscr{M}_{\mathscr{L}}$ with the same number of elements which are not isomorphic. Our analysis of this situation leads us into the subject matter of the next section.

4. Adequate model theory. An X-valued logic $\mathscr{L} = \langle X, \mathscr{C}, \mathscr{Q} \rangle$ is *adequate* if in addition to conditions (1), (2), and (3), the following are also satisfied:

(4) For any two distinct points $x, y \in X$, there exists a function of one argument $t \in \mathscr{C}$ such that $t(x) = 1$ and $t(y) \neq 1$.

(5) For every open set V containing 1, there exists a function of two arguments $k \in \mathscr{C}$ such that $k(1, 1) = 1$ and $\check{k}[1] \subseteq V \times V$.

(6) For every set $Y \in X^*$ such that $1 \notin \overline{Y}$, there exists $q \in \mathscr{Q}$ such that $q(Y) \neq 1$ and for every $Z \in X^*$, $q(Z) = 1$ whenever $1 \in \overline{Z}$.

Conditions (4) and (6) are pointwise conditions on the spaces X and X^*, respectively. Condition (5) is a local condition. The obvious pointwise condition for (5) proves to be inadequate for our purposes. It is clear that l is an adequate logic. It can be shown that l is also an adequate logic.

We can prove that if a logic $\mathscr{L}$ is adequate then Theorems 7 and 8 hold. Hence, adequate model theory contains all the usual results about special models. In adequate model theory we can also obtain the following result, the 2-valued analog of which was proved in Keisler [61e].

Theorem 9. (*Assume the generalized continuum hypothesis.*) *Let $A, B \in \mathscr{M}$. Then $\mathrm{Th}(A) = \mathrm{Th}(B)$ if and only if A, B have isomorphic ultrapowers.*

We shall not give the definitions of ultraproducts and ultrapowers of models of $\mathscr{M}$ here. We should mention, however, that they can be defined whenever X is a compact Hausdorff space and that they have all the usual properties of ultraproducts whenever $\mathscr{L}$ is a continuous logic. In the development of our theory, they play a crucial role in Theorems 2, 4, 6, and, of course, 9. Using Theorem 9 a number of results giving algebraic characterizations of elementary equivalence and elementary classes can be obtained. For 2-valued analogs of such results, see Keisler [61e].

While it is relatively easy to find examples of continuous logics, it is not so certain that many adequate logics can be found. This is because the point $1 \in X$ has to have certain special properties which would preclude X from being an arbitrary compact Hausdorff space. For simplicity let us say that the space X is *adequate* if $\langle X, \mathscr{C}, \mathscr{D} \rangle$ is adequate when $\mathscr{C}$ is the set of all continuous functions on X^n to X and $\mathscr{D}$ is the set of all continuous functions on X^* to X.

Theorem 10. *Suppose that $2 \subseteq X$ and X is a boolean space, that is, X is a compact Hausdorff space in which every member of $\mathscr{S}_0$ is closed. Then X is adequate.*

Theorem 10 includes the case of all finite spaces.

Theorem 11. *Suppose that $2 \subseteq X$, X is compact Hausdorff, and X contains a non-trivial arc (a continuous image of the closed real unit interval $[0, 1]$) terminating at 1. Then X is adequate.*

Theorem 11 includes the case of the logic l. The next theorem shows that new adequate spaces X' can be obtained from known adequate spaces X.

Theorem 12. *Suppose that X is adequate, $2 \subseteq X'$, X' is compact Hausdorff, and the following hold:*

(i) *For every pair of distinct points $x, y \in X'$, there exists a continuous mapping c of X' into X such that $c(x) \neq c(y)$.*

(ii) *For every open set V of X containing 1, there exists a continuous mapping c of X into X' such that $c(1) = 1$ and $\check{c}[1] \subseteq V$.*

Then X' is adequate.

The problem of determining all adequate spaces X remains open. In particular, an intriguing problem is the following: Does every compact

Hausdorff space X contain some point $z \in X$ such that condition (4) holds with 1 replaced by z? This problem does not seem to be easy.

We might add that the following is also true for an adequate logic.

Theorem 13. *Let μ be an ordinal and let (A, a), (B, b) be models of $\mathscr{M}(\mu)$. Then* $\mathrm{Th}((A, a)) = \mathrm{Th}((B, b))$ *if and only if* $\overline{\mathrm{Th}^+((A, a))} = \overline{\mathrm{Th}^+((B, b))}$.

It turns out that each of the Theorems 7, 8 and 9 is provable from Theorem 13. Thus, one might very well investigate all those logics $\mathscr{L}$ for which Theorem 13 holds.

5. Ordered model theory. A large body of results in 2-valued model theory is concerned with the problem of providing syntactic characterizations for classes $\mathscr{K}$ of models which are closed under certain algebraic operations. For instance, we may mention the results of Łoś, Robinson, and Tarski on universal and existential classes, of Łoś–Suszko and Chang on universal-existential classes, of Keisler on sandwiches and Horn classes, and of Lyndon on positive classes. (See for example the survey by Lyndon [59c].) We have found that in each of the above theorems the natural $\leqslant$ ordering relation on 2 plays a crucial role. Thus, to obtain faithful and fruitful generalizations of these results, we must consider logics $\mathscr{L} = \langle X, \mathscr{C}, \mathscr{Q} \rangle$ together with a binary relation H on X.

We say that the binary relation H *orders* X, or X *is ordered by* H, if the following hold:

(7) H is a reflexive and transitive relation on X.

(8) For every open set V of X, both $H[V]$ and $\breve{H}[V]$ are open in X.

(9) For every open set V containing 1, there exists an open U such that $1 \in H[U] \subseteq V$. For every open set V containing 0, there exists an open set U such that $0 \in \breve{H}[U] \subseteq V$.

Notice that conditions (7), (8), and (9) are symmetric with respect to H and $\breve{H}$ and with respect to the points 0 and 1. Furthermore, every X is ordered by the identity relation. The spaces 2 and $[0, 1]$ are both ordered by the natural $\leqslant$ relation.

Let $\mathscr{L} = \langle X, \mathscr{C}, \mathscr{Q} \rangle$ be an X-valued logic. We let

$\mathscr{C}_H = \{c \in \mathscr{C} : c$ is H-preserving, that is, $c(x_1, \ldots, x_n) H c(y_1, \ldots, y_n)$ whenever $x_1 H y_1, \ldots, x_n H y_n\}$,

$\mathscr{Q}_H = \{q \in \mathscr{Q} : q(Y) H q(Z)$ whenever $Z \subseteq H[Y]\}$,

$\mathscr{Q}_{\breve{H}} = \{q \in \mathscr{Q} : q(Y) H q(Z)$ whenever $Y \subseteq \breve{H}[Z]\}$,

and

$$\mathscr{Q}_H^* = \{q \in \mathscr{Q} : q(Y)Hq(Z) \text{ whenever both } Z \subseteq H[Y] \text{ and } Y \subseteq \breve{H}[Z]\}.$$

Suppose that $X = 2$, $\mathscr{C} = \{\vee, \wedge, \neg\}$, and $\mathscr{Q} = \{\exists, \forall\}$. Let H be the $\leqslant$ relation on 2. Then, we see that

$$\mathscr{C}_H = \{\vee, \wedge\},\ \mathscr{Q}_H = \{\forall\},\ \mathscr{Q}_{\breve{H}} = \{\exists\},\ \text{and}\ \mathscr{Q}_{H^*} = \{\exists, \forall\}.$$

This little example provides us with the necessary hint on how to extend some of the 2-valued syntactical notions to the general case.

Let Ψ be an arbitrary set of formulas and let $\mathscr{D} \subseteq \mathscr{C} \cup \mathscr{Q}$. $(\mathscr{D})\Psi$ shall denote the smallest set of formulas containing Ψ and is closed under all connectives and quantifiers in $\mathscr{D}$. Let Λ be the set of atomic formulas of Φ. A formula φ is said to be *existential* if

$$\varphi \in (\mathscr{Q}_{\breve{H}} \cup \mathscr{C}_H)(\mathscr{C})\Lambda.$$

A formula φ is said to be *universal* if

$$\varphi \in (\mathscr{Q}_H \cup \mathscr{C}_H)(\mathscr{C})\Lambda.$$

A formula φ is said to *universal-existential* if

$$\varphi \in (\mathscr{Q}_H \cup \mathscr{C}_H)(\mathscr{Q}_{\breve{H}} \cup \mathscr{C}_H)(\mathscr{C})\Lambda.$$

Finally, a formula φ is said to be *positive* if

$$\varphi \in (\mathscr{Q}_{H^*} \cup \mathscr{C}_H)\Lambda.$$

We say that a logic $\mathscr{L} = \langle X, \mathscr{C}, \mathscr{Q}\rangle$ is *adequately ordered by* H if conditions (1)–(4) hold, X is ordered by H, and the following conditions hold:

(4′) For any $x, y \in X$ if $\langle x, y\rangle \notin H$, then there exists $t \in \mathscr{C}_H$ such that $t(x) = 1$ and $t(y) \neq 1$.

(4″) For any $x, y \in X$ if $\langle x, y\rangle \notin H$, then there exists $t \in \mathscr{C}_H$ such that $t(x) \neq 0$ and $t(y) = 0$.

(5′) For every open set V containing 1, there exists $k \in \mathscr{C}_H$ such that $k(1, 1) = 1$ and $\breve{k}[1] \subseteq V \times V$.

(5″) For every open set V containing 0, there exists a $k \in \mathscr{C}_H$ such that $k(0, 0) = 0$ and $\breve{k}[0] \subseteq V \times V$.

(6′) For every set $Y \in X^*$ such that $1 \notin \overline{Y}$, there exists $q \in \mathscr{Q}_{\breve{H}}$ such that $q(Y) \neq 1$ and for every $Z \in X^*$, $q(Z) = 1$ whenever $1 \in \overline{Z}$.

(6″) For every set $Y \in X^*$ such that $0 \notin \overline{Y}$, there exists $q \in \mathscr{Q}_H$ such that $q(Y) \neq 0$ and for every $Z \in X^*$, $q(Z) = 0$ whenever $0 \in \overline{Z}$.

One can easily verify that, in case of the logic l, both (4′) and (4″) are satisfied by the identity function, (5′) is satisfied by $\mathbf{v}$, (5″) by $\mathbf{\wedge}$, (6′) by $\exists$, and (6″) by $\forall$. In case of the logic $\bar{l}$, one can first verify that the functions $\max(x, y)$ and $\min(x, y)$ belong to $\mathscr{C}\bar{l}$. Thus again (5′), (5″), (6′), and (6″) are satisfied. Without too much difficulty, both (4′) and (4″) can also be verified. Hence, both l and $\bar{l}$ are adequately ordered by their natural ordering relation $\leqslant$. Notice also that if $\mathscr{L}$ is adequately ordered by some relation H then it is certainly already adequate.

A minor result for adequately ordered logics $\mathscr{L}$ which will clarify forthcoming definitions is the following: If H adequately orders $\mathscr{L}$, then $H[Y]$ and $\breve{H}[Y]$ are closed for every closed set Y of X.

A class $\mathscr{K} \subseteq \mathscr{M}$ is said to be an *existential*, *universal*, *universal-existential*, or *positive class* if it is an arbitrary intersection of finite unions of classes of the form

$$\{A \in \mathscr{M} : \varphi[A] \in H[Y]\}$$

where Y is a closed subset of X and φ is an existential, universal, universal-existential, or positive sentence, respectively. If $\mathscr{L}$ is adequately ordered by H, then each of the four classes defined above is, *a posteriori*, an elementary class.

Suppose that $\mathscr{L}$ is adequately ordered by H.

Theorem 14. *Let $\mathscr{K} \in \mathbf{EC}_\Delta$. Then $\mathscr{K}$ is an existential class if and only if $A \in \mathscr{K}$ and $A \subseteq B$ imply $B \in \mathscr{K}$.*

Theorem 15. *Let $\mathscr{K} \in \mathbf{EC}_\Delta$. Then $\mathscr{K}$ is a universal class if and only if $A \in \mathscr{K}$ and $B \subseteq A$ imply $B \in \mathscr{K}$.*

Theorem 16. *Let $\mathscr{K} \in \mathbf{EC}_\Delta$. Then $\mathscr{K}$ is a universal-existential class if and only if $\mathscr{K}$ is closed under unions of chains of members of $\mathscr{K}$.*

There is no difficulty if we wish to extend these results to include the case of n alternations of quantifiers.

Two models $A = \langle R, S\rangle$, $B = \langle R', S'\rangle$ are *H-homomorphic* if there exists a function f mapping R onto R' such that for any $x, y, z \in R$,

$$\langle S(x, y, z),\ S'(f(x),\ f(y),\ f(z))\rangle \in H.$$

Assume now $\langle 0, 1\rangle \in H$ and $\mathscr{L}$ is adequately ordered by H.

Theorem 17. *Let $\mathscr{K} \in \mathbf{EC}_\Delta$. Then $\mathscr{K}$ is a positive class if and only if $A \in \mathscr{K}$ and A is H-homomorphic to B imply $B \in \mathscr{K}$.*

This last theorem may be proved under somewhat less stringent conditions than stated. For instance, $\mathscr{L}$ need not satisfy condition (4) and, in conditions (6′) and (6″), $\mathscr{Q}_H$ and $\mathscr{Q}_{\bar{H}}$ may be replaced by $\mathscr{Q}_{H^*}$.

Combining some of our earlier results with Theorems 14, 15, 16, and 17, we can obtain purely algebraic characterizations of these classes, without the assumption that $\mathscr{K} \in \mathbf{EC}_\Delta$.

A theorem analogous to the above four theorems and which does not appear to carry over to the general case is the previously mentioned result of Keisler on Horn classes and reduced products. This problem is quite interesting because there is a proper generalization of the theorem in the logic l (see Chang [61]).

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INDEPENDENCE RESULTS IN SET THEORY

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The purpose of this paper is to give a brief presentation of the proof that the axiom of choice (AC) and the continuum hypothesis (CH) are independent of the axioms of Zermelo–Fraenkel (Z–F) set theory. Complete proofs will not be presented, but all essential points will be treated and only those details which can be handled by standard methods will be omitted. By Z–F set theory we mean any of the well-known axiomatizations, e.g., Fraenkel–Bar–Hillel [58], which speak of one undefined relation $\in$, without any axiom of choice but with the axiom of regularity and, of course, an axiom of replacement. The latter is an infinite collection of axioms generated by a simply stated rule. The exact statement of the axioms of Z–F is not critical since the method here presented will work with little change for any "reasonable" set of axioms, though of course the notion of "reasonable" must be left rather vague. It would certainly seem to cover any of the axioms of infinity suggested by Azriel Lévy or strengthened versions of the axiom of replacement.

Historically, CH was proposed by Cantor as the simplest answer to the problem of determining the cardinality of the continuum among the various $\aleph_\tau$. Although it has often been used in mathematical arguments, few mathematicians have been willing to accept it as a self-evident axiom but have rather regarded it as a conjecture presumably capable of proof or disproof. In this sense the independence of CH, although not a surprising result (Gödel [47]), leaves the matter in a somewhat unsatisfactory state of affairs from which there does not seem any natural escape. In contrast, AC has been accepted by most mathematicians as an axiom not requiring further proof and thus the independence of AC is primarily of technical interest. AC was first explicitly stated by Zermelo in the course of attempting to prove the well-ordering theorem of Cantor. For some time after their introduction both principles were regarded as ordinary mathematical statements and attempts were made to prove them, or even to disprove them (Sierpiński [47]). This

"naive" attitude came to an end as these attempts proved fruitless and with the axiomatization of set theory the way was opened to discuss their logical relation to the other axioms. As is well known, the greatest breakthrough was made by Gödel in 1938 (see [39], [58]), when he showed that if Z–F is consistent, as is commonly believed, Z–F together with AC and CH and even the Generalized Continuum Hypothesis (GCH) is consistent. In his proof Gödel introduced the very important concept of constructibility, which will play a large role in what follows. In another direction much work was done with the system of Fraenkel–Mostowski and similar systems (Mostowski [39]), and it was proved that AC and GCH were independent with respect to these systems. These systems suffer from an obvious defect and the results obtained did not apply to CH or to the well-ordering of the continuum. However, in the present work the "genuine" sets a_δ to be introduced do play a role somewhat similar to atoms in the Fraenkel–Mostowski system.

The most natural way to proceed in giving a consistency proof is to show explicitly a model with the desired properties. In set theory this leads to a slight complication because the natural model for Z–F is the universe, which is not a set. No specific models were discussed in Gödel [58] but a concept of constructible sets was defined and for each statement Q, the relativized statement Q_L, in which the word set is replaced by constructible set, was associated. It was then shown that the relativized versions of the axioms were true in Z–F as well as the relativized version of GCH and AC. In reality one is speaking of the "model" of constructible sets without explicitly defining it as a set. Let us denote by (Z–F)′ the axioms of Z–F together with the axiom that there is a set $\mathfrak{M}$ such that the axioms of Z–F hold when the word set is interpreted to mean an element of $\mathfrak{M}$. This axiom cannot be deduced from Z–F (Shepherdson [51]). Adjoining such an axiom allows us to state a stronger consistency theorem, namely, that models exist. For the moment we shall work within such a system and later discuss the consistency statements themselves. The word model will henceforth refer only to a set whose elements form a model under the $\in$-relation, i.e., standard models.

Theorem 1. *From the axioms of* (Z–F)′ *it follows that there exist models for* Z–F *in which the following hold*:

(i) GCH *and* AC *hold but there exists a subset* $a \subseteq \omega$ *such that* a *is not constructible.*

(ii) AC *holds and* $2^{\aleph_0} = \aleph_2$.

(iii) *The continuum has no well-ordering.*

In (ii) every set is constructible from just one set so that AC holds in a very strong form. It is well known that the axiom of constructibility implies GCH, and GCH implies AC so that Theorem 1 settles the question of the relative strength of these axioms. Many other examples can be given to prove other consistency theorems. The following remarkable lemma illustrates a very important point which remained unnoticed.

Lemma 1. *From* (Z–F)', *or from any system with which the axiom of constructibility is consistent, one cannot prove the existence of an uncountable model in which* AC *holds and* $2^{\aleph_0} \neq \aleph_1$.

(A word of caution here: by the Skolem–Löwenheim theorem there are uncountable non-standard models; the lemma refers only to standard models.) We may assume that the model is transitive, i.e., $x \in \mathfrak{M}$, $y \in x$ imply $y \in \mathfrak{M}$, since it is easily shown that every model is isomorphic to a transitive model. If $\mathfrak{M}$ is such an uncountable model then $\{\text{rank } x | x \in \mathfrak{M}\}$ either contains all countable ordinals or else has a countable upper bound α. In the second case for some $\beta \in \mathfrak{M}$, the set of x such that rank $x = \beta$, which is a set in $\mathfrak{M}$, is uncountable, and so by AC $\mathfrak{M}$ contains an uncountable ordinal. In either case $\mathfrak{M}$ contains all the countable ordinals. Since the axiom of constructibility is consistent, and only countable ordinals are needed to construct real numbers, it is consistent to assume that all real numbers are constructible in $\mathfrak{M}$, which would imply CH.

From the Skolem–Löwenheim theorem it follows that countable models for Z–F do exist and we let $\mathfrak{M}$ be such a model which is transitive. In constructing new models, the basic requirement which must be observed is to insure that the model be "large" enough to be closed under the construction of the sets demanded by the axioms. AC and CH are not axioms which speak about the size of the model and so we shall adopt the principle that the new models we construct will not contain any ordinals not already in $\mathfrak{M}$. This principle is of great help in deciding on the procedure to be used.

We first consider (i) of Theorem 1. Assume there is a model $\mathfrak{N}$ with such a set a. It is natural to attempt to construct $\mathfrak{N}$ as a "minimal" model containing a, (Cohen [63a], Shepherdson [51], [52], [53]). We review the notion of construction introduced in Gödel [39]. There the

sets $\mathfrak{M}_\alpha$ are defined by $(\bigcup_{\beta<\alpha} \mathfrak{M}_\beta)'$, where $(\mathfrak{S})'$ denotes all subsets of $\mathfrak{S}$ which are definable over $\mathfrak{S}$, i.e., which can be described by a condition $P(x)$ in which all variables and constants take values in $\mathfrak{S}$. A set is called constructible if it belongs to some $\mathfrak{M}_\alpha$. In Gödel [58], a somewhat different definition is given, whereby only one set, denoted by F_α, is constructed at each ordinal α. Functions $N(\alpha), K_1(\alpha), K_2(\alpha), J(i, \alpha, \beta)$ from ordinals to ordinals are introduced such that $0 \leqslant N(\alpha) \leqslant 8$, $K_i(\alpha) \leqslant \alpha$, $K_i(\alpha) < \alpha$ if $N(\alpha) > 0$ and if $J(i, \alpha, \beta) = \gamma$, then $N(\gamma) = i$, $K_1(\gamma) = \alpha$, $K_2(\gamma) = \beta$. The sets F_α are defined by putting

$$F_\alpha = \{F_\beta | \beta < \alpha\} \qquad \text{if } N(\alpha) = 0,$$

$$F_\alpha = \mathfrak{F}_i(F_\beta, F_\gamma) \qquad \text{if } N(\alpha) = i > 0 \text{ and } K_1(\alpha) = \beta,\ K_2(\alpha) = \gamma$$

The operations $\mathfrak{F}_i$ are the fundamental operations from which all conditions can be constructed. For any subset a of ω, define $F_\alpha(a) = F_\alpha$ as above except that $F_\alpha = \alpha$ if $\alpha \leqslant \omega$, $F_{\omega+1} = a$. Put $\mathfrak{N} = \{F_\alpha | \alpha \in \mathfrak{M}\}$. The problem is to find a set a such that $\mathfrak{N}$ satisfies (i) of Theorem 1. To insure that $\mathfrak{N}$ is a model for Z–F a must satisfy certain conditions. For example, a can be regarded as a subset of $\omega \times \omega$ in a natural manner and so may give rise to a well-ordering on ω. For $\mathfrak{N}$ to be a model it is necessary that this countable ordinal be in $\mathfrak{N}$. Rather than determine a directly we shall examine successively every one of the countably many statements in $\mathfrak{N}$ and determine which are to be true. The set a will be determined by the statements concerning $F_{\omega+1}$ which are true in $\mathfrak{N}$. It is not clear what rule we shall use to do this but it is clear that some decisions about a are simpler to observe than others. For example, it is simplest to insure that a does not correspond to an ordering of ω at all in the above example. On the other hand some properties must be true about a simply because they are true for all subsets ω. Intuitively, the answer lies in choosing a to behave as a "generic" subset of ω. The most basic properties of a, namely $n \in a$ for a particular n, cannot be decided on any such basis and must be treated as the basic decisions in terms of which all other properties will be "forced" to hold or not.

Definition 1. *A* limited statement *is a statement, built up from the logical symbols and the individual constants* F_α *standing for the* F_α, *in which each quantifier* $\forall_\alpha$, $\exists_\alpha$ *has an ordinal subscript. In the statements* $\forall_\alpha x Q(x)$, $\exists_\alpha x Q(x)$, x *shall be thought of as ranging over all* F_β *with* $\beta < \alpha$.

An unlimited statement *is a statement of the usual type.*

We shall consider only statements in prenex form, i.e., with all quantifiers preceding a quantifier-free matrix. We shall define a notion of "forcing" for limited and unlimited statements which is to some degree parallel to the usual notion of truth and serves here as a substitute for it. We are in the end concerned only with unlimited statements, but the inductive argument leads us to consider also limited ones. The ordinary notion of truth is defined by induction on the number of quantifiers in the obvious way, until one encounters the statements $\mathsf{F}_\alpha \in \mathsf{F}_\beta$. Since each F_α is constructed in terms of the previous F_β, such elementary statements are equivalent to certain limited statements. This method will now be repeated in the present context treating a as a "generic" subset of ω. The symbol P shall always denote a finite collection of conditions $\{\mathsf{n}_k \in \mathsf{a}, \sim \mathsf{n}_l \in \mathsf{a}\}$ where no n_k equals any n_l. The rank of a limited statement is (α, r) if r is the number of quantifiers, and α is the least ordinal such that $\alpha \geqslant \beta$ if $\exists_\beta$ or $\forall_\beta$ occurs and $\alpha > \beta$ if F_β occurs. Put $(\alpha, r) > (\beta, s)$ if $\alpha > \beta$ or $\alpha = \beta$ and $r > s$.

Definition 2. *Define* P forces *a limited statement* Q *by induction on the rank as follows*:

P *forces* $\exists_\alpha \mathsf{x} \mathrm{R}(\mathsf{x})$ *if for some* $\beta < \alpha$, P *forces* $\mathrm{R}(\mathsf{F}_\beta)$.

P *forces* $\forall_\alpha \mathsf{x} \mathrm{R}(\mathsf{x})$ *if for all* $\mathrm{P}' \supset \mathrm{P}$, P' *does not force* $\sim \mathrm{R}(\mathsf{F}_\beta)$ *for any* $\beta < \alpha$.

If Q *has no quantifiers* P *forces* Q *if it forces each component* $\mathsf{F}_\alpha \in \mathsf{F}_\beta$ *or its negation in such a way as to make* Q *formally true.*

P *forces* $\mathsf{F}_\alpha \in \mathsf{F}_\beta$ *or* $\sim \mathsf{F}_\alpha \in \mathsf{F}_\beta$ *if it forces the corresponding statement of lower rank which is equivalent to it.*

P *forces* $\mathsf{F}_\alpha \in \mathsf{F}_\beta$ *or* $\sim \mathsf{F}_\alpha \in \mathsf{F}_\beta$ *if* $\alpha, \beta \leqslant \omega + 1$ *if the statement is a formal consequence of* P.

Definition 3. P forces *the unlimited statement* $\exists \mathsf{x} \mathrm{Q}(\mathsf{x})$ *if for some* α, P *forces* $\mathrm{Q}(\mathsf{F}_\alpha)$. P *forces* $\forall \mathsf{x} \mathrm{Q}(\mathsf{x})$ *if for all* $\mathrm{P}' \supset \mathrm{P}$ *and* α, P' *does not force* $\sim \mathrm{Q}(\mathsf{F}_\alpha)$. *If* Q *is a statement with no quantifiers then it is a limited statement and Definition* 2 *applies.*

The following lemmas are all easily proved by induction.

Lemma 2. P *never forces* Q *and* $\sim$ Q.

Lemma 3. *If* P *forces* Q *and* $\mathrm{P}' \supset \mathrm{P}$, *then* P' *forces* Q.

Lemma 4. *For all* P *and* Q *there exists* $\mathrm{P}' \supset \mathrm{P}$ *such that either* P' *forces* Q *or* P' *forces* $\sim$ Q.

The next lemma will not be used directly but it illustrates in what sense forcing may be regarded as a notion of truth.

Lemma 5. *If* P *forces* $Q_1, \ldots, Q_n$ *and* R *is a logical consequence of* Q_i *then for some* $P' \supset P$, P' *forces* R. *In particular,* P *does not force* $\sim$ R.

The proof of Lemma 5 is complicated by the fact that only statements in prenex form were considered, so that if P forces Q_1 and Q_2 it need not force Q_1 & Q_2. However by using Lemma 4 we can extend P to P' so that all statements which naturally arise in an analysis of Q_i and R are also forced to be either true or false and then the proof becomes quite simple. We shall use Lemma 5 in the discussion of the arithmetization of the proof.

To define the set a, enumerate all the limited and unlimited statements Q_n, and define P_n as the first extension of P_{n-1} under any natural well-ordering, which forces either Q_n or $\sim Q_n$. Since all statements of the form $\mathsf{n} \in \mathsf{a}$ will eventually be discussed it is clear that the P_n approach in an obvious sense a set $a \subseteq \omega$. Since the sequence Q_n is not definable in $\mathfrak{M}$, the set a need not lie in $\mathfrak{M}$ and by Cohen [63a], Shepherdson [51], [52], [53], definitely cannot do so. With this choice of a, define $\mathfrak{N}$ as the set $\{F_\alpha(a)\}$.

Lemma 6. *In* $\mathfrak{N}$, *a statement* Q *is true if and only if it is forced by some* P_n.

This is again proved by induction on the rank of the statement.

We now let Q_α denote an enumeration of all limited statements where α ranges over the ordinals of $\mathfrak{M}$. It is assumed that this enumeration is done in the natural way, akin to the Gödel numbering, so that the processes of substituting F_α for variables, conjunction of statements, etc., are all expressible in $\mathfrak{M}$ as operations on the subscript α in Q_α. Furthermore we can assume that the ordering of Q_α corresponds to the definition of forcing by induction, so that if $\alpha < \beta$, rank $Q_\alpha \leqslant$ rank Q_β. If this is done it is clear that the following holds:

Lemma 7. *There is a statement about* x *and* y *expressible in* $\mathfrak{M}$ *which says that* $x = P$, $y = \alpha$ *and* P *forces* Q_α.

It is well known that "truth" cannot be defined in a model and analogously there is no obvious way to define forcing within $\mathfrak{M}$ for an enumeration Q_α' of the unlimited statements. Indeed it would not seem too difficult to give a rigorous proof of this fact, although we shall not

need it. For our purposes we need not consider more than one unlimited statement at a time.

Lemma 8. *If* $Q(x_1, \ldots, x_n)$ *is an unlimited formula with* n free *variables, the relation on* P, α_i *which says that* P *forces* $Q(F_{\alpha_1}, \ldots, F_{\alpha_n})$ *is expressible in* $\mathfrak{M}$.

This is proved by induction on the number of quantifiers in Q. The main task before us is to show that $\mathfrak{N}$ is a model for Z–F. Much of the proof follows quite closely the proof in Gödel [58] that the constructible sets form a model for Z–F. The only axioms which need verification are the axiom of replacement and the axiom of the power set. First consider the replacement axiom. Let $Q(x, y)$ be a condition which defines y as a univalent function of x in $\mathfrak{N}$, say $y = \psi(x)$. We wish to show that the range of ψ on a set $Z = F_\alpha$ is also a set.

Lemma 9. *The range of* ψ *on* F_α *is contained in some set of* $\mathfrak{N}$.

Proof. By Lemma 8 the relation on β, α', and P which says β is the least ordinal such that P forces $Q(F_{\alpha'}, F_\beta)$, is definable in $\mathfrak{M}$. From this it follows from the use of the replacement axiom in $\mathfrak{M}$ that the set of all such β corresponding to some $\alpha' < \alpha$ and some P is a set in $\mathfrak{M}$ and hence bounded by some γ. This clearly implies the lemma.

Lemma 10. *Let* $Q(x, y)$ *be an unlimited statement with* n *bound variables* $t_1, \ldots, t_n$ *and let* α_1, α_2 *be two ordinals. There exist ordinals* $\beta_1, \ldots, \beta_n$ *such that if the quantifiers binding the* t_i *are modified by attaching the subscripts* β_i *the resulting limited statement* $Q'(x, y)$ *has precisely the same pairs* $\langle x, y \rangle$ *with* $x \in F_{\alpha_1}$ *and* $y \in F_{\alpha_2}$ *satisfying it as does* $Q(x, y)$.

This is proved by determining successively the ordinals β_i using Lemma 9. Using Lemma 10 it now follows that the range of ψ on a set F_α is actually some F_β, since sets defined by limited conditions can also be constructed using the fundamental operations $\mathfrak{F}_i$. Thus:

Lemma 11. *The axiom of replacement holds in* $\mathfrak{N}$.

The reader will of course notice the formal similarity between our proof and that in Gödel [58]. In Gödel [58] two proofs of the power set axiom are given. The first is as follows: Consider the power set of F_α, $\mathscr{P}(F_\alpha)$. For each $x \in \mathscr{P}(F_\alpha)$ let β be the least ordinal such that β constructs x if x is constructible; otherwise let $\beta = 0$. By taking the

supremum of all such β we can obtain an ordinal γ such that $N(\gamma)=0$ and F_γ contains all the constructible subsets of F_α. This simple argument cannot be carried over directly to the present situation since our new model is not a submodel of the old model. The second proof in Gödel [58] yields much more information. It proves that every constructible subset of F_α is an F_β with $\bar{\bar{\beta}} \leqslant \bar{\bar{\alpha}}$. This fact in turn is used to prove GCH. Since we are also concerned with the latter question it is this proof which is most natural to generalize.

Lemma 12 ("Collapsing lemma"). *Let S be a collection of ordinals such that $\alpha \in S$ if $\alpha \leqslant \omega+1$. Assume S is closed under the functions $J(i, \alpha, \beta)$, K_1, K_2, and is such that if $\alpha \in S$ and $F_\alpha \neq \emptyset$ then for some $\beta \in S$, $\beta < \alpha$ we have $F_\beta \in F_\alpha$. Then there is a map $\alpha \to \alpha'$ which preserves the functions J, K_1, K_2, N and such that $F_\alpha \in F_\beta$ if and only if $F_{\alpha'} \in F_{\beta'}$, and such that $\bar{\bar{\alpha}}' \leqslant \bar{\bar{S}}$ for all α. Furthermore every member of $F_{\alpha'}$ is equal to some $F_{\beta'}$ with $\beta' < \alpha'$.*

This is proved by defining α' by induction on α, proving inductively that $\bar{\bar{\alpha}}'$ is not greater than the cardinality of all β in S less than α. The requirement that the set F_α, $\alpha \in S$, be existential plays an important role in proving the isomorphism $F_\alpha \to F_{\alpha'}$ especially when $N(\alpha)=4$. In working with cardinals in $\mathfrak{M}$ we are assuming AC and GCH in $\mathfrak{M}$, which by Gödel's work is clearly no restriction.

If α, β are infinite ordinals in $\mathfrak{M}$, we shall show how to construct a set S as in Lemma 12, containing β and all $\gamma \leqslant \alpha$ and whose cardinality is $\bar{\bar{\alpha}}$. Introduce the operator $C(\mathrm{P}, \gamma)$ to be the smallest ordinal δ such that P forces $\mathsf{F}_\delta \in \mathsf{F}_\gamma$ if such exists, and otherwise to be zero. Then a set S containing all $\gamma \leqslant \alpha$ and β exists, closed under J, K_1, K_2 and $C(\mathrm{P}, \gamma)$ for all P and $\gamma \in S$, and $\bar{\bar{S}} = \bar{\bar{\alpha}}$. Since every true statement in $\mathfrak{N}$ is forced by some P, the set S satisfies the existential condition of Lemma 12. Furthermore, it is certainly possible to have $\gamma' = \gamma$ for $\gamma \leqslant \alpha$. If $F_\beta \subseteq F_\alpha$, it then follows that $F_\beta = F_{\beta'}$.

Lemma 13. *Any subset of F_α in $\mathfrak{N}$ is equal to F_β for some β with $\bar{\bar{\beta}} \leqslant \bar{\bar{\alpha}}$. Consequently, the axiom of the power set holds in $\mathfrak{N}$.*

For the sake of completeness we now give the analogue of the first proof in Gödel [58] of the power set axiom. This proof was pointed out to me by Robert Solovay. Let α be a fixed ordinal. Let S denote any collection of pairs (P, β) where $\beta < \alpha$. For each γ associate the set $f(\gamma) = S$

of all pairs (P, β), $\beta<\alpha$ such that P forces $\mathsf{F}_\beta \in \mathsf{F}_\gamma$. For each pair (P, S) associate the ordinal $g(P, S)=\gamma$ such that γ is the least ordinal such that $f(\gamma)=S$ and such that P forces $\mathsf{F}_\gamma \subseteq \mathsf{F}_\alpha$; if no such ordinal exists put $g(P, S)=0$. If in the model $\mathfrak{N}$, $F_\beta \subseteq F_\alpha$, let $S=f(\beta)$ and let P be one of the P_n such that P forces $\mathsf{F}_\beta \subseteq \mathsf{F}_\alpha$. If $\beta_1=g(P, S)$, then in $\mathfrak{N}$, $F_{\beta_1}=F_\beta$. Since the range of g, by the axiom of replacement in $\mathfrak{M}$, is bounded by an ordinal in $\mathfrak{M}$, the power set axiom is seen to hold.

Having verified that $\mathfrak{N}$ is a model for Z–F, we shall next show that the set a is not constructible. Recall that the only property used about the sequence P_n is that every statement is forced either true or false by some P_n. Assume some P forced a to be constructible. Starting with $P_0=P$ we could construct a sequence P_n which forces each statement to be true or false, and such that the subsequence P_{2n} is such that the nth ordinal α_n in $\mathfrak{M}$, under some enumeration, constructs a set a_n which specifically does not satisfy all the conditions of P_{2n}. This is possible since knowing α_n completely determines the set a_n, and P_{2n} can always be chosen to include the condition $\mathsf{k} \in \mathsf{a}$ for some $k \notin a_n$, or vice versa. In the model $\mathfrak{N}$ thus obtained a would not be constructible, which is a contradiction. Of course we could have taken this sequence P_n as the original sequence, but it is of some interest to show, as we have done, that all P force a not to be constructible. Finally, since in $\mathfrak{N}$ every set is constructible from a, it follows by the arguments in Gödel [58] that AC and GCH hold. The proof of (i) is now complete.

We next turn to (ii) of Theorem 1. We form a model $\mathfrak{N}$ as before except that rather than introducing one generic set a, for each $\delta \leqslant \aleph_\tau$, we set $F_{\omega+\delta}=a_\delta$, where a_δ is to be thought of as a subset of ω. Then enumerate the unordered pairs $\{a_\delta, a_{\delta'}\}$, the ordered pairs $\langle a_\delta, a_{\delta'}\rangle$ and lastly the set $T=\{\langle a_\delta, a_{\delta'}\rangle | \delta<\delta'\}$. The object is to enlarge the continuum so that in $\mathfrak{N}$ it will have cardinality at least $\aleph_\tau$. The set T must also be introduced to insure that AC will hold. From this point on, the construction of the F_α is as before, with the sole difference that a case $N(\alpha)=9$ is introduced, and F_α for such α is defined as $\{F_\beta | \beta<\alpha \ \& \ N(\beta)=9\}$. The point of this new construction is that the ordinals of $\mathfrak{M}$ will now occur in canonical places independently of what forcing sequence P_n is to be chosen. The conditions P now consist of finitely many conditions of the form $\mathsf{n} \in \mathsf{a}_\delta$ or $\sim \mathsf{n} \in \mathsf{a}_\delta$. The definition of forcing is given much as before, taking into account the new complications introduced. The proofs of the basic lemmas about forcing carry through with very little change. Similarly the definition of the sequence P_n and the model $\mathfrak{N}$

can be given as before. The proof that $\mathfrak{N}$ is a model for Z–F entails no new difficulties.

Lemma 14. *In $\mathfrak{N}$, all sets are constructible from T and hence* AC *holds.*

This is true because from the set T one can recover the sets a_δ and the well-ordering of the a_δ. Since the construction of the F_α can be described once the a_δ are enumerated, the lemma easily follows.

An important new point which arises in the present situation is given by the following:

Lemma 15. *If α and β are infinite ordinals in $\mathfrak{M}$ such that $\bar{\bar{\alpha}} > \bar{\bar{\beta}}$, then in $\mathfrak{N}$ also $\bar{\bar{\alpha}} > \bar{\bar{\beta}}$.*

Here we shall use the fact that the ordinals in $\mathfrak{M}$ are canonically placed in $\mathfrak{N}$. When we speak of an ordinal in this proof we shall only mean those F_α such that $N(\alpha) = 9$. Let $\bar{\bar{\alpha}} > \bar{\bar{\beta}}$ in $\mathfrak{M}$. Assume that some element F_γ in $\mathfrak{N}$ sets up a univalent mapping ψ from β onto α. For fixed $\beta' < \beta$, let $R(\beta')$ be the set of all γ such that for some P, P forces ψ to be univalent and ordinal-valued and $\psi(\beta') = \gamma$. If for each such γ we pick just one associated P, we obtain a set $S(\beta')$ of conditions P such that if P_1, P_2 belong to $S(\beta')$, P_1 and P_2 are incompatible in the sense that $P_1 \cup P_2$ is not an admissible set of conditions. We shall prove below that such a set of conditions is at most countable (in $\mathfrak{M}$). Thus the range of ψ is contained in the union of $R(\beta')$ for $\beta' < \beta$ and since $R(\beta')$ is countable, this range is of cardinality $\leqslant \bar{\bar{\beta}}$ in $\mathfrak{M}$ and thus cannot be all of α which proves the lemma.

Now, let S be an uncountable collection of incompatible P. Since for some k there must be uncountably many P in S with k conditions, we may assume that all the P in S have k conditions. By induction on i we shall show that for all i there are i conditions such that uncountably many P in S contain these conditions. If this is true for some $i \geqslant 0$ we may assume that for all P in S, P contains some fixed i conditions. If P_0 is in S, then every other P in S must contradict P_0 in one of finitely many ways so that uncountably many must have some condition in common which is different from any in P_0 so that they will have $i+1$ fixed conditions in common. Since the P in S have only k conditions, for $i > k$ we obtain a contradiction.

Clearly for $\delta \neq \delta'$, $a_\delta \neq a_{\delta'}$, since they can never be forced equal.

The cardinal $\aleph_\tau$ is the same in $\mathfrak{N}$ as in $\mathfrak{M}$ so that the continuum in $\mathfrak{N}$ has cardinality at least $\aleph_\tau$. To find the cardinality of the continuum more precisely we need to use the argument of the "collapsing lemma". It is not difficult to construct a countable set of ordinals S, such that S contains α, all $\beta \leqslant \omega$, is closed under K_1, K_2, and J, contains β such that $F_\beta = T$ and if $S' = \{F_\beta | \beta \in S\}$, if $\langle a_\delta, a_{\delta'}\rangle$ or $\{a_\delta, a_{\delta'}\}$ belongs to S' so do a_δ and $a_{\delta'}$. We do not give these properties in the precise form in which we actually need them, but the result is that we can find an order-preserving map g of S onto an ordinal γ and a map G of F_β onto sets F'_β, such that each F'_β is a collection of F'_{β_1} with $\beta_1 < \beta$, and if $F_{\beta_1} = \mathfrak{F}_i(F_{\beta_2}, F_{\beta_3})$ then $F_{g(\beta_1)} = \mathfrak{F}_i(F_{g(\beta_2)}, F_{g(\beta_3)})$ and similarly if $N(\beta_1) = 0$, or 9. The sets a_δ and $\{a_\delta, a_{\delta'}\}$ or $\langle a_\delta, a_{\delta'}\rangle$ which occur in S' are mapped onto themselves by G and the set T is mapped onto $\{\langle a_\delta, a_{\delta'}\rangle | a_\delta, a_{\delta'} \in S'\}$. We then have $F_{g(\alpha)} = F_\alpha$ by virtue of the argument of the "collapsing lemma". The number of possibilities for $F_{g(\alpha)}$ depends upon the ordinal γ, the set of a_δ which occur among the F'_β, and the functions N, K_1, K_2, J which are transferred to the ordinal γ by means of the map g. The number of countable subsets of $\aleph_\tau$ is $\aleph_\tau$ if this cardinal is not co-final with ω, $\aleph_{\tau+1}$ if it is. The number of other possibilities does not exceed $2^{\aleph_0} = \aleph_1$ in $\mathfrak{M}$, so that the continuum in $\mathfrak{N}$ has cardinality not exceeding $\aleph_\tau$ or $\aleph_{\tau+1}$ according to the two cases [1]. The well-known König's lemma says that the continuum is not co-final with ω, so that since in $\mathfrak{N}$ the continuum has cardinality at least $\aleph_\tau$ we have:

Lemma 16. *In $\mathfrak{N}$, the continuum has cardinality $\aleph_\tau$ or $\aleph_{\tau+1}$ according as $\aleph_\tau$ is not or is co-final with ω.*

In this connection should be mentioned the work of Solovay who showed that the set of statements $2^{\aleph_n} = \aleph_{g(n)}$ is consistent if $g(n)$ is an increasing integer-valued function. He also obtained more general results and recently William Easton has announced a theorem which would seem to be the most general in this direction, allowing essentially any set of equalities which are not prohibited by a suitable generalization of König's lemma.

We now turn to the proof of (iii). The axiom of choice has many implications and equivalent forms and so there are many consistency questions which are of interest. Our first model will be a simple example

[1] In this argument the set S and the various functions used are definable in $\mathfrak{M}$ and thus we are justified in using $2^{\aleph_0} = \aleph_1$.

in which the axiom of choice for sets of real numbers is violated. To this end we introduce generic sets a_{ij} in the enumeration of F_α following $F_\alpha = \alpha$ for $\alpha \leqslant \omega$. We follow with the introduction of sets $U_i = \{a_{i1}, a_{i2}, \ldots\}$ and finally with a set $V = \{U_1, U_2, \ldots\}$. The later F_α are defined as before according to the case $N(\alpha) = i$, $0 \leqslant i \leqslant 8$, with one important difference. The functions K_1, K_2, J and N are so chosen that if $N(\alpha) = 0$ the set of all previous F_β are symmetric with respect to permutations of the a_{ij} which preserve each U_i. More precisely, let Π denote the group of all permutations of the a_{ij} which move only finitely many a_{ij} and preserve the sets U. The construction of the F_α is to be defined in such a way that Π can be made to act on the F_α satisfying the following conditions:

(a) Π acts on the initial segment of F_α in the obvious manner.

(b) if $F_\alpha = \mathfrak{F}_i(F_\beta, F_\gamma)$ then for $\pi \in \Pi$, $\pi F_\alpha = \mathfrak{F}_i(\pi F_\beta, \pi F_\gamma)$.

(c) if $N(\alpha) = 0$, then the set $\{F_\beta | \beta < \alpha\}$ is invariant under Π and $\pi F_\alpha = F_\alpha$ for all $\pi \in \Pi$.

The construction of the model $\mathfrak{N}$ proceeds as in the previous cases. In considering limited statements it becomes necessary to demand that if $\forall_\alpha$ or $\exists_\alpha$ occur in a statement, $N(\alpha) = 0$. The group Π acts on statements in an obvious way. Let Π_m denote the subgroup of Π consisting of those π which keep U_i for $i \leqslant m$ elementwise fixed.

Lemma 17. *For each α, there exists an m such that Π_m leaves F_α fixed.*

This is easily proved by induction. The group Π also acts on the conditions P and it can again be proved by induction that if P forces Q, πP forces πQ. We shall now prove that there is no choice function for the sets U_i. Assume some P forces F_α to be such a choice function. Let Π_m keep F_α fixed. Some extension P′ of P must force $a_{m+1,j}$ for some j to be chosen by F_α from U_{m+1}. For some π in Π_m, we will have πP′ to be compatible with P′ and $\pi a_{m+1,j} = a_{m+1,k}$, $j \neq k$. Then πP′ forces F_α to choose $a_{m+1,k}$ from U_{m+1} which, since $a_{m+1,j} \neq a_{m+1,k}$, clearly contradicts the fact that P forces F_α to be a choice function.

To give another example of this type, we construct a model in which a countable union of countable sets is uncountable. Introduce generic sets a_{ij}, sets $U_i = \{\langle a_{i1}, 1\rangle, \langle a_{i2}, 2\rangle, \ldots\}$ and $V = \{\langle U_1, 1\rangle, \langle U_2, 2\rangle, \ldots\}$. Define Π and Π_m as before. To define the F_α we enumerate all $\alpha \leqslant \omega$,

a_{ij}, U_i and V and proceed as before. In this case we observe that although Π leaves invariant the sets $\{a_{i1}, a_{i2}, \ldots\}$ it does not leave the set U_i invariant. Thus before allowing $N(\alpha)=0$ we must introduce all the sets πU_i and πV. Let P force F_α to map the integers one-to-one onto the set $\{a_{ij}\}$, let Π_m keep F_α fixed. Let P′ be an extension of P such that it forces F_α to map n onto $a_{m+1,1}$ for some integer n. Then for some $\pi \in \Pi_m$, P′ and πP′ are compatible and $\pi a_{m+1,1} = a_{m+1,j}$ for $j \neq 1$. This means that P′ $\cup$ πP′ would force n to be mapped onto both $a_{m+1,1}$ and $a_{m+1,j}$, which is a contradiction. In this model there is no univalent map of ω onto $\{a_{ij}\}$.

Let τ denote a fixed infinite ordinal in $\mathfrak{M}$. We show how to construct a model $\mathfrak{N}$, having no ordinals not in $\mathfrak{M}$, such that in $\mathfrak{N}$, τ is countable. This is a somewhat different type of construction since Lemma 15 will no longer hold in this construction. Construct the F_α by enumerating all the ordinals $\leqslant\tau$, the unordered pairs and finally ordered pairs of such ordinals. Then introduce a new generic set A which will consist of pairs $\langle n, \alpha\rangle$ for $\alpha<\tau$ and define as before the construction of the F_α from this point on. The forcing conditions P now consist of finitely many pairs $\langle n, \alpha_n\rangle$, with $\alpha_n<\tau$ with the proviso that each integer n is paired with at most one α_n and no $\alpha_m=\alpha_n$ for $m \neq n$. (We observe that since the conditions $\langle 1, \alpha\rangle$ for all $\alpha<\tau$ are incompatible the proof of Lemma 15 breaks down.) Forcing and the model $\mathfrak{N}$ are defined as before. The set A is forced to define a one-to-one function from all of ω onto τ since one can never force a particular $\alpha<\tau$ to be excluded from the range, nor to exclude any integer n from the domain. Thus in $\mathfrak{N}$, the ordinal τ is countable. If we take $\tau=\aleph_1$ in $\mathfrak{M}$, since every counstructible real number in $\mathfrak{M}$ is constructed by a countable ordinal, and since no new constructible sets are introduced in $\mathfrak{N}$ (because no new ordinals are) there are only countably many constructible reals in $\mathfrak{N}$. A still more complicated example has been given by Lévy who showed that it is possible to construct $\mathfrak{N}$ so that $\aleph_n$ for each n is countable in $\mathfrak{N}$ and $\aleph_\omega$ in $\mathfrak{M}$ becomes $\aleph_1$ in $\mathfrak{N}$ and that in $\mathfrak{N}$, $\aleph_1$ is a countable union of countable ordinals. Furthermore, the continuum is a countable union of countable sets. Of course, the countable axiom of choice is violated. A model $\mathfrak{N}$ can also be constructed in which AC fails for a countable set of pairs of sets of reals. This is done by introducing sets $V=\{\langle U_1,1\rangle, \langle U_2,2\rangle,\ldots\}$, $U_n=\{A_n,B_n\}$, $A_n=\{a_{n1},a_{n2},\ldots\}$, $B_n=\{b_{n1},b_{n2},\ldots\}$ and imposing the obvious symmetry conditions.

In the light of Lemma 1, it is interesting to note that uncountable

models in which AC fails can be constructed. The ranks of all sets in the model must have a countable bound, as is seen from the proof of that lemma. This can be achieved by introducing in the construction of the model for (iii), for each U_i a set $\{a_{i\alpha}\}$ where α ranges over a set uncountable in the universe (not just in $\mathfrak{M}$). The $a_{i\alpha}$ consist of all limits of all forcing sequences P_n which can be used to define generic subsets of ω, subject to certain conditions which are too tedious to be spelled out precisely here, but which in essence say that $a = \operatorname{Lim} P_n$ can be made one of an arbitrarily large set $a_1, \ldots, a_n$ which are independent generic sets. The reader should have little difficulty in making this concept precise and proving, by means of a diagonal argument, that uncountably many such $a_{i\alpha}$ exist. A question which seems of some interest to us is whether from (Z–F)′ one can deduce the existence of a (necessarily) non-standard model for Z–F with the axiom of regularity and an automorphism of this model of order two. Such a model immediately proves the independence of the axiom of constructibility since the latter yields the existence of a canonical well-ordering which could not be kept fixed by an automorphism of order two. Theorems concerning automorphisms of general systems are known (Ehrenfeucht–Mostowski [56]) and such an example would illustrate the specific properties of Z–F which make this type of construction possible. A very important problem, from the author's point of view, is the consistency of Z–F and the countable axiom of choice with the statement that all sets of real numbers are measurable. This would be of mathematical interest as well as requiring perhaps some basic elaboration of the ideas of forcing and generic sets. Many other consistency theorems have already been proved, among which may be mentioned those of Feferman [*], but it would seem premature to attempt to list them here.

A word concerning Z–F versus the system G–B of Gödel–Bernays. The two systems are of course essentially equivalent since the classes in G–B correspond to conditions $A(x)$ expressible in Z–F. In G–B the rules concerning class formations essentially allow one to construct a class corresponding to any condition, and the axiom of replacement can now be phrased as a single statement about a class A and a set x. In a proof such as ours one could obtain a consistency proof corresponding to G–B by defining classes in the new model $\mathfrak{N}$ for all the conditions expressible in Z–F. In Gödel [58], however, a short proof using G–B was obtained since the new model of constructible sets is a submodel

of the universe and so the classes can be defined merely as the restriction of the original classes. Such a proof has an obvious aesthetic appeal since one deals with only finitely many axioms. Since in our proof the new models are not submodels it would seem that no such short cut is possible and one has to go through Z–F. In the author's opinion G–B in this regard suffers from the defect of speaking about objects, namely classes, which are primarily fictions introduced so as to simplify the presentation of the axioms.

The last topic to be discussed is the question of arithmetizing the proof and of working within Z–F rather than (Z–F)′. Let $(\text{Z–F})_1$ denote Z–F with the added axioms corresponding to the various cases of Theorem 1, e.g., AC and $2^{\aleph_0}=\aleph_2$. Assume that all formulae built up from $\in$ have been assigned Gödel numbers in the usual manner so that all the logical processes are expressible by elementary functions, more precisely primitive recursive functions. It can easily be shown that the axioms of Z–F or $(\text{Z–F})_1$ correspond to recursive sets. The strongest form of a relative consistency proof would be to prove, using only elementary number theory, the existence of a primitive recursive function f such that if n is the number of a contradiction in $(\text{Z–F})_1$, $f(n)$ is the number of a contradiction in Z–F. (We note that the notion of a contradiction can be expressed arithmetically in terms of the Gödel number.) We sketch the proof of this by means of a series of steps asserting the existence of certain primitive recursive functions. In the following we shall refer back to the notions used in the course of the proof of Theorem 1, with the understanding that the model $\mathfrak{M}$ is to be replaced by the entire universe. For example, a limited statement is one which has arbitrary ordinals attached to its quantifiers. In (ii) the symbols a_δ will have δ ranging over all ordinals $<\aleph_2$, where $\aleph_2$ refers to the actual ordinal in the universe of that description. In the Gödel numbering we also enumerate limited statements with variable indices attached to the quantifiers. The following then hold:

1. The sets of numbers of limited statements, unlimited statements, proofs, etc., are all recursive sets.

2. There is a primitive recursive function which assigns to the number of each statement Q (with ordinal variables $\mathsf{v}_1, \ldots, \mathsf{v}_n$) the number of a formula with an additional variable P, and which expresses that P forces Q (see Lemma 8).

3. There is a primitive recursive function which assigns to the

number of each statement Q a proof from Z–F of the universal closure of the assertions obtained from Lemmas 2–4 by translating "P forces Q" in the way indicated in 2.

4. There is a primitive recursive function which assigns to the number of each axiom of Z–F a proof in Z–F that all P force that axiom.

5. There is a proof in Z–F that all P force $2^{\aleph_0} = \aleph_2$ and all P force AC.

6. There is a primitive recursive function which assigns to the number of an implication $Q_1 \,\&\, \ldots \,\&\, Q_n \rightarrow Q_0$ the number of a proof in Z–F that the existence of a P which forces $Q_1 \,\&\, \ldots \,\&\, Q_n$ implies the existence of a P which forces Q_0 (see Lemma 5).

7. There is a primitive recursive function which assigns to the number of any proof of a contradiction from the axioms of $(\text{Z–F})_1$ the number of a proof from Z–F that there exists a P which forces that contradiction.

8. Finally, there is a primitive recursive function which assigns to the number of a proof of a contradiction from $(\text{Z–F})_1$ the number of a proof of a contradiction in Z–F.

In this manner an extremely strong form of the implication $\text{Con(Z–F)} \rightarrow \text{Con(Z–F)}_1$ can be proved entirely within elementary number theory. The most difficult step is the proof of 4, which must be obtained by examining closely the corresponding proof given in the model $\mathfrak{N}$.

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BOOLEAN NOTIONS EXTENDED TO HIGHER DIMENSIONS[1]

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1. Introduction and Summary. Boole was concerned with sets of dimension or rank 1 and operations on these. Peirce's calculus of binary relations and the classical predicate calculus are sometimes regarded as extensions of Boole's work to dimension 2 and to higher dimensions, respectively. These two calculi fail to distinguish operations with a certain simple kind of behavior from others which involve (at least more strongly) the totality of objects in the given universe A. Now the *Boolean operations*, i.e., those generated by substitution from complementation and intersection applied to sets of dimension 1, are exactly those operations $\mathscr{F}$ on and into sets of dimension 1 which behave simply (in a sense made apparent by T3, T2, and P6 below) and which are *invariant* (on A) in this sense: Whenever π is a one-to-one mapping of A onto A and π^* the induced one-to-one mapping of the appropriate kind, then $\pi^*(\mathscr{F})=\mathscr{F}$. Hence it is natural to ask which invariant operations $\mathscr{F}$ on sets of higher dimensions behave similarly. We consider several characterizations of classes of invariant operations and determine under which conditions they are equivalent. For the classes Υ characterized, we also consider the sets closed under all operations in Υ, and the class of algebraic structures corresponding to Υ. However, we have no algebraic characterization of these classes of structures.

Consider first those operations $\mathscr{F}$ such that, for some finite $r_0, r_1, \ldots, r_m$ depending on $\mathscr{F}$, the range of $\mathscr{F}$ contains only relations of rank r_0 on the given universe A and the domain of $\mathscr{F}$ consists of exactly those $\langle X_1, \ldots, X_m \rangle$ such that $X_1, \ldots, X_m$ are relations of rank $r_1, \ldots, r_m$ respectively on A. The operations $\mathscr{F}$ of this kind are those *of a relational similarity type.*

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[1] The criterion below of being local, semilocal, or continuous, or at any rate a closely related criterion, was suggested to me by Scott, Keisler, and Ryll-Nardzewski, respectively. This study was carried out under Grants G-19286 and GP-1850 of the National Science Foundation. (Added March 23, 1964: Localness also occurs as a criterion in Fraïssé [56a], p. 69.)

For operations $\mathscr{F}$ of a relational similarity type, and for others to be described presently, we consider three criteria of simple behavior. One criterion requires of $\mathscr{F}$ that whether or not a given $\langle a_0, \ldots, a_{r_0-1}\rangle$ is in $\mathscr{F}(X_1, \ldots, X_m)$ depends only on memberships and nonmemberships in $X_1, \ldots, X_m$ of sequences formed from $a_0, \ldots, a_{r_0-1}$ (so that there is no need for a search through the totality of objects in A). More briefly, $\mathscr{F}$ is required to be locally determined or, simply, *local*. For invariant $\mathscr{F}$ of a relational similarity type, localness is equivalent to expressibility by a first-order formula without quantifiers. A second criterion requires of $\mathscr{F}$ that whether or not a given $\langle a_0, \ldots, a_{r_0-1}\rangle$ is in $\mathscr{F}(X_1, \ldots, X_m)$ depends only on a finite number of memberships and nonmemberships in $X_1, \ldots, X_m$. More briefly, $\mathscr{F}$ is required to be *continuous* [2]. A third criterion requires of $\mathscr{F}$ that there be a uniform procedure for converting any m decision methods for any $X_1, \ldots, X_m$ respectively of the appropriate kinds on A into a decision method for $\mathscr{F}(X_1, \ldots, X_m)$. More briefly, $\mathscr{F}$ is required to *(uniformly) preserve decidability*. We shall show, without using Church's thesis, that if A is infinite and $\mathscr{F}$ invariant and of a relational similarity type, then these three criteria are equivalent [P6].

Thus the class $\mathbb{L}_R$ of invariant local operations of a relational similarity type is rather stable. Just as the class of Boolean operations gives rise to the Boolean algebras of sets $X \subseteq A$, so $\mathbb{L}_R$ gives rise to relational structures $\mathfrak{B}$ whose universe B consists of relations X on A and whose relations $G_0, \ldots, G_n, \ldots$ are the restrictions to B of the operations $\mathscr{F}_0, \ldots, \mathscr{F}_n, \ldots$ in $\mathbb{L}_R$. Each such $\mathfrak{B}$ is an $\mathbb{L}_R$-*semialgebra*, since for $\mathscr{F}_n \in \mathbb{L}_R$ and $X_1, \ldots, X_m \in B$, $\langle X_1, \ldots, X_m\rangle$ is not in the domain of $\mathscr{F}_n$ or G_n unless $X_1, \ldots, X_m$ are of appropriate rank.

For many $\mathscr{F}$, for example, complementation, intersection, and many operations induced by a transformation of sequence places, the procedure of converting decision methods for $X_1, \ldots, X_m$ into a decision method for $\mathscr{F}(X_1, \ldots, X_m)$ applies not only to sets of sequences of fixed length $r_1, \ldots, r_m$, respectively, but likewise to sets $X_1, \ldots, X_m$ of sequences of arbitrary finite length. We therefore also consider operations of a second kind, with a much larger domain than before, namely the operations *on and into sets of finite sequences*. Since substitution is no longer restricted

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[2] The topology justifying this terminology will not be used here. If $\mathscr{F}$ is of similarity type $\langle r_0, r_1, \ldots, r_m\rangle$ then for the argument space a set $\mathscr{K}$ is an element of our base of open sets if and only if there are finite W_i^+ and W_i^- such that the members of $\mathscr{K}$ are those arguments $\langle X_1, \ldots, X_m\rangle$ of $\mathscr{F}$ such that $W_i^+ \subseteq X_i$ and $W_i^- \subseteq A^{r_i} - X_i$. Analogously for the image space.

by considerations of rank, operations of this kind yield algebras rather than semialgebras. However, the three criteria are now no longer equivalent. Rather, for A infinite and $\mathscr{F}$ invariant, the first criterion is now less restrictive than the second, and the second less than the third [T5 and $\mathscr{E}_4$, P4 and $\mathscr{E}_{5,g}$]. Hence for sets of finite sequences (sfs) there are at least three analogues of the notion of a Boolean operation. To these there correspond three different notions of a field of sfs [$\mathscr{E}_{7,J}$ and $\mathscr{E}_8$], and hence three nonequivalent notions of an algebra.

A more cautious extension of the notion of a relation is the notion of a set of sequences whose length does not exceed a finite bound. Indeed, since the symmetric difference of bounded sets is bounded (as has been stressed to me by William Hanf), the extension is minimal for embedding the IL_R-semialgebras in Boolean rings. Hence operations on and into sets thus bounded, the bound depending on the set, and likewise the corresponding algebras, may be of interest. However, complementation (with respect to the set of all finite sequences formed from elements of A) applied to a bounded set X does not yield a bounded set. Thus we are led to a further extension of the notion of a relation, minimal for embedding the IL_R-semialgebras in Boolean algebras. More precisely, we let a set be *(co)bounded* if and only if either it or its complement is bounded, and we consider as a third kind of operation those which are *on and into (co)bounded sets* [3]. For invariant operations of this kind and for infinite A, the first, second, and third criteria are, again in that order, increasingly stringent, so that there are again at least three analogues of the notion of a Boolean operation.

Nevertheless, to each of these three notions there corresponds the same notion of a field of (co)bounded sets [P15]. Moreover, there is a simple one-to-one correspondence between these fields and those sets of relations which are closed under all IL_R-operations [T14].

The three notions of an algebra which correspond to our three notions of a Boolean operation on and into (co)bounded sets also turn out to be in a certain sense equivalent [T19]. Moreover, all three are similarly

[3] Our problem is to select operations using criteria which have direct mathematical or philosophical appeal (instead of criteria which are linguistic in origin and thus depend on extrinsic factors). Hence it may be objected that one particular operation, namely complementation, is thus being singled out without the use of such a criterion. The notion of an operation on and into bounded sets does not have this defect. Moreover all of our arguments below also hold with obvious changes for operations on and into bounded, instead of (co)bounded, sets.

equivalent to other notions of an algebra for which there is a more economical set of operations and thus perhaps more hope of algebraic characterizability [T17 and T19]. Such a characterization would probably be useful for investigating the $\mathfrak{L}_R$-semialgebras, since the two classes of structures are closely related [T18].

In conclusion we consider a generalization of localness to *semilocalness*. The closure of the set of semilocal invariant operations under all those substitutions which are not ruled out by considerations of rank is the set of *elementary* operations. If $\mathscr{F}$ is of a relational similarity type, then it is elementary in our sense if and only if it is the restriction to relational structures whose universe is A of an operation which is elementary in the sense of Tarski. For (co)bounded sets the situation described in the last two paragraphs now has certain analogues. Perhaps this will lead to a more satisfactory formulation of classical first-order logic.

(Added August 15, 1964: Instead of considering sets of finite sequences, i.e., of functions whose domain is an initial segment of the integers, Charles M. Howard has proposed to consider sets of functions whose domain is an arbitrary finite subset of the integers. For operations on and into such sets he has achieved, in a thesis in progress, an algebraic formulation of classical first-order logic and a characterization of the corresponding set algebras which (except that there are probably no analogues of T15 and T19 and only partial analogues of T14 and T18) seems much superior to anything one might hope to achieve for the above three kinds of operations.)

2. Operations. We now introduce some notation and make more precise, or slightly qualify, some of the notions already used. $|Y|$ is the cardinality of Y. With or without subscripts, 'i', 'j', 'm', 'n', 'r', 's' shall range over nonnegative integers, 'q' and 't' over positive integers, and 'a', 'b', ... over objects of the *universe* A, a nonempty set kept fixed throughout. We assume that there is a unique *empty sequence* $\langle - \rangle$, and let $A^0 = \{\langle - \rangle\}$. We let A^t be the set of all sequences $\langle a_0, \ldots, a_{t-1} \rangle$, $A^{\underline{t}} = \bigcup_{r<t} A^r$, and $A^{\underline{\omega}} = \bigcup_{r<\omega} A^r$. Similarly, for example, $\{b_0, \ldots, b_{q-1}\}^{\underline{\omega}}$ is the set of all finite sequences formed from $b_0, \ldots, b_{q-1}$. $^-$ is complementation with respect to $A^{\underline{\omega}}$, and $X - Y = X \cap \overline{Y}$. A *relation of rank r* is a subset of A^r, and a *relation* is a subset of some A^r. Thus, for algebraic reasons, we have (two) relations of rank 0. A set *bounded by t* is a subset of $A^{\underline{t}}$. A set X is *(co)bounded by t* if and only if either X or its complement

(with respect to A^{ω}) is bounded by t, i.e., either $X \subseteq A^{\leq t}$ or $A^{\omega} - X \subseteq A^{\leq t}$. A *(co)bounded set* is a set (co)bounded by some t. A *set of finite sequences (sfs)* is a subset of A^{ω}.

With or without subscripts we use '$\mathscr{F}$' for an operation of any one of the three kinds described, and '$\mathscr{E}$' for a specific operation serving as an example. We often assume tacitly that $\mathscr{F}$ is m-ary and that $1 \leqslant i \leqslant m$, and sometimes that $\mathscr{F}$ is 1-ary. For $m = 0$, $\mathscr{F}$ is a particular relation, (co)bounded set, or sfs respectively. We let Dom $\mathscr{F}$ be the domain of $\mathscr{F}$. To repeat, $\mathscr{F}$ is *of a relational similarity type (on and into (co)-bounded sets) (on and into sfs)* shall mean: For some $r_0, r_1, \ldots, r_m$, Dom $\mathscr{F}$ consists of those $\langle X_1, \ldots, X_m \rangle$ such that X_i is a relation of rank r_i (is (co)bounded) (is a sfs), and each value of $\mathscr{F}$ is a relation of rank r_0 (is (co)bounded) (is a sfs). In the first case we shall sometimes mention the particular $r_0, r_1, \ldots, r_m$ and say that $\mathscr{F}$ is of similarity type $\langle r_0, r_1, \ldots, r_m \rangle$. Evidently, each $\mathscr{F}$ of any of the three kinds may be regarded as a class of a certain kind of relational structures $\mathfrak{A}$ all of which have the same universe A. Our treatment can, but will not, be extended to the more general case where A is not always the same.

For illustration, consider now the following formula μ_1 and operation $\mathscr{E}_1$ where $\mathscr{E}_1$ is of similarity type $\langle 2, 1, 3 \rangle$ and its values are as indicated.

$\mu 1$: $$[\neg\, \mathsf{v}_0 \eqsim \mathsf{v}_1 \wedge \mathsf{R}_1^1\, \mathsf{v} \wedge \neg\, \mathsf{R}_2^3\, \mathsf{v}_0\, \mathsf{v}_1\, \mathsf{v}_0] \vee [\mathsf{v}_0 \eqsim \mathsf{v}_1 \wedge \neg\, \mathsf{R}_1^1\, \mathsf{v}_1 \wedge \mathsf{R}_2^3\, \mathsf{v}_0\, \mathsf{v}_0\, \mathsf{v}_0]$$

$$\mathscr{E}_1(X_1, X_2) = \{\langle a_0, a_1 \rangle : [a_0 \neq a_1, a_1 \in X_1, \text{ and } \langle a_0, a_1, a_0 \rangle \notin X_2] \text{ or } [a_0 = a_1,\ a_1 \notin X_1,\ \text{and } \langle a_0, a_0, a_0 \rangle \in X_2]\}.$$

Whenever an operation $\mathscr{F}$ of similarity type $\langle r_0, r_1, \ldots, r_m \rangle$ and a quantifier-free formula μ (whose free individual variables are $\mathsf{v}_0, \ldots, \mathsf{v}_{r_0-1}$ and whose m predicate variables are R with symbols for $r_1, \ldots, r_m$ as superscript and for $1, \ldots, m$ as subscript respectively) are related to each other as $\mathscr{E}_1$ is to μ_1, then μ *expresses* $\mathscr{F}$. We allow 0-placed predicate symbols in μ and assume the appropriate semantic conventions for these.

It is easily seen that $\mathscr{E}_1$ is invariant, local, continuous, and decidability-preserving in the senses introduced earlier and now to be made precise.

$\mathscr{F}$ is *invariant*: For any one-to-one mapping π of A onto A and any $\langle X_1, \ldots, X_m \rangle \in$ Dom $\mathscr{F}$, if $\pi'(Y) = \{\langle \pi(a_0), \ldots, \pi(a_{r-1}) \rangle : \langle a_0, \ldots, a_{r-1} \rangle \in Y\}$ for every $Y \subseteq A^{\omega}$, then $\mathscr{F}(\pi'(X_1), \ldots, \pi'(X_m)) = \pi'(\mathscr{F}(X_1, \ldots, X_m))$.

$\mathscr{F}$ is *local*: For any $\langle a_0, \ldots, a_{r-1}\rangle$, $\langle X_1, \ldots, X_m\rangle \in \mathrm{Dom}\ \mathscr{F}$, and $\langle Y_1, \ldots, Y_m\rangle \in \mathrm{Dom}\ \mathscr{F}$, if $\{a_0, \ldots, a_{r-1}\}^{\omega} \cap X_i = \{a_0, \ldots, a_{r-1}\}^{\omega} \cap Y_i$, then $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1, \ldots, X_m)$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(Y_1, \ldots, Y_m)$ [4].

$\mathscr{F}$ is *continuous*: For any $\langle a_0, \ldots, a_{r-1}\rangle$ and any $\langle X_1, \ldots, X_m\rangle \in \mathrm{Dom}\ \mathscr{F}$ there is a finite set $W \subseteq A^{\omega}$ such that for any $\langle Y_1, \ldots, Y_m \in \mathrm{Dom}\ \mathscr{F}$, if $W \cap X_i = W \cap Y_i$, then $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1, \ldots, X_m)$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(Y_1, \ldots, Y_m)$.

$\mathscr{F}$ (*uniformly*) *preserves decidability*: There is a uniform ["reduction"] procedure whereby, given any m methods [including oracles or reliable physical processes, in this or a conceivable world, often incomputable in Turing's sense and as inaccessible as a permanently sealed box (Hanf)] of deciding whether or not an arbitrarily given $\langle a_0, \ldots, a_{r-1}\rangle \in A^{\omega}$ is in $X_1, \ldots, X_m$ respectively, we obtain a method of deciding whether or not an arbitrarily given $\langle a_0, \ldots, a_{r-1}\rangle \in A^{\omega}$ is in $\mathscr{F}(X_1, \ldots, X_m)$.

For example, the reduction procedure for $\mathscr{E}_1$ may be described as follows. Let any decision methods M_1 for X_1 and M_2 for X_2, and any $\langle a_0, \ldots, a_{r-1}\rangle \in A^{\omega}$ be given. Determine the length r [5]. If $r \neq 2$, then $\langle a_0, \ldots, a_{r-1}\rangle$ is not in $\mathscr{E}_1(X_1, X_2)$. Now assume that $r = 2$. Determine whether or not $a_0 = a_1$ [5]. Consider first the case where $a_0 \neq a_1$. Using M_1 test whether or not $a_1 \in X_1$; and producing $\langle a_0, a_1, a_0\rangle$ [5] and using M_2 test whether or not $\langle a_0, a_1, a_0\rangle \notin X_2$. In this case $\langle a_0, a_1\rangle$ is in $\mathscr{E}_1(X_1, X_2)$ if and only if the answer to both tests is affirmative. In the other case where $a_0 = a_1$, use of M_1 and M_2 again yields a decision. Evidently, $|A|$ or what kinds of objects are in A does not matter.

More generally, by induction on the length of μ we obtain:

Proposition 1. *If some quantifier-free μ expresses $\mathscr{F}$, then $\mathscr{F}$ uniformly preserves decidability.*

Our definitions also yield almost immediately the following two well-known facts.

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[4] Equivalently: For any $\langle a_0, \ldots, a_{r-1}\rangle$ and $\langle X_1, \ldots, X_m\rangle \in \mathrm{Dom}\ \mathscr{F}$, $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1, \ldots, X_m)$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(\{a_0, \ldots, a_{r-1}\}^{\omega} \cap X_1, \ldots, \{a_0, \ldots, a_{r-1}\}^{\omega} \cap X_m)$.

[5] We assume that there is a uniform procedure of effecting the determination or production required.

Theorem 2. *If $\mathscr{F}$ is of a relational similarity type, then the following are equivalent*:

(i) *$\mathscr{F}$ is invariant and local.*

(ii) *Some quantifier-free μ expresses $\mathscr{F}$.*

Theorem 3. *If $\mathscr{F}$ is of similarity type $\langle 1, 1, \ldots, 1\rangle$, then the following are equivalent*:

(i) *$\mathscr{F}$ is Boolean.*

(ii) *Some quantifier-free μ expresses $\mathscr{F}$.*

We now come to our two main implications. The argument for P4 is the usual one depending on the uniformity required of the procedure and on the finitary character of decision methods. A proof of T5 will be sketched below.

Proposition 4. *If $\mathscr{F}$ uniformly preserves decidability, then $\mathscr{F}$ is continuous.*

Remark. A condition which seems in general to be weaker than that of uniformly preserving decidability is the following: (δ) There is a uniform procedure whereby, given any m methods of deciding whether or not an arbitrarily given $\langle a_0, \ldots, a_{r-1}\rangle \in A^\omega$ is in $X_1, \ldots, X_m$, respectively, *and in addition* a method of effectively generating A [or perhaps: a set including A], we obtain a method of deciding whether or not an arbitrarily given $\langle a_0, \ldots, a_{r-1}\rangle \in A^\omega$ is in $\mathscr{F}(X_1, \ldots, X_m)$. For example, let Dom $\mathscr{E}$ consist of the 1-ary functions X mapping A into A, and let $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{E}(X)$ if and only if $r = 2$ and, for some b, $\langle a_0, b\rangle \in X$ and $\langle b, a_1\rangle \in X$. Then $\mathscr{E}$ is invariant and satisfies (δ), but without some method of effectively generating a set including A there seems to be no procedure for preserving decidability. Nevertheless, if $|A| = \aleph_0$ then (δ) also implies that $\mathscr{F}$ is continuous. From this and from T2, P1, and T5 below one can prove: If $|A| = \aleph_0$ and $\mathscr{F}$ is invariant, then (δ) implies that $\mathscr{F}$ uniformly preserves decidability. Thus for $|A| = \aleph_0$ one can add to each of P6, P9, and P10 below another equivalence. $\mathscr{E}$ fails to preserve decidability despite satisfying (δ) because the domain of $\mathscr{E}$ is smaller than that of an operation of similarity type $\langle 2, 2\rangle$, so that $\mathscr{E}$ does not belong to one of the three kinds of operation we consider.

Theorem 5. *If* $|A| \geqslant \aleph_0$ *and* $\mathscr{F}$ *is invariant and continuous, then* $\mathscr{F}$ *is local* [6].

To prove T5 assume its hypothesis. Consider any $\langle a_0, \ldots, a_{r-1}\rangle$ and any $\langle X_1, \ldots, X_m\rangle \in \mathrm{Dom}\,\mathscr{F}$. Let $X_i' = \{a_0, \ldots, a_{r-1}\}^\omega \cap X_i$. Then for some finite $W \in A^\omega$, if (α) $W \cap X_i = W \cap Y_i$, then $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1, \ldots, X_m)$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(Y_1, \ldots, Y_m)$. Likewise for some finite $W' \subseteq A^\omega$, if $(\alpha)'$ $W' \cap Y_i' = W' \cap X_i'$, then $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(Y_1', \ldots, Y_m')$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1', \ldots, X_m')$. Using the invariance of $\mathscr{F}$, T5 can therefore be derived from the following:

Lemma. *If* $|A| \geqslant \aleph_0$ *and if* $\langle a_0, \ldots, a_{r-1}\rangle$, $\langle X_1, \ldots, X_m\rangle$, $\langle X_1', \ldots, X_m'\rangle$, W, *and* W' *are as described, then there are* $\langle Y_1, \ldots, Y_m\rangle$ *and* $\langle Y_1', \ldots, Y_m'\rangle$ *in* $\mathrm{Dom}\,\mathscr{F}$ *such that* $(\alpha), (\alpha)'$, *and* $\pi(Y_1) = Y_1', \ldots, \pi(Y_m) = Y_m'$ *for some one-to-one mapping* π *of* A *onto* A *which keeps each of* $a_0, \ldots, a_{r-1}$ *fixed* [7].

Using T2, P1, P4, and T5 but *not* using Church's thesis we obtain:

Proposition 6. *If* A *is infinite and* $\mathscr{F}$ *of a relational similarity type then the following are equivalent*:

(i) $\mathscr{F}$ *is invariant and local.* ($\mathscr{F} \in \mathsf{IL}_\mathsf{R}$.)

(ii) $\mathscr{F}$ *is invariant and continuous.*

(iii) $\mathscr{F}$ *is invariant and uniformly preserves decidability.*

The condition of invariance cannot be omitted from T5 or P6. For example, let $A = \omega$ and let $\mathscr{E}_2$ be that operation of similarity type $\langle 1, 1\rangle$ such that, for each $X \subseteq \omega$, $\mathscr{E}_2(X) = \{a : |a \cap X| > |a \cap (\omega - X)|\}$, where each $a \in A = \omega$ is the set of its predecessors. Then $\mathscr{E}_2$ preserves decidability and hence is continuous but it is not local. The condition that A is infinite also cannot be omitted from T5 or P6. For if A is finite and $\mathscr{F}$ of a relational similarity type, then $\mathscr{F}$ trivially preserves decidability and is trivially continuous, yet if $|A| > 1$, then for example the projection operation $\mathscr{E}_3$ of similarity type $\langle 1, 2\rangle$ determined by $\mathscr{E}_3(X) = \{\langle a_0\rangle :$ for some a_1, $\langle a_0, a_1\rangle \in X\}$ is not local.

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[6] If $\mathscr{F}$ is of a relational similarity type then evidently localness implies continuity. Thus the equivalence of (i) and (ii) in P6 follows directly from T5.

[7] A closer look at this proof of T5 shows that, for $\aleph_\beta \leqslant |A|$, continuity can be replaced in the hypothesis by $\aleph_\beta$-continuity, obtained from it by requiring only $|W| < \aleph_\beta$ instead of $|W| < \aleph_0$. Conversely, localness evidently implies $\aleph_1$-continuity. Also invariance can be replaced by the weaker hypothesis which requires $\pi^*(\mathscr{F}) = \mathscr{F}$ only for those π such that π is the identity map on some $A - A'$, $|A'| < |A|$.

We now turn to operations on and into sets of finite sequences. Fo) example, let $\mathscr{E}_4$ be that operation on and into sfs such that $\langle - \rangle \in \mathscr{E}_4(X\mathrm{r}$ and $\langle a_0, \ldots, a_{r-1} \rangle \in \mathscr{E}_4(X)$ if and only if $|\{a_0, \ldots, a_{r-1}\}^{\omega} \cap X| \geqslant r$. Then $\mathscr{E}_4$ is invariant and local ($\mathscr{E}_4 \in \mathsf{IL}_S$) but not continuous. For each mapping g of ω into $\omega - \{0\}$ let $\mathscr{E}_{5,g}$ be that operation on and into sfs such that $\langle - \rangle \in \mathscr{E}_{5,g}(X)$ and $\langle a_0, \ldots, a_{r-1} \rangle \in \mathscr{E}_{5,g}(X)$ if and only if $|\{a_0, \ldots, a_{r-1}\}^{g(r)} \cap X| = r$. Then $\mathscr{E}_{5,g}$ is invariant and continuous. Now suppose $\mathscr{E}_{5,g}$ also preserves decidability. Given any $a_0 \in A$ and any r, we can effectively produce, for any s, a set $X_s \subseteq \{a_0\}^{\omega}$ such that $|\{a_0\}^{t} \cap X_s| = r$ if and only if $t = s$. Hence, converting our decision methods for X_s into decision methods for $\mathscr{E}_{5,g}(X_s)$ and applying these to the sequence $\langle a_0, \ldots, a_0 \rangle$ of length r we obtain a method of computing $g(r)$. Hence, by Church's thesis, if $\mathscr{E}_{5,g}$ preserves decidability then g is recursive. Conversely, if g is recursive then $\mathscr{E}_{5,g}$ evidently preserves decidability.

Generalizing from this last example, we consider the following condition on a mapping g of ω into $\omega - \{0\}$: For any $\langle a_0, \ldots, a_{r-1} \rangle$ and $\langle X_1, \ldots, X_m \rangle \in \mathrm{Dom}\, \mathscr{F}$, $\langle a_0, \ldots, a_{r-1} \rangle \in \mathscr{F}(X_1, \ldots, X_m)$ if and only if $\langle a_0, \ldots, a_{r-1} \rangle \in \mathscr{F}(A^{g(r)} \cap X_1, \ldots, A^{g(r)} \cap X_m)$.
If g satisfies this condition, then it *bounds the sequences affecting* $\mathscr{F}$. If $\mathscr{F}$ is 1-ary, then g bounds the sequences affecting $\mathscr{F}$ if and only if, for each r we can associate with $\mathscr{F}$ an operation ${}_r\mathscr{F}$ of relational similarity type $\langle r, 0, \ldots, g(r-1) \rangle$ such that, for any $X \in \mathrm{Dom}\, \mathscr{F}$, $A^r \cap \mathscr{F}(X) = {}_r\mathscr{F}(A^s \cap X)_{s<g(r)}$. Similarly for m-ary $\mathscr{F}$.

Using König's tree lemma one easily proves:

Theorem 7. *If $\mathscr{F}$ is on and into sfs, invariant, continuous, and local, then some g bounds the sequences affecting $\mathscr{F}$.*

T5 and T7 allow us to characterize, for infinite A, the invariant and continuous $\mathscr{F}$ on and into sfs in terms of the invariant and continuous operations of a relational similarity type.

Theorem 8. *If A is infinite and if $\mathscr{F}$ is on and into sfs and for convenience 1-ary, then the following are equivalent:*

(i) *$\mathscr{F}$ is invariant and continuous.* ($\mathscr{F} \in \mathsf{IL}_S$.)

(ii) *For each r there is some $g(r) > 0$ and some ${}_r\mathscr{F}$ of similarity type $\langle r, 0, 1, \ldots, g(r)-1 \rangle$ which is invariant and continuous such that, for any $X \in \mathrm{Dom}\, \mathscr{F}$, $A^r \cap \mathscr{F}(X) = {}_r\mathscr{F}(A^s \cap X)_{s<g(r)}$.*

By T2 and T5, to any ${}_r\mathscr{F}$ in (ii) of T8 there corresponds a quantifier-

free ${}_r\mu$ which expresses it. In terms of these ${}_r\mu$ we now give a mathematical characterization of an intuitive concept. The argument uses Church's thesis and a constructive version of König's lemma.

Proposition 9. *If A is infinite and if $\mathscr{F}$ is on and into sfs and for convenience* 1*-ary then the following are equivalent*:

(i) *$\mathscr{F}$ is invariant and uniformly preserves decidability.* ($\mathscr{F} \in \mathsf{ID}_S$.)

(ii) *There is a recursive function f such that each $f(r)$ is the Gödel number of a quantifier-free ${}_r\mu$ which expresses an ${}_r\mathscr{F}$ of some similarity type $\langle r, 0, \ldots, g(r)-1\rangle$ such that, for any $X \in \mathrm{Dom}\,\mathscr{F}$, $A^r \cap \mathscr{F}(X) = {}_r\mathscr{F}(A^s \cap X)_{s<g(r)}$.*

We now turn to operations on and into (co)bounded sets. Restricting the domain of $\mathscr{E}_4$ to (co)bounded sets we get an invariant operation on and into (co)bounded sets which is local but not continuous, and a cardinality argument shows that there are others which are continuous but do not preserve decidability. Consider now that operation $\mathscr{E}_6$ on (co)bounded sets such that, for any $\langle a_0, \ldots, a_{r-1}\rangle$ and any $X \in \mathrm{Dom}\,\mathscr{E}_6$, $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{E}_6(X)$ if and only if in the sequence $\langle a_0\rangle$, $\langle a_0, a_0\rangle$, $\langle a_0, a_0, a_0\rangle$, ... there occur r consecutive members of X before there occur r consecutive nonmembers of X. Then $\mathscr{E}_6(X)$ or $A^{\omega} - \mathscr{E}_6(X)$ is bounded according to whether X or $A^{\omega} - X$ is bounded, so that $\mathscr{E}_6$ is into (co)bounded sets. Evidently $\mathscr{E}_6$ is invariant and local, and moreover $\mathscr{E}_6$ preserves decidability and hence is continuous. However no g bounds the sequences affecting $\mathscr{E}_6$. Hence T7 has no analogue for operations on and into (co)bounded sets. It also follows for infinite A that $\mathscr{E}_6$ cannot be extended into an invariant and continuous operation on and into sfs.

Failure of the analogue of T7 also causes the characterizations analogous to those in T8 and P9 to be more complex. We shall only state the analogue of P9 which again gives a mathematical characterization of an intuitive concept. The analogue of T8 can be obtained from it by using IC_C instead of ID_C in (i) and by letting f be an arbitrary partial function in (ii) instead of partial recursive.

Proposition 10. *If A is infinite and if $\mathscr{F}$ is on and into (co)bounded sets and for convenience* 1*-ary then the following are equivalent*:

(i) *$\mathscr{F}$ is invariant and uniformly preserves decidability.* ($\mathscr{F} \in \mathsf{ID}_C$.)

(ii) *There is a partial recursive f such that for any r, if $K(L)$ is the set of even (odd) j such that $\langle r, j\rangle \in \mathrm{Dom}\,f$, then, for any j in $K \cup L$,*

$f(r, j)$ is the Gödel number of a quantifier-free formula expressing an ${}_{r,j}\mathscr{F}$ of some similarity type $\langle r, 0, \ldots, g(r, j)-1\rangle$, and, for any $X \in \mathrm{Dom}\ \mathscr{F}$, $A^r \cap \mathscr{F}(X) = \bigcup_{j \in K} {}_{r,j}\mathscr{F}(A^s \cap X)_{s<g(r,j)}$ and $A^r - \mathscr{F}(X) = \bigcup_{j \in L} {}_{r,j}\mathscr{F}(A^s \cap X)_{s<g(r,j)}$.

We now turn to the set IL_C of those invariant operations on and into (co)bounded sets which are local, and to the problem of relating IL_C to IL_R. In T8 or P9 we used an operation ${}_r\mathscr{F}$ serving as a "substitute" for $\mathscr{F}$ on the "factor" determined by A^r of the values of $\mathscr{F}$. We now consider operations ${}_r\mathscr{F}_t$ which thus serve as a "substitute" only on a part of the domain of $\mathscr{F}$. More precisely, for any 1-ary $\mathscr{F}$ on and into (co)bounded sets and any r and t, let ${}_r\mathscr{F}_t$ be the operation of similarity type $\langle r, 0, \ldots, t\rangle$ such that:

(i) If $\{a_0, \ldots, a_{r-1}\}^t \cap Y_t = \emptyset$, then $\langle a_0, \ldots, a_{r-1}\rangle \in {}_r\mathscr{F}_t(Y_s)_{s \leqslant t}$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(\bigcup_{s<t} Y_s)$; and

(ii) If $\{a_0, \ldots, a_{r-1}\}^t \cap Y_t \neq \emptyset$, then $\langle a_0, \ldots, a_{r-1}\rangle \in {}_r\mathscr{F}_t(Y_s)_{s \leqslant t}$ if and only if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}((A^{\omega} - A^t) \cup \bigcup_{s<t} Y_s)$.

For m-ary $\mathscr{F}$, let ${}_r\mathscr{F}_t$ be defined similarly, choosing $\bigcup_{s<t} Y_{i,s}$ as i-th term of the argument of $\mathscr{F}$ if and only if $\{a_0, \ldots, a_{r-1}\}^t \cap Y_{i,t} = \emptyset$. The following two theorems are now easy consequences.

Theorem 11. *If $\mathscr{F}$ is on and into (co)bounded sets and for convenience 1-ary, and if X is (co)bounded by t, then*

(a) *for any r, $A^r \cap \mathscr{F}(X) = {}_r\mathscr{F}_t(A^s \cap X)_{s \leqslant t}$, and hence*

(b) *for some q, either $\mathscr{F}(X) = \bigcup_{r<q} {}_r\mathscr{F}_t(A^s \cap X)_{s \leqslant t}$ or $\mathscr{F}(X) = (A^{\omega} - A^q) \cup \bigcup_{r<q} {}_r\mathscr{F}_t(A^s \cap X)_{s \leqslant t}$.*

Theorem 12. *If $\mathscr{F}$ is on and into (co)bounded sets, then*

(a) *$\mathscr{F}$ is local if and only if each ${}_r\mathscr{F}_t$ is local, and*

(b) *$\mathscr{F}$ is invariant if and only if each ${}_r\mathscr{F}_t$ is invariant.*

Finally, using T11, T12, and König's lemma one can prove:

Theorem 13. *If $\mathscr{F} \in \mathsf{IL}_C$, then for each t there is some q such that, if $X_1, \ldots, X_m$ are (co)bounded by t, then $\mathscr{F}(X_1, \ldots, X_m)$ is (co)bounded by q.*

3. Fields. We now consider problems of extending from dimension 1 to higher dimensions the notion of a field of sets. An IL_R-*semifield* is any set B of relations which is closed under all $\mathscr{F}$ in IL_R. Thus, each 0-ary $\mathscr{F} \in \mathsf{IL}_R$ and hence, among others, each A^r and each $\{\langle a_0, \ldots, a_{r-1}\rangle : a_0 = \ldots = a_{r-1}\}$ is an element of every IL_R-semifield. By P6, no other notion of semifield need concern us here. Similarly, an IL_C-*field* is any set C of (co)bounded sets which is closed under all $\mathscr{F}$ in IL_C. The following theorem gives a simple one-to-one correspondence between the IL_R-semifields and the IL_C-fields. The first sentence follows from T11 and T12. For proof of the second note that, for every X in C, each $A^r \cap X$ is in B so that X is in the closure of B under $\cup$ and $^-$, while C is closed under $\cup$ and $^-$ and hence includes this closure.

Theorem 14. *If B is any IL_R-semifield then the closure C of B under $\cup$ and $^-$ is an IL_C-field such that, evidently, the set of relations in C is B. Also, if C is any IL_C-field, then the set B of relations in C is evidently an IL_R-semifield and the closure of B under $\cup$ and $^-$ is C.*

We can now show that, for (co)bounded sets and A infinite, each of our criteria yields the same notion of field. By P4 and T5 the notion of an IL_C-field is evidently not wider than the others, and by T14 and P6 it is not narrower.

Proposition 15. *If A is infinite and C a set of (co)bounded sets then the three conditions that C be closed under all $\mathscr{F}$ in IL_C, IC_C, or ID_C, respectively, are equivalent.*

For sfs, the analogues of T14 and P15 are false, since there is no finite composition of $\mathscr{F}(X)$ analogous to that expressed in T11b. As for P15, we now show that for sfs each of our criteria yields a different notion of field. Since ID_S is denumerable, there are denumerable sets C of sfs which are closed under all $\mathscr{F}$ in ID_S. On the other hand, if $\mathscr{E}_{7,J} = \bigcup_{r \in J} A^r$ where J is any subset of ω, then every set C closed under all $\mathscr{F}$ in IC_S contains as element each $\mathscr{E}_{7,J}$ and hence is nondenumerable. Now let A be infinite and let $\mathscr{E}_8 \in \mathsf{IL}_S$ be such that $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{E}_8(X)$ if and only if $r = 1$ and $\{a_0\}^\omega \subseteq X$. Let $Y \subseteq A^\omega$ be such that for each $\alpha \leqslant \omega$ there is some b such that $\{b\}^\beta \subseteq Y$ if and only if $0 < \beta \leqslant \alpha$. Then, by T8, $\mathscr{E}_8(Y) \neq \mathscr{F}(Y)$ for every $\mathscr{F} \in \mathsf{IC}_S$. Hence the closure of $\{Y\}$ under all $\mathscr{F}$ in IC_S is not closed under all $\mathscr{F}$ in IL_S.

In contrast to T14, there are distinct sets C and C' of sfs which are closed under all $\mathscr{F}$ in IC_S and which contain the same relations as elements. Let A be infinite, B any IC_R-semifield, C the closure of B under all $\mathscr{F}$ in IC_S, and C' the set of exactly those $X \subseteq A^\omega$ such that $A^r \cap X \in B$ for each r. Then, by T8, C' is closed under all $\mathscr{F}$ in IC_S, and evidently B is the set of relations in C'. Also, by T8, B is the set of relations in C. Now suppose, for example, that B is generated by the relations $\{\langle a_1{}^1\rangle\}$, $\{\langle a_1{}^2, a_2{}^2\rangle\}$, $\{\langle a_1{}^3, a_2{}^3, a_3{}^3\rangle\}$, ... Then the union $\{\langle a_1{}^1\rangle\} \cup \{\langle a_1{}^2, a_2{}^2\rangle\} \cup \{\langle a_1{}^3, a_2{}^3, a_3{}^3\rangle\} \cup \ldots$ of these relations is in C' but, by T8, not in C.

As a partial analogue of T14 one can easily prove:

Theorem 16. *If B is any IL_R-semifield then the closure C of B under all $\mathscr{F}$ in IL_S is such that the set of relations in C is B.*

4. Algebras. We now turn to algebraic questions, concentrating on IL_R, the most fundamental among our sets of operations, and on subsets Υ of IL_C. We shall show that in important respects the choice of Υ matters little. This calls for experimentation as to which Υ, if any, yield a class of structures whose theory is tractable.

Let $\mathscr{F}_0, \ldots, \mathscr{F}_n, \ldots, n<\omega$, be a well-ordering of IL_R, and $\mathscr{F}'_0, \ldots, \mathscr{F}'_\zeta, \ldots$ $\ldots, \zeta<\eta$, a well-ordering of IL_C such that, for each $n<\omega$, $\mathscr{F}_n{}'$ is that extension of $\mathscr{F}_n$ to (co)bounded sets such that if $\langle X_1, \ldots, X_m\rangle \in$ $\in \mathrm{Dom}\,\mathscr{F}_n{}' - \mathrm{Dom}\,\mathscr{F}_n$, then $\mathscr{F}'(X_1, \ldots, X_m) = \emptyset$. A structure $\mathfrak{B} = \langle B, G_n\rangle_{n\in\omega}$ is an IL_R-*semialgebra* if and only if (i) B is a nonempty set of relations which is closed under each $\mathscr{F}_n$ in IL_R, and (ii) each G_n is the restriction to B of $\mathscr{F}_n$, $n \in \omega$. Now consider any $\Upsilon \subseteq \mathsf{IL}_C$. Let $\mathrm{Ord}\,\Upsilon = \{\zeta : \mathscr{F}'_\zeta \in \Upsilon\}$. A structure $\mathfrak{B} = \langle B, G_\zeta\rangle_{\zeta\in \mathrm{Ord}\,\Upsilon}$ is a Υ-*algebra* if and only if (i) B is a nonempty set of (co)bounded sets which is closed under each $\mathscr{F}_\zeta$ in Υ, and (ii) each G_ζ is the restriction to B of $\mathscr{F}_\zeta$, $\zeta \in \mathrm{Ord}\,\Upsilon$.

Let $\mathrm{Sb}\,\Upsilon$ be the set of all $\mathscr{F}'_\zeta$ which are definable from operations $\mathscr{F}'_\zeta$ in Υ by substitution. Let Υ be *adequate* for IL_R if and only if $\omega \subseteq \mathrm{Ord}\,\mathrm{Sb}\,\Upsilon$. One easily observes:

Theorem 17. *There are finite $\Upsilon \subseteq \mathsf{ID}_C$ which are adequate for IL_R. Moreover, Υ can be chosen so that $\cup$ is the only operation in Υ which is not 1-ary or 0-ary.*

For each Υ-algebra $\mathfrak{B}'$ such that Υ is of a certain kind we now define a structure $\mathfrak{B}$ using only abstract structural properties of $\mathfrak{B}'$ in our

definition. For each r let $\mathscr{F}_{k(r)} = \mathscr{F}'_{k(r)}$ be the 0-ary operation A^r and $\mathscr{F}_{j(r)}$ the operation of similarity type $\langle r, r, r\rangle$ such that $\mathscr{F}_{j(r)}(X_1, X_2) = X_1 \cup X_2$ for any $X_1, X_2 \subseteq A^r$. Consider any $\Upsilon \subseteq \mathsf{IL}_C$ adequate for IL_R and any Υ-algebra $\mathfrak{B}' = \langle B', G'_\delta\rangle_{\delta \in \mathrm{Ord}\,\Upsilon}$. Let $\mathfrak{B}''$ be the corresponding Sb Υ-algebra $\langle B', G'_\delta\rangle_{\delta \in \mathrm{Ord}\,\mathrm{Sb}\,\Upsilon}$. Let B be the set of those $X \in B'$ such that $G'_{j(r)}(X, G'_{k(r)}) = G'_{k(r)}$ for some r, and for each $n \in \omega$, if $\langle r_0, r_1, \ldots r_m\rangle$ is the similarity type of $\mathscr{F}_n$, let G_n be the restriction of G_n' to those arguments $\langle X_1, \ldots, X_m\rangle$ such that $G'_{j(r_i)}(X_i, G'_{k(r_i)}) = G'_{k(r_i)}$, $1 \leqslant i \leqslant m$. Then $\mathfrak{B} = \langle B, G_n\rangle_{n \in \omega}$ is *the* IL_R*-reduct* of $\mathfrak{B}'$.

The class of IL_R-semialgebras and the class of Υ-algebras are closely related, whenever $\Upsilon \subseteq \mathsf{IL}_C$ is adequate for IL_R and contains $\cup$ and $^-$ as elements. This is shown by using T11 and T12.

Theorem 18. *Consider any* $\Upsilon \subseteq \mathsf{IL}_C$ *which is adequate for* IL_R.

(a) *The* IL_R*-semialgebras are the* IL_R*-reducts of the* Υ*-algebras.*

(b) *If* $\cup$ *and* $^-$ *are in* Υ *and* $\mathfrak{B}_i$ *is the* IL_R*-reduct of* $\mathfrak{B}_i'$, $i = 1, 2$, *then* (1) $\mathfrak{B}_i$ *contains no proper* Υ*-subalgebra containing* $\mathfrak{B}_i$, *and* (2) *If* g *is an isomorphism of* $\mathfrak{B}_1$ *onto* $\mathfrak{B}_2$ *then there is one and only one* g' *such that* $g \subseteq g'$ *and* g' *is an isomorphism of* $\mathfrak{B}_1'$ *onto* $\mathfrak{B}_2'$ [8].

The class of Υ_1-algebras and that of Υ_2-algebras are in a certain sense equivalent, whenever $\Upsilon_i \subseteq \mathsf{IL}_C$ is adequate and contains $\cup$ and $^-$, $i = 1, 2$. This follows from T11 and T12.

Theorem 19. *Consider any* $\Upsilon_i \subseteq \mathsf{IL}_C$ *such that* Υ_i *is adequate for* IL_R *and* $\cup$ *and* $^-$ *are in* Υ_i, $i = 1, 2$. *Then for any* Υ_1*-algebra* $\langle B_1, G_\zeta\rangle_{\zeta \in \mathrm{Ord}\,\Upsilon_1}$ *there is one and only one* Υ_2*-algebra* $\langle B_2, H_\zeta\rangle_{\zeta \in \mathrm{Ord}\,\Upsilon_2}$ *such that* $B_1 = B_2$.

For sfs, in place of (co)bounded sets, the situation is less favorable. This follows from our remarks about fields. Thus, only T17 and T18a have analogues, the latter following from T16.

5. Semilocal and elementary operations. We now introduce a generalization of localness, which, however, has little direct appeal. $\mathscr{F}$ is *semilocal*: For each r there is some $g(r) < \omega$ such that, for any $\langle a_0, \ldots, a_{r-1}\rangle$ and $\langle X_1, \ldots, X_m\rangle \in \mathrm{Dom}\,\mathscr{F}$, if $\langle a_0, \ldots, a_{r-1}\rangle \in \mathscr{F}(X_1, \ldots, X_m)$ then there are $b_0, \ldots, b_{g(r)-1}$ such that, for any $\langle Y_1, \ldots, Y_m\rangle \in \mathrm{Dom}\,\mathscr{F}$, if $\{a_0, \ldots, a_{r-1}$.

[8] Let B_i and B_i' consist of relations or respectively (co)bounded sets on A_i, $i = 1, 2$. For g and g' to be isomorphisms we require that $|B_1| = |B_2|$ or $|B_1'| = |B_2'|$, respectively, but not that $|A_1| = |A_2|$.

$b_0, \ldots, b_{g(r)-1}\}^{\omega} \cap X_i = \{a_0, \ldots, a_{r-1}, b_0, \ldots, b_{g(r)-1}\}^{\omega} \cap Y_i$, then $\langle a_0, \ldots, a_{r-1}\rangle \in$ $\in \mathscr{F}(Y_1, \ldots, Y_m)$.

For the case where $g(r) = 0$ for all r, this condition is equivalent to localness. Also, for the case where $\mathscr{F}$ is of similarity type $\langle r_0, r_1, \ldots, r_m\rangle$ this condition together with invariance is equivalent to the condition that some $\exists u_0 \ldots \exists u_{g(r_0)-1}\, \mu$ where μ is quantifier-free expresses $\mathscr{F}$ as μ_1 expresses $\mathscr{E}_1$ (so that μ can be chosen to contain $\doteq$ and m distinct predicate symbols of rank $r_1, \ldots, r_m$ respectively, and distinct $u_0, \ldots u_{g(r_0)-1}$, $v_0, \ldots, v_{r_0-1}$ as the only individual variables).

It is easy to prove:

Theorem 20. *If $\mathscr{F}$ is on and into (co)bounded sets, then $\mathscr{F}$ is semilocal if and only if each ${}_r\mathscr{F}_t$ is semilocal.*

Now let E_C be the closure under substitution of the class of invariant semilocal operations on and into (co)bounded sets, and likewise let E_R be the closure under substitution, when defined, of the class of invariant semilocal operations of a relational similarity type [9]. Then by our above remark about $\exists u_0 \ldots \exists u_{g(r_0)-1}\, \mu$ it can be seen that E_R consists of those $\mathscr{F}$ of a relational similarity type which are elementary in the usual sense.

For fields one then gets an analogue of T14 with E_R and E_C in place of IL_R or IL_C respectively. The proof uses T11, T12b, and T20. For algebras one similarly gets analogues of T17, T18, and T19. In the analogue of T18, let E_R-reducts be IL_R-reducts. Evidently, except for T17, a similar argument can be used for any notion in place of semilocalness for which there is an analogue of T20.

[9] Each class considered in Section 2 is closed under such substitutions.

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ELEMENTARY THEORIES WITH MODELS WITHOUT AUTOMORPHISMS

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According to Ehrenfeucht–Läuchli [62], a model M of an elementary theory T is called *rigid* if the identity function is the only automorphism of the model. It was shown in Ehrenfeucht–Mostowski [56] that every theory T which has an infinite model has a non-rigid model in every infinite power $\mathfrak{m}$. We shall discuss here some aspects of the reverse problem.

By the *r-spectrum* of a theory T we mean the class $K(\mathrm{T})$ of all infinite cardinals $\mathfrak{m}$ such that T has a rigid model in the power $\mathfrak{m}$.

Theorem 1. *For every ordinal* $\alpha \in \aleph_1$, *there exists a theory* T_α *with finitely many non-logical constants such that* [1]

$$K(\mathrm{T}_\alpha) = \{\mathfrak{m} : \aleph_0 \leqslant \mathfrak{m} \leqslant \aleph_\alpha\},$$

i.e., *theory* T_α *has models without automorphisms in every power between* $\aleph_0$ *and* $\aleph_\alpha$ *and in no other power.*

Theorem 2. *For every* $\alpha \in \aleph_\beta$ *there is a theory* T_α *with less than* $\aleph_\beta$ *non-logical constants such that*

$$K(\mathrm{T}_\alpha) = \{\mathfrak{m} : \aleph_0 \leqslant \mathfrak{m} \leqslant \aleph_\alpha\}.$$

Note: By Theorem 1 the case of finitely many and of $\aleph_0$ non-logical constants coincide.

The following two problems are still open:

1) Is it true that the spectrum of every theory T has one of the following forms:

$$K(\mathrm{T}) = \{\mathfrak{m} : \mathfrak{m} > \aleph_0\},$$
$$K(\mathrm{T}) = \{\mathfrak{m} : \mathfrak{m} \geqslant \aleph_0\},$$
$$K(\mathrm{T}) = \{\mathfrak{m} : \aleph_0 < \mathfrak{m} \leqslant \aleph_\alpha\},$$
$$K(\mathrm{T}) = \{\mathfrak{m} : \aleph_0 \leqslant \mathfrak{m} \leqslant \aleph_\alpha\}?$$

[1] We assume here the generalized continuum hypothesis. Without this assumption the formula would have the form $K(\mathrm{T}_\alpha) = \{\mathfrak{m} : \beth_0 \leqslant \mathfrak{m} \leqslant \beth_\alpha\}$, where $\alpha \in \aleph_1$ and $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, etc.

2) Is it true that in the last two cases the cardinality of α cannot be bigger than the cardinality of the set of non-logical constants of T, or, in the case of finitely many non-logical constants, α is smaller than $\aleph_1$?

Proof of Theorem 1. We shall define, at first, the basic theories T′ and T whose extensions T_α (in general, not recursive) satisfy the conditions given in Theorem 1.

Theory T′ contains as non-logical constants the unary predicate U( ) and the five binary predicates E(,), C(,), R(,), O(,), P(,), (= is included in the underlying logic).
The axioms are:

a1. $\bigwedge_x \bigwedge_y (E(x, y) \to (\neg U(x) \wedge \neg U(y)))$.

a2. $\bigwedge_x \bigwedge_y (C(x, y) \to (U(x) \wedge U(y)))$.

a3. $\bigwedge_x \bigwedge_y (O(x, y) \to (U(x) \wedge U(y)))$.

a4. $\bigwedge_x \bigwedge_y (R(x, y) \to (\neg U(x) \wedge U(y)))$.

b1. $\bigwedge_x (U(x) \to C(x, x))$.

b2. $\bigwedge_x \bigwedge_y (C(x, y) \to C(y, x))$.

b3. $\bigwedge_x \bigwedge_y \bigwedge_z (C(x, v) \wedge C(y, z) \to C(x, z))$.

[(Thus $\langle M, U, E, C, R, O, P\rangle$ is a model of b1–b3 if and only if C is an equivalence relation on U.]

c1. $\bigwedge_x \bigwedge_y (P(x, y) \to C(x, y))$.

[P can hold only between elements which belong to the same equivalence class of C.]

d1. $\bigwedge_x \bigwedge_y \bigwedge_z \bigwedge_t (O(x, y) \wedge C(x, z) \wedge C(y, t) \to O(z, t))$.

d2. $\bigwedge_x \neg O(x, x)$.

d3. $\bigwedge_x \bigwedge_y (O(x, y) \to \neg O(y, x))$.

d4. $\bigwedge_x \bigwedge_y \bigwedge_z (O(x, y) \wedge O(y, z) \to O(x, z))$.

d5. $\bigwedge_x \bigwedge_y (U(x) \wedge U(y) \to (O(x, y) \vee O(y, x) \vee C(x, y)))$.

[O is a linear ordering between equivalence classes of C.]

e1. $\bigwedge_x \bigwedge_y (\neg U(x) \wedge \neg U(y) \rightarrow (\bigwedge_z (E(z, x) \equiv E(z, y)) \rightarrow x = y))$.

[E is extensional.]

f1. $\bigwedge_x \bigwedge_y \bigwedge_z (R(x, y) \rightarrow (R(x, z) \equiv C(y, z)))$.

f2. $\bigwedge_x \bigvee_y (\neg U(x) \rightarrow R(x, y))$.

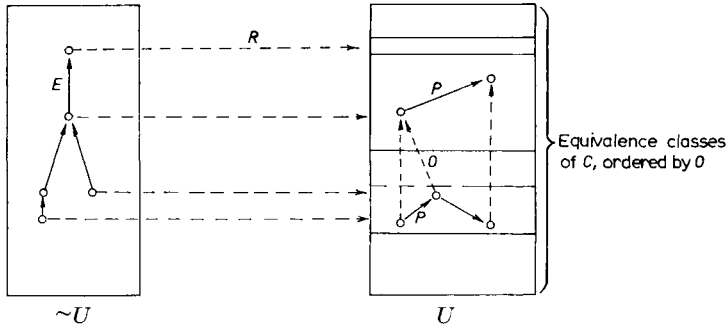
f3. $\bigwedge_x \bigvee_y (U(x) \rightarrow R(y, x))$.

f4. $\bigwedge_x \bigwedge_y \bigwedge_z \bigwedge_t ((E(x, y) \wedge R(x, z) \wedge R(y, t)) \rightarrow O(z, t))$.

f5. $\bigwedge_x \bigwedge_y \bigwedge_z \bigvee_t \bigvee_u (R(x, y) \wedge O(z, y) \rightarrow E(t, x) \wedge R(t, u) \wedge (O(z, u) \vee C(z, u)))$.

[R is a function which maps $\sim U$ onto equivalence classes of C in U; and if $E(x, y)$ (x is an element of y), then x is mapped on an "earlier" equivalence class (and the equivalence class corresponding to y is the least upper bound in the sense of the ordering O of the set of classes corresponding to "elements" of y).]

We can intuitively represent the models of the axioms given above as follows:



From the axioms given above it follows that:

Lemma 1. *If in a model M of an extension of the theory* T′ *the relation P, restricted to an equivalence class, has a non-trivial automorphism, then the model M has a non-trivial automorphism.*

This is clear because P holds only between elements of one class and all other relations hold between classes as a whole.

Now we shall define an extension T of T′ specifying the properties of the relation P.

Lemma 2. *There exists a theory* T* *with one binary predicate* P *which has the following properties*:

(a) T* *has infinitely many finite models*;
(b) *every finite model of* T* *is rigid*;
(c) *no infinite model of* T* *is rigid.*

Let us consider the theory with the following axioms: [2]

(1) $\bigwedge_x \bigwedge_y \bigwedge_z (P(x, y) \wedge P(x, z) \rightarrow y = z)$.

[P is a function.]

Define: $L(x) \equiv_{df} \bigwedge_y \neg P(x, y)$.

[L(x) means: x is the last element.]

$F(x) \equiv_{df} \bigwedge_y \neg P(y, x)$.

[F(x) means: x is a first element.]

(2) $\bigvee_x L(x) \wedge \bigwedge_x \bigwedge_y (L(x) \wedge L(y) \rightarrow x = y)$.

[There is exactly one last element.]

(3) $\bigvee_x \bigvee_y (F(x) \wedge F(y) \wedge x \neq y) \wedge \bigwedge_x \bigwedge_y \bigwedge_z (F(x) \wedge F(y) \wedge F(z) \wedge x \neq y \rightarrow$
$\rightarrow (x = z \vee y = z))$.

[There are exactly two first elements.]

(4) $\bigwedge_x \bigwedge_y \bigwedge_z (\neg L(x) \wedge P(y, x) \wedge P(z, x) \rightarrow y = z)$.

[Every element except the last one has, at most, one predecessor.]

(5) $\bigwedge_x \bigvee_y \bigvee_z \bigwedge_t (L(x) \rightarrow (P(y, x) \wedge P(z, x) \wedge y \neq z \wedge (P(t, x) \rightarrow (t = y \vee t = z))))$.

[The last element has exactly two predecessors.]

Axiom schemata, for $n = 1, 2, \ldots$:

(6n) $\bigwedge_{x_1} \bigwedge_{x_2} \ldots \bigwedge_{x_n} \neg (P(x_1, x_2) \wedge P(x_2, x_3) \wedge \ldots \wedge P(x_{n-1}, x_n) \wedge P(x_n, x_1))$.

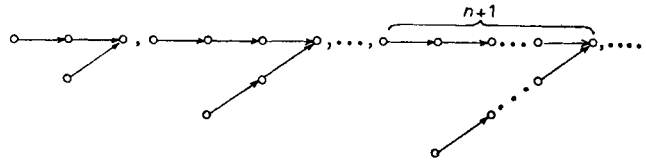
[There are no loops.]

(7n) $\bigvee_{x_1} \bigvee_{x_2} \ldots \bigvee_{x_n} (F(x_1) \wedge L(x_n) \wedge P(x_1, x_2) \wedge \ldots \wedge P(x_{n-1}, x_n)) \rightarrow$
$\rightarrow (\bigvee_{y_1} \ldots \bigvee_{y_{n-1}} (F(y_1) \wedge L(y_{n-1}) \wedge P(y_1, y_2) \wedge \ldots \wedge P(y_{n-2}, y_{n-1})) \vee$
$\vee \bigvee_{y_1} \ldots \bigvee_{y_{n+1}} (F(y_1) \wedge L(y_{n+1}) \wedge P(y_1, y_2) \wedge \ldots \wedge P(y_n, y_{n+1})))$.

[If there is a "chain" of length n, then there is a "chain" of length $n+1$ or $n-1$.]

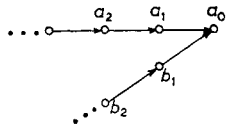
[2] The idea for this construction was given to the author by Hans Läuchli.

It follows from (1)–(7) that the only finite models of T* are the following:

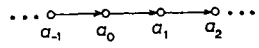


None of them have automorphisms.

Let us take an infinite model. It contains either a part of the form



(and then the function which maps a_i on b_i, and conversely, can be extended to an automorphism) or it contains a part of the form



(and then $a_i \to a_{i+1}$ can be extended to an automorphism).

Theory T is an extension of T′ by axioms g1, ... g7, which are the relativization of (1)–(7) of T* to the equivalence classes of C, and the axiom schemata hn and in. The relativization gk of formula (k) ($k = 1, \ldots, 7$) has the following content: "For every w in U formula (k), relativized to the set of those y for which $C(w, y)$, holds".

For example:

$$\text{g1.}\quad \bigwedge_{w} (U(w) \to \bigwedge_{x} \bigwedge_{y} \bigwedge_{z} (C(w, x) \wedge C(w, y) \wedge C(w, z) \to \to (P(x, y) \wedge P(x, z) \to y = z))).$$

The axiom schemata ($n = 1, 2, \ldots$):

$$\text{h}n.\quad \bigwedge_{x_1} \bigwedge_{x_2} \ldots \bigwedge_{x_n} \bigwedge_{y_1} \ldots \bigwedge_{y_n} [(C(x_1, x_2) \wedge C(x_1, x_3) \wedge \ldots \wedge C(x_1, x_n) \wedge \wedge C(y_1, y_2) \wedge \ldots \wedge C(y_1, y_n) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n \wedge \wedge y_1 \neq y_2 \wedge \ldots \wedge y_{n-1} \neq y_n \wedge \bigwedge_{z} (C(x_1, z) \to (z = x_1 \vee \ldots \vee z = x_n)) \wedge \wedge \bigwedge_{z} (C(y_1, z) \to (z = y_1 \vee \ldots \vee z = y_n))) \to C(x_1, y_1)].$$

[There is at most one equivalence class of C in U which has exactly n elements.]

$in.\ \bigvee_{x_1}\bigvee_{x_2}\dots\bigvee_{x_{2n+2}}(C(x_1,x_2)\wedge\dots\wedge C(x_1,x_{2n+2})\wedge x_1\neq x_2\wedge\dots\wedge x_{2n+1}\neq$

$\neq x_{2n+2}\wedge\bigwedge_{y}(C(x_1,y)\rightarrow(y=x_1\vee\dots\ y=x_{2n+2}))).$

[There is an equivalence class which has exactly $2n+2$ elements.]

From Lemma 1 and the properties of T* and T, it follows that if a model $\mathcal{M}$ of T is rigid then:

1°. Every equivalence class of C is finite.

2°. The number of different classes is countable (each two have a different number of elements).

Now, for every ordinal $\alpha\in\aleph_1$, we shall define an extension T_α of T in such a way that first, the conditions 1 and 2 will be not only necessary but also sufficient for rigidity of a model M of T_α, and second, the models for which 1°–2° hold exist only in powers $\mathfrak{m}$ for $\aleph_0\leqslant\mathfrak{m}\leqslant\aleph_\alpha$.

Let $<_\alpha$ be a well-ordering of the set of natural numbers having the order-type $\omega+\alpha$. For every pair of natural numbers (k_1, k_2) such that $k_1<_\alpha k_2$ we take the axiom which says:

$\mathrm{j}(k_1, k_2)$: if x belongs to the equivalence class which has exactly k_1 elements and y belongs to the equivalence class which has exactly k_2 elements then $O(x, y)$.

Let us notice that axioms $\mathrm{j}(k_i, k_j)$ establish an isomorphism between the ordering O among finite equivalence classes of C and the well-ordering $<_\alpha$.

If conditions 1°–2° hold for a model M then for every automorphism φ: if $U(x)$ then $\varphi(x)=x$, because every element of U is in one of the equivalence classes which does not have any nontrivial automorphism and because all such classes are distinct.

On the other hand, by axioms e1 and f1–f5 if the ordering O is the well-ordering of the type $\omega+\alpha$ then the set of elements x satisfying $\neg U(x)$ in M with the relation E, is isomorphic with a founded family of sets having ranks smaller than $\omega+\alpha$.

Therefore:

$$\text{if } x\notin U \text{ then } \varphi(x)=x,$$

because a founded family of sets has no nontrivial automorphism. Hence, φ is the identity function.

Taking the maximal family of sets of the ranks $<\omega+\alpha$, we obtain

the model M_α of cardinality $\aleph_\alpha$ (continuum hypothesis). Rigid models of smaller cardinalities for T_α can be built as elementary submodels of M_α. q.e.d.

The proof of Theorem 2 is a small modification of the proof of Theorem 1. We can omit all considerations concerning Theory T*. Instead of P we introduce a family of individual constants. Instead of the assumptions hn and in we assume that each equivalence class has exactly two elements. Therefore the equivalence class which does not contain any individual constant has an "intrinsic automorphism". The $<_\alpha$ is, therefore, the ordering between constants.

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COMBINATORIAL THEOREMS FOR THE CONSTRUCTION OF MODELS *

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The following combinatorial lemma is a version of a theorem of Rado [49]. Let $\mathscr{F}$ be a family of functions with domains $\subseteq S$ and range $\{0, 1\}$ such that: (a) for every finite subset E of S there exists $f \in \mathscr{F}$ with dom $f = E$; (b) the restriction of any function in $\mathscr{F}$ to any subset of its domain belongs to $\mathscr{F}$; (c) if f is any function whose domain is a subset of S and whose restrictions to finite subsets of its domain belong to $\mathscr{F}$ then $f \in \mathscr{F}$. Then there exists $f \in \mathscr{F}$ with domain S. This result was rediscovered independently by the author [59b] and by A. Robinson [61b], who both recognized its importance for model theory.

In this paper we make a further study of the interconnections between model theory and combinatorics and attempt a systematic development of this subject.

For many languages L we make the following observations: Through simple devices we may present the structures corresponding to L as algebraic structures $\mathfrak{F} = \langle F, f \rangle$ with one unary operation f. Clearly, the first-order language L_1 associated with the similarity type of $\mathfrak{F}$ is in general inadequate to properly reflect the much richer structures corresponding to L. We observe, however, that we may retain the grammar of L_1 and repair the mentioned inadequacy by choosing a suitable category of maps between the universes of the structures $\mathfrak{F}$ and stipulating a truth definition dependent on this category.

This approach enables us to give a uniform treatment of the model theories of various languages and to discern some basic combinatorial facts (as the above lemma of Rado and our Theorem 3.2) reflected in the theorems of these model theories. We have also found it interesting

* These results were obtained while the author was working on a research project in the foundations of mathematics at the University of California, Berkeley, U.S. National Science Foundation Grant G–19673.

to seek those categories that induce a logic in which an analog of the compactness theorem holds (Theorem 2.4).

In this paper we start by studying in general a first-order grammar with categorical truth definitions. Such an inquiry may be of independent interest. For this exposition we restrict ourselves to the quantifier-free grammar.

1. Categorical truth-definitions. Let $\mathscr{F}$ be a class of algebraic structures $\mathfrak{F}=\langle F, f\rangle$ with one unary operation f. Denote by $\mathscr{F}_0$ the class of all universes of structures in $\mathscr{F}$. We assume that we have an (open) first-order grammar $G(\mathscr{F}_0)$ which provides us, apart from the logical connectives and equality, with the following: (a) a unary operation symbol f; (b) for each $x \in \bigcup \mathscr{F}_0$ uniquely determined symbols x' (an individual variable) and x'' (an individual constant). Let $G(\mathscr{F}_0)$ have no other variables nor constants.

Let $\mathscr{P}$ be a family of mappings with the following properties:

(i) for any $p \in \mathscr{P}$ the identity map of the domain of p belongs to $\mathscr{P}$;

(ii) for any $p, q \in \mathscr{P}$ with range $p = \operatorname{dom} q$ the composition $q \circ p$ belongs to $\mathscr{P}$;

(iii) for any $p \in \mathscr{P}$ which is one-to-one the inverse p^{-1} belongs to $\mathscr{P}$;

(iv) for any $p_1, p_2 \in \mathscr{P}$, if $\operatorname{dom} p_1 \cap \operatorname{dom} p_2$ is not empty then there exists $q \in \mathscr{P}$ such that $\operatorname{dom} q = \operatorname{dom} p_1 \cap \operatorname{dom} p_2$.

(v) for any $p \in \mathscr{P}$ and $S \in \mathscr{P}_0$, if $S \subseteq \operatorname{dom} p$ then the restriction $p \restriction S$ of p to S belongs to $\mathscr{P}$, where $\mathscr{P}_0$ is the family of all domains and ranges of mappings in $\mathscr{P}$.

$\mathscr{F}$ and $\mathscr{P}$ are interrelated by the following assumption:

(vi) $\mathscr{F}_0 = \mathscr{P}_0$.

Our truth definition for formulas in $G(\mathscr{F}_0)$ depends on this category $\mathscr{P}$ and is therefore called a *categorical truth definition* [1]. It may be given as follows: Let $\mathfrak{F}=\langle F, f\rangle$ be a structure in $\mathscr{F}$, let φ be a formula of the grammar $G(\mathscr{F}_0)$ and suppose that for all individual constants x'' occurring in φ we have $x \in F$, (otherwise φ assumes the value false). If φ contains no variables then we write $\models_{\mathfrak{F}} \varphi$ if and only if F satisfies

[1] There is no connection with the "semantical categories" mentioned in Tarski [36a]. But, of course, the classical concepts and methods introduced in this central paper of Tarski have served here as guidelines.

φ in the usual sense. If φ contains variables $x_0', \ldots, x_n'$ (and no others) then we adopt the following definition.

Definition 1.1. $\models_{\mathfrak{F}} \varphi(x_0', \ldots, x_n')$ *if and only if* $\models_{\mathfrak{F}} \varphi((px_0)'', \ldots, (px_n)'')$ *for all* $p \in \mathscr{P}$ *with* $x_0, \ldots, x_n \in \operatorname{dom} p$ *and* $px_0, \ldots, px_n \in F$ *for which the restriction* $p \restriction \{x_0, \ldots, x_n\}$ *is one-to-one.*

The following lemma allows us to effect in some cases a "change of variables".

Lemma 1.2. *If* $q \in \mathscr{P}$, q *is one-to-one,* $x_0, \ldots, x_n \in \operatorname{dom} q$, *and* $\varphi(x_0', \ldots, x_n')$ *contains exactly the variables* $x_0', \ldots, x_n'$ *then for any* $\mathfrak{F} \in \mathscr{F}$ *we have* $\models_{\mathfrak{F}} \varphi(x_0', \ldots, x_n')$ *if and only if* $\models_{\mathfrak{F}} \varphi((qx_0)', \ldots, (qx_n)')$.

Proof.

(a) Suppose that $\models_{\mathfrak{F}} \varphi(x_0', \ldots, x_n')$. Let p' be any element of $\mathscr{P}$ with $qx_0, \ldots, qx_n \in \operatorname{dom} p'$, $p'qx_0, \ldots, p'qx_n \in F$ and $p' \restriction \{qx_0, \ldots, qx_n\}$ one-to-one. Let $S = \operatorname{range} q \cap \operatorname{dom} p'$. Observe that $S \in \mathscr{F}_0$. Let p be $(p' \restriction S) \circ (q^{-1} \restriction S)^{-1}$. Observe that $p \in \mathscr{P}$, $p \restriction \{x_0, \ldots, x_n\}$ is one-to-one, $x_0, \ldots, x_n \in \operatorname{dom} p$ and $px_0, \ldots, px_n \in F$. Thus, since $\models_{\mathfrak{F}} \varphi(x_0', \ldots, x_n')$, we have $\models_{\mathfrak{F}} \varphi((px_0)'', \ldots, (px_n)'')$. Since $px_i = p'\,(q^{-1})^{-1}\, x_i = p'qx_i$ for $i = 0, 1, \ldots, n$, this means that $\models_{\mathfrak{F}} \varphi((p'qx_0)'', \ldots, (p'qx_n)'')$. Hence $\models_{\mathfrak{F}} \varphi((qx_0)', \ldots, (qx_n)')$.

(b) Suppose that $\models_{\mathfrak{F}} \varphi((qx_0)', \ldots, (qx_n)')$. Let p be any element of $\mathscr{P}$ with $x_0, \ldots, x_n \in \operatorname{dom} p$, $px_0, \ldots, px_n \in F$ and $p \restriction \{x_0, \ldots, x_n\}$ one-to-one. Let $S = \operatorname{dom} p \cap \operatorname{dom} q$. Observe that $S \in \mathscr{F}_0$, $p \restriction S \in \mathscr{P}$, $q \restriction S \in \mathscr{P}$. Let p^* be $(p \restriction S) \circ (q \restriction S)^{-1}$. Observe that $p^* \in \mathscr{P}$, $qx_0, \ldots, qx_n \in \operatorname{dom} p^*$, $p^*qx_0, \ldots, p^*qx_n \in F$ and $p^* \restriction \{qx_0, \ldots, qx_n\}$ one-to-one. Thus, since $\models_{\mathfrak{F}} \varphi((qx_0)', \ldots, (qx_n)')$, we have $\models_{\mathfrak{F}} \varphi((p^*qx_0)'', \ldots, (p^*qx_n)'')$. Observe that $p^*qx_i = pq^{-1}qx_i = px_i$, $i = 0, 1, \ldots, n$. Hence $\models_{\mathfrak{F}} \varphi((px_0)'', \ldots, (px_n)'')$ and thus $\models_{\mathfrak{F}} \varphi(x_0', \ldots, x_n')$.

Let $\mathscr{F}$ and $\mathscr{P}$ be given as above. Owing to Definition 1.1 the grammar $G(\mathscr{F}_0)$ is extended to a language, i.e., a grammar with a scheme of interpretation. This language will be denoted by $L(\mathscr{P})$ since it depends only on $\mathscr{P}$ by virtue of property (vi).

If φ is any formula in $L(\mathscr{P})$ and $\mathfrak{F} \in \mathscr{F}$ then we say that $\mathfrak{F}$ is a *model* of φ if $\models_{\mathfrak{F}} \varphi$. A structure $\mathfrak{F}$ is a model of a set Γ of formulas of $L(\mathscr{P})$ if $\models_{\mathfrak{F}} \varphi$ for all $\varphi \in \Gamma$. For any such set Γ let $\mathscr{M}(\Gamma, \mathscr{P})$ be the class of all models of Γ whose universes are domains of mappings in $\mathscr{P}$.

For our purposes it is important to have a necessary and sufficient condition for the existence of a set Γ such that $\mathscr{F} = \mathscr{M}(\Gamma, \mathscr{P})$. Such a condition is given by the following theorem.

Theorem 1.3. *If $\mathscr{F}$ and $\mathscr{P}$ are given as above then there exists a set Γ of formulas of $L(\mathscr{P})$ such that $\mathscr{F}=\mathscr{M}(\Gamma,\mathscr{P})$ if and only if*

(*) *for any $F\in\mathscr{F}_0$ and any unary operation f on F the following condition is sufficient for $\langle F,f\rangle\in\mathscr{F}$: For any finite subset $E\subseteq F$ and any natural number n there exists $\mathfrak{G}\in\mathscr{F}$, $\mathfrak{G}=\langle G,g\rangle$, and $p\in\mathscr{P}$ such that $E\subseteq\operatorname{dom}p\subseteq F$, $pE\subseteq G$, $p\restriction E$ one-to-one, and for any $x,y\in E$ and all $r,s\leqslant n$ we have $f^rx=f^sy$ if and only if $g^rpx=g^spy$.*

Proof. For any finite set $E\subseteq\bigcup\mathscr{F}_0$ and any natural number n we define the formula $\varphi_{E,n}$ as follows: If there exists $\mathfrak{F}=\langle F,f\rangle\in\mathscr{F}$ and $p\in\mathscr{P}$ such that $E\subseteq\operatorname{dom}p$, $pE\subseteq F$, $p\restriction E$ one-to-one, let $\varphi_{p,\mathfrak{F},E,n}$ be the conjunction of all equations (inequalities)

$$f^rx'=f^sy' \text{ or } f^rx'\neq f^sy',\ (r,s\leqslant n,\ x,y\in E).$$

The equation (inequality) is taken in the cases

$$\models_{\mathfrak{F}} f^r(px)''=f^s(py)'',\ \models_{\mathfrak{F}} f^r(px)''\neq f^s(py)'',$$

respectively. Let $\varphi_{E,n}$ be the (finite) disjunction of all such formulas $\varphi_{p,\mathfrak{F},E,n}$. If no such $\mathfrak{F},p$ exist let $\varphi_{E,n}$ be the formula $x'=x'$ for some $x\in E$.

Denote by $\Gamma(\mathscr{F},\mathscr{P})$ the set of all formulas $\varphi_{E,n}$, E finite, $E\subseteq\bigcup\mathscr{F}_0$, $n=0,1,2,\ldots$. We shall show that $\Gamma(\mathscr{F},\mathscr{P})$ serves as the set Γ required in the theorem, i.e., that $\mathscr{M}(\Gamma(\mathscr{F},\mathscr{P}),\mathscr{P})=\mathscr{F}$. The other half of the theorem is straightforward.

(a) If $\mathfrak{F}\in\mathscr{F}$ then $\models_{\mathfrak{F}}\varphi$ for all $\varphi\in\Gamma(\mathscr{F},\mathscr{P})$. Namely, let $\varphi_{E,n}\in\Gamma(\mathscr{F},\mathscr{P})$, let $p\in\mathscr{P}$, $E\subseteq\operatorname{dom}p$, $p\restriction E$ one-to-one and $pE\subseteq F$. Then, by definition of $\varphi_{E,n}$, the conjunction of those equations (inequalities)

$$f^rx'\underset{(\neq)}{=}f^sy';\ r,s\leqslant n;\ x,y\in E$$

is a disjunctive member of $\varphi_{E,n}$ for which

$$\models_{\mathfrak{F}} f^r(px)''\underset{(\neq)}{=}f^s(py)''.$$

Since this is the case for any such p we have $\models_{\mathfrak{F}}\varphi_{E,n}$ by Definition 1.1.

(b) If $\mathfrak{F}=\langle F,f\rangle$ is such that $\models_{\mathfrak{F}}\varphi$ holds for all $\varphi\in\Gamma(\mathscr{F},\mathscr{P})$ and $F\in\mathscr{F}_0$, then $\mathfrak{F}\in\mathscr{F}$. Namely, let E be any finite subset of F and n any natural number. By assumption we have $\models_{\mathfrak{F}}\varphi_{E,n}$. Hence, for any

$q \in \mathscr{P}$ with $E \subseteq \operatorname{dom} q$, $qE \subseteq F$, $q \restriction E$ one-to-one, there exists $\mathfrak{G} = \langle G, g \rangle \in \mathscr{F}$ and $p \in \mathscr{P}$ with $E \subseteq \operatorname{dom} p$, $pE \subseteq G$, $p \restriction E$ one-to-one, such that for all $x, y \in E$ and $r, s \leqslant n$ we have

$$f^r qx = f^s qy \text{ if and only if } g^r px = g^s py.$$

If, in particular, q is the identity on F the above observation implies, by property (*), that $\mathfrak{F} \in \mathscr{F}$.

Theorem 1.3 gives a mathematical characterization of those pairs $\mathscr{F}, \mathscr{P}$ for which $\mathscr{F} = \mathscr{M}(\Gamma, \mathscr{P})$ for some set Γ of formulas of $L(\mathscr{P})$. Thus the properties (i)–(vi) and (*) may be taken as a starting point for an axiomatic development of the model theory of $L(\mathscr{P})$. The following corollary is a, rather easy, example of such a general result.

Corollary 1.4. *If $\mathscr{F}, \mathscr{P}$ satisfy* (i)–(vi) *and* (*) *then $\mathscr{F}$ is closed under isomorphisms from $\mathscr{P}$; i.e., if $\mathfrak{F} = \langle F, f \rangle \in \mathscr{F}$, $p \in \mathscr{P}$, $\operatorname{dom} p = F$ and p one-to-one, then*

$$p\mathfrak{F} = \langle pF, pfp^{-1} \rangle \in \mathscr{F}.$$

Proof. Let E be a finite subset of pF. We wish to apply (*) and show that there exists $\mathfrak{G} = \langle G, g \rangle \in \mathscr{F}$ and $q \in \mathscr{P}$ such that $E \subseteq \operatorname{dom} q \subseteq pF$ and for all $x, y \in E$ and all r, s we have

$$(pfp^{-1})^r x = (pfp^{-1})^s y \text{ if and only if } g^r qx = g^s qy.$$

Let $\mathfrak{G} = \mathfrak{F}$, $q = p^{-1}$. Then obviously $E \subseteq \operatorname{dom} p^{-1} = pF$ and the equations $(pfp^{-1})^r x = (pfp^{-1})^s y$, $pf^r p^{-1} x = pf^s p^{-1} y$, $f^r p^{-1} x = f^s p^{-1} y$, $g^r qx = g^s qy$ are equivalent.

We now wish to substantiate our claim that the structures corresponding to many formal languages may be presented as algebraic structures with one unary operation. For this purpose we will have to produce for the class $\mathscr{C}$ of models of any set of formulas of such a language two things: (a) a class $\mathscr{F}$ of algebraic structures $\mathfrak{F} = \langle F, f \rangle$; (b) a family $\mathscr{P}$ of mappings between universes of structures in $\mathscr{F}$.

Example 1.5. *First-order theories.* Consider relational structures with, e.g., one unary relation P, one binary operation $+$, and one constant a. Let Γ be a set of universal sentences of the first-order language corresponding to P, $+$, a. To any model $\mathfrak{A} = \langle A, P, +, a \rangle$ of the so-given theory determine an algebraic structure $\mathfrak{F}_{\mathfrak{a}} = \langle F_{\mathfrak{a}}, f_{\mathfrak{a}} \rangle$ with one unary operation $f_{\mathfrak{a}}$ as follows. Let $F_{\mathfrak{a}}$ be the union of (three) mutually exclusive

sets $\{c_p, c_\mathfrak{a}, c_{\text{true}}, c_{\text{false}}\} \times A$, $\{c_+\} \times A \times A$, and A. Let $f_\mathfrak{a}$ be the identity on $A \cup \{c_{\text{true}}, c_{\text{false}}\} \times A$. Let $f_\mathfrak{a}(z) = a_\mathfrak{a}$ for all $z \in \{c_\mathfrak{a}\} \times A$, let $f_\mathfrak{a}\langle c_p, x\rangle = \langle c_{\text{true}}, x\rangle$ if $x \in P$, $f_\mathfrak{a}\langle c_p, x\rangle = \langle c_{\text{false}}, x\rangle$ if $x \notin P$ and let $f_\mathfrak{a}\langle c_+, x, y\rangle = x + y$ for all $x, y \in A$. Let $\mathscr{F}$ be the class of all such systems $\mathfrak{F}_\mathfrak{a}$. If $\mathfrak{A}, \mathfrak{B}$ are models of Γ and p_0 is any mapping of A onto B with $p_0(a_\mathfrak{a}) = a_\mathfrak{b}$, let p be the mapping of $F_\mathfrak{a}$ onto $F_\mathfrak{b}$ determined as follows: $p\langle u, v\rangle = \langle u, p_0(v)\rangle$ for $\langle u, v\rangle \in \{c_p, c_{\text{true}}, c_\mathfrak{a}, c_{\text{false}}\} \times A$, $p\langle c_+, x, y\rangle = \langle c_+, p_0(x), p_0(y)\rangle$ and $p(x) = p_0(x)$ for all $x, y \in A$. Let $\mathscr{P}$ be the class of all such mappings p. Clearly $\mathscr{F}, \mathscr{P}$ satisfy (i)–(vi) and (*).

Example 1.6. *Higher order theories.* Consider structures with, say, one binary relation R over a set A, one operation O associating to pairs $\langle x, P\rangle$, where $x \in A$ and $P \subseteq A \times A$, a subset $O(x, P)$ of A, and one unary operation k on A. Let φ be, say, the following universal sentence of the (second-order) language corresponding to these systems:

$$(\forall x)(\forall y)(\forall P)(\langle x, y\rangle \in P \wedge \langle y, k(x)\rangle \in R \rightarrow k(y) \in O(x, P)).$$

To any system $\mathfrak{A} = \langle A, R, O, k\rangle$ satisfying φ we associate an algebraic system $\mathfrak{F}_\mathfrak{a} = \langle F_\mathfrak{a}, f_\mathfrak{a}\rangle$ with one unary operation $f_\mathfrak{a}$ as follows: Let $F_\mathfrak{a}$ be the union of the (mutually exclusive) sets A, $\mathfrak{P}(A \times A)$, $\{c_k, c_{\text{true}}, c_{\text{false}}\} \times A$, $\{c_1\} \times \mathfrak{P}(A \times A) \times A \times A$, $\{c_0\} \times A \times \mathfrak{P}(A \times A)$, $R_\mathfrak{a} (\subseteq A \times A)$, $\{c_R\} \times A \times A$. Let $f_\mathfrak{a}$ be the following operation on $F_\mathfrak{a}$: $f_\mathfrak{a}(x) = x$ for $x \in A \cup \mathfrak{P}(A \times A) \cup \cup \{c_{\text{true}}, c_{\text{false}}\} \times A$, $f_\mathfrak{a}\langle c_1, P, x, y\rangle = \langle c_{\text{true}}, x\rangle$ if $\langle x, y\rangle \in P$ and $f_\mathfrak{a}\langle c_1, P, x, y\rangle = \langle c_{\text{false}}, x\rangle$ if $x, y \notin P$, $f_\mathfrak{a}\langle c_0, x, P\rangle = O(x, P)$ for all $x \in A$ and $P \subseteq A \times A$, $f_\mathfrak{a}\langle c_k, x\rangle = k(x)$ for all $x \in A$, $f_\mathfrak{a}\langle c_R, x, y\rangle = \langle c_{\text{true}}, x\rangle$ if $\langle x, y\rangle \in R_\mathfrak{a}$ and $f_\mathfrak{a}\langle c_R, x, y\rangle = \langle c_{\text{false}}, x\rangle$ if $\langle x, y\rangle \notin R_\mathfrak{a}$. Let $\mathscr{F}$ be the family of all such structures $\mathfrak{F}_\mathfrak{a}$. If $\mathfrak{A}, \mathfrak{B}$ are any structures in $\mathscr{F}$, and p_0 is a map of A onto B such that $R_\mathfrak{b} = \{\langle p_0 x, p_0 y\rangle : \langle x, y\rangle \in R_\mathfrak{a}\}$, let p be the mapping from $F_\mathfrak{a}$ onto $F_\mathfrak{b}$ induced by p_0 as follows: $p(x) = p_0(x)$ for $x \in A$, $p(P) = \{\langle p_0 x, p_0 y\rangle : \langle x, y\rangle \in P\}$ for $P \in \mathfrak{P}(A \times A)$, $p\langle c, x\rangle = \langle c, p_0(x)\rangle$ for $c \in \{c_k, c_{\text{true}}, c_{\text{false}}\}$, $p\langle c_1, P, x, y\rangle = \langle c_1, p(P), p_0(x), p_0(y)\rangle$ for all $P \in \mathfrak{P}(A \times A)$ and all $x, y \in A$, $p\langle c_0, x, P\rangle = \langle c_0, p_0(x), p(P)\rangle$, $p\langle c_R, x, y\rangle = \langle c_R, p_0(x), p_0(y)\rangle$. Let $\mathscr{P}$ be the family of all such mapping p. Then $\mathscr{F}, \mathscr{P}$ satisfy again (i)–(vi) and (*).

It is clear how the above constructions of $\mathscr{F}$ and $\mathscr{P}$ generalize to other higher order theories and that operations corresponding to generalized quantifiers like "there exist at least $\aleph_\varkappa$" can be accommodated.

2. Compactness. Let $\mathscr{P}$ be a family of mappings satisfying (i)–(v) and consider $L(\mathscr{P})$. In analogy to first-order logic we would say that

the language $L(\mathscr{P})$ is compact if and only if for any set Γ of formulas of $L(\mathscr{P})$ we have:

(1) If $\mathscr{M}(\Gamma_0, \mathscr{P}) \neq \emptyset$ for any finite $\Gamma_0 \subseteq \Gamma$ then $\mathscr{M}(\Gamma, \mathscr{P}) \neq \emptyset$.

By virtue of Theorem 1.3 the compactness property (1) can be formulated as a purely combinatorial property (2) of the family $\mathscr{P}$:

Definition 2.1. *A family of mappings $\mathscr{P}$ satisfying* (i)–(v) *is compact if and only if*

(2) *for any collection $\{\mathscr{F}_i : i \in I\}$ of families $\mathscr{F}_i$ such that $\mathscr{F}_i, \mathscr{P}$ satisfy* (i)–(vi) *and* (*) *which has the finite intersection property*

$$\bigcap \{\mathscr{F}_i : i \in I\} \neq \emptyset.$$

Clearly (1) and (2) are equivalent.

For any formula $\varphi(x_0^+, \ldots, x_n^+)$ of $L(\mathscr{P})$, where $x_0^+, \ldots, x_n^+$ are all variables and constants $x_i^+ = x_i'$ or x_i'' occurring in φ, let $\mathscr{F}(\varphi(x_0^+, \ldots, x_n^+))$ be the set of all $\mathfrak{F} = \langle F, f\rangle$ such that $F = \operatorname{dom} p$ for some $p \in \mathscr{P}$ and $\models_{\mathfrak{F}} \varphi(x_0'', \ldots, x_n'')$. Let B be the Boolean algebra of sets $\mathscr{F}(\varphi)$, $\varphi \in L(\mathscr{P})$.

For any set Γ of formulas of $L(\mathscr{P})$ let $D^0(\Gamma)$ be the set of all $\mathscr{F}(\varphi) \in B$ such that $\mathscr{F}(\varphi) \supseteq \bigcap_{\varrho \in \Gamma_0} \mathscr{F}(\varrho)$ for some finite $\Gamma_0 \subseteq \Gamma$.

Corollary 2.2. *If $\bigcap_{\varrho \in \Gamma_0} \mathscr{F}(\varrho) \neq \emptyset$ for all finite $\Gamma_0 \subseteq \Gamma$ then $D^0(\Gamma)$ is a filter on B.*

Proof. It is clear that $0 \notin D^0(\Gamma)$ since each $\mathscr{F}(\varphi) \in D^0(\Gamma)$ contains a non-empty set by definition. If $\mathscr{F}(\varphi) \in D^0(\Gamma)$ and $\mathscr{F}(\psi) \in B$ then $\mathscr{F}(\varphi) \cup \mathscr{F}(\psi) = \mathscr{F}(\varphi \vee \psi) \supseteq \mathscr{F}(\varphi)$ and hence $\mathscr{F}(\varphi) \cup \mathscr{F}(\psi) \in D^0(\Gamma)$. If $\mathscr{F}(\varphi), \mathscr{F}(\psi) \in D^0(\Gamma)$, $\mathscr{F}(\varphi) \supseteq \bigcap_{\varrho \in \Gamma_0} \mathscr{F}(\varrho)$, $\mathscr{F}(\psi) \supseteq \bigcap_{\varrho \in \Gamma_1} \mathscr{F}(\varrho)$, then $\mathscr{F}(\varphi) \cap \cap \mathscr{F}(\psi) = \mathscr{F}(\varphi \wedge \psi) = \bigcap_{\varrho \in \Gamma_0 \cup \Gamma_1} \mathscr{F}(\varrho)$, hence $\mathscr{F}(\varphi) \cap \mathscr{F}(\psi) \in D^0(\Gamma)$.

Let $\mathscr{P}_0$ be the class of all domains of mappings $p \in \mathscr{P}$. For any $S \subseteq \bigcup \mathscr{P}_0$ let $T_0(S) = \{x' : x \in S\} \cup \{x'' : x \in S\}$ and $T(S) = \{f^r x : x \in T_0(S), r = 0, 1, 2, \ldots\}$.

For any filter D on the Boolean algebra B let $[D]$ be the equivalence relation on $T(\bigcup \mathscr{P}_0)$ such that for any $x, y \in T_0(\bigcup \mathscr{P}_0)$ and any r, s we have

$$f^r x/[D] = f^s y/[D] \text{ if and only if } \mathscr{F}(f^r x = f^s y) \in D.$$

Observe that $[D]$ is indeed an equivalence relation. To verify transitivity, suppose $\mathscr{F}(f^r x = f^s y)$, $\mathscr{F}(f^s y = f^t z) \in D$. Clearly $\mathscr{F}(f^r x = f^s y \wedge f^s y = f^t z) \in D$ and $\mathscr{F}(f^r x = f^t z) \supseteq \mathscr{F}(f^r x = f^s y \wedge f^s y = f^t z)$. Hence $\mathscr{F}(f^r x = f^t z) \in D$.

Let $\mathscr{P}$ satisfy (i)–(v) and let $S \in \mathscr{P}_0$.

Definition 2.3. *A set Δ of formulas of $L(\mathscr{P})$ is called a* translation *of a set Γ of formulas of $L(\mathscr{P})$* into *S if to every formula $\varphi \in \Gamma$ there corresponds in a biunique way a formula $\varphi^* \in \Delta$ such that the following condition is satisfied:*

If $x_0', \ldots, x_n', y_0'', \ldots, y_m''$ are all the variables and constants occurring in φ then $y_0, \ldots, y_m \in S$ and there exists $q \in \mathscr{P}$, q one-to-one, $qx_0, \ldots, qx_m \in S$ such that

$$(\varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m''))^* = \varphi((qx_0)', \ldots, (qx_n', y_0'', \ldots, y_m'').$$

It follows from Lemma 1.2 that $\mathscr{M}(\Gamma, \mathscr{P}) = \mathscr{M}(\Delta, \mathscr{P})$ for any Γ, Δ thus connected.

Theorem 2.4. *A family of mappings $\mathscr{P}$ satisfying* (i)–(v) *is compact if, for any set Γ of formulas of $L(\mathscr{P})$ such that $\bigcap_{\varrho \in \Gamma_0} \mathscr{F}(\varrho) \neq \emptyset$ for any finite $\Gamma_0 \subseteq \Gamma$, there exists $S \in \mathscr{P}_0$ and a translation Δ of Γ into S with the following property:*

There exists a refinement $D(\Delta)$ of $D^0(\Delta)$ to an ultrafilter on B such that
(a) *for any $\mathscr{F}(\varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m'')) \in D^0(\Delta)$, where $y_0, \ldots, y_m \in S$, and any $p \in \mathscr{P}$ with $x_0, \ldots, x_n \in \operatorname{dom} p$, $px_0, \ldots, px_n \in S$, we have $\mathscr{F}(\varphi((px_0)', \ldots, (px_n)', y_0'', \ldots, y_n'')) \in D(\Delta)$,*
(b) *any set $t/[D(\Delta)]$, where $t \in T(S)$, contains exactly one $x'' \in T_0(S)$, $x \in S$.*

Proof. Let g_0 be the following operation on the equivalence classes of $T(S)$: For any $t \in T(S)$ let $g_0(t/[D(\Delta)]) = ft/[D(\Delta)]$. Let $G_0 = \langle T(S)/[D(\Delta)], g_0 \rangle$. Observe that g_0 is indeed operating on equivalence classes, for suppose $x, y \in T_0(S)$, $f^r x/[D(\Delta)] = f^s y/[D(\Delta)]$. By definition this means that $\mathscr{F}(f^r x = f^s y) \in D(\Delta)$. Now, for any f, if $f^r x = f^s y$ then $f^{r+1}x = f^{s+1}y$. Hence $\mathscr{F}(f^{r+1}x = f^{s+1}y) \supseteq \mathscr{F}(f^r x = f^s y)$. Thus we have $ff^r x/[D(\Delta)] = ff^s y/[D(\Delta)]$.

Next observe that for any $x_0, \ldots, x_n, y_0, \ldots, y_m \in S$ and any $\varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m'')$ in $L(\mathscr{P})$ we have $\models_{\mathfrak{G}_0} \varphi(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)])$ if and only if $\mathscr{F}(\varphi) = \{\mathfrak{F} : F \in \mathscr{P}_0, \models_{\mathfrak{F}} \varphi(x_0'', \ldots, x_n'', y_0'', \ldots, y_n'')\} \in D(\Delta)$.

We prove this by induction on the structure of φ:

(i) If φ is an equation, e.g., of the form $f^r x' = f^s y'$ then, by definition of g_0 we have $\models_{\mathfrak{G}_0} g_0^r(x'/[D(\Delta)]) = g_0^s(y'/[D(\Delta)])$ if and only if $f^r x'/[D(\Delta)] = f^s y'/[D(\Delta)]$, and hence if and only if $\{\mathfrak{F} : F \in \mathscr{P}_0, \models_{\mathfrak{F}} f^r x'' = f^s y''\} \in D(\Delta)$.

(ii) If φ is of the form

$$\alpha(x_0', \ldots, x_n', y_0'', \ldots, y_m'') \wedge \beta(x_0', \ldots, x_n', y_0'', \ldots, y_m'')$$

then

$$\models_{\mathfrak{G}_0} \alpha(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)]) \wedge \beta(x_0'/[D(\Delta)], \ldots)$$

if and only if both

$$\models_{\mathfrak{G}_0} \alpha(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)])$$

and

$$\models_{\mathfrak{G}_0} \beta(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)]).$$

By induction assumption this is the case if and only if

$$\mathscr{F}(\alpha), \mathscr{F}(\beta) \in D(\Delta).$$

It follows that $\mathscr{F}(\alpha) \cap \mathscr{F}(\beta) = \mathscr{F}(\alpha \wedge \beta) \in D(\Delta)$. Conversely, we have clearly $\mathscr{F}(\alpha), \mathscr{F}(\beta) \in D(\Delta)$ if $\mathscr{F}(\alpha \wedge \beta) \in D(\Delta)$.

(iii) If φ is of the form $\neg\, \alpha$ then $\models_{\mathfrak{G}_0} \neg\, \alpha(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)])$ if and only if not $\models_{\mathfrak{G}_0} \alpha(x_0'/[D(\Delta)], \ldots, y_m''/[D(\Delta)])$, hence by induction assumption if and only if $\mathscr{F}(\alpha) \notin D(\Delta)$. But $\mathscr{F}(\alpha)$ is the complement of $\mathscr{F}(\neg\, \alpha)$, hence $\mathscr{F}(\neg\, \alpha) \in D(\Delta)$, since $D(\Delta)$ is an ultrafilter.

Let now g be the operation on S defined as follows: If $x \in S$ let $t/[D(\Delta)]$ be that equivalence class that contains x''. Let y be the unique (by property (b)) element of S such that y'' is contained in the equivalence class $g_0(t/[D(\Delta)])$ and let $g(x) = y$. Let $\mathfrak{G} = \langle S, g \rangle$.

Observe that for any $x_0, \ldots, y_m \in S$ and any $\varphi(x_0'', \ldots, y_m'')$ in $L(\mathscr{P})$ we have

$$\models_{\mathfrak{G}} \varphi(x_0'', \ldots, y_m'') \text{ if and only if}$$

$$\models_{\mathfrak{G}_0} \varphi(x_0''/[D(\Delta)], \ldots, y_m''/[D(\Delta)]),$$

i.e., if and only if

$$\mathscr{F}(\varphi(x_0'', \ldots, y_m'')) \in D(\Delta).$$

Now, we have

$$=_{\mathfrak{G}} \varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m'') \text{ if and only if}$$

$$\models_{\mathfrak{G}} \varphi((px_0)'', \ldots, (px_n)'', y_0'', \ldots, y_m''),$$

for all $p \in \mathscr{P}$, $p \restriction \{x_0, \ldots, x_n\}$ one-to-one, $x_0, \ldots, x_n \in \operatorname{dom} p$ and $px_0, \ldots, px_n, y_0, \ldots, y_m \in S$; and hence, if and only if

$$\mathscr{F}(\varphi((px_0)', \ldots, (px_n)', y_0'', \ldots, y_m'')) \in D(\Delta)$$

for all such p.

We want to show that $\models_{\mathfrak{B}} \varphi$ for all $\varphi \in \Gamma$. By Lemma 1.2 it is sufficient to show that $\models_{\mathfrak{B}} \varphi$ for all $\varphi \in \Delta$. For any $\varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m'')$ in Δ we have

$$\mathscr{F}(\varphi(x_0', \ldots, x_n', y_0'', \ldots, y_m'')) \in D^0(\Delta);$$

hence by property (a) we have

$$\mathscr{F}(\varphi(px_0)', \ldots, (px_n)', y_0'', \ldots, y_m'')) \in D(\Delta)$$

for all $p \in \mathscr{P}$ with $x_0, \ldots, x_n \in \operatorname{dom} p$, $px_0, \ldots, px_n$, $y_0, \ldots, y_m \in S$ and $p \restriction \{x_0, \ldots, x_n\}$ one-to-one.

Example 2.5. The family $\mathscr{P}$ of Example 1.5 (first-order theories) is compact.

Example 2.6. For non-empty sets A, C with $C \subseteq A$ let $\mathscr{P}_0$ be the family of set S such that $C \subseteq S \subseteq A$. Let $\mathscr{P}$ be the family of all mappings p with $\operatorname{dom} p$, $\operatorname{range} p \in \mathscr{P}_0$, $px = x$ for all $x \in C$. Then $\mathscr{P}$ is compact[2].

3. Using compactness. Since the compactness of first-order logic is one of the most successfully used properties in model theory, it is natural to investigate the role of compactness also in our more general framework. This leads to a program of finding simple combinatorial analogues of the main results of first-order model theory. Clearly, it is to be expected that some such results, because of their more general character, will become of independent interest.

We illustrate the above remarks by considering the well-known theorem of Ehrenfeucht and Mostowski [56] about the existence of models with many automorphisms.

Let $\mathscr{P}$ be a family of mappings satisfying (i)–(v) and such that all $p \in \mathscr{P}$ have as their domains and ranges a fixed infinite set S.

Definition 3.1. *An operation h on S is called* weakly homogeneous (with respect to S) *if for any finite E, $F \subseteq S$ with* $\operatorname{card} E = \operatorname{card} F$ *there exists exactly one mapping p of E onto F such that $hpx = hx$ for all $x \in E$ and $p = q \restriction E$ for some $q \in \mathscr{P}$, q one-to-one.*

[2] The lemma of Rado mentioned in the introduction follows easily from this observation.

Assume that $\mathscr{F}$ is given such that $\mathscr{F}$, $\mathscr{P}$ satisfy (i)–(vi) and (*), and that $\mathscr{P}$ is compact. Suppose furthermore that for all $\mathfrak{F} = \langle S, f\rangle \in \mathscr{F}$ and for all $q \in \mathscr{P}$, q one-to-one, we have $qf = f$.

Theorem 3.2. *If $\mathscr{G} \subseteq \mathscr{P}$ is a group of permutations of S, h is weakly homogeneous and $hg = h$ for all $g \in \mathscr{G}$, then there exists $\mathfrak{F} \in \mathscr{F}$ such that $fg = f$ for all $g \in \mathscr{G}$.*

Proof. Let $\mathscr{F} = \mathscr{M}(\Gamma, \mathscr{P})$, let

$$\Delta = \{fx'' = y'' \leftrightarrow f(gx)'' = y'' : g \in \mathscr{G}, x, y \in S\}.$$

We wish to show that $\mathscr{M}(\Gamma \cup \Delta, \mathscr{P}) \neq \emptyset$. Since $\mathscr{P}$ is compact it is sufficient to show that $\mathscr{M}(\Gamma \cup \Delta_0, \mathscr{P}) \neq \emptyset$ for any Δ_0 of the form

$$\Delta_0 = \{fx_i'' = y_i'' \leftrightarrow f(g_i x_i)'' = y_i'' : x_i, y_i \in S, g_i \in \mathscr{G}, y_i = fz_i$$

for some $\langle S, f\rangle \in \mathscr{F}$, $z_i \in S$, $i = 1, 2, \ldots, n\}$.

Let $E = \{x_1, \ldots, x_n\}$. Choose any $\mathfrak{F}_0 \in \mathscr{F}$ and let $C_{\varepsilon_1, \ldots, \varepsilon_n}$, $\varepsilon_i = 0$ or 1, be a partition of the n-element subsets of S into 2^n mutually exclusive classes defined as follows: $E' \in C_{\varepsilon_1, \ldots, \varepsilon_n}$ if the mapping p of E onto E' for which $hpx = hx$ for all $x \in E$ has the property that

$$f_0 p x_i = y_i \text{ if } \varepsilon_i = 1, \; f_0 p x_i \neq y_i \text{ if } \varepsilon_i = 0.$$

By Ramsey's theorem there exists an infinite subset $S' \subseteq S$ and $\varepsilon_1', \ldots, \varepsilon_n'$ such that $E' \in C_{\varepsilon_1', \ldots, \varepsilon_n'}$ for all $E' \subseteq S'$ with card $E' = n$.

Now consider the set

$$F = \{x_1, \ldots, x_n\} \cup \{g_j(x_i) : i, j = 1, 2, \ldots, n\}.$$

Let card $F = m$. Select any m elements $z_1, \ldots, z_m \in S'$. Let $G = \{z_1, \ldots, z_m\}$. Let p be the mapping of F onto G such that $hpx = hx$ for all $x \in F$. Let

$$H_j = \{pg_j(x_i) : i = 1, 2, \ldots, n\}.$$

Obviously $H_j \subseteq S'$ and card $H_j = n$ for any $j = 1, 2, \ldots, n$. Hence $H_j \in C_{\varepsilon_1', \ldots, \varepsilon_n'}$ and we have a mapping r_j of E onto H_j with $hr_j x = hx$ for all $x \in E$. This mapping has the property that

$$f_0 r_j x_i = y_i \text{ if } \varepsilon_i' = 1, \; f_0 r_j x_i \neq y_i \text{ if } \varepsilon_i' = 0, \; i = 1, 2, \ldots, n.$$

Observe that the mapping $p \circ (g_j \restriction E)$ has the property that

$$hpg_j x = hx \text{ for all } x \in E.$$

Namely, $hpg_jx = hg_jx = hz$ by our assumptions on p and g_j. By unicity of r_j we have therefore $r_j = pg_j$ on E and hence

$$f_0pg_jx_i = y_i \text{ if } \varepsilon_i' = 1,\ f_0pg_jx_i \neq y_i \text{ if } \varepsilon_i' = 0,\ i = 1, 2, \ldots, n.$$

Observe furthermore that p has the property that $hpx = hx$ for all $x \in E$, hence again by unicity $f_0px_i = y_i$ if $\varepsilon_i' = 1$, $f_0px_i \neq y_i$ if $\varepsilon_i' = 0$, $i = 1, 2, \ldots, n$. Thus for any $i = 1, 2, \ldots, n$ we have

$$f_0px_i = y_i \text{ if and only if } f_0pg_ix_i = y_i.$$

Let $q \in P$ be one-to-one, $p = q \restriction F$. Let $\mathfrak{F} = q\mathfrak{F}_0 = \langle S, q^{-1}f_0q\rangle$. $\mathfrak{F} \in \mathscr{F}$ by Corollary 1.4. Observe that

$$fx_i = y_i, \text{ if and only if } fg_ix_i = y_i.$$

Namely, the equations $fx_i = y_i$, $q^{-1}f_0qx_i = y_i$, $f_0px_i = qy_i$, $f_0px_i = y_i$, $f_0pg_ix_i = y_i$, $q^{-1}f_0qg_ix_i = q^{-1}y_i$, $q^{-1}f_0qg_ix_i = y_i$ and $fg_ix_i = y_i$ are equivalent. Thus $\mathfrak{F} \in \mathscr{M}(\Gamma \cup \Delta_0, \mathscr{P})$ and, by compactness, $\mathscr{M}(\Gamma \cup \Delta, \mathscr{P}) \neq \emptyset$. From this the conclusion of the theorem is immediate.

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SOME APPLICATIONS OF THE NOTIONS OF FORCING AND GENERIC SETS (SUMMARY) *

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The notions of *forcing* and of *generic* sets were introduced by Paul Cohen [*] to settle long-outstanding logical independence problems relative to Zermelo–Fraenkel set theory. In this paper we extend these notions to other contexts, namely that of number theory and analysis, and obtain applications there as well as some new applications in set theory. The following presentation gives a summary of the basic notions and results, with full details to appear elsewhere.

The general setting is as follows. One starts with a language L, formulated in type theory or set theory, whose structures $\mathfrak{M}$ we are interested in. Let $0<\delta\leqslant\omega$. L is extended to an auxiliary language $L^*=L^*(\boldsymbol{S}_0, \ldots, \boldsymbol{S}_n, \ldots)$ containing a (finite or infinite) sequence of symbols $\boldsymbol{S}_0, \ldots, \boldsymbol{S}_n, \ldots$ $(n<\delta)$ for the generic sets $S_0, \ldots, S_n, \ldots$ of natural numbers to be defined. L* also contains the means to denote the members of a structure $\mathfrak{M}^*=\mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$ of objects constructed in certain ways from the $S_0, \ldots, S_n, \ldots$. In the applications below, this is accomplished by taking L* to be a *ramified type theory* with variables $X^{\alpha,\beta}$ of (possibly transfinite) type $\beta<\beta_0$ and ramification rank $\alpha<\alpha_0$. The objects of $\mathfrak{M}^*$ are the sets defined by the completely ranked formulas of L*. This corresponds to the definition of constructibility first given by Gödel [39], rather than to the more publicized definition in Gödel [58].

For applications, one considers different choices of α_0, β_0. Namely, in the case of number theory we take $\beta_0=1$, $\alpha_0=0$; thus $\mathfrak{M}^*$ consists just of the natural numbers in this case and we can express only arithmetical relationships between the $S_0, \ldots, S_n, \ldots$. In the case of analysis we take $\beta_0=2$, $\alpha_0=$ a limit number, i.e., ramified analysis in

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ranks $< \alpha_0$; of special interest here is the case $\alpha_0 =$ least non-constructive ordinal. In the case of set theory we take, following Cohen [*], $\alpha_0 = \beta_0 =$ a denumerable ordinal such that the sets constructible in less than α_0 steps form a model of Zermelo–Fraenkel set theory. However, if one is interested only in the case of set theory it is simpler to use a ramified type-free language L^* as in Gödel [39].

The number-theoretical part of the languages considered is determined by using variable symbols x, y, z, ..., ranging over ω, distinct constant symbols $\bar{n}$, $n = 0, 1, 2, \ldots$, relation symbols $\mathbf{R}_i(x_1, \ldots, x_{l_i})$ corresponding to some fixed arithmetically definable relations R_i (including identity, successor, sum, product), and basic logical symbols $\sim, \mathbf{v}, \mathbf{V}$. Atomic formulas involving the unary predicate symbols or set constants $\mathbf{S}_n$ are written $(t \in \mathbf{S}_n)$, t a variable or constant. By a *finite set of conditions* we mean a finite consistent set of formulas of the form $(\bar{k} \in \mathbf{S}_n)$ or $\sim(\bar{k} \in \mathbf{S}_n)$; we use the letters Q, Q', to range over such. We denote by $Diag\,(S_0, \ldots, S_n, \ldots)$ the formal diagram of the sets $S_0, \ldots, S_n, \ldots$, associating $\mathbf{S}_n$ with S_n.

Dana Scott suggested to us the following modified definition of the notion of forcing, which has the advantage of not being restricted to prenex sentences, as in Cohen [*].

Definition 1. *The relation,* Q forces $\mathscr{F}$, *in symbols,* $Q \Vdash \mathscr{F}$, *is defined inductively for arbitrary finite sets of conditions* Q *and sentences* $\mathscr{F}$ *of* L^* *as follows:*

(i) $Q \Vdash (\bar{k} \in \mathbf{S}_n) \Leftrightarrow (\bar{k} \in \mathbf{S}_n)$ *is in* Q;

(ii) $Q \Vdash \mathbf{R}_i(\bar{k}_1, \ldots, \bar{k}_{l_i}) \Leftrightarrow \langle k_1, \ldots, k_{l_i} \rangle \in R_i$;

(iii) $Q \Vdash \mathscr{F} \mathbf{v} \mathscr{G} \Leftrightarrow Q \Vdash \mathscr{F} \vee Q \Vdash \mathscr{G}$;

(iv) $Q \Vdash \sim \mathscr{F} \Leftrightarrow \neg (\exists Q')\{Q' \supseteq Q \;\&\; Q' \Vdash \mathscr{F}\}$;

(v) $Q \Vdash \mathbf{V}x\mathscr{F}(x) \Leftrightarrow (\exists k)\, Q \Vdash \mathscr{F}(\bar{k})$.

This definition is to be regarded as being further supplemented to take care of higher-order languages. However, the only change for these cases is to bring in additional clauses for the higher-order existential quantifiers. We delay for the moment considering the exact nature of these clauses.

It is easily seen that if $\mathscr{F}$ is an arithmetical sentence (containing no $\mathbf{S}_n$)

then $Q \Vdash \mathscr{F} \Leftrightarrow \mathscr{F}$ is true in the natural numbers. Further,

$$Q \Vdash \sim(\bar{k} \in \mathbf{S}_n) \Leftrightarrow \sim(\bar{k} \in \mathbf{S}_n) \text{ is in } Q.$$

Definition 2. *A sequence* $S_0, \ldots, S_n, \ldots (0 \leqslant n < \delta)$ *is said to be* generic *if for every sentence* $\mathscr{F}$ *of* L* *there exists* $Q \subseteq Diag\ (S_0, \ldots, S_n, \ldots)$ *such that* $Q \Vdash \mathscr{F}$ *or* $Q \Vdash \sim\mathscr{F}$.

The basic results of Cohen [*] about these notions are given in the next theorem; they are easily derived from Definitions 1 and 2.

Theorem 3.

(i) $Q \Vdash \mathscr{F}\ \&\ Q \subseteq Q' \Rightarrow Q' \Vdash \mathscr{F}$.

(ii) $(\forall Q)(\forall \mathscr{F})\{\mathscr{F}$ *a sentence of* $\mathrm{L}^* \Rightarrow (\exists Q')[Q' \supseteq Q\ \&\ (Q' \Vdash \mathscr{F} \vee Q' \Vdash \sim\mathscr{F})]\}$.

(iii) *There exist generic sequences* $S_0, \ldots, S_n, \ldots (0 \leqslant n < \delta)$.

(iv) *If* $S_0, \ldots, S_n, \ldots$ *is a generic sequence,* $D = Diag\ (S_0, \ldots, S_n, \ldots)$, *and* $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$ *then for any sentence* $\mathscr{F}$,

$$\models_{\mathfrak{M}^*} \mathscr{F} \Leftrightarrow (\exists Q)[Q \subseteq D\ \&\ Q \Vdash \mathscr{F}].$$

To prove the independence of the axiom of choice, Cohen [*] introduced certain symmetries into the model he constructed by considering permutations π of ω. Corresponding to any such π there is an induced permutation of the basic symbols, $\mathbf{S}_n$ into $\mathbf{S}_{\pi(n)}$, and then induced transformations $\pi(Q)$ and $\pi(\mathscr{F})$ of finite sets of conditions Q and sentences $\mathscr{F}$ of L*. These have the property, $Q \Vdash \mathscr{F} \Leftrightarrow \pi(Q) \Vdash \pi(\mathscr{F})$.

In this paper we make use of a different kind of transformation. Let $\tau(k, n)$ be a function defined for any $k, n < \omega$ taking only the values 0 and 1. Given any sequence $S_0, \ldots, S_n, \ldots (0 \leqslant n < \delta)$, we consider the sequence $S_0^{(\tau)}, \ldots, S_n^{(\tau)}, \ldots$ obtained from it by taking

$$k \in S_n^{(\tau)} \Leftrightarrow \begin{cases} k \in S_n \text{ if } \tau(k, n) = 1 \\ k \notin S_n \text{ if } \tau(k, n) = 0. \end{cases}$$

We call τ a *switching transform* (since it switches certain elements of S_n and its complement). It is easily seen how to define a corresponding transform $\tau(Q)$ of sets of conditions Q. This has the properties, $Q \subseteq Q' \Rightarrow \Rightarrow \tau(Q) \subseteq \tau(Q')$ and $\tau(\tau(Q)) = Q$. In order to define $\tau(\mathscr{F})$ for sentences $\mathscr{F}$ of L*, the relation $\tau(k, n) = i$ must be formally definable and, for applications, "absolute", i.e., arithmetically definable. Then the following is easily seen.

Theorem 4. *If τ is an arithmetically definable switching transform then $Q \Vdash \mathscr{F} \Leftrightarrow \tau(Q) \Vdash \tau(\mathscr{F})$.*

Theorems 3 and 4 hold also for the higher-order languages considered, when Definition 1 is appropriately extended.

For a general application of Theorem 4 with interesting consequences in the various special cases, consider the following notions. We say a formula $\mathscr{F}$ of L^* is $\mathbf{S}$-*free* iff it contains no symbol $\mathbf{S}_n$. (This is not the same as saying that $\mathscr{F}$ is a formula of L, since it may contain ranked variables not in L when L is of higher order.) Consider any $S_0, \ldots, S_n, \ldots$ and $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$. Let $i > 0$ and let $B, C_1, \ldots, C_{i-1}$ be subsets of ω (in $\mathfrak{M}^*$). We say that B is $\mathfrak{M}^*$-*dependent on* $C_1, \ldots, C_{i-1}$ iff for some $\mathbf{S}$-free formula $\mathscr{F}(X, Y_1, \ldots, Y_{i-1})$, $B =$ the unique X (in $\mathfrak{M}^*$) such that $\models_{\mathfrak{M}^*} \mathscr{F}(X, C_1, \ldots, C_{i-1})$. We say that a sequence of sets is $\mathfrak{M}^*$-*independent* iff no term of it is $\mathfrak{M}^*$-dependent on any finite number of other terms. In the case of the number-theoretical language with $\beta_0 = 1$, $\alpha_0 = 0$, we drop the clauses "(in $\mathfrak{M}^*$)". Thus for B to be $\mathfrak{M}^*$-dependent on $C_1, \ldots, C_{i-1}$ in this case simply means that it is the unique solution X of some predicate $A(X, C_1, \ldots, C_{i-1})$ arithmetical in $C_1, \ldots, C_{i-1}$.

Theorem 5. *Any generic sequence of sets $S_0, \ldots, S_n, \ldots$ $(0 \leqslant n < \delta)$ is $\mathfrak{M}^*$-independent for $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$.*

For the idea of the proof, suppose $\mathscr{F}$ is $\mathbf{S}$-free and $\models_{\mathfrak{M}^*} \mathscr{F}(S_0, S_1, \ldots, S_{i-1})$. Let $D = Diag\ (S_0, \ldots, S_n, \ldots)$. Then for some $Q \subseteq D$, $Q \Vdash \mathscr{F}(\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_{i-1})$. Let k_0 be a natural number with neither $(\bar{k}_0 \in \mathbf{S}_0)$ nor $\sim(\bar{k}_0 \in \mathbf{S}_0)$ in Q. Define a switching transform τ which is 0 only in the case $n = 0$, $k = k_0$. Then by Theorem 4, $Q \Vdash \tau(\mathscr{F}(\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_{i-1}))$. This expresses that $\mathscr{F}$ holds between $S_0^{(\tau)}, S_1^{(\tau)}, \ldots, S_{i-1}^{(\tau)}$; but $S_0^{(\tau)} \neq S_0$, $S_n^{(\tau)} = S_n$ for $n > 0$.

A minor modification in this argument leads to the following more useful result.

Theorem 6. *Suppose that $\delta = 1$ and that S is generic (relative to $L^* = L^*(\mathbf{S})$). Let $\mathfrak{M}^* = \mathfrak{M}^*(S)$. Then the sequence of sets $S^{(n)} = \{k : \langle k, n\rangle \in S\}$, $0 \leqslant n < \omega$, is $\mathfrak{M}^*$-independent.*[1]

It is seen that Theorems 5 and 6 depend only on Theorems 3 and 4, and hence will apply to all languages considered here.

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[1] The sequence $S^{(0)}, \ldots, S^{(n)}, \ldots$ $(n < \omega)$ is much like a generic sequence for $\delta = \omega$. In fact, it *is* a generic sequence in the case of the number-theoretical language $(\beta_0 = 1)$. However, we don't know if this latter extends to higher-order languages.

We now consider the special case of the number-theoretical language ($\beta_0 = 1$, $\alpha_0 = 0$). Assuming a standard effective Gödel-numbering of expressions and finite sequences and sets of expressions, we can obtain the following.

Theorem 7. *Let* $\beta_0 = 1$, $\alpha_0 = 0$.

(i) *The relation* $Q \Vdash \mathscr{F}$ *is hyperarithmetical.*

(ii) *There exists a hyperarithmetical sequence* $S_0, \ldots, S_n, \ldots$ $(0 \leqslant n < \delta)$ *of generic sets.*

We immediately conclude from Theorem 5 (or 6) and Theorem 7(ii) that *there exists a hyperarithmetical set* S_0 *which is not the unique solution* X *of any arithmetical predicate* $A(X)$; this answers a question raised by Kreisel.

We next turn to the case $\beta_0 = 2$, α_0 a denumerable limit number. We use $X, Y, \ldots$ for the second-order variables of L, and in addition, for each $\alpha < \alpha_0$, $\mathrm{X}^\alpha, \mathrm{Y}^\alpha, \ldots$ for the second-order variables of rank α in L*. We say that a formula $\mathscr{F}$ of L* is *ranked* if each second-order variable in $\mathscr{F}$ is of this latter kind. We denote by $\varrho(\mathscr{F})$ the maximum of all $\alpha + 1$ such that a variable X^α is bound in $\mathscr{F}$, or 0 if there are no such.

The model $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$ is obtained inductively from models $\mathfrak{M}^*_\alpha$, where $\mathfrak{M}^*_\alpha$ consists of all sets of the form $\{n : \mathscr{G}(\bar{n})$ is true in $\langle \mathfrak{M}_\beta \rangle_{\beta < \alpha}\}$ such that $\mathscr{G}(\mathrm{x})$ is ranked with $\varrho(\mathscr{G}(\mathrm{x})) \leqslant \alpha$; we take $\mathfrak{M}^* = \mathfrak{M}^*_{\alpha_0}$.

Definition 1 (*Extended to* $\beta_0 = 2$, α_0 *a denumerable limit ordinal*). *The relation,* $Q \Vdash \mathscr{F}$, *for* $\mathscr{F}$ *a ranked sentence of* L*, *is defined inductively by means of* (i)–(v) *and*

(vi) $Q \Vdash \mathsf{V}\mathrm{X}^\alpha \mathscr{F}(\mathrm{X}^\alpha) \Leftrightarrow$ *for some ranked formula* $\mathscr{G}(\mathrm{x})$ (*with just* x *free*) *such that* $\varrho(\mathscr{G}(\mathrm{x})) \leqslant \alpha$, *we have* $Q \Vdash \mathscr{F}(\hat{\mathrm{x}}\mathscr{G}(\mathrm{x}))$.[2]

The relation, $Q \Vdash \mathscr{F}$, *is then defined for arbitrary sentences* $\mathscr{F}$ *of* L* *by using*

(vii) $Q \Vdash \mathsf{V}\mathrm{X}\mathscr{F}(\mathrm{X}) \Leftrightarrow (\exists \alpha)\{\alpha < \alpha_0 \;\&\; Q \Vdash \mathsf{V}\mathrm{X}^\alpha \mathscr{F}(\mathrm{X}^\alpha)\}$.

As we stated earlier, the Theorems 3–6 continue to hold for this extended definition. For interesting applications, we take $\alpha_0 = \omega_1 =$ the least non-constructive ordinal, in the sense of Church–Kleene. Using notations in O, we can again number the expressions of L*.

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[2] Here $\mathscr{F}(\hat{\mathrm{x}}\mathscr{G}(\mathrm{x}))$ is taken to mean the result of substituting the formula $\mathscr{G}(\mathrm{t})$ for each occurrence $(\mathrm{t} \in \mathrm{U})$ in $\mathscr{F}(\mathrm{U})$.

Theorem 8. *Let* $\beta_0 = 2$, $\alpha_0 = \omega_1$.

(i) *The relation* $Q \Vdash \mathscr{F}$, *for* $\mathscr{F}$ *a ranked sentence of* L*, *is* $\prod^1_1$.

(ii) *The relation* $Q \Vdash \mathscr{F}$, *for* $\mathscr{F}$ *an arbitrary sentence of* L*. *is hyperarithmetic in* O.

(iii) *There exists a generic sequence of sets* $S_0, \ldots, S_n, \ldots$ $(0 \leqslant n < \delta)$ *which is hyperarithmetical in* O.

(iv) *If* $S_0, \ldots, S_n, \ldots$ $(0 \leqslant n < \delta)$ *is a generic sequence of sets then* $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$ *is a model of the hyperarithmetic comprehension axiom and, in fact, of the* $\sum^1_1$*-axiom of choice*.[3] *Further, if* $B, C_1, \ldots, C_{i-1} \in \mathfrak{M}^*$ *and* B *is hyperarithmetical in* $C_1, \ldots, C_{i-1}$ *then* B *is* $\mathfrak{M}^*$*-dependent on* $C_1, \ldots, C_{i-1}$. *Finally,* $S_0, \ldots, S_n, \ldots$ *are hyperarithmetically incomparable.*

This last is obtained from the preceding parts of (iv) by applying Theorem 5. Hence we obtain a new proof of Spector's incomparability result [58], originally obtained by measure-theoretic arguments.

To conclude, we consider the case that L is the language of set theory and where $\alpha_0 = \beta_0 =$ a denumerable ordinal such that the sets constructible in $< \alpha_0$ steps form a model for Z–F (Zermelo–Fraenkel) set theory. We assume this ordinal fixed in the remainder. Without going into details, Definition 1 can be further extended, much as in clauses (vi) and (vii), to cover this case in such a way that Theorems 3–6 continue to apply. We further assume now that $S_0, \ldots, S_n, \ldots$ $(0 \leqslant n < \delta)$ is a generic sequence and $\mathfrak{M}^* = \mathfrak{M}^*(S_0, \ldots, S_n, \ldots)$. In case $\delta = 1$, we take $S = S_0$ and $S^{(n)} = \{k : \langle k, n \rangle \in S\}$. By adapting the arguments from Cohen [*] to the present formulation, the following can be shown.

Theorem 9.

(i) $\mathfrak{M}^*$ *is a model of* Z–F.

(ii) *If* δ *is finite it is also a model of the axiom of choice and the generalized continuum hypothesis.*

Theorem 6 can now be applied to give (i) of the following; part (ii) can be proved by techniques similar to those used in a proof of Theorem 6.

Theorem 10. *Let* $\delta = 1$.

(i) *No* $S^{(n)}$ *is set-theoretically definable in terms of any of the other* $S^{(m)}$*'s relative to* $\mathfrak{M}^*$.

[3] For a definition of these axioms see Kreisel [62].

(ii) *There is no set-theoretically definable well-ordering of the continuum in* $\mathfrak{M}^*$.

It follows from (i) that *the axiom of constructibility is false in* $\mathfrak{M}^*$, thus extending the result of Cohen [*]. (i) is also of interest with respect to the result of Addison [59, p. 354], according to which, if one assumes the axiom of constructibility, any two sets of natural numbers have comparable $\prod^1_2$ degrees. (ii) bears on questions dealt with by Myhill and Scott in unpublished work on ordinal definability.

Theorem 11. *For* $\delta = \omega$, *the prime ideal theorem for Boolean algebras is false in* $\mathfrak{M}^*$. *In particular, every prime ideal in the algebra of all subsets of* ω *in* $\mathfrak{M}^*$ *is principal.*

Let $X \equiv Y$ (for $X, Y \subseteq \omega$) mean that X, Y differ at only a finite number of places. Then Theorem 11 is proved from Theorem 4 by using appropriate switching functions τ which make $S_n^{(\tau)} \equiv \omega - S_n$ for some n, $S_m^{(\tau)} \equiv S_m$ for other m.

It follows from Theorem 11 that (for $\delta = \omega$) *the well-ordering of the continuum is false in* $\mathfrak{M}^*$; this is proved without introducing any special symmetries as in Cohen [*]. Dana Scott pointed out to us that, more generally than Theorem 11 and by similar arguments, there is no choice set in $\mathfrak{M}^*$ which selects an element from each set of the form $\{[X], [\omega - X]\}$, where $[X]$ is the $\equiv$-class of X. This also has Cohen's [*] result, that *the axiom of choice for unordered pairs is false in* $\mathfrak{M}^*$, as a consequence.

As in the conclusion of Cohen's [*], all of the arguments for Theorems 9–11 can be formalized to give finitist independence results if we assume the consistency of Z–F as hypothesis.

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A HYPOTHESIS CONCERNING THE EXTENSION OF FINITE RELATIONS AND ITS VERIFICATION FOR CERTAIN SPECIAL CASES*

First part

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We recall in Section 1 some known definitions and results on the restrictions and extensions of a relation (or of a multirelation). In Section 2, the notion of p-equivalence between multirelations is introduced: existence of the same restrictions, up to isomorphism, to $\leqslant p$ elements, or equivalently, satisfaction of the same logical formulas of the first degree with $\leqslant p$ individual variables, all bound. We then state, in 3.1, a hypothesis which can be summarized as follows: every multirelation with a large enough finite base admits a p-equivalent extension to any set. The verification of this hypothesis for unary multirelations and binary multirelations with $p=2$ is easy (see 3.2 and 3.3). Also the case of binary multirelations with p arbitrary can be reduced to the case $p=3$ (see Section 4). Two weakenings of the hypothesis are derived in 2.4; if we replace the words "large enough finite base" by "infinite base" we get statement 1.3, which is a known consequence of the axiom of choice. In a second part, which will be published soon, we split the hypothesis in an infinity of special cases and we verify some of these cases: reflexive and symmetric relations, equivalence relations, reflexive and antisymmetric relations, ordering relations, multirelations consisting of a simple ordering and some unary relations, multirelations consisting of two simple orderings, and binary multirelations which can be split into two constant restrictions.

1. Multirelation; restriction, extension, isomorphism. Let us recall that an *n-ary relation*, or a relation of *signature* n (a positive integer), and of *base* A, is a function which associates with each element of A^n the value $+$ or $-$. If $B \subseteq A$, the *restriction* of the relation R to B, denoted $R|B$, is the relation of base B which takes the same value as R for each element of B^n. A bijection f of A on A' is an *isomorphism* of R onto R' (of base A') if and only if $R(x_1, \ldots, x_n) = R(f(x_1), \ldots, f(x_n))$ for all

* Translated from the French by Michel Jean.

$x_1, \ldots, x_n \in A$. A ν-ary *multirelation*, or a relation of *signature* $\nu = (n_1, \ldots, n_h)$ (for h a positive integer) and of *base* A, is a sequence $M = (R_1, \ldots, R_h)$ of h relations of signature $n_1, \ldots, n_h$ (positive integers) and of common base A. When $h = 1$, we identify $M = (R_1)$ with the relation R_1. A multirelation is said to be *finite*, *infinite*, *denumerable*, or *of cardinality* a, according to whether its base is finite, infinite, denumerable, or of cardinality a. The *restriction* to $B \subseteq A$ of $M = (R_1, \ldots, R_h)$ is $N = M|B = (R_1|B, \ldots, R_h|B)$; we say that M is an *extension* of N to A. A bijection of A on A' is an *isomorphism* of M onto $M' = (R_1', \ldots, R_h')$, of base A', if and only if it is an isomorphism of R_i onto R_i' for $i = 1, \ldots, h$.

1.1. If p is a positive integer and $\nu = (n_1, \ldots, n_h)$ is a sequence of positive integers, then *there are only a finite number of ν-ary multirelations on a base of p elements*, for example on the integers $1, \ldots, p$; there are exactly 2 to the power $p_1{}^{n_1} + \ldots + p_h{}^{n_h}$ such multirelations. Therefore any infinite set of ν-ary multirelations no two of which are isomorphic contains either infinite multirelations or multirelations of arbitrarily large finite cardinality.

1.2. *Given an infinite sequence of ν-ary multirelations M_i $(i = 1, 2, \ldots)$, where M_i is of finite cardinality a_i, strictly increasing with i:*

1°. *There exists a denumerable ν-ary multirelation M such that every finite restriction of M is isomorphic to a finite restriction of some M_i.*

2°. *If furthermore each M_i is an extension of a given multirelation L,* or of an isomorph of L, *then we can take an M which is also an extension of L.*

Proof of 2°. Let p be the cardinality of L; we can assume that the base of L is the set $\{1, \ldots, p\}$ and that the base of M_i is $\{1, \ldots, p_i\}$, where p_i is $\geqslant p$ and strictly increasing with i. Since there are only a finite number of ν-ary multirelations on a finite base, there are infinitely many indices i for which the restrictions $M_i|\{1, \ldots, p+1\}$ are all the same. Let us denote by L_1, this common restriction which is also an extension of L, and by $M_{1,1}, \ldots, M_{1,j}, \ldots$ those M_i which are extensions of L_1. Again there are infinitely many indices j, for which the restrictions $M_{1,j}|\{1, \ldots, p+2\}$ are all the same extension of L_1, say L_2; let $M_{2,1} \ldots, M_{2,k}, \ldots$ be those $M_{1,j}$ which admit the restriction L_2. By iteration we obtain $L_3, L_4 \ldots$ and for M we take the common extension of the L_i's $(i = 1, 2, \ldots)$, defined on the set of positive integers.

1.3. *Extension of an infinite multirelation.* Let M be an infinite multirelation with base A and let $A' \supseteq A$; from the axiom of choice, or more precisely from the existence of an ultrafilter containing a given filter, we can prove that *there exists an extension M' of M to A' such that any finite restriction of M' is isomorphic to a finite restriction of M.* In other words M and M' have, up to isomorphism, the same finite restrictions.

Summary of the proof. To each finite subset F of A', we associate the set U_F of all extensions M' of M to A', such that there exists at least one isomorphism f of $M'|F$ onto some restriction of M, with $f(x)=x$ for all $x \in A \cap F$. We see that U_F is nonvoid for all F, and that $G \supseteq F$ implies $U_G \subseteq U_F$; it follows that the sets U_F constitute a filter. We take an ultrafilter containing that filter; it associates to each F one and only one multirelation N_F of base F, such that the set of common extensions of M and N_F to A' is an element of the ultrafilter. Then M and all the N_F's admit a common extension to A' which has the required property (cf. Fraïssé [54a], [54b]).

2. p-equivalence. Two multirelations M and M' are said to be *p-equivalent if and only if every restriction of M to $\leqslant p$ elements is isomorphic to a restriction of M' and conversely.* We see easily that it is indeed an equivalence relation (reflexive, transitive and symmetric), that p-equivalence implies q-equivalence for $1 \leqslant q \leqslant p$, and that two isomorphic multirelations are p-equivalent for all p; on the other hand two simple orderings of infinite bases are p-equivalent for all p without necessarily being isomorphic. If M is of cardinality $<p$, then all multirelations p-equivalent to M are isomorphic to M.

The following statements are less obvious.

2.1. *There are only a finite number of equivalence classes of ν-ary multirelations determined by the relation of p-equivalence.*

Proof. Each one of these classes is determined by a set of isomorphism types of ν-ary multirelations on $\leqslant p$ elements. But there are only a finite number of distinct isomorphism types of ν-ary multirelations on $\leqslant p$ elements (see 1.1).

2.2. *Given a finite sequence ν of integers, and an integer p, there exists an integer $r(\nu, p)$, such that any ν-ary multirelation admits a p-equivalent restriction of cardinality $\leqslant r(\nu, p)$.*

Proof. Let M be a v-ary multirelation; the p-equivalence class to which M belongs is determined by a finite set of isomorphism types of v-ary multirelations on $\leqslant p$ elements; let us take a restriction of M as a representative for each one of these isomorphism types. Let B be the union of the bases of these representatives; then $M|B$ is p-equivalent to M. On the other hand the cardinality of B is bounded by the product of p by the number of these isomorphism types. And this last number is also bounded by the number of v-ary multirelations the bases of which are subsets of $\{1, \ldots, p\}$; finally that number depends only on v and p (see 1.1).

2.3. 1°. *If a p-equivalence class contains multirelations of arbitrarily large finite cardinality, then it contains a denumerable multirelation.*

2°. *Given a finite multirelation L, if the class of p-equivalent multirelations which are extensions of L, contains multirelations of arbitrarily large finite cardinality, then it contains a denumerable multirelation.*

Proof. Let us take an infinite sequence of p-equivalent multirelations M_i, whose cardinality is strictly increasing with i. For each M_i let us take a restriction N_i of cardinality $\leqslant r(v, p)$ which is p-equivalent to M_i (see 2.2). There are an infinite number of indices i for which the N_i are all isomorphic to some multirelation N. By 1.2.2° there is a denumerable multirelation M which is an extension of N and such that every finite restriction of M is isomorphic to a restriction of some M_i; it follows that M is p-equivalent to each M_i. In order to prove 2°, we replace each N_i by an extension of L and of N_i of cardinality equal to the cardinality of L increased by at most $r(v, p)$ elements.

2.4. *Given a finite sequence v of positive integers, and the positive integer p:*

1°. *There exists an integer $s(v, p)$ such that every v-ary multirelation of cardinality $\geqslant s(v, p)$, admits a denumerable p-equivalent multirelation.*

Proof. There are only a finite number of classes of v-ary multirelations defined by the relation of p-equivalence (see 2.1). For each one of these classes U, either there exists an integer $s(U)$ strictly greater than the cardinality of the multirelations of the class U; or there are in U multirelations of arbitrarily large finite cardinality, and hence denumerable multirelations (see 2.3.1°). It suffices to take for $s(v, p)$ the maximum of all $s(U)$.

2°. *There exists an integer $s'(v, p)$ such that every v-ary multirelation of cardinality $\geqslant s'(v, p)$ has a p-equivalent restriction which admits a denumerable p-equivalent extension.*

Proof. From 2.2, there exists an integer $r(v, p)$ such that each p-equivalence class of v-ary multirelations contains a multirelation of cardinality $\leqslant r(v, p)$. Let us take one of these multirelations, say L. Either there are p-equivalent extensions of L of arbitrarily large finite cardinality, and hence a denumerable p-equivalent extension of L (see 2.3.2°); or there is an integer $s(L)$ strictly larger than the cardinality of all p-equivalent extensions of L. There are up to isomorphism only a finite number of these multirelations L (see 1.1); it suffices to take $s'(v, p) =$ maximum of the $s(L)$'s.

2.5. *Logical interpretation of the notion of p-equivalence; statements* 2.3 *are consequences of the compactness theorem for elementary classes.* A formula of the predicate calculus with equality is said to be of *first degree* if and only if it is a propositional combination of universal formulas (i.e., of quantifier-free formulas prefixed by universal quantifiers). The multirelations M and M' of signature $(n_1, \ldots, n_h)$ are p-equivalent if and only if they satisfy the same sentences of the first degree with predicates $\varrho_1, \ldots, \varrho_h$ (where ϱ_i is n_i-ary for $i = 1, \ldots, h$) and containing $\leqslant p$ individual variables, all bound.

An *elementary class* is a class of multirelations which satisfy a first-order sentence. The compactness theorem says that if a set of elementary classes has a void intersection, then there is a finite subset of these classes which has a void intersection. Consequently if an elementary class contains multirelations of arbitrary large finite cardinality, it also contains an infinite multirelation. The statements 2.3.1° and 2° then follow from the fact that a p-equivalence class and the class of p-equivalent multirelations which are extensions of a given finite multirelation or of its isomorphic images, are elementary classes.

3. The hypothesis; its verification for unary multirelations and for binary multirelations with $p = 2$. Statements 2.4 suggest the following statement, which we have neither been able to prove nor to disprove:

3.1. *Given a sequence v of positive integers and a positive integer p, there exists an integer $t(v, p)$ such that every v-ary multirelation of cardinality $\geqslant t(v, p)$ has a denumerable p-equivalent extension, or equivalently, has a p-equivalent extension of arbitrarily large finite cardinality.*

The hypothesis is related to statement 1.3, which has been proven from the axiom of choice and which can be formulated as follows: *every multirelation M with an infinite base A admits, for all sets A' containing A, an extension to A' which is p-equivalent to M for all p.*

3.2. *Verification for unary multirelations.* Let $M=(R_1, \ldots, R_h)$, where each R_i, for $i=1, \ldots, h$, is a unary relation; and let p be a positive integer. Let us divide the base A of M into 2^h sets, each one consisting of the elements x of A such that each $R_i(x)$ $(i=1, \ldots, h)$ takes on some given value, $+$ or $-$. Let us take $t(v, p)=(p-1)2^h+1$; then at least one of these subsets of A, say B, has $\geqslant p$ elements. It suffices to add to A a denumerable number of new elements for which each R_i takes the same value as in B, in order to obtain a denumerable p-equivalent extension of M.

3.3. *Verification for binary multirelations, with $p=2$.* Let $M=(R_1, \ldots, R_h)$, where each R_i is a binary relation. The same proof would be valid if the R_i's were binary or unary. Let us take $t(v, 2)=2^h+1$; then there are two distinct elements a, b of the base A such that $M|\{a\}$ is isomorphic to $M|\{b\}$. Let us adjoin to A the new element o and let us define the extension $M'=(R_1', \ldots, R_h')$ of M to $A \cup \{o\}$ as follows: for each $x \in A$, $x \neq a$, let $R_i'(o, x)=R_i(a, x)$; $R_i'(x, o)=R_i(x, a)$; $R_i'(o, o)=R_i(a, a)$ for each $i=1, \ldots, h$; therefore $M'|\{o, x\}$ is isomorphic to $M|\{a, x\}$. Furthermore let $R_i'(o, a)=R_i(b, a)$; $R_i'(a, o)=R_i(a, b)$; let us note that we already have $R_i'(o, o)=R_i(a, a)=R_i(b, b)$ and therefore $M'|\{o, a\}$ is isomorphic to $M|\{b, a\}$. The extension M' of M is 2-equivalent to M; and by iteration of the above argument we obtain p-equivalent extensions of M of arbitrarily large finite cardinality.

We were not able to verify the hypothesis for other signatures or other integers p; in light of the following result (Section 4) it would be important to know if the hypothesis is true for binary relations and multirelations with $p=3$.

3.4. Let us note $M' \geqslant M$ when M is isomorphic to a restriction of M'; obviously $\geqslant$ is a preordering relation (reflexive and transitive) between multirelations of the same signature. If M, M' are finite, then $M \geqslant M' \geqslant M$ implies M isomorphic to M'; but this does not hold for M, M' infinite: take for instance the ordering of rational numbers, and the same after addition of a first or a last element. Now we have the following corollary of hypothesis 3.1:

Given a finite multirelation M, there exists an integer $t(M)$ such that every finite multirelation M' of cardinality $\geqslant t(M)$ and such that $M' \not\geqslant M$ has a denumerable extension $\not\geqslant M$ (take p = cardinal of M, r = signature of M and use 3.1 with $t(M) = t(r, p)$).

This hypothesis seems easier to verify in particular cases than 3.1; for instance, it is obviously true if M is a binary reflexive relation and is easily provable if M is any binary relation of cardinality 1, 2, 3.

4. The hypothesis for binary multirelations with arbitrary p can be reduced to the case $p=3$. More precisely, for given integers h and $p \geqslant 3$, there exists an integer $k(h, p)$ such that: if the hypothesis is true for multirelations consisting of k binary relations, with the integer 3, it is also true for multirelations consisting of h unary or binary relations, with the integer p.

Proof of statement 4. For simplicity let us assume that M is a multirelation consisting of h binary relations of base A. We will adjoin to A new elements o_i $i = 1, \ldots, p-2$; we will define an extension M' of M to that new base, which is p-equivalent to M. Let I be a simple ordering of base A; let B be the set of all subsets of A of cardinality $p-2$.

Given four elements X, Y, X', Y' of B we say that (X, Y) is equivalent to (X', Y') if the subsets $X \cup Y$ and $X' \cup Y'$ of A have the same cardinality and if the bijection of the first onto the second which preserves the ordering relation I, maps X onto X', Y onto Y' and is an isomorphism of $M|X \cup Y$ onto $M|X' \cup Y'$. It follows immediately from the definition that (X, Y) is equivalent to (X', Y') if and only if (Y, X) is equivalent to $(Y' X')$, in which case (X, X) is equivalent to (X', X') and (Y, Y) is equivalent to (Y', Y'). The number of equivalence classes of ordered pairs of elements of B is bounded by a function $k(h, p)$; indeed it is at most equal to the number of pairs of multirelations (which are sequences of h binary relations) defined on subsets of the set of integers: $1, 2, \ldots, 2(p-2)$.

We define a multirelation N of base B, consisting of k binary relations S_i $(i = 1, \ldots, k)$, such that to each equivalence class of pairs (X, Y) of elements of B, there exists at least one relation S_i and $S_i(X, Y) = +$ or $-$ according to whether (X, Y) belongs to that class or not; and conversely, for each S_i, $i = 1, \ldots, k$, there corresponds an equivalence class of pairs of elements of B. It follows from previous remarks that two pairs (X, Y) and (X', Y') of elements of B are equivalent if and only if the trans-

formation which sends X on X' and Y on Y' is an isomorphism of $N|\{X, Y\}$ onto $N|\{X', Y'\}$.

The hypothesis is assumed to be true for the signature ν of N and the integer 3; therefore, if the cardinality of A is large enough so that the cardinality of B is $\geqslant t(\nu, 3)$, we can adjoin to B an element O and we can define on the base $B \cup \{O\}$ a multirelation N' which is a 3-equivalent extension of N. We consider O as a set of $p-2$ elements o_i $(i = 1, \ldots, p-2)$; we identify some o_i, but not all, with some elements of A; and we define on $A \cup \{o_1, \ldots, o_{p-2}\}$ a p-equivalent extension M' of M, by the following conditions:

(1) Consider an element o_i $(i = 1, \ldots, p-2)$ and an element $x \in A$. Let X be a subset of A of cardinality $p-2$ such that $x \in X$. Let j be the rank of x in X with respect to the ordering I. Since N' is 3-equivalent, and a fortiori 2-equivalent to N, there are two elements $Y, Z \in B$ such that the transformation which sends O on Y and X on Z is an isomorphism of $N'|\{O, X\}$ onto $N|\{Y, Z\}$. Let y be the ith element of Y and z the jth element of Z with respect to the ordering I. Let $o_i = x$ or $\neq x$ according to whether $y = z$ or $\neq z$; for each relation R of M, its extension R' is defined by $R'(o_i, x) = R(y, z)$, $R'(x, o_i) = R(z, y)$, $R'(o_i, o_i) = R(y, y)$.

(2) Consider now two elements o_i, o_j $(i, j = 1, \ldots, p-2$ and $i \neq j)$. Since N' is 3-equivalent and hence 1-equivalent to N, there is a $Y \in B$ such that $N'|\{O\}$ is isomorphic to $N|\{Y\}$. Let y be the ith element and z the jth element of Y with respect to the ordering I. Let $o_i \neq o_j$; for each relation R of M its extension R' is defined by $R'(o_i, o_j) = R(y, z)$, $R'(o_j, o_i) = R(z, y)$, $R'(o_i, o_i) = R(y, y)$, $R'(o_j, o_j) = R(z, z)$.

Statement 4 will then follow from the next five statements.

4.1. *The preceding conditions*, concerning the identity or non-identity between elements of o and elements of A, and the relations R', *are independent of the choice of* X, Y, Z.

Proof. In the above condition (1) let us first keep X fixed and let us replace Y by Y_1 and Z by Z_1. The transformation which sends Y on Y_1 and Z on Z_1 is an isomorphism of $N|\{Y, Z\}$ onto $N|\{Y_1, Z_1\}$; therefore (Y, Z) is equivalent to (Y_1, Z_1) and there is one and only one bijection of $Y \cup Z$ onto $Y_1 \cup Z_1$ which preserves the ordering I, which maps Y onto Y_1, and Z onto Z_1 and which is an isomorphism of $M|Y \cup Z$ onto $M|Y_1 \cup Z_1$. Therefore the condition $y = z$ or $y \neq z$ and the values of $R(y, z)$, $R(z, y)$, $R(y, y)$, $R(z, z)$ remain unchanged if y and z

denote the ith element of Y_1 and the jth element of Z_1 instead of the ith element of Y and the jth element of Z. The same argument will hold for condition (2) if we take $Z = Y$ and $Z_1 = Y_1$.

In condition (1) let us replace now X by X_1, where x is still a member of X_1. Let j and j_1 be the rank of x in X and X_1, respectively. Since N' is 3-equivalent to N, there are elements $Y, Z, Z_1 \in B$ such that the transformation which sends O on Y, X on Z, and X_1 on Z_1 is an isomorphism of $N'|\{O, X, X_1\}$ onto $N|\{Y, Z, Z_1\}$. But then (X, X_1) is equivalent to (Z, Z_1); the jth element of Z is identical to the j_1 element of Z_1; the condition $y = z$ or $y \neq z$ and the values of $R(y, z)$, $R(z, y)$, $R(y, y)$, $R(z, z)$ remain unchanged when y denotes the ith element of Y, and z the j_1th element of Z_1 instead of the jth element of Z.

4.2. *The preceding conditions are consistent.*

Proof. Throughout Section 4 we can obviously take $p \geqslant 4$. Let us assume that by condition (1) the element o_i is identified with the elements x and x_1 of A. As $p - 2 \geqslant 2$, let us take X_0 containing x and x_1; by 4.1, if o_i has been identified with x by means of some X, Y, Z, then o_i is also identified with x by means of X_0 and some Y_0, Z_0. Also o_i is identified with x_1 by means of X_0, Y_0, Z_0. Then if j and j_1 are the ranks of x and x_1 in X_0, the ith element of Y_0 is equal to the jth and to the j_1th elements of Z_0; thus $j = j_1$ and $x = x_1$. Similarly we see that if o_i and o_j are identified by condition (1) to the same element of A, then $i = j$.

4.3. *At least one o_i is not identified to any element of A.*

Proof. Let us assume on the contrary that every element of O is identified with some element of A; let X be the set of these elements of A; then $X \in B$. There are $Y, Z \in B$ such that the transformation which sends O on Y and X on Z is an isomorphism of $N'|\{O, X\}$ onto $N|\{Y, Z\}$. But by condition (1) and the established hypothesis, the elements of Y are identical to the elements of Z; so $Y = Z$ but $O \neq X$, a contradiction.

4.4. *Let $X, Y, Z \in B$; if the transformation which maps O on Y and X on Z is an isomorphism of $N'|\{O, X\}$ onto $N|\{Y, Z\}$, then the transformation which maps each o_i on the ith element of Y and each element of X on the element of the same rank in Z is an isomorphism of $M'|O \cup X$ onto $M|Y \cup Z$.*

Proof. It follows from conditions (1) and (2) that the considered transformation is a bijection of $O \cup X$ on $Y \cup Z$. We now show that it is also an isomorphism of $M'|O \cup X$ onto $M|Y \cup Z$. Let R be an arbitrary binary relation of M and R' its extension. We consider three cases. (1) Let x, x_1 be two elements of X and z, z_1 the corresponding elements of Z. Since $N'|\{X\} = N|\{X\}$ is isomorphic to $N|\{Z\}$, the pairs (X, X) and (Z, Z) are equivalent; therefore the transformation which sends x on z and x_1 on z_1 is an isomorphism of $R'|\{x, x_1\} = R|\{x, x_1\}$ onto $R|\{z, z_1\}$. (2) Let $o_i, o_j \in O$ and let y, z be the corresponding elements of Y; from condition (2) the map which sends o_i on y and o_j on z is an isomorphism of $R'|\{o_i, o_j\}$ onto $R|\{y, z\}$. (3) Let $o_i \in O$, $x \in X$ and let $y \in Y$, $z \in Z$ be the corresponding elements; from condition (1) $R'(o_i, x) = R(y, z)$, $R'(x, o_i) = R(z, y)$, $R'(o_i, o_i) = R(y, y)$. As in the preceding cases, (X, X) is equivalent to (Z, Z) and $R'(x, x) = R(x, x) = R(z, z)$. Therefore the map which sends o_i on y and x on z is an isomorphism of $R'|\{o_i, x\}$ onto $R|\{y, z\}$.

4.5. *If $X, X_1 \in B$, there exist $Y, Z, Z_1 \in B$ such that the transformation which maps each o_i on the ith element of Y, each element of X on the element of the same rank in Z, and each element of X_1 on the element of the same rank in Z_1, is an isomorphism of $M'|O \cup X \cup X_1$ onto $M|Y \cup Z \cup Z_1$.*

Proof. By the hypothesis, there exist $Y, Z, Z_1 \in B$ such that the transformation which sends O, X, X_1 respectively on Y, Z, Z_1 is an isomorphism of $N'|\{O, X, X_1\}$ onto $N|\{Y, Z, Z_1\}$. Let f be the transformation described in the above statement. From 4.4, this f is a bijection of $O \cup X$ onto $Y \cup Z$ and of $O \cup X_1$ onto $Y \cup Z_1$. Since (X, X_1) is equivalent to (Z, Z_1), then f is also a bijection of $X \cup X_1$ onto $Z \cup Z_1$. Finally f maps each of the sets O, X, X_1 onto Y, Z, Z_1, respectively; therefore it is a bijection of $O \cup X \cup X_1$ onto $Y \cup Z \cup Z_1$. Always by 4.4 and the fact that (X, X_1) and (Z, Z_1) are equivalent, f is an isomorphism of $M'|O \cup X$ onto $M|Y \cup Z$, of $M'|O \cup X_1$ onto $M|Y \cup Z_1$, and of $M'|X \cup X_1 = M|X \cup X_1$ onto $M|Z \cup Z_1$. Since M is binary, it follows that f is an isomorphism of $M'|O \cup X \cup X_1$ onto $M|Y \cup Z \cup Z_1$.

We conclude the proof of statement 4 as follows. Let us consider p arbitrary elements of $A \cup \{o_1, \ldots, o_{p-2}\}$; either they are all in A, or some of them are contained in O and the others are contained in two subsets $X, X_1 \subseteq A$, of cardinality $p-2$. Then $X, X_1 \in B$ and by 4.5 the restriction of M' to the given elements is isomorphic to a restriction

of M. Thus M' is a p-equivalent extension of M. Using a similar argument we could have extended I to a relation I' of base $A \cup \{o_1, \ldots, o_{p-2}\}$ which is p-equivalent to I; since $p \geqslant 3$, I' would have been also a simple ordering.

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THE THEORIES OF FUNCTORS AND MODELS

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My talk today shall really be two talks: I shall attempt to describe a little of the theory of abelian categories and the model-theoretic results therein; secondly, and more importantly, I shall attempt to describe the main existence theorem in the theory of adjoint functors and its applications to model theory. The latter will be illustrated by three examples: a characterization of those theories for which "free models" exist; a theorem which says in effect that inverse limits of models for a given theory exist if and only if direct limits exist; and a theorem which tells us when "restricted free products" exist (cf. Łoś [*]). Of course I must define all of the above notions.

Before I define categories allow me to describe a specific functor. Let X be a topological space and $x_0 \in X$ a point in X called its base point, and let $\pi(X, x_0)$ be the fundamental group of X (based at x_0). π is a function from the class of models for the axioms of topology to the class of models for the axioms of groups. So far π is not a functor. But recall that for every continuous map $f : X \to Y$ such that $f(x_0) = y_0$ there is induced a group homomorphism $\pi(f) : \pi(X, x_0) \to \pi(Y, y_0)$. Now π is a functor. Its domain is the *category* of topological-spaces-with-base-points-and-continuous-base-point-preserving-maps, and its range is the category of groups-and-group-homomorphisms. π sends the "objects" of the first category into objects in the second (spaces into groups) and it sends "maps" in the first into maps in the second (continuous functions into group homomorphisms). If 1_X is the identity map on X then $\pi(1_X)$ is the identity map on $\pi(X, x_0)$. And if a continuous map f is the composition of maps g and h then $\pi(f)$ is the composition of $\pi(g)$ and $\pi(h)$.

It is precisely these properties that are formalized into the definition of functor. But first we must have a definition of category. The shortest known definition is so short that it is misleading. For this lecture allow me to say that a category is a collection of "objects" (undefined) which shall be denoted by capital Latin letters, and a collection of "maps"

(undefined) which run between the objects. $A \xrightarrow{x} B$ refers to a map from "domain" A to "range" B. Part of the data of a category is a "composition" operator not everywhere defined. We may compose $A \xrightarrow{x} B$ and $B \xrightarrow{y} C$ to obtain $A \xrightarrow{yx} C$. If B is not B' then $A \xrightarrow{x} B$ and $B' \xrightarrow{y} C$ do not compose. Finally, for every object A there is an "identity" map $A \xrightarrow{1} A$ which acts as an identity whenever it can: $A \xrightarrow{1} A \xrightarrow{x} B = A \xrightarrow{x} B$ and $B \xrightarrow{z} A \xrightarrow{1} A = B \xrightarrow{z} A$.

The objects of a category are usually thought of as sets with some sort of structure and the maps as functions which in some way preserve the structure. But note that the definition of category is a first-order definition. If we formalize the above description we would need a unary predicate to distinguish objects and maps; a pair of functions from the maps to the objects which assign domains and ranges; a function which assigns to pairs of composing maps their compositions; a function from objects to maps which assign identities. We have thrown away the elements of the objects [1]. One of the pleasant surprises of the subject is that in most concrete cases we can recover all the original structure — we only seem to have thrown it away.

Let me define what I shall mean by a *category of models*: Given a first-order theory $\mathbb{T}$ let $\mathscr{A}$ be the category whose objects are the models of the theory $\mathbb{T}$. For the maps of $\mathscr{A}$ we need a little more data. Let $\mathbb{F}$ be a family of predicates (formulas) which may be obtained by combining the predicates of $\mathbb{T}$. If A and B are models and $g : |A| \to |B|$ is a function, we shall say that g is an $\mathbb{F}$-map if it preserves all the predicates of $\mathbb{F}$ in the positive sense, that is, if for every $F \in \mathbb{F}$ and $x_1, \ldots, x_n \in |A|$ such that $F(x_1, \ldots, x_n)$ is true in A it is the case that $F(g(x_1), \ldots, g(x_n))$ is true in B. Letting the maps of $\mathscr{A}$ be all the $\mathbb{F}$-maps we obtain what I shall call a category of models. I shall always assume that $\mathscr{A}$ is non-empty and that the cardinality of its objects is unbounded. If $\mathbb{F}$ is the family of all possible formulas then $\mathscr{A}$ is the category of elementary extensions (or more correctly, $A \xrightarrow{g} B \in \mathscr{A}$ if g is an isomorphism of A onto an elementary submodel of B). If $\mathbb{F}$ is empty then $\mathscr{A}$ is essentially no different from the category of sets.

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[1] Indeed, we can throw away the objects, too. The collection of maps together with the composition predicate are sufficient: one axiom to state associativity and another axiom to insure the existence of enough identities. The vanished objects are present in the form of identity maps.

As an example, if we began with the theory of groups and let $\mathbf{F}$ consist of the single predicate $xy=z$, then $\mathscr{A}$ is the category of groups and group homomorphisms. If we enlarge $\mathbf{F}$ to include $x \neq y$ and the infinite collection of unary predicates $\{\forall_y(y^n \neq x)\}_{n=1,2,\ldots}$ then $\mathscr{A}$ becomes the category of *pure* group extensions ($g : A \to B \in \mathscr{A}$ if g is an isomorphism onto a pure subgroup of B).

In order to do much of interest it is necessary to develop a long list of categorical definitions. An *isomorphism* in a category is a map $A \to B$ such that there exists a map $B \to A$ such that $A \to B \to A = 1_A$ and $B \to A \to B = 1_B$. A *monomorphism* in a category is a map $A \to B$ such that for any pair $C \xrightarrow{x} A$, $C \xrightarrow{y} A$, $C \xrightarrow{x} A \to B = C \xrightarrow{y} A \to B \Rightarrow x = y$. In most categories that arise in nature this definition is coextensive with the definition of one-to-one function.

Dually, $A \to B$ is an *epimorphism* if and only if for any pair $B \xrightarrow{x} C$, $B \xrightarrow{y} C$, $A \to B \xrightarrow{x} C = A \to B \xrightarrow{y} C \Rightarrow x = y$. In the categories of sets, groups, and partially ordered sets, epimorphisms are onto functions.

We define a *product* of two objects A and B to be an object P together with maps $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that for all pairs $X \xrightarrow{x} A$, $X \xrightarrow{y} B$ there exists a unique map $X \xrightarrow{(x,y)} P$ such that $X \xrightarrow{(x,y)} P \xrightarrow{p_1} A = x$ and $X \xrightarrow{(x,y)} P \xrightarrow{p_2} B = y$. In most categories of nature, products may be constructed by taking cartesian products. Dually we define a *sum* of A and B to be an object S together with maps $A \xrightarrow{u_1} S$, $B \xrightarrow{u_2} S$ such that for every pair $A \xrightarrow{x} X$, $B \xrightarrow{y} X$ there is a unique $S \xrightarrow{\binom{x}{y}} X$ such that $A \xrightarrow{u_1} S \xrightarrow{\binom{x}{y}} X = x$ and $B \xrightarrow{u_2} S \xrightarrow{\binom{x}{y}} B = y$. Most categories do have sums but there is no uniform construction. Letting $A + B$ denote the sum of A and B we may note that in the categories of sets, partially ordered sets and topological spaces $|A + B|$ is the disjoint union of $|A|$ and $|B|$; in the category of commutative rings $A + B$ is the tensor product of A and B; in the category of groups $A + B$ is the "free product" of A and B; in the category of abelian groups $A + B$ coincides with the cartesian product of A and B.

For a pair of maps $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ we define their *difference kernel* as an object K together with a map $K \to A$ such that

(1) $K \to A \xrightarrow{x} B = K \to A \xrightarrow{y} B$

(2) for any map $X \to A$ such that $X \to A \xrightarrow{x} B = X \to A \xrightarrow{y} B$ there

is a unique map $X \to K$ such that $X \to K \to A = X \to A$. It is easily proved that $K \to A$ is a monomorphism. In most categories in nature the difference kernel of two maps is constructed as the set of elements in their domain on which the two maps agree. The dual notion, that of *difference cokernel* does not have a standard construction. (For dual definitions just reverse all the arrows.)

Let me examine an easy case where the category definition of product is not coextensive with cartesian product. Consider the empty theory for one unary predicate U. If we let $\mathbb{F}$ consist of the single formula $U(x)$ the corresponding category of models has as objects $\{\langle A, U_A\rangle | A$ is a set, $U_A \subseteq A\}$. A map from $\langle A, U_A\rangle$ to $\langle B, U_B\rangle$ is a function $f : A \to B$ such that for all $x \in A$, $U_A(x) \Rightarrow U_B(f(x))$. In this case products are cartesian. The product of $\langle A, U_A\rangle$ and $\langle B, U_B\rangle$ is $\langle A \times B, U_A \times U_B\rangle$.

Now enlarge $\mathbb{F}$ to be $\{U(x), \neg U(x)\}$. The objects of the corresponding category of models may still be represented as pairs $\langle A, U_A\rangle$ but the maps must be such that $U_A(x) \Leftrightarrow U_B(f(x))$. The product of $\langle A, U_A\rangle$ and $\langle B, U_B\rangle$ exists in this category and may be constructed as $\langle \{(x, y) \in \in A \times B | U_A(x) \Leftrightarrow U_B(y)\}, \{(x, y) | U_A(x) \wedge U_B(y)\}\rangle$ that is, as a *proper* subset of the cartesian product.

Further examination of the category removes most of the mystery. We could alternatively represent its objects as pairs $\langle A_1, A_2\rangle$ where A_1 and A_2 are disjoint sets. The maps from $\langle A_1, A_2\rangle$ to $\langle B_1, B_2\rangle$ are *pairs* of functions $\{\langle f_1, f_2\rangle | f_1 : A_1 \to B_1, f_2 : A_2 \to B_2\}$. The product of $\langle A_1, A_2\rangle$ and $\langle B_1, B_2\rangle$ is $\langle A_1 \times B_1, A_2 \times B_2\rangle$. Indeed, the category is representable as the cartesian product of the category of sets with itself and the construction of products within the category is the standard cartesian construction in each coordinate separately.

The fact that these definitions do not always agree with the usual notions has turned out to be not a disadvantage but one of the main attractions of category theory. Indeed, the subject came of age when Grothendieck [57] observed that categories of sheaves were nice enough to have good cohomology theories, or formally stated, were abelian categories. In his categories, epimorphisms turned out to be not quite the same as onto functions. I will not say that the present theory of sheaves would have been impossible without the notion of abelian category but I think it is certain that it would have taken so long to develop the theory that someone would have invented abelian categories in the meantime.

Heuristically, abelian categories are those in which there is a good notion of *exact sequence*. Recall that an exact sequence of abelian groups $\ldots \to A_{n-1} \to A_n \to A_{n+1} \ldots$ is a sequence of homomorphisms such that for all n the kernel of $(A_n \to A_{n+1})$ is the same as the image of $(A_{n-1} \to A_n)$. (The kernel of $A \xrightarrow{x} B$ is the difference kernel of x and the zero-map.) Through the work of Eilenberg, MacLane, Steenrod, and others, it was known long ago that the language of exact sequences is a very powerful language.

An abelian category is a category with a zero-object (exactly one map to and from every other object) and hence zero-maps (those which factor through zero-objects); products and sums for every pair of objects; kernels and cokernels for every map; and most characteristically, with the property that every monomorphism appears as a kernel and every epimorphism as a cokernel. From the above listed properties one is able to construct an additive structure: for $A \xrightarrow{x} B$ and $A \xrightarrow{y} B$ there is assigned $A \xrightarrow{x+y} B$. The addition is commutative, associative, distributes with composition and enjoys inverses (zero-maps are neutral). Products and sums of objects coincide. Moreover one may define exact sequences.

MacLane [50] first drew attention to abelian categories. He observed that a great variety of theorems about abelian groups could be stated categorically. (We can prove, in fact, that all theorems can be so stated.) The advantage of doing so is that categorical statements allow dualization, and categorical proofs, if based on self-dual axioms, allow dualization.

Buchsbaum [55] in his dissertation refined the notion of abelian category and gave a convincing demonstration that almost all of homological algebra, as done, say, in Cartan–Eilenberg [56], could be done in abelian categories. (I should say that the axioms for abelian categories until recently were far more complicated than those I have just listed. The additive structure, for example, was always hypothesized and its connection with the composition operator required many axioms.) As I said, Grothendieck justified the existence of abelian categories and since that time many other non-trivial examples have been found.

The obvious question arose: without elements in the objects how do we know how many of the lemmas needed for homological algebra are true in all abelian categories. The first set of answers was supplied, separately but within a span of a year, by Lubkin [60], Heron [60], and Freyd [60]. Our methods were dissimilar, but each of us embedded arbitrary small abelian categories into the category of abelian groups

in a fashion that represented them as universally-elementary submodels of the category of abelian groups in which we allow the entire collection of original predicates (composition, addition, etc.) together with a predicate of exactness (the definition of which requires an existential quantifier). By "universally-elementary" I mean that every universal statement true in the big model is true in the small.

Mitchell and I (see Freyd [64]) improved this to a representation into categories of modules over rings in such a way as to catch a larger class of statements. Before I explicate let me prepare for the strongest known theorem in this direction.

In recent years the subject of *relative* homological algebra has become important. The definitions of homological algebra are in the language of exact sequences. Suppose we interpret exact sequence in other than the standard way? There are perhaps as many applications of such interpretations as there are for the "absolute" interpretation. Moreover, there are categories which are not abelian but which do seem to have a good family of exact sequences, the easiest being the category of Banach spaces and bounded linear transformations. We may define an *exact category* as a category with an additive structure and an exactness predicate satisfying a short list of axioms which I will not here state. It suffices for now to say that the axioms are obviously necessary for the forthcoming metatheorem and that abelian categories with the absolute exactness predicate are examples. I define a *fill-in* statement as a first-order statement in prenex normal form in which the only existential quantifiers refer to *maps* between objects which have previously been quantified. As an example, consider the connecting-homomorphism-theorem:

If the commutative diagram has exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K' & \to & K & \to & K'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A' & \to & A & \to & A'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B' & \to & B & \to & B'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F' & \to & F & \to & F'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

then there exists a map $K' \to F''$ such that $K \to K' \to F'' \to F$ is exact.

In general the lemmas needed in applications are fill-in statements.

Theorem. *A fill-in statement is true in all exact categories iff it is true for the category of countable modules over the ring freely generated by a countable family of (non-commuting) indeterminates, in which category "exactness" is interpreted in the standard absolute manner.*

The proof is for the most part intensely algebraic, involving heavily the entire theory of functors and the construction of some very abstract abelian categories in which it is no longer useful to interpret the objects as sets or the maps as functions. The algebra yields a weaker meta-theorem than that above: all rings and their modules must be tested rather than the specific above-mentioned category. At this point the Löwenheim–Skolem theorem becomes useful. By cutting down to countable exact categories we ind that they are isomorphic to certain very nice categories of *cyclic* modules. By formalizing the theory of categories of cyclic modules into a theory about rings and ideals (which become unary predicates) we apply Löwenheim–Skolem again to cut down to countable rings. We then return to the functor theory to cut down to the specific ring described in the metatheorem.

Thus ends Part One of this talk. Because, in fact, this is not a talk and I do not have a captive audience I shall put a sign post for the readers whom I lost when I began to be abelian:

Part Two. Applications to Model Theory

I need two more preliminary definitions. Let $\mathscr{A}$ be a category and A, B two objects. The notation (A, B) shall refer to the set of maps from A to B. If $B_1 \xrightarrow{x} B_2$ is a map in $\mathscr{A}$ there is induced a function $(A, B_1) \to (A, B_2)$, called (A, x), defined by composition. $A \to B_1$ is sent to $A \to B_1 \xrightarrow{x} B_2$. Fixing A we obtain a functor $(A, -) : \mathscr{A} \to \mathscr{S}$ to the category of sets.

Given a map $A_1 \xrightarrow{y} A_2$ there is induced a function $(A_2, B) \to (A_1, B)$ and the order of the indices is not an error. $A_2 \to B$ is sent to $A_1 \xrightarrow{y} A_2 \to B$ and the function that does it is called (y, B). Fixing B we obtain a *contravariant* functor $(-, B) : \mathscr{A} \to \mathscr{S}$. The features of a *covariant* (not contravariant) functor T may be summarized by saying that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ & {\scriptstyle z}\searrow & \downarrow {\scriptstyle y} \\ & & C \end{array}$$

the diagram in the range

$$\begin{array}{ccc} T(A) & \overset{T(x)}{\rightarrow} & T(B) \\ & {\scriptstyle T(z)}\searrow \quad \downarrow & {\scriptstyle T(y)} \\ & T(C) & \end{array}$$

makes sense and is commutative. For a contravariant functor T the diagram in the range

$$\begin{array}{ccc} T(A) & \overset{T(x)}{\leftarrow} & T(B) \\ & {\scriptstyle T(z)}\nwarrow \quad \uparrow & {\scriptstyle T(y)} \\ & T(C) & \end{array}$$

makes sense and is commutative.

The best-known contravariant functor is the *dual-space functor* which assigns to a vector space V the space V^* of scalar-valued linear functions on V. Because V^* is a vector space and not just a set this is not quite a special case of $(-, B)$.

Category theory is an embodiment of Klein's dictum that it is the maps that count in mathematics. If the dictum is true, then it is the functors between categories that are important, not the categories. And such is the case. Indeed, the notion of category is best excused as that which is necessary in order to have the notion of functor. But the progression does not stop here. There are maps between functors, and they are called natural transformations. And it was in order to define these that Eilenberg and MacLane [45] first defined functors. To borrow from their original paper let me first examine an "unnatural" equivalence.

When V is a finite-dimensional vector space there exists an isomorphism from V to V^*. But there is no particular isomorphism. We say that V and V^* are equivalent but not naturally so. On the other hand, there is a "natural" isomorphism from V to its second dual V^{**}. Now "natural" in its previous public usage always seemed best analyzed as meaning choice-free constructibility. It is remarkable that the Eilenberg-MacLane definition has nothing to do with the intension (sic) of the previous public usage but has proved to be so consistently co-extensive.

They observed first that for linear $V_1 \xrightarrow{f} V_2$ there is induced a map $V_2^* \xrightarrow{f^*} V_1^*$ and then a map $V_1^{**} \xrightarrow{f^{**}} V_2^{**}$. In other words, there is a second dual *functor*. Next if $\Phi_1 : V_1 \to V_1^{**}$ and $\Phi_2 : V_2 \to V_2^{**}$ are

the "natural" maps of antiquity, then for any $V_1 \xrightarrow{f} V_2$ the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2}. \\ V_1{}^{**} & \xrightarrow[f^{**}]{} & V_2{}^{**} \end{array}$$

This commutativity became the definition of *natural.* A natural transformation is not a single map but a collection of maps relating two functors, in the above case the identity functor and the second dual functor. In general, if $T_1 : \mathscr{A} \to \mathscr{B}$ and $T_2 : \mathscr{A} \to \mathscr{B}$ are functors, then a natural transformation $\eta : T_1 \to T_2$ is a function η : (Objects of $\mathscr{A}$) $\to$ $\to$ (Maps of $\mathscr{B}$) such that for all $A_1 \xrightarrow{x} A_2 \in \mathscr{A}$ the diagram

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(x)} & T_1(A_2) \\ {\scriptstyle \eta(A_1)}\downarrow & & \downarrow{\scriptstyle \eta(A_2)} \\ T_2(A_1) & \xrightarrow[T_2(x)]{} & T_2(A_2) \end{array}$$

makes sense and is commutative [2].

Skipping the rather straightforward definitions of functors on several variables and natural transformations between them, I can now define the central concept, first formalized by Kan [58], of *adjoint functor.* Let $S : \mathscr{A} \to \mathscr{B}$ and $T : \mathscr{B} \to \mathscr{A}$ be functors. We say that S is the *left-adjoint* of T (and T is the *right-adjoint* of S) if there exists a natural equivalence from $(S(A), B)$ to $(A, T(B))$. That is, for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$ there is a one-to-one correspondence between the sets of maps $(S(A), B) \subseteq \mathscr{B}$ and $(A, T(B)) \subseteq \mathscr{A}$ and the collection of these is natural, i.e., yields commutative diagrams such as:

$$\begin{array}{ccc} (S(A), B_1) & \xrightarrow{(S(A), x)} & (S(A), B_2) \\ \downarrow & & \downarrow \\ (A, T(B_1)) & \xrightarrow[(A, T(x))]{} & (A, T(B_2)) \end{array} \qquad \begin{array}{ccc} (S(A_2), B) & \xrightarrow{(S(y), B)} & (S(A_1), B) \\ \downarrow & & \downarrow \\ (A_2, T(B)) & \xrightarrow[(y, T(B))]{} & (A_2, T(B)) \end{array}$$

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[2] The progression has not ended. We can now talk about the category of functors from $\mathscr{A}$ to $\mathscr{B}$. The objects are functors and the maps are natural transformations. Indeed, such functor categories are the main tool in the proof of the metatheorem of Part One.

As an example, let $\mathscr{A}$ be the category of metric spaces and uniformly continuous functions and $\mathscr{B}$ the subcategory of complete spaces. Then the functor $S : \mathscr{A} \to \mathscr{B}$ which assigns to each metric space its completion is the left-adjoint of the inclusion functor $\mathscr{B} \to \mathscr{A}$.

Let $\mathscr{B}$ be a category of models as defined earlier and let $T : \mathscr{B} \to \mathscr{S}$ be the forgetful functor into the category of sets. That is $T(B) = |B|$. Suppose $S : \mathscr{S} \to \mathscr{B}$ is a left-adjoint of T. For any set A we obtain a natural equivalence between $(S(A), B)$ and $(A, |B|)$. An example: let $\mathscr{B}$ be the category of groups and define $S : \mathscr{S} \to \mathscr{B}$ to be the functor such that $S(A)$ is the group freely generated by the set A. Then S is the left-adjoint of the forgetful functor T.

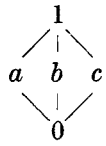
In general suppose $\mathscr{B}$ is such that there exist "free" models. A more direct analysis of the usage of "the free model over a set A" is a model B "generated" by $A \subseteq |B|$ such that for any other model B' and *function* $f : A \to |B|$ there is an $\mathbf{F}$-map $\bar{f} : B \to B'$ such that $\bar{f}|A = f$. First, I shall remove the word "generate" as follows. B is free over A if $A \subseteq |B|$ and if for every *function* $f : A \to |B'|$ there is a *unique* $\mathbf{F}$-map $\bar{f} : B \to B'$ such that $\bar{f}|A = f$.

Now if $\mathscr{B}$ is a category of models such that for every set A there does exist a free model over A then by choosing one for each set we obtain a functor $S : \mathscr{S} \to \mathscr{B}$ which is the left-adjoint of the forgetful functor $T : \mathscr{B} \to \mathscr{S}$.

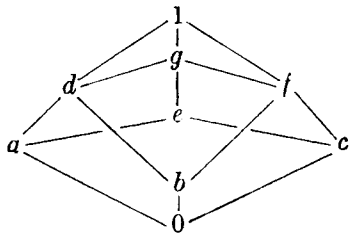
Conversely, if $S : \mathscr{S} \to \mathscr{B}$ is a left-adjoint of the forgetful functor then free models exist. The equivalence between $(S(A), S(A))$ and $(A, |S(A)|)$ sends the identity map on $S(A)$ to a function $A \to |S(A)|$. Assuming only that the objects of $\mathscr{B}$ are of unbounded cardinality we can prove that $A \to |S(A)|$ is one-to-one. It is an elementary exercise to replace S by a naturally equivalent functor (and hence still a left-adjoint of the forgetful functor) such that $A \to |S(A)|$ becomes the inclusion, that is $A \subseteq |S(A)|$. Finally it is easy to prove that for any model $B \in \mathscr{B}$ the equivalence $(S(A), B) \to (A, |B|)$ is none other than that engendered by the unique extensions of functions $A \to |B|$. All of which says that we could have defined free models as the values of the left-adjoint of the forgetful functor.

Please observe that the class $\mathbf{F}$ of formulas affects not only the existence of free models but their structure. In algebraic theories $\mathbf{F}$ usually just runs through the original predicates. But consider the theory of lattices with extremal elements "0" and "1". Let $A = \{a, b, c\}$. If $\mathbf{F}$ is empty then all the six possible orderings of A become isomorphic

(any function is an $\mathbf{F}$-map) and all are free over A. If $\mathbf{F} = \{x = 0, x \leqslant y, y = 1\}$ then the free model over A is



If $\mathbf{F} = \{x = 0,\ x \cup y = z,\ x = 1\}$ then the free model over A looks like



If $\mathbf{F} = \{x = 0,\ x \cup y = z,\ x \cap y = z,\ x = 1\}$ then the free model over A is infinite. If $\mathbf{F}$ is the set of all formulas then there is no free model over A.

The adjoint functor theorem which I shall shortly use characterizes functors with left-adjoints when the functors are defined on *complete categories*. A category is *left-complete* iff it has difference kernels for every pair of maps and products for every indexed set of objects. (We have defined products only where the indexing set is $\{1, 2\}$. The extension of the definition to the infinite case is straightforward.) These two conditions imply that the category has arbitrary "inverse limits"; this justifies the word "complete". If a functor $T : \mathscr{B} \to \mathscr{A}$ has a left-adjoint it may be proved that T preserves difference kernels and arbitrary products. In the case that T is the forgetful functor from a category of models, to say that T preserves difference kernels is the same as saying that the standard construction of difference kernels works. That is, for any two $\mathbf{F}$-maps $A \xrightarrow{g} B$, $A \xrightarrow{h} B$ we can define predicates on $K = \{x \in |A| \mid g(x) = f(x)\}$ such that the inclusion $K \subseteq A$ is an $\mathbf{F}$-map and such that for any model X, a function $|X| \to K$ is an $\mathbf{F}$-map iff $X \to K \subseteq A$ is an $\mathbf{F}$-map. To say that the forgetful functor preserves products is the same as saying that the standard cartesian construction of products works.

Because of a lack of space I cannot here state the adjoint functor theorem (Freyd [64, Exercise 3–J]) which says that a functor $T : \mathscr{A} \to \mathscr{B}$

has a left-adjoint if and only if it preserves left-roots and satisfies certain set-theoretical conditions. The domain category must be not only left-complete but must be *well-powered*, a condition always true for categories of models (Freyd [64, Exercise 3–0]). The proofs of these side conditions on $\mathscr{A}$ and T, in the case under discussion, use Tarski's extension of the Löwenheim–Skolem theorem.

Theorem. *A left-complete category of models enjoys free models if and only if the forgetful functor preserves difference kernels and products (that is, the standard constructions of difference kernels and products work).*

A question is raised: whereas it is undecidable whether a given theory is such that the standard construction of products works, is it undecidable whether the standard constructions of both products and difference kernels work? The above theorem allows one to switch the question to that of the existence of free models.

For the next application consider a left-complete category of models $\mathscr{B}$ and the cartesian product category $\mathscr{B} \times \mathscr{B}$. The diagonal functor $T : \mathscr{B} \to \mathscr{B} \times \mathscr{B}$ automatically preserves difference-kernels and products. Supposing such to be the case, the adjoint functor theorem tells us there is a functor $S : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ which is a left-adjoint of $T : \mathscr{B} \to \mathscr{B} \times \mathscr{B}$. That is, for models B_1, B_2, B, $(S\langle B_1, B_2\rangle, B) \simeq (\langle B_1, B_2\rangle, \langle B, B\rangle) \simeq$ $\simeq (B_1, B) \times (B_2, B)$. That is, the maps from $S\langle B_1, B_2\rangle$ to B correspond naturally to pairs of maps from B_1 and B_2 to B. In other words, $S\langle B_1, B_2\rangle$ is a *sum* of B_1 and B_2. (The pair $\langle u_1, u_2\rangle$ demanded by definition of *sum* corresponds to the identity of $S\langle B_1, B_2\rangle$ under the equivalence $(S\langle B_1, B_2\rangle, S\langle B_1, B_2\rangle) \simeq (\langle B_1, B_2\rangle, \langle S\langle B_1, B_2\rangle, \langle S\langle B_1, B_2\rangle\rangle)$.)

One of the conditions, I'll call it condition (0), of the adjoint functor theorem requires in this case that for every $\langle B_1, B_2\rangle \in \mathscr{B} \times \mathscr{B}$ there exists $B \in \mathscr{B}$ and a map (any map) from $\langle B_1, B_2\rangle$ to $\langle B, B\rangle$. That is, for every pair of models $B_1, B_2 \in \mathscr{B}$ there is a third model $B \in \mathscr{B}$ and a pair of $\mathbf{B}$-maps $B_1 \to B$, $B_2 \to B$.

Thus assuming that $\mathscr{B}$ is left-complete and that condition (0) is satisfied we have shown that $\mathscr{B}$ has binary sums of objects.

Condition (0) is going to be a bother. Allow me to adjoin a couple of ideal objects, O_L and O_R, to $\mathscr{B}$. For every $B \in \mathscr{B}$, (O_L, B) and (B, O_R) have exactly one element each, the only map into O_L is its own (abstract) identity map and the only map from O_R is its identity map.

By a straightforward extension of the above we find that left-completeness of $\mathscr{B}$ implies that $\mathscr{B}$ has arbitrary *sums*.

By working with slightly more complicated derived categories than cartesian product categories we can show that left-completeness implies the existence of difference cokernels. In other words, left-completeness implies right-completeness. This is true for categories of models to which we have adjoined the two ideal objects O_L, O_R. I remind you that I have skipped completely an important condition of the adjoint functor theorem, a condition which requires verification, a verification which requires the Tarski–Löwenheim–Skolem theorem.

We may dualize the whole matter to prove that right-completeness implies left-completeness and obtain:

Theorem. *A category of models is left-complete if and only if it is right-complete.*

I should explain that the addition of the ideal objects does not really have much effect. The situation is similar to that of the real numbers where instead of requiring that every set have a least upper bound we require only that sets with upper bounds have least upper bounds. O_L and O_R act as $-\infty$ and $+\infty$. The above theorem remains true without assuming the ideal objects if we redefine completeness. On the left we would say that a category is left-complete iff every set $\{B_i\}_I$ has a product if there is at least one family of the form $\{B \to B_i\}_I$, and every pair $A \xrightarrow{x} B$, $A \xrightarrow{y} B$ has a difference kernel if there is any $C \to A$ with $C \to A \xrightarrow{x} B = C \to A \xrightarrow{y} B$.

The third application is to the problem of restricted free products, which I would prefer to call restricted sums. For examples, I refer you to Łoś [*].

Suppose that $\mathscr{B}$ is a category of models and that $\mathbf{T}$, $\mathbf{F}$ are the defining classes of $\mathscr{B}$. Define a class of *restrictions* $\mathfrak{R}$ as a subset of $\{\langle G_1, G_2, F\rangle | G_1, G_2, F$ are formulas, $F \in \mathbf{F}\}$. I define an $\mathfrak{R}$-*restricted pair* of maps $B_1 \xrightarrow{f_1} B$, $B_2 \xrightarrow{f_2} B$ to be a pair such that for all $\langle G_1, G_2, F\rangle \in \mathfrak{R}$ and $x_1, \ldots, x_n \in B_1$, $y_1, \ldots, y_m \in B_2$ it is the case that

$$G_1(x_1, \ldots, x_n) \wedge G_2(y_1, \ldots, y_m) \Rightarrow F(f_1(x_1), \ldots, f_1(x_n), f_2(y_1), \ldots, f_2(y_m)).$$

S is an $\mathfrak{R}$-restricted sum of B_1 and B_2 if there is an $\mathfrak{R}$-restricted pair $B_1 \xrightarrow{u_1} S$ $B_2 \xrightarrow{u_2} S$ such that for every $\mathfrak{R}$-restricted pair $B_1 \xrightarrow{f_1} X$, $B_2 \xrightarrow{f_2} X$ there is a unique $\mathbf{F}$-map $S \to X$ such that $B_1 \xrightarrow{u_1} S \to X = f_1$ and $B \xrightarrow{u_2} S \to X = f_2$.

For example, let $\mathscr{B}$ be the category of rings-with-units and unit-preserving ring homomorphisms. Let $\mathfrak{R}$ consist of the single triple $\langle x=x, y=y, xy=yx\rangle$. The $\mathfrak{R}$-restricted sum of two rings B_1 and B_2 is then $B_1 \otimes B_2$, the torsion product. Note that if the set of third coordinates of $\mathfrak{R}$, when universally quantified, are theorems for the objects of $\mathscr{B}$ then $\mathfrak{R}$-restricted sums are ordinary sums. Hence for the category of commutative rings, the torsion product is the sum.

The general philosophy of the application of the adjoint functor theorems is that every universal mapping problem gives rise to a pair of adjoint functors. Sometimes a little ingenuity is needed to find the categories and functors. But once found, we can then read off the conditions of the adjoint functor theorem, translate them to the special case at hand, and obtain a result. In this case they reduce to the following, where $\mathfrak{R}_3$ is the set of third coordinates of $\mathfrak{R}$.

(R 1) *For every* $F \in \mathfrak{R}_3$, *product* $\{P \xrightarrow{p_i} B_i\}_I$ *and* $x_1, \ldots, x_n \in P$ *it is the case that* $\bigvee_I [F(p_i(x_1), \ldots, p_i(x_n)] \Rightarrow F(x_1, \ldots, x_n)$.

(R 2) *For every* $F \in \mathfrak{R}_3$ *and map* $K \xrightarrow{f} B$ *that appears as a difference kernel and for every* $x_1, \ldots, x_n \in K$ *it is the case that* $F(f(x_1), \ldots, f(x_n)) \Rightarrow$ $\Rightarrow F(x_1, \ldots, x_n)$.

In the event that $\mathscr{B}$ has free models, i.e., that the standard construction of products and difference kernels works, then both conditions are automatically true.

Theorem. *If* $\mathscr{B}$ *is a left-complete category of models and* $\mathfrak{R}$ *is a family of restrictions then* $\mathfrak{R}$*-restricted sums exist iff the above conditions* (R1) *and* (R_2) *are satisfied.*

Note that my definition of $\mathfrak{R}$-restricted sums did not insist that $B \xrightarrow{u_1} S$, $B \xrightarrow{u_2} S$ be a pair of one-to-one maps. It can be an interesting problem to determine when they are, but a problem best attacked, I think, by knowing what to look for if they aren't. u_1 is a one-to-one map if and only if for every $x, y \in B_1$ $x \neq y$, there exists an $\mathfrak{R}$-restricted pair $B_1 \xrightarrow{f_1} B$, $B_2 \xrightarrow{f_2} B$ such that $f_1(x) \neq f_1(y)$.

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LANGUAGES WITH ADDED QUANTIFIER "THERE EXIST AT LEAST $\aleph_\alpha$"

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1. Let $\mathbf{L}$ be a first-order language and α an ordinal. With $\mathbf{L}$ and α a new language is associated which – following the notation of Professor Mostowski's opening lecture – will be denoted by $\mathbf{Q}_\alpha$.

The grammar of $\mathbf{Q}_\alpha$ is obtained from the grammar of $\mathbf{L}$ by adding to the list of symbols of $\mathbf{L}$ a new symbol Q and treating Q as a quantifier. Thus, if φ is a formula of $\mathbf{Q}_\alpha$ and v is a variable, then $Q\mathsf{v}\varphi$ is a formula of $\mathbf{Q}_\alpha$, in which v no longer occurs free. Note that the grammar of $\mathbf{Q}_\alpha$ does not depend on the particular α.

Let φ be any sentence of $\mathbf{L}$ and R a binary predicate of $\mathbf{L}$. The following are examples of sentences of $\mathbf{Q}_\alpha$:

(a) $\varphi \wedge Q\mathsf{v}_0\mathsf{R}\mathsf{v}_0\mathsf{v}_0 \wedge \forall \mathsf{v}_0 \neg Q\mathsf{v}_1\mathsf{R}\mathsf{v}_1\mathsf{v}_0$;

(b) $\varphi \wedge \neg Q\mathsf{v}_0\mathsf{R}\mathsf{v}_0\mathsf{v}_0$;

(c) $Q\mathsf{v}_0\exists\mathsf{v}_1\mathsf{R}\mathsf{v}_0\mathsf{v}_1 \rightarrow [\exists\mathsf{v}_1 Q\mathsf{v}_0\mathsf{R}\mathsf{v}_0\mathsf{v}_1 \vee Q\mathsf{v}_1\exists\mathsf{v}_0\mathsf{R}\mathsf{v}_0\mathsf{v}_1]$.

The semantical rules of $\mathbf{Q}_\alpha$ are obtained from the semantical rules of $\mathbf{L}$ by adding to them a clause which makes Q read "there are at least $\aleph_\alpha$". More precisely, let $\mathscr{C}_\alpha$ be the class of all structures $\langle A, X\rangle$ with $X \subseteq A$ and $\aleph_\alpha \leqslant |X|$, where $|X|$ is the cardinality of X. Given a relational structure $\mathfrak{A}$ (of appropriate similarity type) and a sequence $\mathbf{x} \in A^\omega$: $\mathbf{x}$ satisfies a formula $Q\mathsf{v}_n\varphi$ of $\mathbf{Q}_\alpha$ in $\mathfrak{A}$ iff $\langle A, X\rangle \in \mathscr{C}_\alpha$, where $X = \{a \in A |$ $\mathbf{x}(n/a)$ satisfies φ of $\mathbf{Q}_\alpha$ in $\mathfrak{A}\}$ and $\mathbf{x}(n/a)$ is $\mathbf{x}$ with its nth term replaced by a. Instead of saying "$\mathbf{x}$ satisfies a formula φ of $\mathbf{Q}_\alpha$ in $\mathfrak{A}$" we shall also say "$\mathbf{x}$ $\mathbf{Q}_\alpha$-satisfies φ in $\mathfrak{A}$". Similarly with other semantical notions like truth and validity.

To illustrate the semantics of $\mathbf{Q}_\alpha$ let φ first be a first-order sentence expressing that R is a linear ordering. Then the sentence (a) expresses that R is a linear ordering whose field is of power $\aleph_\alpha$ while every proper initial segment is of smaller power. Such linear orderings will play a role later and – for lack of a better name – we shall call them $\aleph_\alpha$-like linear orderings. An $\aleph_0$-like linear ordering is, of course, a linear ordering

of type ω. In general, however, an $\aleph_\alpha$-like linear ordering need not be a well-ordering.

Next let φ be a first-order sentence expressing that R is a dense linear ordering without end points. Then, in the case $\alpha = 1$, the sentence (b) expresses that R is a linear ordering of type η, while in the case $\alpha = 0$, (b) has no model, so its negation is $\mathbf{Q}_0$-valid.

Finally, the sentence (c) is $\mathbf{Q}_\alpha$-valid if and only if $\aleph_\alpha$ is regular.

The study of these languages was suggested in Mostowski [57] and Mostowski [59b]. The suggestion was taken up by Craig, who in turn caused several other people to get interested in the subject, including the author. Mostowski and Craig consider, however, a more general situation: they admit several quantifiers, quantifiers binding more than one variable, and, in particular, allow also other classes besides the $\mathscr{C}_\alpha$'s in the semantical rules. The special case treated here recommends itself by its simpler intuitive content.

The main problem concerning the $\mathbf{Q}_\alpha$'s which Mostowski raised is the following: Let V_α be the set of all sentences of $\mathbf{Q}_\alpha$ which are $\mathbf{Q}_\alpha$-valid. Is V_α recursively enumerable (assuming $\mathbf{Q}_\alpha$ to be arithmetized)? Actually there are (at least) two ways of defining $\mathbf{Q}_\alpha$-validity. One way is to call a sentence $\mathbf{Q}_\alpha$-valid iff it is $\mathbf{Q}_\alpha$-true in every relational structure. The second way is to call a sentence $\mathbf{Q}_\alpha$-valid iff it is $\mathbf{Q}_\alpha$-true in every relational structure of power at least $\aleph_\alpha$. Since in a relational structure of power less than $\aleph_\alpha$ the $\aleph_\alpha$-quantifier acts trivially (much as the ordinary existential quantifier acts trivially in the empty domain), the second alternative is somewhat simpler and we shall adopt it here[1]. Thus $Q\mathsf{v}_0\, \mathsf{v}_0 \doteq \mathsf{v}_0$ is counted among the $\mathbf{Q}_\alpha$-valid sentences. Accordingly, only relational structures of power at least $\aleph_\alpha$ will be considered as candidates for being $\mathbf{Q}_\alpha$-models.

A second problem is the question about the relationship between the various V_α's for different α.

2. Mostowski answered his own question in the negative for the case $\alpha = 0$ (assuming that $\mathbf{Q}_0$ has a sufficient supply of nonlogical con-

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[1] In Fuhrken [b] and Vaught [a] the first alternative was taken. A sentence φ of $\mathbf{Q}_\alpha$ is $\mathbf{Q}_\alpha$-valid in the first sense if and only if φ is $\mathbf{Q}_\alpha$-valid in the second sense and $\neg\, Q\mathsf{v}_0 \mathsf{v}_0 \doteq \mathsf{v}_0 \to \varphi$ is $\mathbf{Q}_\alpha$-valid in the first sense. The latter is equivalent, for $\alpha \neq 0$, to the validity of a certain first-order sentence (obtained from φ by replacing subformulas of the form $Q\mathsf{v} \ldots$ by $\bot$ (falsity).

stants; this assumption underlies several of the following results without being mentioned explicitly).

Call an ordinal ξ *characterizable* in $\mathbf{Q}_\alpha$ by $\mathfrak{m}$ sentences iff there is a set Σ of sentences of $\mathbf{Q}_\alpha$ and a formula φ of $\mathbf{Q}_\alpha$ with two free variables such that (i) $|\Sigma| \leqslant \mathfrak{m}$; (ii) Σ has $\mathbf{Q}_\alpha$-models; and (iii) in every $\mathbf{Q}_\alpha$-model of Σ, φ defines a well-ordering of type ξ.

Taking for φ in (a) of Section 1 a first-order sentence saying that R is a linear ordering of the universe one obtains

(1) *ω is characterizable in $\mathbf{Q}_0$ by a single sentence.*

From (1) it follows by a general argument that first-order arithmetic is interpretable in a categorical, finitely axiomatizable theory formulated in $\mathbf{Q}_0$ – the axioms are the sentence characterizing ω and the recursive definitions of addition and multiplication – and hence Mostowski's result.

There are further consequences of (1) for $\mathbf{Q}_0$. Call $\mathbf{Q}_\alpha$ $(\mathfrak{m}, \mathfrak{n})$-*compact* iff for every set Σ of sentences of $\mathbf{Q}_\alpha$ of power at most $\mathfrak{m}$, if every subset of Σ of power less than $\mathfrak{n}$ has a $\mathbf{Q}_\alpha$-model, then Σ itself has a $\mathbf{Q}_\alpha$-model. Then

(2) *$\mathbf{Q}_0$ is $(\aleph_0, \aleph_0)$-incompact.*

This is a consequence of (1) and the general observation that a language in which an infinite ordinal ξ is characterizable by $\mathfrak{m}$ sentences is $(\mathfrak{m} + |\xi|, |\xi|)$-incompact.

Denote by $\mathfrak{h}_\alpha(\mathfrak{m})$ the least cardinal $\mathfrak{n}$ such that for every set Σ of sentences of $\mathbf{Q}_\alpha$ of power at most $\mathfrak{m}$, if Σ has a $\mathbf{Q}_\alpha$-model of power at least $\mathfrak{n}$, then Σ has $\mathbf{Q}_\alpha$-models in all such powers. By a general argument of Hanf [60], it follows from the downward Löwenheim–Skolem theorem for $\mathbf{Q}_\alpha$ that such a cardinal $\mathfrak{n}$ exists.

(3) $\aleph_0 < \mathfrak{h}_0(1)$,

or more generally

(3′) $\aleph_n < \mathfrak{h}_0(1)$, *even* $\beth_n < \mathfrak{h}_0(1)$ *for* $n \in \omega$.

Proposition (3) follows from (1). Proposition (3′) follows from (3) by repeated application of two devices. The idea of the first device is as follows: Let $\langle A, X, \ldots \rangle$ be a relational structure of power $\aleph_\beta$; let R be a linear ordering with field B such that every proper initial R-segment of B has power at most $\aleph_\beta$; for each $b \in B$ let F_b be a biunique correspondence between the R-segment of B determined by b and a subset of A;

and let $G=\bigcup\{\{b\}\times F_b|b\in B\}$. Then $\langle A\cup B, B, G, A, X, \ldots\rangle$ is a relational structure of power at most $\aleph_{\beta+1}$ and it may be of that power.

The idea of the second device – due to R. M. Robinson – is as follows: Let $\langle A, X, \ldots\rangle$ be as before. Let furthermore B be a set of subsets of A. Finally let $E=\{\langle a, b\rangle|a\in b\in B\}$. Then $\langle A\cup B, B, E, A, X, \ldots\rangle$ is a relational structure of power at most $2^{\aleph_\beta}$ and it may be of that power.

The two devices can be iterated into the transfinite when infinite ordinals are characterizable. More precisely:

(4) *If ξ is characterizable in $\mathbf{Q}_0$ by $\mathfrak{m}$ sentences and $\mathfrak{n}<\beth_0(\mathfrak{m})$, then $\aleph_\xi(\mathfrak{n})<\beth_0(\mathfrak{m})$ or even $\beth_\xi(\mathfrak{n})<\beth_0(\mathfrak{m})$, where $\aleph_\xi(\mathfrak{n})$ is the ξth aleph after $\mathfrak{n}$ and $\beth_\xi(\mathfrak{n})$ is the ξth beth after $\mathfrak{n}$.*

Note also that, if there is a set of power $\mathfrak{m}$ of sentences of $\mathbf{Q}_\alpha$ which has models of power $\mathfrak{n}$ but of no higher power, then $\mathbf{Q}_\alpha$ is $(\mathfrak{m}+\mathfrak{n}^+, \mathfrak{n}^+)$-incompact.

Propositions (1)–(4) generalize from $\alpha=0$ to arbitrary α, provided that (1) is rephrased to:

(1′) *The class of $\aleph_0$-like linear orderings is characterizable in $\mathbf{Q}_0$ by a single sentence.*

3. These and other results suggest the following question: For what α are what ordinals characterizable in $\mathbf{Q}_\alpha$ by how many sentences?

First there are some general results which apply to the $\mathbf{Q}_\alpha$'s.

(5) (i) *If ω is characterizable in $\mathbf{Q}_\alpha$ by a single sentence, then so is every constructive ordinal.* (ii) *If ω is characterizable in $\mathbf{Q}_\alpha$ by $\mathfrak{m}$ sentences, then*
(a) *every ordinal ξ less than the first weakly inaccessible ordinal is characterizable in $\mathbf{Q}_\alpha$ by $\mathfrak{m}+|\xi|$ sentences* (Craig);
(b) *every ordinal ξ less than the first strongly inaccessible ordinal is characterizable in $\mathbf{Q}_\alpha$ by $\mathfrak{m}+2^{|\xi|}$ sentences* (Craig and Hanf [62]);
(c) *every ordinal ξ less than the first measurable ordinal is characterizable in $\mathbf{Q}_\alpha$ by $\mathfrak{m}+2^{|\xi|}$ sentences* (Keisler).

Yasuhara [62a] has shown that (i) cannot be improved, i.e., that the converse of (i) is also true. The question whether (ii) can be improved remains open.

In view of (1), (5) applies to $\mathbf{Q}_0$. In addition we have:

(6) *If* (a) *$\aleph_\alpha$ is cofinal with ω; or* (b) *for some cardinal $\mathfrak{m}$, $\mathfrak{m}<\aleph_\alpha$ but $\aleph_\alpha\leqslant\mathfrak{m}^{\aleph_0}$, then ω is characterizable in $\mathbf{Q}_\alpha$ by $\aleph_\alpha$ sentences* (Fuhrken [62c]).

To see this in the case $\alpha = \omega$, let E and F be binary predicates and, for each $\xi \in \omega_\omega$, let c_ξ be an individual constant. Let Σ be the set of all first-order sentences (in these symbols) which are true when (i) the universe of discourse is ω_ω; (ii) E denotes the membership relation; (iii) c_ξ denotes ξ, for $\xi \in \omega_\omega$; (iv) F denotes $\{\langle \xi, \omega_\xi \rangle | \xi \in \omega\}$. The set $\Sigma \cup \{\forall \mathsf{v}_0 \neg Q \mathsf{v}_1 \mathsf{E} \mathsf{v}_1 \mathsf{v}_0\}$ – actually a much more perspicuous subset of this set – characterizes ω by means of the formula $\mathsf{E}\mathsf{v}_1\mathsf{c}_\omega \wedge [\mathsf{E}\mathsf{v}_0\mathsf{v}_1 \vee \mathsf{v}_0 = \mathsf{v}_1]$. It is clear how the argument for (a) runs in general. The argument for (b) is similar, though somewhat more involved.

One can try to improve (6) in two directions: one direction is to reduce the number of sentences necessary for the characterization; the other direction is to extend the result to other ordinals α. The first problem – except for the cases $\alpha = 0$ (by (1)) and $\alpha = 1$ (by (18)) – remains open. The second problem has a negative answer. This can be seen by applying the method of ultraproducts [2].

Call a cardinal $\mathfrak{n}$ *small* for $\aleph_\alpha$ iff for every family $\langle \mathfrak{m}_t | t \in T \rangle$ of cardinals, $|T| \leqslant \mathfrak{n}$ and $\mathfrak{m}_t < \aleph_\alpha$, for each $t \in T$, implies $\prod \langle \mathfrak{m}_t | t \in T \rangle < \aleph_\alpha$. Note that $\aleph_0$ is small for $\aleph_\alpha$ if and only if neither (a) nor (b) of (6) is satisfied.

The basic result of Łoś about the preservation of first-order properties under ultraproducts extends to properties expressible in $\mathfrak{Q}_\alpha$ with the proviso that the number of factors is small for $\aleph_\alpha$. More precisely:

(7) *Given a family $\langle \mathfrak{A}_t | t \in T \rangle$ of (similar) relational structures and an ultrafilter $\mathscr{D}$ on T. If $|T|$ is small for $\aleph_\alpha$, then for every sentence φ of $\mathfrak{Q}_\alpha$, φ is $\mathfrak{Q}_\alpha$-true in the ultraproduct $\mathscr{P}_{\mathscr{D}} \langle \mathfrak{A}_t | t \in T \rangle$ if and only if $\{t \in T | \varphi$ is $\mathfrak{Q}_\alpha$-true in $\mathfrak{A}_t\} \in \mathscr{D}$* (Fuhrken [62c]).

As in the case of first-order properties one obtains from (7):

(8) *If $\aleph_0$ is small for $\aleph_\alpha$, then every relational structure $\mathfrak{A}$ has a $\mathfrak{Q}_\alpha$-elementary extension such that every infinite definable subset is properly increased. Hence under the same assumption, ω is not characterizable in $\mathfrak{Q}_\alpha$ by any number of sentences.*

As a bonus one obtains a compactness theorem:

(9) *If $\aleph_\beta$ is small for $\aleph_\alpha$, then $\mathfrak{Q}_\alpha$ is $(\aleph_\beta, \aleph_0)$-compact.*

Due to the uniform nature of the ultraproduct construction, (7)–(9) extend to languages which have several generalized quantifiers, provided each quantifier satisfies the hypothesis about smallness.

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[2] For the notion of ultraproduct see, e.g., Frayne–Morel–Scott [62].

The present line of attack, starting from Mostowski's result, though leading to some results concerning the $\mathbf{Q}_\alpha$'s, did not lead to the solution of any more cases of Mostowski's problem. We shall now turn to a different line of attack.

4. The languages $\mathbf{Q}_\alpha$ furnish a means to generalize certain first-order Löwenheim–Skolem theorems. The prototype of the theorems we have in mind is Vaught's Löwenheim–Skolem theorem for two cardinals.

(10) *Let Γ be a countable set of first-order sentences and U and V unary predicates. If Γ has a model $\mathfrak{B}$ with $|\mathsf{V}^{\mathfrak{B}}| \geqslant \aleph_\alpha > |\mathsf{U}^{\mathfrak{B}}|$, then Γ has a model $\mathfrak{B}$ with $|\mathsf{V}^{\mathfrak{B}}| \geqslant \aleph_1 > |\mathsf{U}^{\mathfrak{B}}|$* (Morley and Vaught [1]) [3].

Proposition (10) can be rephrased as:

(10′) *Let Γ be as before. If $\Gamma \cup \{Q\mathsf{v}_0 \mathsf{V}\mathsf{v}_0, \neg Q\mathsf{v}_0 \mathsf{U}\mathsf{v}_0\}$ has a $\mathbf{Q}_\alpha$-model, then it has a $\mathbf{Q}_1$-model.*

Proposition (10′) generalizes to:

(11) *Let α be a successor ordinal and let Σ be a countable set of sentences of $\mathbf{Q}_\alpha$. If Σ has a $\mathbf{Q}_\alpha$-model, then Σ has a $\mathbf{Q}_1$-model.*

(Some restriction on α is necessary in view of example (c) of Section 1.)

As a corollary to (11) one obtains:

(12) $\mathrm{V}_1 \subseteq \mathrm{V}_\alpha$ *for any successor ordinal α.*

The proof of (11) consists in reducing it to (10) with the help of a *Skolem-type normal form*. The idea behind this normal form is as follows: Let $\mathfrak{A}$ be a $\mathbf{Q}_\alpha$-model of Σ. Choose subsets X and Y of A of power $\aleph_\alpha$ and $\aleph_\beta$ respectively, where $\alpha = \beta + 1$. Instead of saying "a subset of A has power at least $\aleph_\alpha$" we shall say "there is a biunique correspondence between a subset of the set and the set X"; instead of saying "a subset of A has power less than $\aleph_\alpha$" we shall say "there is a biunique correspondence between the set and a subset of Y". The collection of these unary functions (for subsets of A which are definable with the help of parameters) is replaced by one binary function, and thus the above second-order phrase is replaced by a first-order phrase.

To carry this idea out, let U and V be new unary predicates and F

[3] The theorem is stated there for $\mathrm{V}^{\mathfrak{A}} = \mathrm{A}$. The present generalization is immediate. Similar remarks apply to other results mentioned.

a new ternary predicate. Define recursively φ^* for every formula φ of $\mathbf{Q}_\alpha$ by

(*) $\quad (Qv_n\varphi)^*$ is $\exists v_m \forall v_{m+1}\,[Vv_{m+1} \to \exists v_n\,[Fv_m v_{m+1} v_n \wedge \varphi^*]]$,

where v_m is the first variable after all the variables of φ^*, while requiring that * commutes with the other logical connectives. Clearly, for every sentence φ of $\mathbf{Q}_\alpha$, φ^* is a first-order sentence.

Furthermore, let Δ be the set of the following first-order sentences:

(i) $\forall v_0[\forall v_1 \exists v_2 \forall v_3[Fv_0 v_1 v_3 \to v_2 \doteq v_3] \wedge \forall v_1 \exists v_2 \forall v_3[Fv_0 v_3 v_1 \to v_2 \doteq v_3]]$;

(ii) the universal closure of

$$\exists v_m \forall v_{m+1}[Vv_{m+1} \to \exists v_n[Fv_m v_{m+1} v_n \wedge \varphi^*]] \vee \exists v_m \forall v_n[\varphi^* \to \exists v_{m+1}[Fv_m v_{m+1} v_n \wedge Uv_{m+1}]],$$

where φ is any formula of $\mathbf{Q}_\alpha$ (involving the symbols of Σ) and v_m is as in (*). Note that neither φ^* nor Δ depends on the particular α.

Now one can prove:

(13) *Let Σ be a set of sentences of $\mathbf{Q}_\alpha$ of power at most $\aleph_\alpha$. A relational structure $\mathfrak{A}$ is a $\mathbf{Q}_\alpha$-model of Σ if and only if there are X, Y, and Z such that $\aleph_\alpha \leqslant |X|$, $|Y| < \aleph_\alpha$, and $(\mathfrak{A}, X, Y, Z)$ is a model of $\Sigma^* \cup \Delta$, where $\Sigma^* = \{\varphi^* | \varphi \in \Sigma\}$. (There is much freedom in choosing X and Y; e.g., if $|A| = \aleph_\alpha$, then one can take $X = A$.)*

Combining (10) and (13) yields (11). In the same way one can generalize a (partial) converse of (10) due to C. C. Chang (the proof of which, however, requires the general continuum hypothesis (GCH)).

(14) (GCH) *Let Γ, $\cup$ and $\vee$ be as in (10). If Γ has a model $\mathfrak{B}$ with $|V^{\mathfrak{B}}| \geqslant \aleph_1 > |U^{\mathfrak{B}}|$, then Γ has a model $\mathfrak{B}$ with $|V^{\mathfrak{B}}| \geqslant \aleph_\alpha > |U^{\mathfrak{B}}|$ where α is the successor of a regular ordinal.*

From (14) one obtains:

(15) (GCH) *Assume α is the successor of a regular ordinal. Let Σ be as in (11). If Σ has a $\mathbf{Q}_1$-model, then Σ has a $\mathbf{Q}_\alpha$-model.*

And as a corollary we have:

(16) (GCH) $V_1 = V_\alpha$ *for any successor α of a regular ordinal.*

The question remains open whether in (14), (15) or (16) the assumption that α has to be the successor of a *regular* ordinal is necessary.

There are further Löwenheim–Skolem theorems for two cardinals due to Chang and Keisler [62]. They do not lend themselves to a generalization of the above kind. They suggested (7), however.

5. A closer look at (the proof of) (10) and (13) leads to further results, in particular to a positive answer to Mostowski's problem for the case $\alpha = 1$.

Vaught's argument for (10) runs, roughly, as follows: Let $\mathfrak{B}$ be a model of Γ with $|V^{\mathfrak{B}}| \geqslant \aleph_\alpha > |U^{\mathfrak{B}}|$. By Tarski's form of the Löwenheim–Skolem theorem one obtains denumerable models $\mathfrak{B}'$ and $\mathfrak{B}''$ of Γ such that $\mathfrak{B}''$ is an elementary extension of $\mathfrak{B}'$ having the same U and being a proper extension with respect to V. $\mathfrak{B}'$ and $\mathfrak{B}''$ can be replaced via a diagram argument by models $\mathfrak{B}_0$ and $\mathfrak{B}_1$ which have the additional property of being homogeneous and having the same kinds of elements. Thus by a Cantor-type argument $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are isomorphic. Hence $\mathfrak{B}_1$ has an elementary extension $\mathfrak{B}_2$ which is related to $\mathfrak{B}_1$ as $\mathfrak{B}_1$ is to $\mathfrak{B}_0$. Repeating this one obtains a chain of elementary extensions, all having the same U and being isomorphic to $\mathfrak{B}_0$. The union of such a countable chain has again these properties. The process can thus be continued into the transfinite. After ω_1 steps one arrives at the desired model.

To state the essential step as a theorem, let W be a new unary predicate and let Θ be the set of the following first-order sentences:

(i) the universal closure of $Wv_0 \wedge \ldots \wedge Wv_{n-1} \to [\varphi \leftrightarrow \varphi^{(W)}]$ for every formula φ (involving the symbols of Γ), where $\varphi^{(W)}$ is obtained from φ by relativising quantifiers to W, and $v_0, \ldots, v_{n-1}$ are all the free variables of φ;

(ii) $\exists v_0 Wv_0 \wedge \exists v_0[\neg\, Wv_0 \wedge Vv_0]$;

(iii) $\forall v_0[Uv_0 \to Wv_0]$.

(17) *The following propositions are equivalent*:

(a) *Γ has a model $\mathfrak{B}$ with* $|V^{\mathfrak{B}}| \geqslant \aleph_1 > |U^{\mathfrak{B}}|$;

(b) *Γ has a model which has an elementary extension with the same U and which is proper extension with respect to V;*

(c) *$\Gamma \cup \Delta$ is consistent.*

Using (17) and analysing the transition from Σ to $\Sigma^* \cup \Delta$ $(= \Gamma)$ and from there to $\Sigma^* \cup \Delta \cup \Theta$ one obtains the following theorems from the first-order compactness theorem:

(18) $\mathbf{Q}_1$ *is* $(\aleph_0, \aleph_0)$*-compact* (Fuhrken [b]).

(19) *For every sentence φ of $\mathbf{Q}_1$, $\varphi \in V_1$ if and only if $\Delta \cup \Theta \vdash \varphi^*$; hence V_1 is recursively enumerable* (Vaught [a]).

The (rather vaguely defined) task remains to find an axiomatization of V_1 of a more usual kind. Yasuhara [62] has given a Gentzen-type axiomatization for the subset of V_1 consisting of all sentences of V_1 which do not contain the ordinary quantifiers $\exists$ and $\forall$.

6. The normal form used in (13) can be adapted to the case of arbitrary – not necessarily successor – α. The single standard set U of lower power than $\aleph_\alpha$ has to be replaced by a collection of sets of lower power than $\aleph_\alpha$. Instead of saying "a subset of A has power less than $\aleph_\alpha$" we shall say "there is a biunique correspondence between the subset and a set of this collection". The collection is combined to one binary relation and thus the above second-order phrase is replaced by a first-order phrase. The most useful choices for such a binary relation seem to be the $\aleph_\alpha$-like linear orderings, taking at the same time their field as V.

To carry the idea out, let U be a new binary predicate and F a new ternary predicate. Define recursively φ^* for every formula φ of $\mathbf{Q}_\alpha$ by

$$(*) \qquad (Q v_n \varphi)^* \text{ is } \exists v_m \forall v_{m+1} [U v_{m+1} v_{m+1} \to \exists v_n [F v_m v_{m+1} v_n \wedge \varphi^*]],$$

where v_n is the first variable after all variables of φ^*, while requiring that * commutes with the other logical connectives. For every sentence φ of $\mathbf{Q}_\alpha$, φ^* is a first-order sentence.

Furthermore, let Δ be the set of the following first-order sentences:

(i) as in the definition preceding (13);

(ii) the universal closure of

$$\exists v_m \forall v_{m+1} [U v_{m+1} v_{m+1} \to \exists v_n [F v_m v_{m+1} v_n \wedge \varphi^*]]$$
$$\vee \exists v_m \exists v_{m+1} \forall v_n [\varphi^* \to \exists v_{m+2} [F v_m v_{m+2} v_n \wedge U v_{m+2} v_{m+1}]],$$

where φ is any formula of $\mathbf{Q}_\alpha$ (involving the symbols of Σ) and v_m is as in (*);

(iii) a sentence expressing that U is a linear ordering.

Now one can prove:

(20) *Let Σ be a set of sentences of $\mathbf{Q}_\alpha$ of power at most $\aleph_\alpha$. A relational structure $\mathfrak{A}$ is a $\mathbf{Q}_\alpha$-model of Σ if and only if there are X and Y such that X is an $\aleph_\alpha$-like linear ordering and $(\mathfrak{A}, X, Y)$ is a model of $\Sigma^* \cup \Delta$, where $\Sigma^* = \{\varphi^* | \varphi \in \Sigma\}$. (There is much freedom in choosing X; e.g., one can choose X to be a well-ordering.)*

In order to use this normal form in a similar way as the previous one, one needs theorems of the following form:

($\neq$) Let Γ be a (countable) set of first-order sentences and $\cup$ a binary predicate. If Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is a linear ordering of type ω_α and α satisfies ..., then Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is an $\aleph_\beta$-like linear ordering and β satisfies ....

The following are such theorems:

(21) *Let Γ and $\cup$ be as above. If Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is a linear ordering of type ω, then for any ordinal α, Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is an $\aleph_\alpha$-like linear ordering* (MacDowell and Specker [61]).

(22) *Let Γ and $\cup$ be as above. If Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is a linear ordering of type θ, θ a weakly inaccessible ordinal, then for every β, $\beta<\theta$, there is an α, $\beta\leqslant\alpha<\theta$ and α cofinal with ω, for which Γ has a model $\mathfrak{B}'$ where $\cup^{\mathfrak{B}'}$ is a linear ordering of type ω_α; in fact, $\mathfrak{B}'$ can be chosen as an elementary substructure of $\mathfrak{B}$ whose universe forms an initial segment* (Dana Scott).

(23) *Let Γ and $\cup$ be as above. If Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is an $\aleph_\alpha$-like linear ordering, $\aleph_\alpha$ being regular, then Γ has a model $\mathfrak{B}$ where $\cup^{\mathfrak{B}}$ is an $\aleph_1$-like linear ordering* (Fuhrken [b]).

From (21), (22), and (23) one obtains with the help of (20):

(24) *Let Σ be a countable set of sentences of $\mathbf{Q}_\alpha$.*

(a) *If Σ has a $\mathbf{Q}_0$-model, then Σ has a $\mathbf{Q}_\alpha$-model for any ordinal α.*

(b) *If Σ has a $\mathbf{Q}_0$-model, θ a weakly inaccessible ordinal, then for every β, $\beta<\theta$, there is an α, $\beta\leqslant\alpha<\theta$, such that α is cofinal with ω and Σ has a $\mathbf{Q}_\alpha$-model.*

(c) *If Σ has a $\mathbf{Q}_\alpha$-model, α being such that $\omega_\varkappa$ is regular, then Σ has a $\mathbf{Q}_1$-model.*

From (24) in turn one obtains as a corollary:

(25) (a) $\mathrm{V}_\alpha \subseteq \mathrm{V}_0$ *for any ordinal α*; (b) $\mathrm{V}_1 \subseteq \mathrm{V}_\varkappa$ *for any ordinal $\varkappa$ such that $\omega_\varkappa$ is regular.*

7. Examples show that $\mathrm{V}_\varkappa$ is not completely independent of $\varkappa$. E.g., sentence (b) from the introduction shows that $\mathrm{V}_0 \nsubseteq \mathrm{V}_\alpha$, α a non-zero ordinal; sentence (c) shows that $\mathrm{V}_\alpha \nsubseteq \mathrm{V}_\beta$, α and β such that $\omega_\varkappa$ is regular

and ω_β is singular. These and other examples leave the following conjecture:

($\neq$) $V_\alpha = V_\beta$ *whenever* α *and* β *are both*

(a) *successor ordinals*; *or*

(b) *such that* ω_α *and* ω_β *are singular*; *or*

(c) *weakly, but not strongly inaccessible*; *or*

(d) *strongly inaccessible.*

A positive answer to this conjecture would reduce Mostowski's problem to three open cases.

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MODEL-THEORETIC METHODS IN THE STUDY OF ELEMENTARY LOGIC

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In order to study the structural and recursive properties of first-order theories, we introduce the notion of isomorphism of theories. Roughly speaking, two theories are isomorphic if there is a one-to-one recursive correspondence between sentences of one theory and sentences of the other which preserves implications valid in the theories. The main result of the paper is that there exists a finitely axiomatizable theory isomorphic to a theory constructed as a product of a fixed theory and an arbitrary axiomatizable theory. From this result, finitely axiomatizable theories can be constructed which reflect certain properties of arbitrary axiomatizable theories. One consequence is that there exists a sentence of elementary logic whose consequences form a recursively enumerable set of any prescribed degree of undecidability.

In Section 1 the basic notion of isomorphism is defined and some preliminary results are formulated (without proof). A simple theory H is given which is characterized up to isomorphism by the property of being decidable and being such that no finite extension is complete. The theory H also has the property that any axiomatizable theory is isomorphic to some axiomatizable extension of H. Section 1 also gives a definition of product of theories and states a necessary and sufficient condition for a theory to be isomorphic to the product of two other theories. Further basic results concerning isomorphism of theories are given in the abstract Hanf [62a]. In particular, a result stated there implies that the theories constructed in this paper could be formulated with one binary predicate symbol.

Section 2 gives a fairly detailed outline of the proof of the main result that given any axiomatizable theory T there exists a finitely axiomatizable theory F which is isomorphic to the product of T and the theory H mentioned above. The proof, which is model-theoretic in nature, makes use of a representation of a Turing machine in each model of F.

Section 3 states some consequences of the result and methods of

Section 2. Theorem 3.1, together with a well-known result of Friedberg and Muchnik, gives an example of a finitely axiomatizable theory of intermediate degree of undecidability. This solves a problem of Feferman [57]; the result was first stated in Hanf [62]. As pointed out by Rabin, it also gives an example of a decidable finitely axiomatizable theory whose decision method is not primitive recursive. The theory constructed in Theorem 3.1 has the additional property that no finite extension is essentially undecidable. This solves a problem of Tarski which appears as Problem 4 in Vaught [60]. This shows that the method of showing theories undecidable by proving them to be compatible with some finitely axiomatizable essentially undecidable theory does not apply to all finitely axiomatizable undecidable theories. Theorem 3.2 solves another problem of Tarski by showing that the above-mentioned method of showing undecidability also does not apply to all essentially hereditarily undecidable theories. Theorem 3.3 improves a result of Shoenfield by giving a finitely axiomatized example of an essentially undecidable theory of intermediate degree. Similarly Theorem 3.4, which was suggested by Cobham, improves a result of Ehrenfeucht by giving a finitely axiomatized example of an essentially undecidable theory which is recursively separable. Kreisel suggested that the author's method could be used to solve a problem stated by Feferman [60, footnote 23] as to whether there exists a decidable theory whose consistency cannot be proved in arithmetic. This turned out to be the case and the result is given in Theorem 3.5 and Corollary 3.6. Finally, in Section 3, one of a number of open problems concerning the existence of finitely axiomatizable theories is stated (many further problems can be stated in terms of the Lindenbaum–Tarski algebras of such theories). Two alternative conjectures are formulated in terms of the notion of isomorphism. The first conjecture implies that the variety of such theories is very broad and the second conjecture implies that the variety is very limited (at least for those theories that have countably many complete extensions). The verification of either conjecture would provide the solution to a number of open problems.

The author is indebted to a number of logicians for conversations at various stages of this research. In particular, the applications mentioned in the previous paragraph were suggested when only a general outline of a proof of Theorem 3.1 was known. The method of Theorem 2.1 was developed later in order to permit a unified presentation of the various results. The author is especially indebted to Professor Vaught,

whose questions and observations during the early stages of the research helped to pinpoint the key results and the principal obstacles to their proof. Finally, it should be mentioned that the author made essential use of the ideas of Minsky [61] in earlier versions of the proof of Theorem 3.1.

1. Isomorphism of theories. In this paper we consider theories formulated in first-order predicate logic with equality. The similarity type $\mu = \langle \mu_1, \mu_2, \ldots, \mu_n \rangle$ of a theory specifies the number n of predicate symbols and gives the number μ_i of places of the ith predicate symbol. We denote by L_μ the set of all sentences (formulas without free variables) formulated in similarity type μ. For any theory T, we denote by $K(T)$ the set of all relational structures of similarity type μ which satisfy all the theorems of T. Given two theories S and T of similarity type μ and ν, we say that S is isomorphic to T if there exist functions F and G such that

(i) F is a function mapping $K(S)$ one-to-one onto $K(T)$.

(ii) G is a recursive function mapping L_μ one-to-one onto L_ν.

(iii) For every sentence α of L_μ and relational structure $\mathfrak{A}$ of $K(S)$, α is true of $\mathfrak{A}$ just in case $G(\alpha)$ is true of $F(\mathfrak{A})$.

Note that if we drop the condition that G be recursive, the definition can be formulated for any language for which the notion of model and truth is defined. Dana Scott has pointed out that in the case of first-order logic the notion can be formulated in terms of sentences only. Two first-order theories are isomorphic if there exists a function G satisfying (ii) and such that

(iv) For all sentences α and β of L_μ, the sentence $\alpha \to \beta$ is a theorem of S just in case $G(\alpha) \to G(\beta)$ is a theorem of T.

Let H be the theory with a single binary relation symbol (thought of as a successor relation) and axioms expressing that every element has a unique successor and at most one predecessor and that there is exactly one element with no predecessor. Models of H all have a standard part isomorphic to the positive integers. In addition, they may have a number of non-standard parts each of which is isomorphic to all integers or is a loop consisting of a finite number of points. Every sentence is equivalent in H to a Boolean combination of sentences $\lambda(k, n)$ which

state that there are at least k loops of n elements each. (This fact is quite elementary but can also be derived using the methods of Section 2.) Since the implications among the sentences $\lambda(k, n)$ are straightforward, H is decidable.

Theorem 1.1. *Suppose T is decidable and no finite extension of T is complete. Then T is isomorphic to H.*

Theorem 1.2. *Any axiomatizable theory T is isomorphic to some axiomatizable extension of H.*

Note: Theorem 1.2 remains true if "axiomatizable" is deleted in both of its occurrences.

Given two theories T_0 and T_1 of similarity types μ and ν, we construct the theory $T_0 \times T_1$ as follows: The similarity type of $T_0 \times T_1$ is $\langle 1 \rangle \frown \mu \frown \nu$; that is, $T_0 \times T_1$ has all the predicate symbols of T_0 and T_1 (assumed distinct) together with a new unary predicate P. P divides the universe of each model of $T_0 \times T_1$ into two parts A_0 and A_1. The axioms of $T_0 \times T_1$ insure that A_0 and A_1 are both non-empty, that no elements of A_0 satisfy predicates of T_1 or vice versa, and, finally, that all the theorems of T_0 hold when their quantifiers are restricted to A_0 and similarly for T_1 and A_1.

Lemma 1.3. *Suppose that S, T_0, and T_1 are theories and Γ_0 and Γ_1 are sets of sentences of S such that every sentence of S is equivalent in S to a Boolean combination of sentences of Γ_0 and Γ_1 and suppose that for $i = 0, 1$:*

(i) *For every $\alpha \in \Gamma_i$, there is a sentence of Γ_i equivalent in S to the negation of α.*

(ii) *There exists a one-to-one recursive function G_i mapping Γ_i onto the set of all sentences of T_i such that, for any $\alpha, \beta \in \Gamma_i$, $\alpha \to \beta$ is a theorem of S if and only if $G(\alpha) \to G(\beta)$ is a theorem of T_i.*

(iii) *If $\alpha \in \Gamma_i$ and $\beta \in \Gamma_{1-i}$ and $\alpha \to \beta$ is a theorem of S, then α is refutable or β is provable in S.*

Then S is isomorphic to $T_0 \times T_1$.

2. The main theorem.

Theorem 2.1. *If T is any axiomatizable theory, then there exists a finitely axiomatizable theory F such that $F \cong T \times H$.*

The proof of this theorem proceeds by first constructing a Turing machine M which "accepts" tapes which have written on them a description of a complete extension of T (Lemma 2.2). The theory F is then constructed so that each model of F incorporates a "diagram" of the machine M including an initial tape and subsequent states of the tape. The theory F requires that the machine accept the initial tape. Hence the initial tape represented in the model must have written on it a description of a complete extension of T. This permits a correspondence to be set up between an arbitrary sentence α of T and a sentence of F which says that α is a sentence in the complete extension represented by the initial tape of the machine. These sentences, although they completely determine the standard part of the model, do not completely describe the model. Additional sentences describing non-standard parts of the model are made to correspond to sentences of the theory H. Using Lemma 1.3, these two correspondences establish the required isomorphism.

The Turing machine we will construct has a single two-way infinite tape. The machine has a finite number m of internal states and each square of the tape can have one of n symbols, including *, 0, and 1, written on it. The machine is defined by a set M of quintuples $\langle i,j,p,q,t\rangle$ where $i, p<m$; $j, q<n$; and $t=0, 1$. For each state i and symbol j, there is at most one quintuple $\langle i, j, p, q, t\rangle$ in M. If the machine is in the ith state and reading the jth symbol, it will change to the pth state and will write the qth symbol in place of the symbol it read. If $t=0$, it will continue moving the tape in the same direction in which it was going and if $t=1$, the tape motion will be reversed. However, if there is no quintuple corresponding to i, j, the machine will halt.

Let g be a fixed recursive function mapping the set of all sentences formulated in the theory T one-to-one onto the positive integers. (Later in this section we will prescribe some further conditions on g in order to make a certain part of the proof work.) A tape for machine M will be said to *represent* an extension T' of T iff the following conditions hold:

(i) Exactly one * appears on the tape.

(ii) Only 0's appear to the right of the *.

(iii) For each sentence σ, if $n=g(\sigma)$, then the nth square of the tape to the left of the * has a 1 written on it if σ is a theorem of T' and has 0 written on it otherwise.

Lemma 2.2. *Given an axiomatizable theory T, there exists a Turing machine M such that*

(i) *If the initial tape does not represent a complete extension of T, then M halts after some finite number of steps.*

(ii) *If the initial tape represents a complete extension T' of T (and the machine is started so that it first reads the square just to the right of the *), then M never halts and every quintuple of M is used infinitely many times.*

Proof. The construction of machine M is simply a matter of combining some well-known techniques. Figure 2.1 shows the overall plan of the program for M. The program is based on the observation that, if T' is not a complete extension of T, then there exists a positive integer n such that either

(a) n is the Gödel number of a sentence σ such that neither σ nor $\neg\sigma$ is in T' or

(b) n is the Gödel number of a proof of the contradiction $(\forall x)(x \neq x)$ from premises taken from T' and the axioms of T.

The program looks successively at each possible n and halts if n satisfies (a) or (b). On the other hand, it is easily seen that if no n satisfies (a) or (b) then T' must be a complete extension of T and in this case the program continues in its loop forever.

Notice that each box of the flow chart in Figure 2.1 represents a simple recursive decision or operation. For example, there is a well-known recursive procedure for determining whether an integer n is the Gödel number of some sentence of a given similarity type (that of the theory T). The question "Is σ in T'?" is simply a matter of checking whether or not the $g(n)$th symbol to the left of the * is a 1. To illustrate how machine M might actually work, let us describe more fully the block which says "Is σ_j an axiom of T?". Upon arriving at this block the tape might have the following configuration:

$$\underset{T'}{\ldots 0\,1\,1\,0\,1\,0} * \overbrace{1\,1\,1\,1\,1\,1\,1\,1\,1\,0}^{n}\,0\,\overbrace{1\,1\,1\,1\,1}^{\sigma_0}\,0\,\overbrace{1\,1\,1}^{\sigma_1}\,0\,0\,0\ldots$$

To the left of the *, the tape has its original value representing a set T' of sentences. To the right is first a series of n 1's representing the number of times through the main loop. Let us suppose that n has been found to

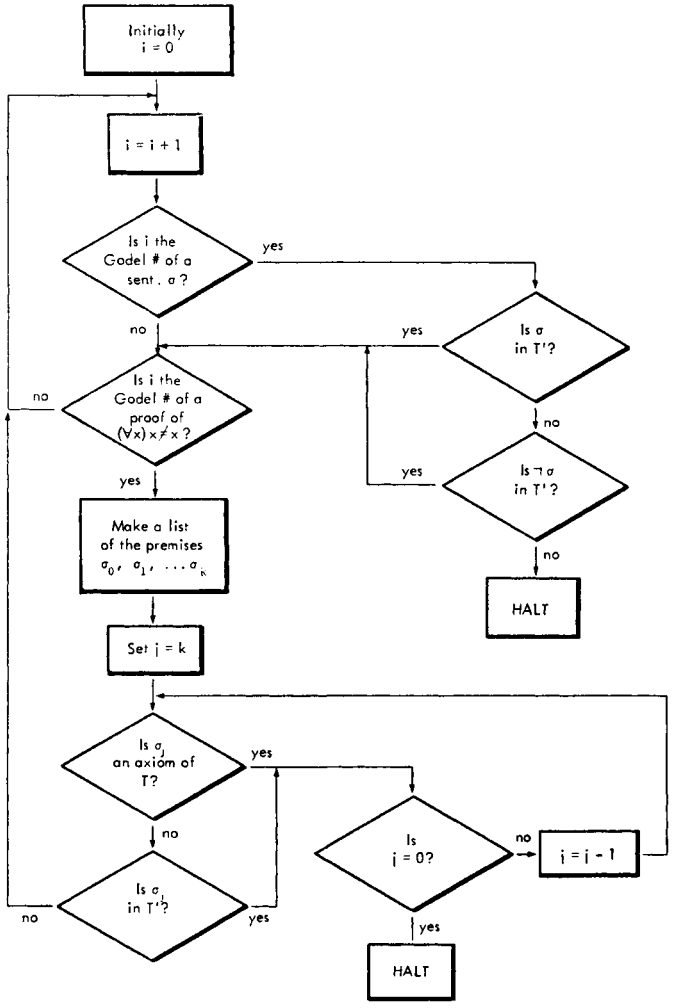


Fig. 2.1

be the Gödel number of a proof of the contradiction $(\forall x)(x \neq x)$ from premises σ_0, σ_1, and σ_2 and that σ_2 has already been found to be in T' and so has been removed from the list of premises. Thus σ_0 and σ_1 remain on the list of premises and we are ready to ask the question "Is σ_1 an axiom of T?". To answer the question, we simply compute the value of the characteristic function of the set of axioms of T. This is done by a sub-program which uses for its working space the portion of the tape to the right of the value σ_1. Thus the computation is not

affected at all by other information on the tape. This sub-program is designed so that if anything but 0's were written in its working area, it would halt. If the value of the characteristic function is zero, the program goes on to determine whether σ_1 is written on the tape to the left of the *. If, on the other hand, the value is one, premise σ_1 is erased and the program goes on to examine σ_0.

It should now be clear how the details of the construction of machine M could be carried out. Notice that this can be done in such a way that each state-symbol combination that is used at all will be used infinitely many times. In the case of the computation of the characteristic function described above, this follows from the fact that each argument is used infinitely many times. It should also be pointed out that the machine M (and consequently the theory F) can be obtained recursively in terms of some standard method of presenting the axioms of T. This might require that the sub-program computing the characteristic function of the set of axioms be written interpretively so that the state-symbol combinations used in the sub-program can be effectively determined.

The theory F which will be constructed has two binary predicates R and S, m unary predicates $P_0, P_1, \ldots P_{m-1}$ corresponding to the m states of the machine M, and n unary predicates $Q_0, Q_1, \ldots Q_{n-1}$ corresponding to the n symbols that can be written on the tape. Figure 2.2 shows a part of a model of F. The dots, small circles, and small

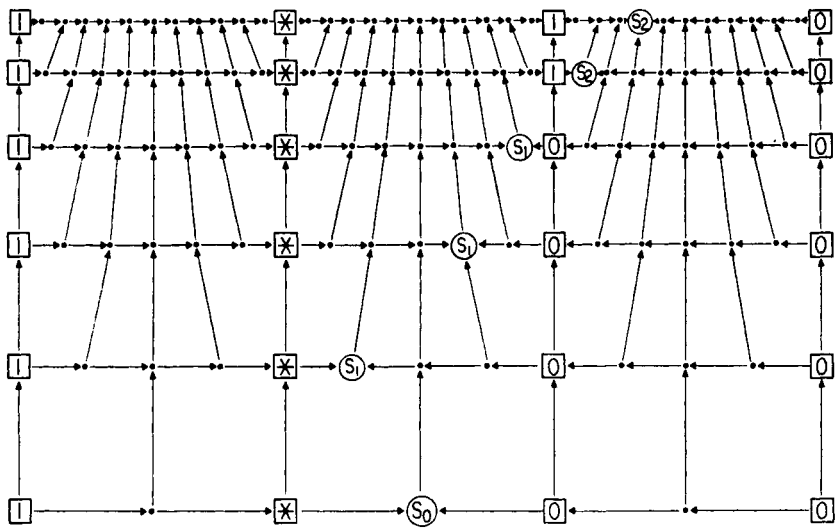


Fig. 2.2

squares represent elements of the model; the elements represented by circles are assumed to satisfy the unary predicate corresponding to the state indicated inside the circle and similarly, the elements represented by squares satisfy the predicate corresponding to the symbol written in the square; the elements represented by dots do not satisfy any unary predicate. The horizontal arrows indicate which elements are in the relation R and the vertical arrows indicate which elements are in the relation S. From the axioms which will be described later, it follows that any model of F has a set of elements which are grouped into a rectangular array by the relations R and S. The elements strung together by R form the rows of the array. Each row extends to infinity in both directions. There is an initial row in which every other element satisfies one of the predicates Q_j. Exactly one of the elements in this row satisfies the initial state predicate P_0. Each element in this initial row is related by S to a unique element of the second row and the same holds for each row and its succeeding row. As in figure 2.2, each "square" element (i.e., each element satisfying $Q_0, \ldots, Q_{n-1}$) in a row is succeeded in the next row by a square element. Furthermore, in each row, there are, in addition to the elements S related to the elements of the preceding row, new elements introduced on each side of each square element. Thus the number of filler elements between each pair of square elements increases from row to row; for example, in the third row, every sixth element is square. If we neglect the filler elements and look only at the square elements in a given row, we see that they form an image of a tape for machine M since each square element has one of the n symbols written on it (i.e., satisfies one of $Q_0, \ldots Q_{n-1}$). Also, in each row, there is exactly one element which satisfies one of the state predicates $P_0, \ldots P_{m-1}$. (This state element is also unique in that it is the R-successor of two elements; all the other elements in the row have one R-predecessor and one R-successor.) Thus each row determines not only a tape for machine M but also a state of M and a position of the reading head on the tape (determined by the position of the state element).

The fundamental property of the theory F is that in any model, the successive rows of the array accurately reflect the operations machine M would go through if started with the tape represented by the first row of the array. Actually, the 2nd, 5th, 14th, 41st, etc., rows of the array represent the state of the machine M and its tape after the first, second, third, fourth, etc. steps of its operation. (The distance between

significant rows of the array increases as we proceed through successive rows because the state element must pass over more and more intervening filler elements. Because new filler elements are added in each row, the state element must move over two filler elements in order to make any progress at all.) Furthermore, because the axioms do not allow the machine to halt, we can conclude from Lemma 2.1 that the initial row of the array corresponds to a tape which represents a complete extension of the theory T. Thus at this point it is clear that for each theorem α of T, there will be a corresponding theorem of F which states that a certain element in the initial row of the array (of any model) satisfies Q_1. From this correspondence it is clear that F will have a degree of undecidability at least as great as that of T. Notice, however, that if the filler elements or some equivalent device had not been introduced, there would be no hope of proving that the degree of undecidability of F is at most that of T. For it could happen that even though T is decidable, the decision procedure given us for its axiom set could involve as intermediate steps the examination of proofs within, say, some system of arithmetic. In this case, it might be possible to formulate sentences of F which are provable just in case the corresponding sentences of arithmetic are provable.

We now formulate a notion of distance within a model and state a lemma which allows us to show elementary equivalence for various models of F. In Figure 2.2, the distance between any pair of elements is simply the smallest number of arrows that must be followed (either in the forward or reverse direction) to get from one element to the other. More precisely, given a model with binary relations R and S, we define the (open) sphere $Sp(n, a)$ of radius n about a point a recursively as follows:

$$\begin{aligned} Sp(1, a) &= a \\ Sp(n+1, a) &= \text{the set of points } x \text{ such that, for some } y \in Sp(n,a), \\ &\quad \text{either } xRy,\ yRx,\ xSy,\ ySx, \text{ or } x=y. \end{aligned}$$

The n-type τ of an element a of a model $\mathfrak{A}$ we define to be the isomorphism type of the sphere of radius n about a; i.e., the isomorphism type of $\mathfrak{A} \restriction Sp(n, a)$.

Lemma 2.3. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models of the same similarity type (having two binary and arbitrarily many unary relations) and that each sphere of finite radius in $\mathfrak{A}$ or $\mathfrak{B}$ contains finitely many points. Then*

$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent provided that, for each integer n and n-type τ either

(i) *Both $\mathfrak{A}$ and $\mathfrak{B}$ have infinitely many elements of n-type τ or*

(ii) *$\mathfrak{A}$ and $\mathfrak{B}$ have the same finite number of elements of n-type τ.*

Proof. We make use of the method of Ehrenfeucht [61, Theorem 7]. Given a length k of sequence, the "strategy" for picking elements is to insure that after partial sequences $a_0, \ldots, a_{j-1}$ and $b_0, \ldots, b_{j-1}$ have been picked, the following condition holds:

$$\text{(I)} \qquad \mathfrak{A} \restriction \bigcup_{i<j} Sp(3^{k-j}, a_i) \cong \mathfrak{B} \restriction \bigcup_{i<j} Sp(3^{k-j}, b_i).$$

To show elementary equivalence, one needs to show only that if (I) holds, and a_j is any element of $\mathfrak{A}$ then there exists an element b_j of $\mathfrak{B}$ such that (I) holds with j replaced by $j+1$. There are two cases: If

$$a_j \in \bigcup_{i<j} Sp(2 \cdot 3^{k-j-1}, a_i),$$

pick b_j to be the corresponding element (given by the isomorphism in (I)) of $\mathfrak{B}$. If not, pick b_j to be any element of $\mathfrak{B}$ which is outside

$$\bigcup_{i<j} Sp(2 \cdot 3^{k-j-1}, b_i)$$

and has the same 3^{k-j-1}-type as a_j.

The axioms of F are all expressed as "local" properties of the models. In fact they describe the 3-types of elements which can appear in the models. First there is an axiom which states that each open sphere of radius 3 contains less than 16 elements. This insures that there are only a finite number of 3-types to consider in further axioms. Then for any 3-type which cannot appear in the kind of two-dimensional array which we have described, there is an axiom stating that there is no element of that 3-type. This insures, for example, that in the array, whenever the state element lies next to a square element, the state element and square element in the next row of the array have the proper unary predicates to satisfy some quintuple in the definition of M. Finally, there is an axiom which states that there is exactly one square element which has no S predecessor and satisfies the unary predicate corresponding to the symbol *.

Using this last axiom as a starting point, we make use of the fact

that each element has two neighbors in the horizontal direction and a successor in the vertical direction (elements not having this property were excluded by the earlier axioms) to show that each model of F has a two-dimensional array of the kind described earlier. This we call the standard part of the model. From the properties of machine M, we know that the initial row of this standard part corresponds to an initial tape which represents a complete extension of T. Furthermore, this initial row uniquely determines the entire two-dimensional array of the standard part of the model. The non-standard parts of a model may consist of any of the following: (i) a two-dimensional infinite array of filler elements, with or without a stripe of square elements running through it and with or without a series of state elements, (ii) a two-dimensional infinite array with an initial edge consisting of alternate square elements and filler elements (each square element has 0 or 1 written on it), (iii) a field of elements having an initial cyclical edge consisting of alternate square elements and filler elements with each successive row having the same number of square elements but more and more filler elements, (iv) fields of fillers which are cyclic in one direction but infinite in the other direction (those that are cyclic in the vertical direction may have a single stripe of square elements), and (v) fields of pure fillers which are cyclic in both directions.

Let Γ_0 be the set of sentences α_n where α_n states that the nth square element to the left of the * in the initial row of the model has a 1 written on it. Let Γ_1 be the set of all Boolean combinations of a set of sentences which describe the various types of cycles (as in (iii), (iv) and (v) above) that can occur in the non-standard parts of models. For example, corresponding to (iii) there are sentences $\beta(k, n, s)$ which state that there are at least k fields of elements having an initial edge with a cycle of n square elements (the unary predicates satisfied by these square elements are given by s which is an n-termed sequence of 0's and 1's). Using Lemma 2.3, we show that any two models of F are either elementarily equivalent or are distinguished by sentences of $\Gamma_0 \cup \Gamma_1$. For example, to show that a model having a non-standard part of the kind described in (ii) is elementarily equivalent to a model having no such part, we simply verify that each n-type of element occurring in the non-standard part also occurs infinitely many times in the standard part of any model. To make this true, we should have chosen the function g introduced above so that it forces every possible finite sequence of 0's and 1's to appear infinitely many times on the initial tape. This

can be done by choosing a sequence of theorems of T (for example, the theorems $(\forall x)x=x$, $(\forall xy)(x=x \wedge y=y)$, ...) and a sequence of sentences refutable in T and mapping them in an appropriate way.

Now it is a simple matter to verify the hypotheses of Lemma 1.3 (taking F, T and H for S, T_0, and T_1). The fact that every sentence of F is equivalent to a Boolean combination of sentences of Γ_0 and Γ_1 follows from the fact that any two distinguishable models of F can be distinguished by sentences of $\Gamma_0 \cup \Gamma_1$. Condition (i) is obvious. The function G_0 of (ii) is obtained directly from the function g and the function G_1 is obtained by the method used to prove Theorem 1.1 (there is a straightforward decision method for sentences of Γ_1). Finally, condition (iii) follows from the fact that we can construct models of F having any combination of standard part and non-standard parts.

3. Conclusions and problems.

Theorem 3.1. *Given any recursively enumerable set E, there exists a finitely axiomatizable theory F such that:*

(i) *The set of theorems of F has the same truth table degree as E.*

(ii) *No finite extension of F is essentially undecidable.*

Proof. Apply Theorem 2.1 to the example of Feferman [57].

Theorem 3.2. *There exists an essentially hereditarily undecidable axiomatizable theory which is not compatible with any finitely axiomatizable essentially undecidable theory.*

Proof. Let E_1 and E_2 be disjoint recursively enumerable sets which are recursively inseparable. Let T be the extension of theory H obtained by adding axioms $\lambda(1, n)$ for every $n \in E_1$. Let F be a finitely axiomatizable theory isomorphic to $T \times H$. Finally, for each $n \in E_2$, add as an additional axiom of F the sentence which corresponds in the isomorphism to the sentence $\neg\, \lambda(1, n)$ of T. The resulting infinitely axiomatized theory has all the desired properties.

Theorem 3.3. *There exists an essentially undecidable finitely axiomatizable theory of any prescribed recursively enumerable degree of undecidability.*

Proof. Apply Theorem 2.1 to the example given by Shoenfield [58].

Theorem 3.4. *There exists a finitely axiomatizable essentially undecidable theory whose set of theorems is recursively separable from the set of sentences which are negations of theorems.*

Proof. Apply Theorem 2.1 to the example given by Ehrenfeucht [61a].

Theorem 3.5. *Given a formula Con(T) expressing the consistency of an arbitrary consistent axiomatizable theory T (in the terminology of Feferman* [60], *Con(T) can be the sentence Con_α for any* RE*-numeration α of the axioms of T), there exists a finitely axiomatizable decidable theory F such that the sentence*

$$\mathrm{Con}(T) \leftrightarrow \mathrm{Con}(F)$$

is provable in arithmetic.

Proof. Let M be a Turing machine which starts with a blank tape and examines all possible proofs to check that no contradiction can be derived from the axioms of T. M halts if a contradiction is found. The theory F is then constructed from M as in Section 2, and shown to be decidable by the methods given there. The proof that $\mathrm{Con}(T)$ is equivalent to $\mathrm{Con}(F)$ is elementary in view of the direct connections between a contradiction in T, the halting of M, and the resulting inconsistency of F. Kreisel has stated that it should even be possible to carry out the proof in primitive recursive (quantifier-free) arithmetic.

Corollary 3.6. *There exists a decidable consistent theory which can not be shown in Peano's arithmetic P to be consistent.*

Proof. In Theorem 3.5, take T to be P. If we had $\vdash_P \mathrm{Con}(F)$, we would have $\vdash_P \mathrm{Con}(P)$ contrary to Gödel's second underivability theorem.

Problem. *Does there exist a finitely axiomatizable undecidable theory with countably many complete extensions?*

Conjecture I. *Every axiomatizable theory is isomorphic to a finitely axiomatizable theory.*

Conjecture II. *Every finitely axiomatizable theory with countably many complete extensions is isomorphic to a finitely axiomatizable theory formulated with a finite number of unary predicates.*

Theorem 3.7. *Conjecture I implies a positive answer and Conjecture II implies a negative answer to the problem given above.*

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EXTENSIONS OF RELATIONAL STRUCTURES *

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1. Introduction. When we speak of classes of structures of a specified kind, e.g., of equational classes of algebras, and when we prove theorems that apply to all such classes, we are in a sense engaged in axiomatic investigations on a second level. For just as we single out certain structures by means of the group axioms in group theory, we here single out certain classes of structures by means of a property that is characteristic of these classes. Model theory has abundantly demonstrated that such investigations are worthwhile and can lead to interesting results. It is only natural that one should there be primarily concerned with elementary classes, or at least with L-classes for some language L, and that the axioms should be in some sense of a model-theoretic character. However, certain properties of a different nature – we shall refer to them as structural properties – play a role in these investigations, and it seems reasonable that in some cases it might be profitable to start with such properties as axioms characterizing the classes to be investigated. Of course, only time can tell which properties deserve to be studied in this manner, but there are some that have already shown up in so many different contexts that they seem worthy of special attention. This discussion will center around two such properties, the embedding property and the amalgamation property.

We shall confine ourselves to first-order structures with operations and relations of finite rank, corresponding to a first-order language with terms and predicates. It is, of course, possible to treat operations as relations of a special kind, but there are certain advantages in not making such an identification, because in defining various auxiliary concepts one must treat operations differently from other relations. For instance, when we speak of a structure $\mathfrak{A}$ as being a substructure

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of a structure $\mathfrak{B}$, we require that the universe $|\mathfrak{A}|$ of $\mathfrak{A}$ be a subset of the universe $|\mathfrak{B}|$ of $\mathfrak{B}$, that $|\mathfrak{A}|$ be closed under all the operations of $\mathfrak{B}$, and that each operation and relation of $\mathfrak{A}$ be obtained by confining the corresponding operation and relation of $\mathfrak{B}$ to the set $|\mathfrak{A}|$.

It is frequently necessary to exclude from the discussion certain trivial classes of structures, and we shall assume throughout that all the classes under consideration have members of arbitrarily large order. It will also be assumed that all the classes $\mathscr{K}$ are closed under isomorphisms,

$$\mathfrak{A} \cong \mathfrak{B} \in \mathscr{K} \Rightarrow \mathfrak{A} \in \mathscr{K}.$$

The following closure properties will also be frequently used, but they will not be assumed unless this is explicitly stated:

Property (S). *Every substructure of a member of $\mathscr{K}$ belongs to $\mathscr{K}$.*

Property (P). *Every direct product of members of $\mathscr{K}$ belongs to $\mathscr{K}$.*

Property (H). *Every homomorphic image of a member of $\mathscr{K}$ belongs to $\mathscr{K}$.*

Property (U). *Every union of a directed subfamily of $\mathscr{K}$ belongs to $\mathscr{K}$.*

2. The embedding property and the amalgamation property. If $\mathfrak{A}$ and $\mathfrak{B}$ are similar structures, let us write $f : \mathfrak{A} \to \mathfrak{B}$ to mean that f maps $\mathfrak{A}$ isomorphically into $\mathfrak{B}$. The two properties referred to above can then be formulated as follows:

Property (E). *For any $\mathfrak{B}_0, \mathfrak{B}_1 \in \mathscr{K}$ there exist $\mathfrak{C} \in \mathscr{K}$ and functions g_0, g_1 such that $g_0 : \mathfrak{B}_0 \to \mathfrak{C}$ and $g_1 : \mathfrak{B}_1 \to \mathfrak{C}$.*

Property (A). *For any $\mathfrak{A}, \mathfrak{B}_0, \mathfrak{B}_1 \in \mathscr{K}$ and functions f_0, f_1 with $f_0 : \mathfrak{A} \to \mathfrak{B}_0$ and $f_1 : \mathfrak{A} \to \mathfrak{B}_1$, there exist $\mathfrak{C} \in \mathscr{K}$ and functions g_0, g_1 such that $g_0 : \mathfrak{B}_0 \to \mathfrak{C}$, $g_1 : \mathfrak{B}_1 \to \mathfrak{C}$ and $g_0 f_0 = g_1 f_1$.*

The two properties (E) and (A) are, of course, closely related to each other. In fact, if we agree to include the null structure $\mathfrak{N}$ among our structures, and if this structure belongs to the class $\mathscr{K}$, then (E) is obviously just the special case of (A) obtained by taking $\mathfrak{A} = \mathfrak{N}$. On the other hand, $\mathscr{K}$ has Property (A) if and only if, for each $\mathfrak{A} \in \mathscr{K}$, the class

$$\mathscr{K}^{\mathfrak{A}} = \{(\mathfrak{B}, f(a))_{a \in |\mathfrak{A}|} \mid f : \mathfrak{A} \to \mathfrak{B} \in \mathscr{K}\}$$

has the embedding property. As regards their applications, however,

there is a very significant practical difference between the two properties, for in the case of most of the familiar classes investigated by algebraists it turns out to be easy to check whether or not the embedding property holds, while this is by no means true of the amalgamation property. It is known that the amalgamation property holds for each of the following classes: groups, groups with a given operator domain, abelian groups, fields, differential fields of characteristic 0, partially ordered sets, lattices, Boolean algebras, locally finite-dimensional cylindric algebras of a given infinite dimension. On the other hand, the property is known to fail for the class of all semi-groups and for the class of all rings, and it is still an open problem whether or not the amalgamation property holds for the class of all modular lattices or for the class of all cylindric algebras of a given dimension. In connection with this last example we should like to raise the following more general question: Does there exist an equational class of Boolean algebras with operators that does not have the amalgamation property?

Looking at the above examples one does not see much hope of finding general results that assert that if an elementary class is characterized by axioms of such and such a form, then this class has the amalgamation property. Also, if one looks at the proofs of the amalgamation property in the cases where it does turn out to hold, one sees that the procedure is rarely the same in two cases, and the known arguments, therefore, give no key to a general method. At the present the sole light in this darkness is the following important result: If $\mathcal{K}$ is an elementary type, then the classes $\mathcal{K}^*$ and $\mathcal{S}(\mathcal{K}^*)$ have the amalgamation property. Here $\mathcal{K}^*$ is defined as follows: Let Φ be the set of all formulas in the language of $\mathcal{K}$, for $\varphi \in \Phi$ and $\mathfrak{A} \in \mathcal{K}$ let $\varphi^{\mathfrak{A}}$ be the relation in $\mathfrak{A}$ that represents φ, and for $\mathfrak{A} \in \mathcal{K}$ let $\mathfrak{A}^* = (|\mathfrak{A}|, \varphi^{\mathfrak{A}})_{\varphi \in \Phi}$. Then $\mathcal{K}^* = \{\mathfrak{A}^* \mid \mathfrak{A} \in \mathcal{K}\}$. $\mathcal{S}(\mathcal{K}^*)$ is, of course, the class of all substructures of members of $\mathcal{K}^*$.

References. The amalgamation property was first formulated explicitly by Fraïssé (cf. Fraïssé [54a]) in connection with certain embedding problems, and it has played a role in connection with other such problems in Jónsson [62], [60], [61] and [56], and Morley and Vaught [62]. The results, positive and negative, on the amalgamation property for particular classes of structures are scattered throughout the literature, and in some cases are not even explicitly stated. For groups with or without operators and for Abelian groups free amalgamated products are known to exist, for fields the property follows from standard

theorems about field extensions, and for differential fields of characteristic 0 it follows from Theorems 2.2 and 2.5 in Kaplansky [57]. For partially ordered sets and lattices the result is explicitly formulated in Jónsson [56], and for locally finite-dimensional cylindric algebras of a given infinite dimension in Daigneault [2], Theorem 6.3. This last result implies the corresponding result for Boolean algebras, although a much simpler direct proof is also possible. The negative result for semi-groups is given in Kimura [57], and another counterexample can be found in Howie [62]. These examples can be used to construct corresponding counterexamples for rings. Finally, the theorem about $\mathscr{N}^*$ and $\mathscr{S}(\mathscr{N}^*)$ is given in Morley and Vaught [62].

3. Elementary consequences of the amalgamation property. If $\mathscr{K}$ has Property (U), then it is easy to show that the two structures $\mathfrak{B}_0$ and $\mathfrak{B}_1$ in Property (A) can be replaced by arbitrarily many structures $\mathfrak{B}_i$:

Theorem 3.1. *Suppose $\mathscr{K}$ has Properties* (U) *and* (A). *If $\mathfrak{A} \in \mathscr{K}$ and $f_i : \mathfrak{A} \to \mathfrak{B}_i \in \mathscr{K}$ $(i \in I)$, then there exist $\mathfrak{C} \in \mathscr{K}$ and isomorphisms $g_i : \mathfrak{B}_i \to \mathfrak{C}$ $(i \in I)$ such that $g_i f_i = g_j f_j$ for all $i, j \in I$.*

It is sometimes more convenient to formulate this property in terms of extensions of structures.

Definition 3.2. *Two extensions $\mathfrak{B}$ and $\mathfrak{C}$ of a structure $\mathfrak{A}$ are said to be* $\mathfrak{A}$-equivalent *if and only if there exists an isomorphism of $\mathfrak{B}$ onto $\mathfrak{C}$ that maps each element of $|\mathfrak{A}|$ onto itself.*

Corollary 3.3. *Suppose $\mathscr{K}$ has Properties* (U) *and* (A). *If all the structures $\mathfrak{B}_i \in \mathscr{K}$ $(i \in I)$ are extensions of the structure $\mathfrak{A} \in \mathscr{K}$, then there exists an extension $\mathfrak{C} \in \mathscr{K}$ of $\mathfrak{A}$ such that each $\mathfrak{B}_i$ $(i \in I)$ is $\mathfrak{A}$-equivalent to a structure $\mathfrak{B}_i'$ with $\mathfrak{A} \subseteq \mathfrak{B}_i' \subseteq \mathfrak{C}$.*

Slightly less trivial are the following two theorems concerning extensions of "partial" isomorphisms:

Theorem 3.4. *Suppose $\mathscr{K}$ has Properties* (U) *and* (A). *If $\mathfrak{A}_i, \mathfrak{B}_i$, $\mathfrak{C} \in \mathscr{K}$, $\mathfrak{A}_i \subseteq \mathfrak{B}_i$ and $f_i : \mathfrak{A}_i \to \mathfrak{C}$ $(i \in I)$, then there exist an extension $\mathfrak{D} \in \mathscr{K}$ of $\mathfrak{C}$ and isomorphisms $g_i : \mathfrak{B}_i \to \mathfrak{D}$ with $f_i \subseteq g_i$ $(i \in I)$.*

Theorem 3.5. *Suppose $\mathscr{K}$ has Properties* (U) *and* (A). *If $\mathfrak{A}_i$, $\mathfrak{B} \in \mathscr{K}$, $\mathfrak{A}_i \subseteq \mathfrak{B}$ and $f_i : \mathfrak{A}_i \to \mathfrak{B}$ $(i \in I)$, then there exist an extension $\mathfrak{C} \in \mathscr{K}$ of $\mathfrak{B}$ and automorphisms g_i of $\mathfrak{C}$ with $f_i \subseteq g_i$ $(i \in I)$.*

References. The simple observations contained in this section are presumably known to everyone who has worked with the amalgamation property. They are explicitly formulated in Jónsson [62]. For the special case of groups, 3.5 is proved in Higman, Neumann and Neumann [49]. Actually they show that all the isomorphisms between subgroups of $\mathfrak{B}$ are induced by inner automorphisms of an extension of $\mathfrak{B}$, but this can easily be obtained by observing that under the usual representation of a group $\mathfrak{G}$ by permutations of $|\mathfrak{G}|$ all the automorphisms of the image group are induced by inner automorphisms of the full permutation group of $|\mathfrak{G}|$.

4. Free products of algebras. Assuming now that $\mathscr{K}$ is a class of similar algebras, one defines the notions of $\mathscr{K}$-free algebras, $\mathscr{K}$-free extensions, $\mathscr{K}$-free products and $\mathscr{K}$-free amalgamated products in a familiar manner in terms of homomorphisms. Since we will here be primarily concerned with $\mathscr{K}$-free products and $\mathscr{K}$-free extensions, we recall the definitions of these two concepts.

Definition 4.1. *We say that* $\mathfrak{B}$ *is a* $\mathscr{K}$-free extension of $\mathfrak{A}$ by X *if and only if* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$, $X \subseteq |\mathfrak{B}|$, $\mathfrak{B}$ *is generated by* $|\mathfrak{A}| \cup X$, *and for every* $\mathfrak{C} \in \mathscr{K}$, *every homomorphism* f *of* $\mathfrak{A}$ *into* $\mathfrak{C}$, *and every map* g *of* X *into* $|\mathfrak{C}|$, $f \cup g$ *extends to a homomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$.

Definition 4.2. *We say that* $\mathfrak{B}$ *is a* $\mathscr{K}$-free product of $\mathfrak{A}_i$ $(i \in I)$ *if and only if* $\mathfrak{A}_i, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A}_i \subseteq \mathfrak{B}$ $(i \in I)$, $\mathfrak{B}$ *is generated by* $\bigcup_{i \in I} |\mathfrak{A}_i|$ *and, for all* $\mathfrak{C} \in \mathscr{K}$, *every system of homomorphisms of* $\mathfrak{A}_i$ *into* $\mathfrak{C}$ $(i \in I)$ *have a common extension that maps* $\mathfrak{B}$ *homomorphically into* $\mathfrak{C}$.

If $\mathfrak{B}$ is a $\mathscr{K}$-free product of its subalgebras $\mathfrak{A}_i$ $(i \in I)$, we write

$$\mathfrak{B} = \prod_{i \in I}^{*} \mathfrak{A}_i,$$

and if $\mathfrak{B}$ is a $\mathscr{K}$-free product of two subalgebras $\mathfrak{A}$ and $\mathfrak{A}'$, we write $\mathfrak{B} = \mathfrak{A} \star \mathfrak{A}'$. If $X \subseteq |\mathfrak{B}|$, let $[X]$ be the subalgebra of $\mathfrak{B}$ that is generated by X, and if $\mathfrak{A} \subseteq \mathfrak{B}$ and $X \subseteq |\mathfrak{B}|$, let $\mathfrak{A}(X) = [X \cup |\mathfrak{A}|]$.

The basic existence theorem for free extensions and free products can be formulated as follows:

Theorem 4.1. *Suppose* $\mathscr{K}$ *has Properties* (S), (P) *and* (E).

(i) *For any* $\mathfrak{A} \in \mathscr{K}$ *and for any cardinal* m, *there exists an algebra* $\mathfrak{B}$ *that is a* $\mathscr{K}$*-free extension of* $\mathfrak{A}$ *by a set of order* m.

(ii) *For any* $\mathfrak{A}_i \in \mathscr{K}$ $(i \in I)$, *there exist* $\mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A}_i' \subseteq \mathfrak{B}$ $(i \in I)$ *such that*

$$\mathfrak{B} = \prod_{i \in I}^{*} \mathfrak{A}_i' \text{ and } \mathfrak{A}_i \cong \mathfrak{A}_i' \ (i \in I).$$

There does not seem to be much hope of obtaining a description of all subalgebras of a $\mathscr{K}$-free product of given algebras $\mathfrak{A}_i$, that would be comparable to Kurosh's theorem about subgroups of a free product of groups. However, our next result may be regarded as a generalization of a trivial part of that theorem.

Theorem 4.2. *Suppose* $\mathscr{K}$ *has Properties* (S), (P), (E) *and* (A). *If*

$$\mathfrak{B} = \prod_{i \in I}^{*} \mathfrak{A}_i, \ \mathfrak{A}_i' \subseteq \mathfrak{A}_i \ (i \in I), \ \mathfrak{B}' = [\bigcup_{i \in I} |\mathfrak{A}_i'|],$$

then

$$\mathfrak{B}' = \prod_{i \in I}^{*} \mathfrak{A}_i'.$$

This theorem has several interesting consequences.

Corollary 4.3. *Suppose* $\mathscr{K}$ *has Properties* (S), (P), (E) *and* (A). *If* $\mathfrak{B}$ *is a* $\mathscr{K}$*-free product of subalgebras of* $\mathscr{K}$*-free algebras, then* $\mathfrak{B}$ *is a subalgebra of a* $\mathscr{K}$*-free algebra.*

Corollary 4.4. *Suppose* $\mathscr{K}$ *has Properties* (S), (P), (E) *and* (A). *If* $\mathfrak{C}$ *is a* $\mathscr{K}$*-free extension of* $\mathfrak{B}$ *by* X, *and if* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\mathfrak{A}(X)$ *is a* $\mathscr{K}$*-free extension of* $\mathfrak{A}$ *by* X.

Corollary 4.5. *Suppose* $\mathscr{K}$ *has Properties* (S), (P), (E) *and* (A). *If* $\mathfrak{A}$ *is* $\mathscr{K}$*-freely generated by an* n*-element set and has a subalgebra that is* $\mathscr{K}$*-freely generated by an* $(n+1)$*-element set, then* $\mathfrak{A}$ *has a subalgebra that is* $\mathscr{K}$*-freely generated by an infinite set.*

Corollary 4.6. (The interpolation theorem). *Suppose* $\mathscr{K}$ *has Properties* (S), (P), (H), (E) *and* (A). *If* $\mathsf{r}_0, \ldots, \mathsf{r}_{m-1}, \mathsf{s}_0, \ldots, \mathsf{s}_{m-1}, \mathsf{t}, \mathsf{u}$ *are terms such that*

$$\models_{\mathscr{K}} (\bigwedge_{i<m} \mathsf{r}_i(x, y) = \mathsf{s}_i(x, y)) \to \mathsf{t}(y, z) = \mathsf{u}(y, z),$$

then there exist terms $\mathsf{r}_0', \mathsf{r}_1', \ldots, \mathsf{r}'_{n-1}, \mathsf{s}_0', \mathsf{s}_1', \ldots, \mathsf{s}'_{n-1}$ *such that*

$$\models_{\mathscr{K}} (\bigwedge_{i<m} \mathsf{r}_i(x, y) = \mathsf{s}_i(x, y)) \to \mathsf{r}_j'(y) = \mathsf{s}_j'(y) \ (j<n), \text{ and}$$

$$\models_{\mathscr{K}} (\bigwedge_{j<n} \mathsf{r}_i'(y) = \mathsf{s}_i'(y)) \to \mathsf{t}(y, z) = \mathsf{u}(y, z).$$

To derive 4.3–4.5 from 4.2 we make use of the connection between $\mathscr{K}$-free algebras and $\mathscr{K}$-free extensions on the one hand and $\mathscr{K}$-free products on the other:

Lemma. *Suppose $\mathscr{K}$ has Property* (S). *If X is the union of pairwise disjoint sets Y_i $(i \in I)$, then the following conditions are equivalent*:

(i) $\mathfrak{B}$ *is $\mathscr{K}$-freely generated by X.*

(ii) $\mathfrak{B} = \prod_{i \in I}^{*} [Y_i]$ *and* $[Y_i]$ *is $\mathscr{K}$-freely generated by Y_i $(i \in I)$.*

Lemma. *Suppose $\mathscr{K}$ has Property* (S). *Then the following conditions are equivalent*:

(i) $\mathfrak{B}$ *is a $\mathscr{K}$-free extension of $\mathfrak{A}$ by X.*

(ii) $\mathfrak{B} = \mathfrak{A}^{*}[X]$ *and* $[X]$ *is $\mathscr{K}$-freely generated by X.*

To prove Corollary 4.6 we consider the $\mathscr{K}$-algebra $\mathfrak{C} = [x, y, z]$ with defining relations $r_i(x, y) = s_i(x, y)$ $(i < m)$, and the subalgebras $\mathfrak{B} = [x, y]$ and $\mathfrak{A} = [y]$. Then $\mathfrak{C}$ is a $\mathscr{K}$-free extension of $\mathfrak{B}$ by $\{z\}$ and hence, by 4.4, $[y, z]$ is a $\mathscr{K}$-free extension of $\mathfrak{A}$ by $\{z\}$. From this the conclusion readily follows.

References. Theorem 4.2 and Corollaries 4.3 and 4.4 are given in Jónsson [61], while Corollary 4.5 is new. Corollary 4.6 is one of several analogues of the well-known interpolation theorem due to Craig (cf. Craig [57]). The earliest proof, due to H. J. Keisler, has not appeared in print, but it was apparently based on essentially the same ideas as the one outlined here.

5. Homogeneous-universal structures. No discussion of the role of the embedding property and the amalgamation property would be complete if it failed to include an account of the basic results on homogeneous-universal structures. However, since these results are relatively well known among logicians, thanks to the work of Morley and Vaught, we shall confine ourselves to a statement of the basic definitions and the fundamental existence and uniqueness theorem.

Definition 5.1. $\mathfrak{A}$ *is* $\mathscr{K}$-homogeneous *if and only if $\mathfrak{A} \in \mathscr{K}$ and for any $\mathfrak{B} \in \mathscr{K}$ with $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} < \mathfrak{A}$, every isomorphism of $\mathfrak{B}$ into $\mathfrak{A}$ extends to an automorphism of $\mathfrak{A}$.*

Definition 5.2. $\mathfrak{A}$ *is* $\mathscr{K}$-universal *if and only if* $\mathfrak{A} \in \mathscr{K}$ *and for every* $\mathfrak{B} \in \mathscr{K}$ *with* $\overline{\overline{\mathfrak{B}}} \leqslant \overline{\overline{\mathfrak{A}}}$, *there exists an isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{A}$.

Theorem 5.3. *Suppose* $\mathscr{K}$ *has the Properties* (U), (E) *and* (A), *and suppose* m *is a regular cardinal such that*:

(i) *For every cardinal* $n < m$, $2^n \leqslant m$.

(ii) *The number of non-isomorphic structures* $\mathfrak{A} \in \mathscr{K}$ *with* $\overline{\overline{\mathfrak{A}}} < m$ *does not exceed* m.

(iii) *For any* $\mathfrak{A} \in \mathscr{K}$ *and* $X \subseteq |\mathfrak{A}|$ *with* $\overline{\overline{X}} < m$, *there exists* $\mathfrak{B} \in \mathscr{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{A}$, $X \subseteq |\mathfrak{B}|$ *and* $\overline{\overline{\mathfrak{B}}} < m$.

Under these conditions there exists up to isomorphism exactly one $\mathscr{K}$-*homogeneous,* $\mathscr{K}$-*universal structure of order* m.

If the Generalized Continuum Hypothesis holds, then (i) is, of course, satisfied by every infinite cardinal m. Assuming that (i) is satisfied, (ii) obviously holds for every class $\mathscr{K}$ whose members have fewer than m relations and operations. Finally, assuming that $\mathscr{K}$ is an elementary class whose members have fewer than m relations and operations, the condition (iii) holds by the Löwenheim–Skolem theorem. The most serious restriction in the above theorem is, therefore, the requirement that m be regular, but easy examples show that this condition cannot be omitted. To remedy this situation, the notion of a $\mathscr{K}$-special structure was introduced. This concept is somewhat more involved, and intuitively less attractive, but it permits us to remove from Theorem 5.3 the assumption that m be regular. More precisely, if $\mathscr{K}$ has the properties (U), (E) and (A), and if m is any infinite cardinal that satisfies (i)–(iii), then there exists up to isomorphism exactly one $\mathscr{K}$-special structure of order m. Furthermore, if m is regular, then this structure is $\mathscr{K}$-homogeneous and $\mathscr{K}$-universal.

References. A somewhat weaker form of Theorem 5.3 can be found in Jónsson [60], and the present formulation is given in Morley and Vaught [62]. The form of the Löwenheim–Skolem theorem alluded to in the discussion following 5.3 is given in Tarski and Vaught [57]. The notion of a special $\mathscr{K}$-structure is introduced and investigated in Morley and Vaught [62].

6. Algebraic extensions of relational structures. There exist in the literature several generalizations of the classical theory of field extensions

or of certain portions of that theory. A critical point in any attempt at such a generalization is, of course, the choice of definitions of the basic concepts, particularly of the notion of an algebraic extension. One way of motivating this choice would be to decide on certain fundamental properties that this concept must have, and to see how this limits the choice of a definition.

Consider a fixed class $\mathscr{K}$ of relational structures and two binary relations **A** and **N** between members of $\mathscr{K}$. If we think of $\mathfrak{A}\mathbf{A}\mathfrak{B}$ and $\mathfrak{A}\mathbf{N}\mathfrak{B}$ as meaning, respectively, that $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$ and that $\mathfrak{B}$ is a normal extension of $\mathfrak{A}$, then we may reasonably require these two relations to have the following four properties:

I. *If* $\mathfrak{A}\mathbf{A}\mathfrak{B}$, *then* $\mathfrak{A} \subseteq \mathfrak{B}$.

II. *If* $\mathfrak{A}\mathbf{N}\mathfrak{B}$, *then* $\mathfrak{A}\mathbf{A}\mathfrak{B}$.

III. *If* $\mathfrak{A}\mathbf{A}\mathfrak{B}$, *then there exists* $\mathfrak{C} \in \mathscr{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{C}$ *and* $\mathfrak{A}\mathbf{N}\mathfrak{C}$.

IV. *If* $\mathfrak{A}\mathbf{N}\mathfrak{B} \subseteq \mathfrak{C} \in \mathscr{K}$, *then every* $\mathfrak{A}$*-isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$ *maps* $\mathfrak{B}$ *into itself.*

Under some reasonable conditions on the class $\mathscr{K}$ these conditions very much limit the possible choice of definition for the relation **A**:

Theorem 6.1. *Suppose* $\mathscr{K}$ *has the Properties* (U) *and* (A), *and assume that every ultraproduct of members of* $\mathscr{K}$ *is a member of* $\mathscr{K}$. *Assume further that* **A** *and* **N** *are binary relations in* $\mathscr{K}$ *that satisfy* I–IV. *Under these conditions, if* $\mathfrak{A}\mathbf{A}\mathfrak{B}$, *then for every* $b \in |\mathfrak{B}|$, *and for every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{A}$, *the set consisting of the images of* b *under all the* $\mathfrak{A}$*-isomorphisms of* $\mathfrak{B}$ *into* $\mathfrak{C}$ *is finite.*

Roughly speaking, therefore, an element of an algebraic extension of $\mathfrak{A}$ should never have more than finitely many "conjugate" elements in any given extension of $\mathfrak{A}$. We take this condition as the defining property for algebraic extensions. In the list of theorems that follows it is assumed that $\mathscr{K}$ has Properties (S), (U), and (A).

Definition 6.2. *Suppose* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathscr{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$ *and* $\mathfrak{A} \subseteq \mathfrak{C}$. *An element* $b \in |\mathfrak{B}|$ *is said to be* $\mathfrak{A}$-equivalent to *an element* $c \in |\mathfrak{C}|$ *if and only if there exists an* $\mathfrak{A}$*-isomorphism of* $\mathfrak{A}(b)$ *onto* $\mathfrak{A}(c)$ *that maps* b *onto* c.

Definition 6.3. *Suppose* $\mathfrak{A}$, $\mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A} \subseteq \mathfrak{B}$.

(i) *An element* $b \in |\mathfrak{B}|$ *is said to be* algebraic over $\mathfrak{A}$ *if and only if for*

every extension $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{A}$ *there are only finitely many elements of* $|\mathfrak{C}|$ *that are* $\mathfrak{A}$*-equivalent to* b.

(ii) *A subset* U *of* $|\mathfrak{B}|$ *is said to be* algebraic over $\mathfrak{A}$ *if and only if every member of* U *is algebraic over* $\mathfrak{A}$.

(iii) $\mathfrak{B}$ *is said to be an* algebraic extension of $\mathfrak{A}$ *if and only if* $|\mathfrak{B}|$ *is algebraic over* $\mathfrak{A}$.

Theorem 6.4. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A} \subseteq \mathfrak{B}$. *If* $U \subseteq |\mathfrak{B}|$ *and* U *is algebraic over* $\mathfrak{A}$, *then* $\mathfrak{A}(U)$ *is an algebraic extension of* $\mathfrak{A}$.

Theorem 6.5. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{B}$ *is a finite algebraic extension of* $\mathfrak{A}$ (*i.e.*, $\mathfrak{B} = \mathfrak{A}(U)$ *where* U *is finite and algebraic over* $\mathfrak{A}$). *Then there exists a positive integer* n *such that for every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{A}$ *there are at most* n *distinct* $\mathfrak{A}$*-isomorphisms of* $\mathfrak{B}$ *into* $\mathfrak{C}$.

Definition 6.6. *Under the hypothesis of Theorem* 6.5, *the smallest positive integer* n *that satisfies the conclusion of that theorem is called the* reduced degree of $\mathfrak{B}$ over $\mathfrak{A}$ — *in symbols,* $n = (\mathfrak{B}, \mathfrak{A})_0$. *If* $\mathfrak{B} = \mathfrak{A}(u)$, *then* n *is also called the* reduced degree of u over $\mathfrak{A}$.

Theorem 6.7. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$, $U \subseteq |\mathfrak{B}|$ *and* $b \in |\mathfrak{B}|$. *If* U *is algebraic over* $\mathfrak{A}$ *and* b *is algebraic over* $\mathfrak{A}(U)$, *and if* U *is finite, then* b *is algebraic over* $\mathfrak{A}$.

Theorem 6.8. *Suppose* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathscr{K}$. *If* $\mathfrak{B}$ *is a finite algebraic extension of* $\mathfrak{A}$ *and* $\mathfrak{C}$ *is a finite algebraic extension of* $\mathfrak{B}$, *then* $\mathfrak{C}$ *is a finite algebraic extension of* $\mathfrak{A}$ *and*

$$(\mathfrak{C}, \mathfrak{A})_0 = (\mathfrak{C}, \mathfrak{B})_0 \cdot (\mathfrak{B}, \mathfrak{A})_0.$$

Definition 6.9. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A} \subseteq \mathfrak{B}$.

(i) *An element* $b \in |\mathfrak{B}|$ *is said to* split in $\mathfrak{B}$ over $\mathfrak{A}$ *if and only if* b *is algebraic over* $\mathfrak{A}$, *and the number of elements of* $|\mathfrak{B}|$ *that are* $\mathfrak{A}$*-equivalent to* b *is equal to the reduced degree of* b *over* $\mathfrak{A}$.

(ii) $\mathfrak{B}$ *is called a* splitting extension of the element $b \in |\mathfrak{B}|$ over $\mathfrak{A}$ *if and only if* b *splits in* $\mathfrak{B}$ *over* $\mathfrak{A}$, *and there exists no extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{C} \subseteq \mathfrak{B}$, $\mathfrak{C} \neq \mathfrak{B}$, *and* b *splits in* $\mathfrak{C}$ *over* $\mathfrak{A}$.

(iii) *A set* $U \subseteq |\mathfrak{B}|$ *is said to* split in $\mathfrak{B}$ over $\mathfrak{A}$ *if and only if every member of* U *splits in* $\mathfrak{B}$ *over* $\mathfrak{A}$.

(iv) $\mathfrak{B}$ *is called a* splitting extension of U over $\mathfrak{A}$ *if and only if* U *splits*

in $\mathfrak{B}$ *over* $\mathfrak{A}$, *and there exists no extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{C} \subseteq \mathfrak{B}$, $\mathfrak{C} \neq \mathfrak{B}$, *and* U *splits in* $\mathfrak{C}$ *over* $\mathfrak{A}$.

Definition 6.10. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{A} \subseteq \mathfrak{B}$. *We say that* $\mathfrak{B}$ *is a* normal extension *of* $\mathfrak{A}$ *if and only if* $|\mathfrak{B}|$ *splits in* $\mathfrak{B}$ *over* $\mathfrak{A}$.

Theorem 6.11. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$, *and* $\mathfrak{B} = \mathfrak{A}(U)$ *where* U *is algebraic over* $\mathfrak{A}$. *Then there exists an extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{B}$ *such that* $\mathfrak{C}$ *is a splitting extension of* U *over* $\mathfrak{A}$. $\mathfrak{C}$ *is unique up to* $\mathfrak{B}$*-isomorphism and is a normal extension of* $\mathfrak{A}$.

Theorem 6.12. *If* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ *and* $\mathfrak{B}$ *is an algebraic extension of* $\mathfrak{A}$, *then the following conditions are equivalent*:

(i) $\mathfrak{B}$ *is a normal extension of* $\mathfrak{A}$.

(ii) *For every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{B}$, *every* $\mathfrak{A}$*-isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$ *maps* $\mathfrak{B}$ *onto itself.*

(iii) *For every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{B}$, *every* $\mathfrak{A}$*-isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$ *maps* $\mathfrak{B}$ *into itself.*

(iv) *For every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{B}$, *every* $\mathfrak{A}$*-automorphism of* $\mathfrak{C}$ *maps* $\mathfrak{B}$ *onto itself.*

(v) *For every extension* $\mathfrak{C} \in \mathscr{K}$ *of* $\mathfrak{B}$, *every* $\mathfrak{A}$*-automorphism of* $\mathfrak{C}$ *maps* $\mathfrak{B}$ *into itself.*

Corollary 6.13. *Suppose* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathscr{K}$, $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$, *and* $\mathfrak{C}$ *is a normal extension of* $\mathfrak{A}$. *Then*:

(i) $\mathfrak{C}$ *is a normal extension of* $\mathfrak{B}$.

(ii) *Every* $\mathfrak{A}$*-isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$ *extends to an* $\mathfrak{A}$*-automorphism of* $\mathfrak{C}$.

It does not follow from Properties (S), (U) and (A) alone that an algebraic extension of an algebraic extension of a member $\mathfrak{A}$ of $\mathscr{K}$ is an algebraic extension of $\mathfrak{A}$, and consequently we cannot prove the existence of algebraically closed algebraic extensions of $\mathfrak{A}$, although we are able to show that $\mathfrak{A}$ has a maximal algebraic extension $\mathfrak{B}$, and that $\mathfrak{B}$ is unique up to isomorphism. We therefore impose an additional condition on $\mathscr{K}$ that will in any case be satisfied if $\mathscr{K}$ is an elementary class.

Theorem 6.14. *Suppose every ultraproduct of members of* $\mathscr{K}$ *is a member of* $\mathscr{K}$. *Then*:

(i) *If* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$, $U \subseteq |\mathfrak{B}|$ *and* $b \in |\mathfrak{B}|$, *and if* b *is algebraic over*

$\mathfrak{A}(U)$, *then there exists a finite subset* V *of* U *such that* b *is algebraic over* $\mathfrak{A}(V)$.

(ii) *For all* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in \mathscr{K}$, *if* $\mathfrak{B}$ *is an algebraic extension of* $\mathfrak{A}$ *and* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{B}$, *then* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{A}$.

(iii) *For every* $\mathfrak{A} \in \mathscr{K}$ *there exists* $\mathfrak{B} \in \mathscr{K}$ *such that* $\mathfrak{B}$ *is an algebraically closed algebraic extension of* $\mathfrak{A}$. $\mathfrak{B}$ *is unique up to* $\mathfrak{A}$*-isomorphism.*

Although these results apply to many of the familiar classes of relational structures, they should not be expected to give in general as clear insight into the variety of possible extensions of a structure $\mathfrak{A} \in \mathscr{K}$ as they do in the case when $\mathscr{K}$ is the class of all fields. This is partly due to the fact that in the classical case any two elements that are not algebraic over $\mathfrak{A}$ are $\mathfrak{A}$-equivalent, but for other classes $\mathscr{K}$ this is usually not the case. In fact, it turns out that if $\mathscr{K}$ is, for instance, the class of all groups, the class of all Abelian groups, the class of all lattices, or the class of all Boolean algebras, then every member of $\mathscr{K}$ is algebraically closed. This suggests that while our terminology was motivated by a desire to carry over certain known theorems, a further classification of simple extensions would be desirable. In fact, to some extent such a classification has been started by Morley. However, his investigations are carried only as far as they are needed for the particular problem under consideration, and both the basic concepts and the methods are model-theoretic. It would be interesting to see if there are other more structural properties that can be used to classify the non-algebraic extensions, or at least to single out what might be considered to be "almost algebraic" extensions.

References. Theorem 6.1 is a more precise formulation of some informal remarks in Jónsson [62], and all the remaining results in this section are derived in that paper. In A. Robinson [51a] the notion of an algebraic predicate is introduced, and a number of results are derived that generalize known theorems about algebraic fields. There is a considerable overlap between these theorems and the ones presented here, but because of the difference in the approach it is difficult to make a precise comparison. Morley's classification of transcendental elements is contained in his dissertation (see Morley [a]).

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FINITE APPROXIMATIONS OF INFINITELY LONG FORMULAS

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This paper deals with a language L which has infinitely long formulas built up from atomic formulas by applying negations, finite and infinite conjunctions and disjunctions, and arbitrary sequences of quantifiers; the quantifiers may alternate between $\exists$ and $\forall$ infinitely many times, for example $(\exists v_0 \forall v_1 \exists v_2 \forall v_3 \ldots)$. Languages like L have been considered by Henkin in [61]. Other types of infinitary logic are discussed elsewhere in this volume, and references are given.

Throughout the paper we use the notion of a finite approximation θ of a formula φ of L; the definition is such that θ will "look like" φ in a certain syntactical sense. In our results in Section 2 and Section 3 we show some relationships between the models of a sentence φ of L and the models of its approximations. A tool which is especially useful here is the notion of a saturated (i.e., homogeneous universal) structure (see Morley–Vaught [62]). Speaking very roughly, the two main theorems of this paper, Theorems 2.3 and 3.2, are as follows: if every finite approximation of a sentence φ of L holds in a saturated structure $\mathfrak{A}$, then φ holds in $\mathfrak{A}$; if $\mathfrak{A}$, $\mathfrak{B}$ are saturated structures, φ holds in $\mathfrak{A}$, and the dual sentence φ^* (see Section 1) holds in $\mathfrak{B}$, then there is a finite approximation θ of φ which holds in $\mathfrak{A}$ but not in $\mathfrak{B}$. In these theorems, however, φ cannot be arbitrary, and we have to assume that φ is an admissible sentence (in the sense of Section 1 below).

The main theorems have a number of consequences which have nothing to do with saturated structures. For instance, in Corollary 3.4 we show that if a sentence ψ of L is (logically) equivalent to a finite sentence σ, and the dual sentence ψ^* is equivalent to the negation of σ, then ψ is already equivalent to one of its finite approximations.

The results of this paper were first motivated by a problem for finite sentences. In the theory of models for the ordinary finitary first order logic there is a large class of theorems, which one might call "preservation theorems" [1], of which the following two results are typical (see Lyndon

[1] The author first heard the name "preserva tion theorems" from R. L. Vaught.

[59c] for an expository account). A sentence φ is preserved under substructures iff φ is equivalent to some universal sentence (Łoś [55a] and Tarski [54]); φ is preserved under homomorphic images iff φ is equivalent to some positive sentence (Lyndon [59d]). The preservation theorems were first proved by a variety of methods. However, it has recently become apparent that most of them can be proved using saturated structures (e.g., see Morley–Vaught [62]). The problem which led to this paper is to find, using saturated structures, a single theorem which captures the common part of the proofs of a large number of preservation theorems; our candidate is Theorem 3.2. Due to lack of space, it is not possible to illustrate in this paper the connection between Theorem 3.2 and preservation theorems. Instead, a sequel to this paper will be published elsewhere with the title "Some applications of infinitely long formulas"; we shall there apply Theorem 3.2 in the proofs of some new preservation theorems which were announced in Keisler [63c], [63d].

1. Preliminaries. We shall use the usual set-theoretic notation. If X is a class, we denote by $S(X)$ the class of all *non-empty* subsets of X. Arbitrary ordinals are denoted by $\alpha, \beta, \gamma, \delta, \xi$, and cardinals $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}$ are identified with initial ordinals. The letters m, n, p, q, r always range over the natural numbers. We assume throughout that $\mathfrak{m}$ is a fixed infinite cardinal, $\mathfrak{m}^+$ is its successor cardinal, π is a fixed ordinal with $\pi < \mathfrak{m}^+$, and $\tau \in \omega^\pi$.

The language L, which depends on $\mathfrak{m}$ and τ, has the following symbols:

individual variables v_α, $\alpha < \mathfrak{m}^+$;
the identity symbol, $\doteq$;
predicate symbols P_β, $\beta < \pi$;
finite connectives $\neg$, $\wedge$, $\vee$;
infinite connectives $\bigwedge$, $\bigvee$;
quantifier symbols $\exists$, $\forall$.

The *atomic formulas* are the expressions $x_1 \doteq x_2$ and $P_\beta x_1 \dots x_{\tau(\beta)}$, where $\beta < \pi$ and $x_1, x_2, \dots, x_{\tau(\beta)}$ are variables. By a *quantifier of length* α, we mean a pair (Qx), where $Q \in \{\exists, \forall\}^\alpha$ and x is an α-termed sequence of distinct individual variables (thus $x = \langle x_\beta \rangle_{\beta<\alpha} = \langle v_{f(\beta)} \rangle_{\beta<\alpha}$ for some one-to-one function f on α into $\mathfrak{m}^+$). We also write $(Q_0 x_0 Q_1 x_1 \dots Q_\beta x_\beta \dots)_{\beta<\alpha}$

for the quantifier $(Q\mathsf{x})$. The class Φ of all *formulas* of L is defined [2] as the least class Ψ such that:

(1.1) every atomic formula belongs to Ψ;

(1.2) if $\varphi \in \Psi$, then $\neg\varphi \in \Psi$;

(1.3) if $\varphi, \psi \in \Psi$, then $\varphi \vee \psi$, $\varphi \wedge \psi \in \Psi$;

(1.4) if $\Theta \in S(\Psi)$, then $\bigvee\Theta$, $\bigwedge\Theta \in \Psi$;

(1.5) if $\varphi \in \Psi$ and $(Q\mathsf{x})$ is a quantifier of length $\leqslant \mathfrak{m}^+$, then $(Q\mathsf{x})\,\varphi \in \Psi$.

The abbreviations $\rightarrow$, $\leftrightarrow$ are used in the usual way.

By a structure for L we mean a system $\mathfrak{A} = \langle A, R_\beta\rangle_{\beta<\pi}$ where A is a non-empty set and each $R_\beta \subseteq A^{\tau(\beta)}$. The notation $\mathfrak{A} \models \varphi[a]$ means that the formula φ is satisfied in the structure $\mathfrak{A}$ by the sequence $a \in A^{\mathfrak{m}^+}$. If φ is atomic then $\mathfrak{A} \models \varphi[a]$ has the usual meaning, and one passes over the connectives in the usual way, e.g., if $\Theta \in S(\Phi)$ then $\mathfrak{A} \models \bigwedge\Theta[a]$ means that $\mathfrak{A} \models \theta[a]$ for all $\theta \in \Theta$. Quantifiers of finite length are also interpreted as usual, and in extending the interpretation to quantifiers of infinite length we follow Henkin [61].

The intuitive idea of the meaning of $\mathfrak{A} \models (Q\mathsf{x})\,\varphi[a]$ may be described as follows. We shall describe a game G played by two players, $(\exists)$ and $(\forall)$. First, an element $c_0 \in A$ is chosen by the player (Q_0), then player (Q_1) chooses $c_1 \in A$, and so on, until a sequence $c \in A^\alpha$ is chosen in this manner, where α is the length of $(Q\mathsf{x})$. Let $b \in A^{\mathfrak{m}^+}$ be defined by

$$b_\gamma = \begin{cases} a_\gamma & \text{if } \mathsf{v}_\gamma \notin \text{range } (\mathsf{x}); \\ c_\beta & \text{if } \mathsf{v}_\gamma = \mathsf{x}_\beta. \end{cases}$$

The player $(\exists)$ is declared the winner of the game G if $\mathfrak{A} \models \varphi[b]$, and $(\forall)$ is declared the winner if $\mathfrak{A} \models \neg\varphi[b]$. Finally, we say that $\mathfrak{A} \models (Q\mathsf{x})\,\varphi[a]$ iff there is a winning strategy for $(\exists)$ in the game G.

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[2] The definition of the class of formulas of L can be made precise in one of several ways. For instance, we could take the symbols of L to be appropriate ordinal numbers, and adopt the convention that the concatenation of n letters denotes and ordered n-tuple. We do not admit quantifiers of length zero or conjunctions or disjunctions over the empty set.

As defined in the text, Φ will be a proper class. However, there are a number of ways in which we could alter the definition so that Φ is a set, without making any extensive changes in the rest of the paper. For instance, if we require in (1.4) that the rank of the set Θ is less than an appropriate fixed ordinal, then Φ will be a set.

We now give a precise definition. Let α be the length of (Qx) and let $y=\{\beta<\alpha : Q_\beta=\exists\}$. By a *winning strategy* for $\mathfrak{A} \models (Qx), \varphi[a]$ we mean a set $\{g_\beta : \beta \in y\}$ of functions g_β on $A^{\beta-\nu}$ into A such that, for all $c \in A^{\beta-\nu}$, we have $\mathfrak{A} \models \varphi[b]$, where $b \in \mathfrak{A}^{\mathfrak{m}^+}$ is defined by

$$b_\gamma = \begin{cases} a_\gamma, & \text{if } v_\gamma \notin \text{range } (x); \\ c_\beta, & \text{if } \beta \in \alpha - y \text{ and } v_\gamma = x_\beta; \\ g_\beta(c \restriction \beta - y), & \text{if } \beta \in y \text{ and } v_\gamma = x_\beta. \end{cases}$$

Then we define $\mathfrak{A} \models (Qx)\varphi[a]$ iff there exists a winning strategy for $\mathfrak{A} \models (Qx), \varphi[a]$.

Games which, like the game G, have infinitely many moves are studied, for example, in Gale–Stewart [53] and Mycielski–Zieba [55]. In each of those papers, examples are given of games G in which neither player has a winning strategy (in fact, with $\mathfrak{A}$ a structure of power 2, φ an open formula, and (Qx) of length ω). It is easily seen, however, that both players can never have a winning strategy for the same game G, because otherwise one could construct a sequence $b \in A^{\mathfrak{m}^+}$ such that $\mathfrak{A} \models \varphi[b]$ and $\mathfrak{A} \models \neg\, \varphi[b]$.

The set $V(\varphi)$, $V(\varphi) \subseteq \{v_\beta : \beta < \mathfrak{m}^+\}$, of *free variables of* φ, is defined in the natural way. Thus whether $\mathfrak{A} \models \varphi[a]$ holds depends only on the restriction of a to the set $\{\beta : v_\beta \in V(\varphi)\}$. Let Σ be the class $\Sigma = \{\varphi \in \Phi : V(\varphi) = 0\}$ of sentences of L. If $\varphi \in \Sigma$ we write $\mathfrak{A} \models \varphi$ for $\mathfrak{A} \models \varphi[a]$.

Let us write $\exists^* = \forall$, $\forall^* = \exists$, and when $Q \in \{\exists, \forall\}^\alpha$, $Q^* = \langle (Q_\beta)^* \rangle_{\beta<\alpha}$. For $\varphi \in \Phi$ we define φ^*, the dual of φ, as follows: if φ is atomic then $\varphi^* = \neg\, \varphi$; $(\neg\, \psi)^* = \psi$; $(\psi \wedge \theta)^* = \psi^* \vee \theta^*$; $(\psi \vee \theta)^* = \psi^* \vee \theta^*$; $(\bigwedge \Theta)^* = \bigvee \{\theta^* : \theta \in \Theta\}$; $(\bigvee \Theta)^* = \bigwedge \{\theta^* : \theta \in \Theta\}$; $((Qx)\psi)^* = (Q^*x)(\psi^*)$. If $\Psi \subseteq \Phi$, we write $\Psi^* = \{\psi^* : \psi \in \Psi\}$. Notice that we always have $V(\varphi) = V(\varphi^*)$ and $V(\Psi) = V(\Psi^*)$.

The dual is closely related to the negation. Indeed, $\mathfrak{A} \models (Q^*x) \neg\, \varphi[a]$ means that player $(\forall)$ has a winning strategy for the game G. In finitary logic we have, by DeMorgan's laws, that $\models \theta^* \leftrightarrow \neg\, \theta$. More generally, if θ belongs to the least class Ψ satisfying conditions (1.1)–(1.4) and such that $(Qx)\varphi \in \Psi$ whenever $\varphi \in \Psi$ and Q is constant, then we have $\models \theta^* \leftrightarrow \neg\, \theta$. It follows that in a restricted infinitary logic such as the system in Tarski [58b], we have $\models \theta^* \leftrightarrow \neg\, \theta$.

Lemma 1.1. (i) *For all* $\varphi \in \Phi$, *we have* $\models \neg\, (\varphi \wedge \varphi^*)$ *and* $\models \varphi \rightarrow \neg\, (\varphi^*)$.

(ii) *There exists* $\varphi \in \Sigma$ *and a structure* $\mathfrak{A}$ *of power* 2 *such that* $\mathfrak{A} \models \neg(\varphi \vee \varphi^*)$.

Proof. (i) is an easy induction using the fact that both players cannot have a winning strategy for the same game. (ii) is a restatement of the result that there is a game in which neither player has a winning strategy.

The following special subclasses of Φ will be of particular importance.

By the class Φ_F of *finite formulas* we mean the least Ψ which satisfies (1.1), (1.2), (1.3), and:

(1.5F) if $\varphi \in \Psi$ and (Qx) is a quantifier of length one, then $(Qx)\varphi \in \Psi$.

The class Φ_A of *admissible formulas* is the least Ψ such that $\Phi_F \subseteq \Psi$ and Ψ satisfies (1.3), (1.4), and:

(1.5A) if $\psi \in \Psi$, $\varphi = (Qx)\psi$, and $V(\varphi)$ has power $|V(\varphi)| \leqslant \mathfrak{m}$, then $\varphi \in \Psi$.

Notice that, by virtue of (1.4), there are admissible formulas which have $\mathfrak{m}^+$ free variables, but by (1.5A) no such formula begins with a quantifier. The class Φ_A is not closed under negation.

The class Φ_C of *conjunctive formulas* is the least Ψ such that $\Phi_F \subseteq \Psi$, (1.3) and (1.5A) hold, and:

(1.4C) if $\Theta \in S(\Psi)$ then $\bigwedge \Theta \in \Psi$.

Several simple remarks are collected in the following lemma.

Lemma 1.2. (i) If $\varphi \in \Phi_F$, *then* $\models \varphi^* \leftrightarrow \neg\varphi$ *and* $V(\varphi)$ *is finite.*

(ii) *If* $\varphi \in \Phi_A$, *then* $\varphi^{**} = \varphi$.

(iii) $\Phi_F = \Phi_F^*$ *and* $\Phi_A = \Phi_A^*$.

(iv) $\Phi_F \subseteq \Phi_C \cap \Phi_C^*$, *and* $\Phi_C \cup \Phi_C^* \subseteq \Phi_A$.

We shall write Σ_F, Σ_C, Σ_A for $\Sigma \cap \Phi_F$, $\Sigma \cap \Phi_C$, $\Sigma \cap \Phi_A$, respectively. The phrase "$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent", in symbols "$\mathfrak{A} \equiv \mathfrak{B}$", has the usual meaning, i.e., that $\mathfrak{A} \models \theta$ and $\theta \in \Sigma_F$ implies $\mathfrak{B} \models \theta$.

A structure $\mathfrak{A}$ is said to be $\mathfrak{m}^+$-*saturated* iff, for any $\Theta \in S(\Phi_F)$ such that $V(\Theta) \subseteq \{v_\beta : \beta < \mathfrak{m}\}$, we have

$$\mathfrak{A} \models (\forall v_0 \forall v_1 \ldots \forall v_\beta \ldots)_{\beta<\mathfrak{m}} \\ (\bigwedge\{(\exists v_0)(\theta_0 \wedge \ldots \wedge \theta_n) : \theta_0, \ldots, \theta_n \in \Theta\} \rightarrow (\exists v_0) \bigwedge \Theta).$$

Obviously any finite structure is $\mathfrak{m}^+$-saturated. $\mathfrak{m}^+$-saturated structures are studied in detail (but with a different terminology) in Morley–Vaught [62] [3]. The following result is proved there (it can be seen from the remark (4) on p. 45 and Theorems 2.5, 2.10, 3.2, and 3.3 of that paper).

Lemma 1.3. *For any infinite structure* $\mathfrak{B}$ *and any cardinal* $\mathfrak{n}$ *such that* $1 < \mathfrak{n} = \mathfrak{n}^{\mathfrak{m}}$, *there is an* $\mathfrak{m}^+$*-saturated structure* $\mathfrak{A}$ *of power* $\mathfrak{n}$ *with* $\mathfrak{A} \equiv \mathfrak{B}$.

2. Conjunctive formulas. The compactness theorem may be formulated in the following way: if $\Theta \in S(\Sigma_{\mathsf{F}})$ and each finite subset of Θ has a model, then $\bigwedge \Theta$ has a model. In this section we shall give a generalization of the compactness theorem in which the sentence $\bigwedge \Theta$ is replaced by an arbitrary conjunctive sentence[4]. First we state a fundamental definition.

Let $\varphi \in \Phi_{\mathsf{C}}$. The set $\mathscr{A}(\varphi)$ of (*finite*) *approximations of* φ is defined recursively as follows:

(2.1) if $\varphi \in \Phi_{\mathsf{F}}$, then $\mathscr{A}(\varphi) = \{\varphi\}$;

(2.2) if $\varphi = \psi \wedge \theta$, $\mathscr{A}(\varphi) = \{\sigma \wedge \eta : \sigma \in \mathscr{A}(\psi),\ \eta \in \mathscr{A}(\theta)\}$;

(2.2*) if $\varphi = \psi \vee \theta$, $\mathscr{A}(\varphi) = \{\sigma \vee \eta : \sigma \in \mathscr{A}(\psi),\ \eta \in \mathscr{A}(\theta)\}$;

(2.3) if $\varphi = \bigwedge \Theta$, $\mathscr{A}(\varphi) = \{\sigma_0 \wedge \ldots \wedge \sigma_n : \theta_0, \ldots, \theta_n$ are distinct members of Θ and $\sigma_0 \in \mathscr{A}(\theta_0), \ldots, \sigma_n \in \mathscr{A}(\theta_n)\}$;

(2.4) if $\varphi = (Q_0 x_0 \ldots Q_\beta x_\beta \ldots)_{\beta<\alpha}\, \psi$, $\mathscr{A}(\varphi) = \{\theta : V(\theta) \subseteq V(\varphi)$ and $\theta = (Q_{\beta_0} x_{\beta_0}) \ldots (Q_{\beta_n} x_{\beta_n})\sigma$, for some $\sigma \in \mathscr{A}(\psi)$ and $\beta_0 < \ldots < \beta_n < \alpha\}$.

Dually, for $\varphi \in \Theta_{\mathsf{C}}{}^*$ we define $\mathscr{A}(\varphi)$ in a similar way but replacing (2.3) by its dual (2.3*), which has $\bigvee, \vee$ in place of $\bigwedge, \wedge$.

Lemma 2.1. (i) *Let* $\varphi \in \Phi_{\mathsf{C}} \cup \Phi_{\mathsf{C}}{}^*$. *Then* $\mathscr{A}(\varphi) \in S(\Phi_{\mathsf{F}})$, $V(\mathscr{A}(\varphi)) = V(\varphi)$, *and* $\mathscr{A}(\varphi^*) = (\mathscr{A}(\varphi))^*$.

(ii) Let $\varphi \in \Phi_{\mathsf{C}}$. *Then* $\models \varphi \to \bigwedge \mathscr{A}(\varphi)$, $\models \bigvee \mathscr{A}(\varphi^*) \to \varphi^*$, *and for all* $\theta_0, \ldots, \theta_n \in \mathscr{A}(\varphi)$ *there exists* $\theta \in \mathscr{A}(\varphi)$ *such that* $\models \theta \to \theta_0 \wedge \ldots \wedge \theta_n$.

Lemma 2.2. *Suppose* $\mathfrak{A}$ *is* $\mathfrak{m}^+$*-saturated,* $\Theta \in S(\Phi_{\mathsf{F}})$, $|V(\Theta)| \leqslant \mathfrak{m}$, *and* $\gamma < \mathfrak{m}^+$. *Then*

$$\mathfrak{A} \models (\forall v_0\, \forall v_1 \ldots \forall v_\beta \ldots)_{\beta<\mathfrak{m}^+} (\bigwedge \mathscr{A}((\exists v_\gamma) \bigwedge \Theta) \to (\exists v_\gamma) \bigwedge \Theta).$$

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[3] The term "saturated structure" is due to Vaught; e.g., see Vaught [63b].

[4] Engeler [63] considered the compactness theorem and the possibility of its generalization from a viewpoint somewhat similar to ours.

Proof. The result follows at once from the definition of $\mathfrak{m}^+$-saturated structures, by rearranging the variables.

Theorem 2.3. *Suppose* $\mathfrak{A}$ *is* $\mathfrak{m}^+$*-saturated. Then for every conjunctive formula* φ *we have*

$$\mathfrak{A} \models (\forall v_0\, \forall v_1 \dots \forall v_\beta \dots)_{\beta<\mathfrak{m}^+} (\bigwedge \mathscr{A}(\varphi) \to \varphi). \tag{1}$$

Proof. Let Ψ be the class of all $\varphi \in \Phi_C$ such that (1) holds. We shall show that $\Psi = \Phi_C$, i.e., that $\Phi_F \subseteq \Psi$ and (1.3), (1.4C), and (1.5A) hold.

It is obvious that $\Phi_F \subseteq \Psi$. It is also seen at once that (1.4C) holds and that ψ, $\theta \in \Psi$ implies $\psi \wedge \theta \in \Psi$.

Suppose $\varphi = \psi \vee \theta$, $a \in A^{\mathfrak{m}^+}$, and $\mathfrak{A} \models \bigwedge \mathscr{A}(\varphi)[a]$. If $\mathfrak{A} \models \neg \bigwedge \mathscr{A}(\psi)[a]$ and $\mathfrak{A} \models \neg \bigwedge \mathscr{A}(\theta)[a]$, then there exist $\sigma \in \mathscr{A}(\psi)$, $\eta \in \mathscr{A}(\theta)$ such that $\mathfrak{A} \models \neg (\sigma \vee \theta)[a]$; since $\sigma \vee \eta \in \mathscr{A}(\varphi)$, it would follow that $\mathfrak{A} \models \neg \bigwedge \mathscr{A}(\varphi)[a]$ which contradicts our assumption. Therefore either $\mathfrak{A} \models \bigwedge \mathscr{A}(\psi)[a]$ or $\mathfrak{A} \models \bigwedge \mathscr{A}(\theta)[a]$. Hence either $\mathfrak{A} \models \psi[a]$ or $\mathfrak{A} \models \theta[a]$, and thus $\mathfrak{A} \models \varphi[a]$. This verifies (1.3).

Now let (Qx) be a quantifier of length $\alpha \leqslant \mathfrak{m}^+$, $\varphi = (Qx)\psi$, $\psi \in \Psi$, and $|V(\varphi)| \leqslant \mathfrak{m}$. Assume that $a \in A^{\mathfrak{m}^+}$ and $\mathfrak{A} \models \bigwedge \mathscr{A}(\varphi)[a]$. We shall construct by transfinite induction a winning strategy for $\mathfrak{A} \models (Qx)$, $\psi[a]$, and this will show that $\varphi \in \Psi$ and verify (1.5A). Let $y = \{\beta < \alpha : Q_\beta = \exists\}$ and, for each $\beta < \alpha$, let

$$(Qx)^{(\beta)} = (Q_\beta x_\beta Q_{\beta+1} x_{\beta+1} \dots Q_{\beta+\gamma} x_{\beta+\gamma} \dots)_{\beta+\gamma<\alpha}.$$

It suffices to prove that for each $\beta < \alpha$ the following condition holds (with the convention $(Qx)^{(\alpha)}\psi = \psi$): (2) *$\{g_\gamma : \gamma \in \beta \cap y\}$ is a set of functions g_γ on $A^{\gamma-y}$ into A such that, for all $c \in A^{\beta-y}$, we have*

$$\mathfrak{A} \models \bigwedge \mathscr{A}((Qx)^{(\beta)}\psi)[b],$$

where $b \in A^{\mathfrak{m}^+}$ is defined by

$$b_\xi = \begin{cases} c_\gamma, & \text{if } \gamma \in \beta - y \text{ and } v_\xi = x_\gamma; \\ g_\gamma(c \restriction \gamma - y), & \text{if } \gamma \in \beta \cap y \text{ and } v_\xi = x_\gamma; \\ a_\xi, & \text{otherwise.} \end{cases}$$

Suppose $\delta \leqslant \alpha$ is a limit ordinal and (2) holds for all $\beta < \delta$. For $\beta = 0$, (2) holds by hypothesis. If $\delta > 0$ then each $\theta \in \mathscr{A}((Qx)^{(\delta)}\psi)$ belongs to $\mathscr{A}((Qx)^{(\gamma)}\psi)$ for some $\gamma < \delta$ and it follows easily that (2) holds when $\beta = \delta$.

Now suppose $\delta<\alpha$ and (2) holds when $\beta=\delta$. If $\delta\notin y$ then (2) obviously holds when $\beta=\delta+1$. Assume $\delta\in y$, let $c\in A^{(\delta+1)-\nu}=A^{\delta-\nu}$, and let $b\in A^{\mathfrak{m}+}$ be defined as in (2) with $\beta=\delta$. Then

$$\mathfrak{A}\models\bigwedge\mathscr{A}((\mathsf{Q}\mathsf{x})^{(\delta)}\psi)[b]. \tag{3}$$

For brevity, let $\eta=(\mathsf{Q}\mathsf{x})^{(\delta+1)}\psi$.
We have $|V(\eta)|\leqslant\mathfrak{m}$. Then by Lemma 2.2,

$$\mathfrak{A}\models(\forall\mathsf{x}_0\,\forall\mathsf{x}_1\ldots\forall\mathsf{x}_\gamma\ldots)_{\gamma<\delta}\,(\bigwedge\mathscr{A}((\exists\mathsf{x}_\delta)\wedge\mathscr{A}(\eta))\rightarrow(\exists\mathsf{x}_\delta)\wedge\mathscr{A}(\eta)). \tag{4}$$

If $\sigma\in\mathscr{A}((\exists\mathsf{x}_\delta)\wedge\mathscr{A}(\eta))$, then we have $\sigma=(\exists\mathsf{x}_\delta)(\theta_1\wedge\ldots\wedge\theta_n)$ for some $\theta_1,\ldots,\theta_n\in\mathscr{A}(\eta)$. By Lemma 2.1, there exists $\theta\in\mathscr{A}(\eta)$ such that $\models\theta\rightarrow(\theta_1\wedge\ldots\wedge\theta_n)$, and hence $\models((\exists\mathsf{x}_\delta)\theta)\rightarrow\sigma$. However, it follows from (2.4) that $(\exists\mathsf{x}_\delta)\theta\in\mathscr{A}((\mathsf{Q}\mathsf{x})^{(\delta)}\psi)$. Therefore, by (3), we have $\mathfrak{A}\models\sigma[b]$, and hence

$$\mathfrak{A}\models\bigwedge\mathscr{A}((\exists\mathsf{x}_\delta)\wedge\mathscr{A}(\eta))[b]. \tag{5}$$

From (4) and (5) we conclude that

$$\mathfrak{A}\models(\exists\mathsf{x}_\delta)\wedge\mathscr{A}(\eta)[b]. \tag{6}$$

This means that there exists $b'\in A^{\mathfrak{m}+}$ such that $b_\gamma'=b_\gamma$ whenever $\mathsf{v}_\gamma\neq\mathsf{x}_\delta$, and $\mathfrak{A}\models\bigwedge\mathscr{A}(\eta)[b']$. We define the function g_δ on $A^{\delta-\gamma}$ into A so that $g_\delta(c)=b_\gamma'$, where $\mathsf{v}_\gamma=\mathsf{x}_\delta$. Then (2) holds when $\beta=\delta+1$. This completes the induction and the proof.

Corollary 2.4. *If $\varphi\in\Sigma_{\mathsf{C}}$ and $\mathfrak{A}$ is $\mathfrak{m}^+$-saturated, then $\mathfrak{A}\models\varphi\vee\varphi^*$.*

Proof. Suppose $\mathfrak{A}\models\neg\varphi$. Then by Theorem 2.3, $\mathfrak{A}\models\neg\theta$ for some $\theta\in\mathscr{A}(\varphi)$. But then $\mathfrak{A}\models\theta^*$, and $\theta^*\in\mathscr{A}(\varphi^*)$, and hence by Lemma 2.1 we have $\mathfrak{A}\models\varphi^*$.

We now state our generalization of the compactness theorem.

Corollary 2.5. *Let $\varphi\in\Sigma_{\mathsf{C}}$. If each approximation of φ has a model, then φ has a model.*

Proof. By Lemma 1.3, 2.1, and the compactness theorem, $\mathscr{A}(\varphi)$ has an $\mathfrak{m}^+$-saturated model $\mathfrak{A}$. Then by Theorem 2.3, $\mathfrak{A}$ is a model of φ.

We also have some theorems of the Löwenheim–Skolem type for $\varphi\in\Sigma_{\mathsf{C}}$. For instance, we see from Lemma 1.3 that if φ has an infinite model and if $1<\mathfrak{n}=\mathfrak{n}^{\mathfrak{m}}$, then φ has a model (in fact an $\mathfrak{m}^+$-saturated model) of power $\mathfrak{n}$. Furthermore, it can be shown by an easy but tedious

induction that, for any $\varphi \in \Sigma_{\mathsf{C}}$, the class of all models of φ is closed under ultraproducts. This result yields various upward Löwenheim–Skolem type theorems, using the methods of Frayne–Morel–Scott [62], Rabin [59], and Chang–Keisler [62]; for instance, if φ has a model of infinite power $\mathfrak{n}$, then for all $\mathfrak{p}$, φ has a model of power $\mathfrak{n}^{\mathfrak{p}}$.

The class of models of a sentence $\varphi \in \Sigma_{\mathsf{C}}$ is not necessarily a PC_{Δ} class, and need not even be closed under limit ultrapowers (see, e.g., Keisler [63b]). A counterexample is provided by the sentence

$$\varphi = (\forall v_0\, \forall v_1 \ldots \forall v_n \ldots)_{n<\omega}\, (\exists v_\omega) \bigwedge \{\mathsf{P}(v_n, v_\omega) : n < \omega\}.$$

(Cf. in this connection Svenonius [*]).

3. Admissible formulas. We first define the notion of a finite approximation for arbitrary admissible formulas.

Let $\varphi \in \Phi_{\mathsf{A}}$. We define the set $\mathscr{A}(\varphi)$ of (*finite*) *approximations of* φ in the following way. We repeat the conditions (2.1), (2.2), (2.2*), (2.3), and (2.3*). If $\varphi \in \Phi_{\mathsf{C}} \cup \Phi_{\mathsf{C}^*}$, then $\mathscr{A}(\varphi)$ has already been defined in Section 2. Finally, for $\psi \in \Phi_{\mathsf{A}} - (\Phi_{\mathsf{C}} \cup \Phi_{\mathsf{C}}{}^*)$, we write:

(3.1) $\mathscr{A}((\exists v_\beta)\psi) = \{(\exists v_\beta)(\theta_0 \wedge \ldots \wedge \theta_n) : \theta_0, \ldots, \theta_n \in \mathscr{A}(\psi)\}$;

(3.1*) $\mathscr{A}((\forall v_\beta)\psi) = \{(\forall v_\beta)(\theta_0 \vee \ldots \vee \theta_n) : \theta_0, \ldots, \theta_n \in \mathscr{A}(\psi)\}$;

(3.2) If $\varphi = (\mathsf{Q}x)\psi$ where $(\mathsf{Q}x)$ is of length α, then $\mathscr{A}(\varphi) = \{\theta : \mathsf{V}(\theta) \subseteq \mathsf{V}(\varphi)$ and $\theta \in \mathscr{A}((\mathsf{Q}_{\beta_0} x_{\beta_0}) \ldots (\mathsf{Q}_{\beta_n} x_{\beta_n})\psi)$ for some $\beta_0 < \beta_1 < \ldots < \beta_n < \alpha\}$.

Note that for all $\varphi \in \Phi_{\mathsf{A}}$ we have $\mathscr{A}(\varphi) \in S(\Phi_{\mathsf{F}})$, $\mathsf{V}(\mathscr{A}(\varphi)) = \mathsf{V}(\varphi)$, and $\mathscr{A}(\varphi^*) = (\mathscr{A}(\varphi))^*$.

Theorem 3.1. *Suppose* $\mathfrak{A}$ *is* $\mathfrak{m}^+$*-saturated. Then for all* $\varphi \in \Phi_{\mathsf{A}}$ *we have*

$$\mathfrak{A} \models (\forall v_0\, \forall v_1 \ldots \forall v_\beta \ldots)_{\beta<\mathfrak{m}^+} (\bigwedge \mathscr{A}(\varphi) \rightarrow \varphi),$$

and

$$\mathfrak{A} \models (\forall v_0\, \forall v_1 \ldots \forall v_\beta \ldots)_{\beta<\mathfrak{m}^+} (\varphi \rightarrow \bigvee \mathscr{A}(\varphi)).$$

Proof. Suppose Φ_{A} is well ordered. For each $\psi \in \Phi_{\mathsf{A}}$ we define a new formula $\psi_0 \in \Phi_{\mathsf{C}}$ as follows. If $\psi \in \Phi_{\mathsf{F}}$, let $\psi_0 = \psi$. Let $(\psi \wedge \theta)_0 = \psi_0 \wedge \theta_0$, $(\psi \vee \theta)_0 = \psi_0 \vee \theta_0$, $(\bigwedge \Theta)_0 = \bigwedge \{\theta_0 : \theta \in \Theta\}$, and $((\mathsf{Q}x)\psi)_0 = (\mathsf{Q}x)(\psi_0)$. Finally, if $\Theta \in S(\Phi_{\mathsf{A}})$, then let $(\bigvee \Theta)_0 = \theta_0$ where θ is the first element of Θ. It is easily seen that, for all $\psi \in \Phi_{\mathsf{A}}$, we have $\psi_0 \in \Phi_{\mathsf{C}}$, $\mathscr{A}(\psi_0) \subseteq \mathscr{A}(\psi)$, and $\models \psi_0 \rightarrow \psi$.

Now let $a \in \mathscr{A}^{\mathfrak{m}^+}$. If $\mathfrak{A} \models \bigwedge \mathscr{A}(\varphi)[a]$, then $\mathfrak{A} \models \bigwedge \mathscr{A}(\varphi_0)[a]$, and hence

$\mathfrak{A} \models \varphi_0[a]$ by Theorem 2.3, and thus $\mathfrak{A} \models \varphi[a]$. Therefore $\mathfrak{A} \models (\bigwedge \mathscr{A}(\varphi) \rightarrow \rightarrow \varphi)[a]$. On the other hand, if $\mathfrak{A} \models \varphi[a]$ then $\mathfrak{A} \models \neg \varphi^*[a]$, so $\mathfrak{A} \models \neg \bigwedge \mathscr{A}(\varphi^*)[a]$, $\mathfrak{A} \models \neg \bigwedge (\mathscr{A}(\varphi)^*)[a]$, and hence $\mathfrak{A} \models \bigvee \mathscr{A}(\varphi)[a]$. Thus $\mathfrak{A} \models (\varphi \rightarrow \bigvee \mathscr{A}(\varphi))[a]$.

Observe that Theorem 3.1 is a generalization of Theorem 2.3 to arbitrary admissible formulas. However, the result appears to be less significant for Φ_A than for Φ_C, because $\models \varphi \rightarrow \bigwedge \mathscr{A}(\varphi)$ holds for all $\varphi \in \Phi_C$ but not for all $\varphi \in \Phi_A$. We use Theorem 3.1 in the proof of our main theorem, to which we now turn.

Theorem 3.2. *Let $\mathfrak{A}, \mathfrak{B}$ be $\mathfrak{m}^+$-saturated structures. Then for any $\varphi \in \Sigma_A$ such that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \varphi^*$, there exists $\theta \in \mathscr{A}(\varphi)$ such that $\mathfrak{A} \models \theta$ and $\mathfrak{B} \models \neg \theta$.*

Proof. Let Ψ be the class of all formulas $\varphi \in \Phi_A$ which have the property that, for all $a \in A^{\mathfrak{m}^+}$ and $b \in B^{\mathfrak{m}^+}$, if $\mathfrak{A} \models \varphi[a]$ and $\mathfrak{B} \models \varphi^*[b]$ then there exists $\theta \in \mathscr{A}(\varphi)$ such that $\mathfrak{A} \models \theta[a]$ and $\mathfrak{B} \models \theta^*[b]$. We shall show that $\Psi = \Phi_A$; i.e., that $\Phi_F \subseteq \Psi$ and (1.3), (1.4), and (1.5A) hold.

Obviously $\Phi_F \subseteq \Psi$. Suppose that $\varphi \in \Phi_C$, $\mathfrak{A} \models \varphi[a]$, and $\mathfrak{B} \models \varphi^*[b]$. Then $\models \varphi \rightarrow \bigwedge \mathscr{A}(\varphi)$, so $\mathfrak{A} \models \bigwedge \mathscr{A}(\varphi)[a]$. Also, $\mathfrak{B} \models \neg \varphi[b]$ (by Lemma 1.1), so by Theorem 2.3 we have $\mathfrak{B} \models \neg \bigwedge \mathscr{A}(\varphi)[b]$. Therefore we have $\mathfrak{A} \models \theta[a]$ and $\mathfrak{B} \models \neg \theta[b]$ for some $\theta \in \mathscr{A}(\varphi)$, and so $\varphi \in \Psi$, and $\Phi_C \subseteq \Psi$. By a similar argument, $\Phi_C{}^* \subseteq \Psi$.

Suppose $\psi, \theta \in \Psi$, $\varphi = \psi \wedge \theta$, $\mathfrak{A} \models \varphi[a]$, and $\mathfrak{B} \models \varphi^*[b]$. Then we have $\mathfrak{A} \models \psi[a]$, $\mathfrak{A} \models \theta[a]$, and either $\mathfrak{B} \models \psi^*[b]$ or $\mathfrak{B} \models \theta^*[b]$, say $\mathfrak{B} \models \psi^*[b]$. Since $\psi \in \Psi$, there exists $\sigma \in \mathscr{A}(\psi)$ such that $\mathfrak{A} \models \sigma[a]$ and $\mathfrak{B} \models \neg \sigma[b]$. By Theorem 3.1, there is an $\eta \in \mathscr{A}(\theta)$ such that $\mathfrak{A} \models \eta[a]$. Therefore $\mathfrak{A} \models (\sigma \wedge \eta)[a]$ and $\mathfrak{B} \models \neg (\sigma \wedge \eta)[b]$. Since $\sigma \wedge \eta \in \mathscr{A}(\varphi)$, we have shown that $\varphi \in \Psi$. By similar arguments we may show that $\psi, \theta \in \Psi$ implies $\psi \vee \theta \in \Psi$, so (1.3) holds, and also show that (1.4) holds.

Now, to verify (1.5A), let $\varphi = (Qx)\psi$, where (Qx) has length $\alpha \leqslant \mathfrak{m}^+$, $\psi \in \Psi$, and $|V(\varphi)| \leqslant \mathfrak{m}$. If $\psi \in \Phi_C \cup \Phi_C{}^*$, then $\varphi \in \Phi_C \cup \Phi_C{}^* \subseteq \Psi$. Assume $\psi \in \Phi_A - (\Phi_C \cup \Phi_C{}^*)$ and let $\mathfrak{A} \models \varphi[a]$, $\mathfrak{B} \models \varphi^*[b]$, and $y = \{\beta < \alpha : Q_\beta = \exists\}$. Then there are winning strategies $\{g_\beta : \beta \in y\}$ for $\mathfrak{A} \models (Qx), \psi[a]$, and $\{h_\gamma : \gamma \in \alpha - y\}$ for $\mathfrak{B} \models (Q^*x), \psi^*[b]$. Let $\Gamma = \{\theta \in \mathscr{A}(\varphi) : \mathfrak{A} \models \theta[a]\}$. We wish to prove that $\mathfrak{B} \models \neg \theta[b]$ for some $\theta \in \Gamma$.

Assume that $\mathfrak{B} \models \bigwedge \Gamma[b]$. We shall arrive at a contradiction. For each $\beta < \alpha$, define

$$(Qx)^{(\beta)} = (Q_\beta x_\beta Q_{\beta+1} x_{\beta+1} \cdots Q_{\beta+\gamma} x_{\beta+\gamma} \cdots)_{\beta+\gamma<\alpha}.$$

We shall construct by transfinite induction two sequences $c \in A^\alpha$, $d \in B^\alpha$ such that for all $\beta \leqslant \alpha$ we have (with $(Qx)^{(\alpha)}\psi = \psi$):

(1) if $\gamma \in \beta \cap y$, then $c_\gamma = g_\gamma(c \restriction \gamma - y)$;

(2) if $\gamma \in \beta - y$, then $d_\gamma = h_\gamma(d \restriction \gamma \cap y)$;

(3) if $\mathfrak{A} \models \theta[a^{(\beta)}]$ and $\theta \in \mathscr{A}((Qx)^{(\beta)}\psi)$, then $\mathfrak{B} \models \theta[b^{(\beta)}]$;

here $a^{(\beta)} \in A^{m+}$ is defined by

$$a_\xi^{(\beta)} = \begin{cases} c_\gamma, & \text{if } \gamma < \beta \text{ and } v_\xi = x_\gamma, \\ a_\xi, & \text{otherwise,} \end{cases}$$

and the sequence $b^{(\beta)}$ is defined analogously. Conditions (1)–(3) when $\beta = \alpha$ will give us our contradiction, because (1), (2) give $\mathfrak{A} \models \psi[a^{(\alpha)}]$ and $\mathfrak{B} \models \psi^*[b^{(\alpha)}]$, and then (3) contradicts our assumption $\psi \in \Psi$. This contradiction will show that $\varphi \in \Psi$, thus verifying (1.5A) and completing our proof.

We now carry through the induction. Since $\mathfrak{B} \models \bigwedge \Gamma[b]$, (1)–(3) hold for $\beta = 0$. It follows from (3.3) that if $\delta \leqslant \alpha$ is a limit ordinal and (1)–(3) hold for all $\beta < \delta$, then (1)–(3) hold for $\beta = \delta$. Suppose $\delta < \alpha$ and (1)–(3) hold for $\beta = \delta$. Assume first that $\delta \in y$. We wish to choose c_δ, d_δ so that (1)–(3) hold for $\beta = \delta + 1$. By (1), we must have $c_\delta = g_\delta(c \restriction \delta - y)$. Let

$$\Delta_\beta = \{\theta \in \mathscr{A}((Qx)^{(\beta)}\psi) : \mathfrak{A} \not\models \theta[a^{(\beta)}]\}$$

for $\beta = \delta$, $\delta + 1$. Then we have, by (3), that $\mathfrak{B} \models \bigwedge \Delta_\delta[b^{(\delta)}]$. Moreover, by (3.1) and (3.3) we have $\Delta_\delta \supseteq \mathscr{A}((\exists x_\delta) \bigwedge \Delta_{\delta+1})$. By Lemma 2.2,

$$\mathfrak{B} \models (\exists x_\delta) \bigwedge \Delta_{\delta+1}[b^{(\delta)}].$$

Therefore we may choose an element $d_\delta \in B$ such that $\mathfrak{B} \models \bigwedge \Delta_{\delta+1}[b^{(\delta+1)}]$. Hence (1)–(3) hold for $\beta = \delta + 1$. In the remaining case, $\delta \in \alpha - y$, we argue in exactly the same way with the roles of $\mathfrak{A}$ and $\mathfrak{B}$ reversed. Our induction, and hence the proof, is complete.

For formulas $\varphi \in \Phi_A$, let $\mathscr{A}^c(\varphi)$ be the closure of $\mathscr{A}(\varphi)$ under finite conjunction and disjunction, i.e. the least class $\Psi \supseteq \mathscr{A}(\varphi)$ such that (1.3) holds.

Theorem 3.3. (*Separation theorem*). *Suppose* $\{\varphi_i : i \in I\} \subseteq \Sigma_F$, $\{\psi_i : i \in I\} \subseteq \Sigma_A$, $\models \bigvee \{\psi_i^* : i \in I\}$, *and* $\models \varphi_i \to \psi_i$ *for each* $i \in I$. *Then there exist* $\theta_i \in \mathscr{A}^c(\psi_i)$, $i \in I$, *such that* $\models \bigvee \{\neg \theta_i : i \in I\}$ *and* $\models \varphi_i \to \theta_i$.

Proof. For each $i \in I$, let $\Theta_i = \{\theta \in \mathscr{A}^c(\psi_i) : \models \varphi_i \to \theta\}$. It suffices to show that there exist $\theta_i \in \Theta_i$, $i \in I$, such that $\models \bigvee \{\neg \theta_i : i \in I\}$. Suppose there exists a model $\mathfrak{A}$ of $\bigcup_{i \in I} \Theta_i$. We may assume, by Lemma 1.3, that $\mathfrak{A}$ is $\mathfrak{m}^+$-saturated. For each i, the set

$$\Gamma_i = \{\sigma : \mathfrak{A} \models \sigma \text{ and } \sigma^* \in \mathscr{A}^c(\psi_i)\}$$

is closed under finite conjunction, and for no $\sigma \in \Gamma_i$ does $\sigma^* \in \Theta_i$. Hence for each $\sigma \in \Gamma_i$ there exists a model of $\varphi_i \wedge \sigma$. By the compactness theorem and Lemma 1.3, there is for each i an $\mathfrak{m}^+$-saturated model $\mathfrak{B}_i$ of $\Gamma_i \cup \{\varphi_i\}$. Then $\mathfrak{B}_i \models \theta$ and $\theta \in \mathscr{A}^c(\psi_i)$ implies $\mathfrak{A} \models \theta$. Moreover, since $\models \varphi_i \to \psi_i$, $\mathfrak{B}_i$ is a model of ψ_i. Hence by Theorem 3.2 we have $\mathfrak{A} \models \neg \psi_i^*$ for each $i \in I$. But this contradicts our hypothesis $\models \bigvee \{\psi_i^* : i \in I\}$. We thus conclude that $\bigcup_{i \in I} \Theta_i$ has no models. Each Θ_i is closed under finite conjunction; thus by the compactness theorem there exist $\theta_i \in \Theta_i$, $i \in I$, such that $\{\theta_i : i \in I\}$ has no models, i.e., $\models \bigvee \{\neg \theta_i : i \in I\}$.

The following corollary justifies our use of the word "approximation".

Corollary 3.4. (i) (*Interpolation theorem*). *Suppose* $\sigma, \eta \in \Sigma_{\mathsf{F}}$, $\psi \in \Sigma_{\mathsf{A}}$, $\models \sigma \to \psi$, *and* $\models \neg \psi^* \to \eta$. *Then there is a* $\theta \in \mathscr{A}^c(\psi)$ *such that* $\models \sigma \to \theta$, *and* $\models \theta \to \eta$.

(ii) *If* $\psi \in \Sigma_{\mathsf{A}}$ *and if* $\models \psi \leftrightarrow \sigma$, $\models \psi^* \leftrightarrow \sigma^*$ *for some* $\sigma \in \Sigma_{\mathsf{F}}$, *then we have* $\models \psi \leftrightarrow \theta$ *for some* $\theta \in \mathscr{A}^c(\psi)$.

Proof. (i) Apply Theorem 3.3 with $I = \{1, 2\}$, $\varphi_1 = \sigma$, $\psi_1 = \psi$, and $\varphi_2 = \psi_2 = \eta^*$.

(ii) Apply (i) with $\eta = \sigma$.

It is an open question whether the above corollary can be simplified by replacing $\neg \psi^*$ by ψ in (i), or by deleting the hypothesis $\models \psi^* \leftrightarrow \sigma^*$ from (ii). However, in the special case $\psi \in \Sigma_{\mathsf{C}} \cup \Sigma_{\mathsf{C}}^*$ one can make those two simplifications, and can also replace $\mathscr{A}^c(\psi)$ by $\mathscr{A}(\psi)$, in Corollary 3.4; this follows from Corollary 2.4. Three other open problems are:

(1) Suppose $\psi_1, \psi_2, \psi_3 \in \Sigma_{\mathsf{A}}$, $\models \neg \psi_1^* \to \psi_2$, and $\models \neg \psi_2^* \to \psi_3$. Does there exist $\theta_i \in \mathscr{A}^c(\psi_i)$, $i = 1, 2, 3$, such that $\models \theta_1 \to \theta_2$ and $\models \theta_2 \to \theta_3$?

(2) To what extent can our results be generalized to apply to non-admissible formulas?

(3) Do our results hold for the richer languages considered by Henkin in [61] which have arbitrary simply ordered quantifiers (our definition of $\mathscr{A}(\varphi)$ still makes sense in that case), or even directed systems of quantifiers?

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TOPICS IN THE THEORY OF DEFINITION

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In the subject I wish to discuss here, one is interested in finding out when a given concept or predicate is definable in terms of a set of other predicates. As a convenient starting point we might begin with a test for definability going back to Padoa [02, 03]. Padoa's method, when described in modern terms by way of the predicate calculus, consists of the following remark. Let Σ be a set of elementary statements in the predicate symbols $P_1, \ldots, P_n, R$. If $\mathfrak{A} = \langle A, \mathscr{P}_1, \ldots, \mathscr{P}_n, \mathscr{R} \rangle$ is a model of Σ, we write $\mathfrak{A}(\hat{R})$ for the relational system $\langle A, \mathscr{P}_1, \ldots, \mathscr{P}_n \rangle$ obtained from $\mathfrak{A}$ by deleting the relation $\mathscr{R}$. The system $\mathfrak{A}(\hat{R})$ is called the R-reduct of $\mathfrak{A}$. Now suppose that it is deducible from Σ that the predicate R is syntactically definable in terms of the predicates $P_1, \ldots, P_n$ in the sense that

$$\Sigma \vdash (\forall x_1)(\forall x_2) \ldots (\forall x_m)(R \equiv \psi),$$

where ψ is an elementary formula in $P_1, \ldots, P_n$, and $x_1, \ldots, x_m$ are the variables occurring in R. It follows that if $\mathfrak{A}$ and $\mathfrak{A}'$ are models of Σ with identical R-reducts, then $\mathfrak{A}$ and $\mathfrak{A}'$ are identical (or, equivalently, if $\sigma : \mathfrak{A} \to \mathfrak{A}'$ is an onto map which is an isomorphism of the R-reducts of $\mathfrak{A}$ and $\mathfrak{A}'$, then σ is an isomorphism). Padoa's method consists of the contraposition of this remark, namely, if we can find two different models of Σ which have the same R-reducts, then there can be no syntactic definition of R in terms of $P_1, \ldots, P_n$.

The situation here is similar to the one for provability. Since a statement X which is decidable (i.e., provable or refutable) from a set Σ of axioms must have the same truth values in all models of Σ, it is sufficient to find two models of Σ in which X does not have the same truth value in order to show that X is not decidable from Σ. Now in this case, this method of showing non-decidability must always work, that is, there will always be two models for which an undecidable statement has different truth values. This of course is just a rewording of the completeness theorem for the predicate calculus.

In 1953 Beth [53] answered the corresponding question for definability by proving the converse of Padoa's result. Thus the predicate calculus is complete for definability as well as for validity: the syntactic and semantic notions of definability coincide.

Further developments in this field arose primarily through weakening the semantic (or syntactic) notion of definability, and then determining the corresponding syntactic (respectively, semantic) notion. Another extension lay in abandoning the assumption that the statements of Σ are elementary. An interesting extension of the first kind is a result due to Svenonius.

Suppose that we weaken the definition of semantic definability by requiring that any two *isomorphic* models $\mathfrak{A}$ and $\mathfrak{A}'$ with identical R-reducts are identical (or equivalently if $\sigma : \mathfrak{A} \to \mathfrak{A}'$ is an isomorphism which is an automorphism of the R-reducts of $\mathfrak{A}$ and $\mathfrak{A}'$, then σ is an automorphism). In what syntactic sense is R defined? Now, if there exist a finite number of formulas $\psi_1, \psi_2, \ldots, \psi_k$ (in which no variables other than $x_1, \ldots, x_n$ occur free) in the predicates $P_1, \ldots, P_n$ such that

$$\Sigma \vdash \bigvee_{i \leqslant k} ((\forall x_1) \ldots (\forall x_m)(R \equiv \psi_i)),$$

then $\mathscr{R}$ may be defined by one ψ_i in a model $\mathfrak{A}$ of Σ and by a different ψ_j in another model $\mathfrak{A}'$, so that $\mathfrak{A}$ and $\mathfrak{A}'$ may be different models although their R-reducts are the same. However, if $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic and have the same R-reducts, then $\mathscr{R}$ and $\mathscr{R}'$ can both be defined by the same ψ_i so that $\mathfrak{A} = \mathfrak{A}'$. What Svenonius showed is that the converse is true.

These and similar problems have been studied through various techniques. Beth [53] used Gentzen's Hauptsatz; Craig [57] applied his methods of linear reasoning; Robinson [56e] and Svenonius [59a] used the model-theoretic methods of diagrams and elementary extensions; Daigneault [59] applied the techniques of polyadic algebras. I wish to consider these problems from the point of view of ultraproducts. Aside from giving a new technique with which to study such problems, the approach by ultraproducts allows us to weaken the hypothesis on Σ to include a class of second-order statements. In addition this approach leads to certain natural topological questions which are equivalent to the logical ones. For instance, the theorem of Svenonius may be viewed as the problem of finding cross sections to a certain class of mappings. It is possible to use this technique to obtain further results in the theory

of definition, but in this paper we shall only address ourselves to the theorems of Beth and Svenonius.

Let me first describe the tools to be used. The ultraproduct construction is now sufficiently well known for me to dispense with its definition. (See, for example, Frayne–Morel–Scott [62] or Kochen [61].) A special case of the ultraproduct is the ultrapower $\mathfrak{A}^I/D$ of a system $\mathfrak{A}$, with D an ultrafilter over the set I. There is a natural imbedding map $\mathfrak{A} \to \mathfrak{A}^I/D$. The ultralimit construction extends the notion of ultrapower. Given a sequence $\{\langle I_i, D_i\rangle | i<\omega\}$ with each D_i an ultrafilter over the set I_i, we may form the sequence

$$\mathfrak{A}^{I_1}/D_1 \to (\mathfrak{A}^{I_1}/D_1)^{I_2}/D_2 \to \ldots.$$

The direct limit $\mathfrak{A}_\omega$ of this sequence of systems is called an ultralimit of $\mathfrak{A}$. The ultralimit $\mathfrak{A}_\omega$ shares with $\mathfrak{A}^I/D$ the property that it is elementarily equivalent to $\mathfrak{A}$, i.e., $\mathfrak{A}_\omega \equiv \mathfrak{A}$. Now, if $\mathfrak{A}$ and $\mathfrak{A}'$ have isomorphic ultralimits then it follows immediately that $\mathfrak{A} \equiv \mathfrak{A}'$. The converse is also true, and this is sufficient for proving Beth's theorem. But to obtain the theorem of Svenonius, the following stronger result is necessary (see Kochen [61] for a proof). Given a cardinal α there exists a universal ultrafilter sequence $\{\langle I_i, D_i\rangle | i<\omega\}$ such that for any two systems $\mathfrak{A}$, $\mathfrak{A}'$ of cardinality less than α, $\mathfrak{A} \equiv \mathfrak{A}'$ if and only if $\mathfrak{A}_\omega \simeq \mathfrak{A}'_\omega$ where $\mathfrak{A}_\omega$, $\mathfrak{A}'_\omega$ are the ultralimits of $\mathfrak{A}$, $\mathfrak{A}'$ with respect to this ultrafilter sequence. We shall use the notation introduced in Kochen [61]. In addition we shall let Π denote the operation of taking ultraproducts, Ω that of taking ultralimits, and $\mathscr{I}$ that of taking isomorphic images.

Let Σ be a set of elementary sentences as before. The set L of R-reducts of models of Σ need not lie in AC_δ (i.e., need not be an intersection of elementary classes), but L is the set of models of a second-order statement in a language in which infinite conjunctions are allowed. If we let $\bigwedge\Sigma$ denote the (possibly infinite) conjunction of formulas of Σ and regard R as a predicate variable, then L is the set of models of $(\exists R) \bigwedge\Sigma$. In consequence, as is easily checked, L is closed under Π, Ω, and $\mathscr{I}$. Now, by the hypothesis of Beth's theorem, for every model in L there is a unique relation $\mathscr{R}$ defined satisfying Σ. In the terminology of Kochen [61] we may define a model function G which assigns tc each element of L the unique relation $\mathscr{R}$. In Kochen [61] necessary and sufficient conditions were given (Theorem 12.1) for a model function to be defined by an elementary formula ψ as required for Beth's theorem. The criterion there, however, required that the domain of the model function lie in

AC_δ, and so does not apply to this case [1]. In what follows we outline a proof of an extension of Theorem 12.1 of Kochen [61] which does apply to this case. Let L be a class of systems which is closed under Π, Ω, and $\mathscr{I}$.

Definition 1. $K \in AC(L)$ if and only if $K = L \cap M$, for some $M \in AC$. We define the relativized classes $AC_\delta(L)$ and $AC_\sigma(L)$ similarly.

Theorem 1. *$K \in AC_\delta(L)$ if and only if $K \subseteq L$, K is closed under Π, Ω and $\mathscr{I}$, and $L-K$ is closed under Ω.*

The proof is entirely similar to the proof of Theorem 11.6(a) of Kochen [61].

To obtain the corresponding generalization of Theorem 11.6(b) we must have a generalization of the compactness theorem for AC's for which the following result of Frayne–Morel–Scott [62] is suitable.

Lemma 1. *Let $\mathscr{L}$ be a family of classes each of which is closed under Π and $\mathscr{I}$. If the intersection of all elements of $\mathscr{L}$ is empty then there is a finite subset of $\mathscr{L}$ the intersection of which is empty.*

Lemma 2. $AC(L) = AC_\delta(L) \cap AC_\sigma(L)$.

Theorem 2. *$K \in AC(L)$ if and only if K is contained in L and K and L–K are closed under Π, Ω, and $\mathscr{I}$.*

Definition 2. *Let G be a model function with domain L. $G \in AF(L)$ if and only if there is an elementary formula $\psi(x_1, \ldots, x_m)$ such that for all systems $\mathfrak{A}$ in L the relation $G(\mathfrak{A}) = \{\langle a_1, \ldots, a_m\rangle | \mathfrak{A} \models \psi(a_1, \ldots, a_m)\}$.*

Defining $U_G(\mathfrak{A}) = \langle \mathfrak{A}, G(\mathfrak{A})\rangle$, we have finally:

Theorem 3. *$G \in AF(L)$ if and only if U_G commutes with the operations Π, Ω, and $\mathscr{I}$.*

This is our required extension of Theorem 12.1. The proofs of these results may be obtained without difficulty from the proofs of the corresponding theorems in Kochen [61] by relativizing to L at the appropriate places. For the purpose of extending the result to cover the theorem of Svenonius, it is convenient to express Theorem 3 in a somewhat different form. Let L^* be a class of systems of the form $\langle A, \mathscr{P}, \ldots, \mathscr{P}_n, \mathscr{R}\rangle$ where $\mathscr{R}$ is an m-ary relation with m fixed. Denote by L

[1] I am indebted to Dana Scott and R. L. Vaught for pointing out that Theorem 12.1 yields only a weakened form of Beth's theorem. Theorem 12.4 of [61] should have the added hypothesis that the set of P-reducts of $\mathfrak{M}(T, \Sigma(P))$ lies in AC_δ.

the set of R-reducts of L^*. By the *reduct function* W of L^* we mean the function $W: L^* \to L$ given by $W(\mathfrak{A}) = \mathfrak{A}(\hat{R})$. A formula $\psi(x_1, \ldots, x_m)$ not involving the predicate R for which the equivalence $R \equiv \psi$ is valid in all systems of L defines, as we have seen above, a particular L^*, and hence a reduct function $W: L^* \to L$. For a fixed class L we let $\mathscr{AF}(L)$ denote the set of all such reduct functions. The following is an immediate consequence of Theorem 3.

Theorem 4. *For a given set L^* the following two conditions are equivalent:*

(1) *L is closed under Π, Ω, and $\mathscr{I}$ and the reduct function W of L^* lies in $\mathscr{AF}(L)$.*

(2) *L^* is closed under Π, Ω, and $\mathscr{I}$ and $\mathfrak{A}(\hat{R}) = \mathfrak{A}'(\hat{R})$ implies $\mathfrak{A} = \mathfrak{A}'$, for any two systems $\mathfrak{A}$, $\mathfrak{A}'$ in L^*.*

Beth's theorem now follows immediately upon letting L be the set of R-reducts of models of Σ. In fact, it is not here necessary to assume that Σ consists of a set of elementary sentences. We need only assume that the set of models of Σ is closed under Π, Ω, and $\mathscr{I}$. In particular we could let Σ be a set of sentences of the form

$$(\exists X_1)(\exists X_2) \ldots (\exists X_s)\, \mathsf{T}\,(X_1, \ldots, X_s),$$

where $X_1, \ldots, X_s$ are second-order variables and T is an elementary formula or an infinite conjunction of elementary formulas in the predicates $X_1, \ldots, X_s$. This result contains the generalization of Beth's theorem considered in Craig [57].

Now we shall prove a similar generalization of the theorem of Svenonius.

Theorem 5. *For a given set L^* the following two conditions are equivalent:*

(1) *L is closed under Π, Ω, and $\mathscr{I}$ and there exists a finite partition $\{E_1, \ldots, E_k\}$ of L^* such that the restriction $W|E_i$ of the reduct function W of L^* to each set E_i lies in $\mathscr{AF}(E_i)$.*

(2) *L^* is closed under Π, Ω, and $\mathscr{I}$ and $\mathfrak{A}(\hat{R}) = \mathfrak{A}'(\hat{R})$ implies $\mathfrak{A} = \mathfrak{A}'$, for any two isomorphic systems $\mathfrak{A}$, $\mathfrak{A}'$ in L^*.*

Proof. It is easily seen that condition (1) implies (2). We prove the converse implication. That L is closed under Π, Ω, and $\mathscr{I}$ is an immediate consequence of the fact that L^* is closed under these operations. Next

we prove that if $\mathfrak{A} \equiv \mathfrak{A}'$ (i.e. $\mathfrak{A}$ and $\mathfrak{A}'$ are elementarily equivalent) and $\mathfrak{A}(\widehat{R}) = \mathfrak{A}'(\widehat{R})$, then $\mathfrak{A} = \mathfrak{A}'$. By a theorem mentioned earlier, if $\mathfrak{A} \equiv \mathfrak{A}'$, then there exists a single ultrafilter sequence such that the ultralimits $\mathfrak{A}_\omega$ and $\mathfrak{A}'_\omega$ of $\mathfrak{A}$ and $\mathfrak{A}'$ with respect to this sequence are isomorphic. Since $\mathfrak{A}_\omega(\widehat{R}) = \mathfrak{A}'_\omega(\widehat{R})$, we have by hypothesis that $\mathfrak{A}_\omega = \mathfrak{A}'_\omega$, from which it immediately follows that $\mathfrak{A} = \mathfrak{A}'$.

Now consider the set $E_{\mathfrak{A}} = \{\mathfrak{A}' | \mathfrak{A}' \in L^*, \mathfrak{A}' \equiv \mathfrak{A}\}$ for $\mathfrak{A} \in L^*$. It is easily seen that $E_{\mathfrak{A}}$ is closed under Π, Ω, and $\mathscr{I}$. Furthermore, if $\mathfrak{A}'$ and $\mathfrak{A}''$ are systems in $E_{\mathfrak{A}}$ with $\mathfrak{A}'(\widehat{R}) = \mathfrak{A}''(\widehat{R})$, then we have proved that $\mathfrak{A}' = \mathfrak{A}''$. Hence, by Theorem 4, $W|E_{\mathfrak{A}} \in \mathscr{AF}(E_{\mathfrak{A}})$. Thus we can partition L^* into a family $\{E_{\mathfrak{A}} | \mathfrak{A} \in L^*\}$ of sets such that $W|E_{\mathfrak{A}} = \mathscr{AF}(E_{\mathfrak{A}})$. But now a compactness argument (using Lemma 1) shows that we may choose a finite set $E_1, \ldots, E_k$ for which $W|E_i \in \mathscr{AF}(E_i)$.

I wish finally to indicate briefly how the results just described can be put into a topological setting. The classes $AC(L^*)$ and $AC(L)$ form Boolean algebras. This allows us to form their associated Stone spaces $U(L^*)$ and $U(L)$. A point $[\mathfrak{A}]$ in L^* consists of the set of all systems in L elementary equivalent to a given system $\mathfrak{A}$ in L; a point $[\mathfrak{A}(\widehat{R})]$ in $U(L)$ is similarly described. The dual of the natural imbedding $AC(L) \to AC(L^*)$, is the onto map w: $U(L^*) \to U(L)$ given by $w([\mathfrak{A}]) = [W(\mathfrak{A})]$.

Now from Theorem 11.7 of Kochen [61] we know that the closure under Π of a set of systems in L^* (or L) corresponds to the topological closure of the set of points in $U(L^*)$ (respectively, $U(L)$) associated with the systems. By using Theorem 3 (or Theorem 4) we can conclude that $w \in \mathscr{AF}(L)$ (i.e., R is defined in terms of $P_1, \ldots, P_n$ in the sense of Beth), if and only if w is a one-to-one continuous map of $U(L^*)$ onto $U(L)$ (and hence a homeomorphism, since the spaces involved are compact). Similarly, by using Theorem 5 we have that R is defined in terms of $P_1, \ldots, P_n$ in the sense of Svenonius if and only if w is a Boolean map with a finite number of cross sections [2].

At this point this approach to the result of Svenonius makes contact with some results in Halmos [56]. There it is proved that there is a natural correspondence between Boolean maps from the Stone space X of a Boolean algebra A and existential quantifiers (in the polyadic

[2] A Boolean map π is an open and continuous map from one Boolean space X onto another Y. A cross section of π is a continuous map σ: $Y \to X$ such that $\pi\sigma y = y$ for all y in Y.

algebra sense) on A. Further, a correspondence is given between cross sections of a Boolean map and constants (again in the polyadic algebra sense) of A. Applied to our case with A the Boolean algebra $AC(L^*)$, we find that the Boolean map w corresponds to the second-order quantifier $(\exists R)$ and the cross sections of w correspond precisely to the formulas $\psi_1, \ldots, \psi_k$ supplied by the theorem of Svenonius.

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LOGICAL STRUCTURES ARISING IN QUANTUM THEORY *

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1. The logical structures studied in this paper are generalizations of the propositional calculus. The classical propositional calculus is essentially Boolean algebra or, alternatively, the theory of functions on an arbitrary set S with values in a two-element set. The generalization consists in allowing partial functions on the set S, i.e., functions defined on certain subsets of S, and defining an equivalence relation among these functions such that any two constant functions with the same constant value belong to the same equivalence class. The generalization is equally natural for functions with values in the field of real numbers and we shall consider this case first.

The admissible partial functions and the equivalence relation are determined by a given structure on the set S. In Section 2, we shall introduce and discuss in some detail the simplest such structure, viz. graphs of a certain type; for, though rather removed from applications, they are our most fruitful source of examples. A system of partial functions closer to applications is the following: Let S be the set of unit vectors in unitary n-space U^n. Functions of the system are real-valued functions whose domain of definition is a unitary basis $(\beta_1,\ldots,\beta_n)$. With such a function f we associate a linear map f^*: $U^n \to U^n$ by defining

$$f^*(\sum x_i\beta_i) = \sum x_i f(\beta_i)\beta_i.$$

Two functions f_1, f_2 are equivalent if and only if the corresponding maps f_1^*, f_2^* are equal.

Systems of partial functions on a set S can be correlated to physical theories in the following way: Elements of S correspond to (pure) states, equivalence classes of functions correspond to observables. (The term observable will therefore be used for such classes.) If $a \in S$ and if f is an element of the observable q, then $f(a)$ is the value of the observable

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q for the physical system in state a. In classical theories, every observable has a value for all states—the functions are defined for the whole set S; in quantum theory, an observable has a (fixed) value only for certain states—the functions are partial functions. (Probability distributions associated with observables and states will not be considered in this paper.)

Sum and product of two observables q_1, q_2 are only defined if there exist functions $f_i \in q_i$ $(i=1,2)$ having the same domain (for which we say "q_1 and q_2 are commeasurable"). In this case, the sum q_1+q_2 is the equivalence class of the functions f_1+f_2 ($f_i \in q_i$, $i=1, 2$) and similarly for the product. The set Q of observables is thus made into a "partial algebra". Partial algebras will also be defined independently of a system of functions. They are structures $\langle A; \downarrow; +, \cdot, \cdot', \mathbf{1}\rangle$; $\downarrow$ is a binary relation (commeasurability); $+, \cdot$ are partial binary operations ($q_1+q_2, q_1 \cdot q_2$ being defined iff $q_1 \downarrow q_2$); $\cdot'$ is the multiplication of an element of A by a real number; $\mathbf{1}$ is the unit element of A.

The subset of idempotent elements of a partial algebra forms a "partial Boolean algebra" $\mathfrak{B}$. The operations in $\mathfrak{B}$ are defined in the usual way: $q_1 \vee q_2 = (q_1+q_2)-(q_1 \cdot q_2)$ etc. The notion of validity of a formula α of the propositional calculus is defined by associating a mapping with α. Consider, e.g., the associative law α

$$[(x_1 \vee x_2) \vee x_3] \leftrightarrow [x_1 \vee (x_2 \vee x_3)].$$

A triple $\langle q_1, q_2, q_3\rangle$ of elements of $\mathfrak{B}$ is in the domain of α if all the operations in α can be performed for q_1, q_2, q_3 (this requires five commeasurabilities). α is said to be "Q-valid" if the element

$$[(q_1 \vee q_2) \vee q_3] \leftrightarrow [q_1 \vee (q_2 \vee q_3)]$$

is the unit element $\mathbf{1}$ of $\mathfrak{B}$ for all triples in the domain of α and all partial Boolean algebras $\mathfrak{B}$.

It may well be that all formulas of propositional calculus in Whitehead-Russell [25], (1.2 to 5.75), are Q-valid. The simplest formula (known to us) which is classically valid but not Q-valid is

$$[(x_1 \leftrightarrow x_2) \leftrightarrow (x_3 \leftrightarrow x_4)] \leftrightarrow [(x_1 \leftrightarrow x_4) \leftrightarrow (x_2 \leftrightarrow x_3)].$$

We shall axiomatize the notion of Q-validity and outline the corresponding completeness proof. We do not know whether the set of Q-valid formulas is recursive.

The notion of validity considered in this paper is based on the class of all partial Boolean algebras. It is equally natural to base corresponding notions on certain subclasses, e.g., the class of transitive partial Boolean algebras. (A partial Boolean algebra is transitive iff $a \subseteq b$, i.e., $a \wedge b = a$, and $b \subseteq c$ implies $a \mathbin{\circ\!\!-\!\!\circ} c$ and therefore $a \subseteq c$). Another natural subclass is the class of partial Boolean algebras associated with n-dimensional Euclidean or unitary space ($\mathfrak{B}(E^n)$, $\mathfrak{B}(U^n)$ as defined in Section 5, Example (1)) or with Hilbert space. It has been shown in Specker [60] that $\mathfrak{B}(E^3)$ cannot be imbedded into a Boolean algebra. This is an immediate consequence of the theorem – not stated in [60] – that some classically valid formula does not hold in $\mathfrak{B}(E^3)$. The relation of the notions of validity, imbeddability, and the connection of these notions with the problem of hidden variables will be discussed in another paper.

2. Let $\mathfrak{G}$ be a graph, i.e., a structure $\langle G, R\rangle$ on an underlying non-empty set G where R is a binary symmetric and irreflexive relation. Elements of G are called "vertices". $R(a, b)$ is read as "a and b are connected". A graph $\mathfrak{G}$ satisfies condition C iff it has the following properties:

1°. Any two connected vertices belong to exactly one triangle. Formally: For all a, b if $R(a, b)$ then there exists exactly one c such that $R(a, c)$ and $R(b, c)$.

2°. G contains at least one pair of connected vertices.

Examples of graphs satisfying condition C:

(a) $G = (a, b, c)$, $R(x, y)$ iff $x \neq y$ ($\mathfrak{G}$ is a triangle).

(b) Graph in Figure 1.

(c) G is the set of all lines through the origin of 3-dimensional Euclidean [alternatively: unitary] space; two lines are connected iff they are [alternatively: unitarily] orthogonal.

We associate a class F of functions with a graph satisfying condition C: $f \in F$ iff the values of f are real numbers and the domain of f – dom f – is a set of three vertices of $\mathfrak{G}$ any two of which are connected.

We define a relation E on $F \times F$: $E(f, g)$ holds iff one of the following conditions is satisfied:

1°. $f = g$.

2°. The sets dom f and dom g have one element in common, say dom $f = (a, b, c)$, dom $g = (a, b', c')$ and we have $f(a) = g(a)$ and

$$f(b) = f(c) = g(b') = g(c').$$

3°. $f(x_1)=g(x_2)$ for all $x_1 \in \operatorname{dom} f$, $x_2 \in \operatorname{dom} g$. ($f$ and g are both constant functions with the same constant value.)

The equivalence classes of the relation E are called "observables"; Q is the set of all observables.

Two observables q_1, q_2 are said to be "commeasurable" ($q_1 \,\underset{\circ}{|}\, q_2$) if there exist functions $f_i \in q_i$ ($i=1, 2$) such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$. Sum and product of commeasurable observables are defined as follows: q_1+q_2 is the equivalence class of the functions f_1+f_2, $q_1 \cdot q_2$ is the equivalence class of the functions $f_1 \cdot f_2$, where $f_i \in q_i$, $i=1, 2$, and $\operatorname{dom} f_1 = \operatorname{dom} f_2$. (One verifies that the equivalence classes do not depend on the choice of the functions.) If q is an observable and a is a real number then all the functions af for $f \in q$ belong to the same equivalence class which is by definition the class $a \cdot' q$.

3. With these definitions, the set Q of observables is made into what we shall call a "partial algebra": $\mathfrak{A}=\langle A; \,\underset{\circ}{|}\,; +, \cdot, \cdot'; \mathbf{1}\rangle$. A partial algebra $\mathfrak{A}$ is given by a nonempty set A, a binary relation denoted by $\,\underset{\circ}{|}\,$, two binary partial functions from A into A (sum $+$, product $\cdot$), a function from $R \times A$ into A (R field of reals) and an element $\mathbf{1}$ of A. The properties are as follows:

1°. The relation $\,\underset{\circ}{|}\,$ is symmetric and reflexive.

2°. For all q in A, $q \,\underset{\circ}{|}\, \mathbf{1}$. (The constant function 1 is commeasurable with all observables.)

3°. The partial functions sum and product are defined exactly for those pairs $\langle q_1, q_2\rangle$ of $A \times A$ for which $q_1 \,\underset{\circ}{|}\, q_2$.

4°. If any two of the observables q_1, q_2, q_3 are commeasurable (i.e., $q_i \,\underset{\circ}{|}\, q_j$ for $i, j=1, 2, 3$) then $(q_1+q_2) \,\underset{\circ}{|}\, q_3$, $(q_1 \cdot q_2) \,\underset{\circ}{|}\, q_3$ and $a \cdot' q_1 \,\underset{\circ}{|}\, q_3$ (a a real number).

5°. If any two of q_1, q_2, q_3 are commeasurable then the polynomials in q_1, q_2, q_3 form a commutative algebra over the field of real numbers. (This condition is equivalent to a longer but more elementary one: If $q_1 \,\underset{\circ}{|}\, q_2$ then $q_1+q_2=q_2+q_1$ etc.)

Remarks.

(1) If any two of $q_1, \ldots, q_n$ are commeasurable then the polynomials in $q_1, \ldots, q_n$ form a commutative algebra over the field of real numbers.

(2) If $\mathfrak{G}$ is the graph considered in example (c) then the associated

partial algebra is isomorphic to the following algebra $\mathfrak{A}$: A is the set of 3×3 real symmetric [alternatively: Hermitian] matrices; $M_1 \mathrel{\mathop{\circ}\limits^{|}} M_2$ iff $M_1 M_2 = M_2 M_1$ (i.e., if the matrices commute); sum and product are the usual sum and product of matrices.

4. Let $\mathfrak{A}$ be a partial algebra and let P_n be the set of polynomials (with real coefficients) containing no other variables than $x_1, \ldots, x_n$. We define recursively the domain $D_{\varphi,n}$ of a polynomial $\varphi \in P_n$ and a map φ^* corresponding to φ. $D_{\varphi,n}$ will be a subset of the n-fold Cartesian product A^n of A, φ^* will be a map from $D_{\varphi,n}$ into A; we put $\langle q_1, \ldots, q_n\rangle = \mathbf{q}$.

1°. If φ is the polynomial 1 then $D_{\varphi,n} = A^n$ and $\varphi^*(\mathbf{q}) = 1$.
2°. If φ is the polynomial x_i $(i = 1, \ldots, n)$ then $D_{\varphi,n} = A^n$ and $\varphi^*(\mathbf{q}) = \varphi^*(q_1, \ldots, q_n) = q_i$.
3°. If $\varphi = a\,\psi$ (a a real number) then $D_{\varphi,n} = D_{\psi,n}$ and $\varphi^*(\mathbf{q}) = a \cdot' \psi^*(\mathbf{q})$.
4°. If $\varphi = \psi \otimes \chi$ (where $\otimes$ is either $+$ or $\cdot$) then $\mathbf{q} \in D_{\varphi,n}$ iff $\mathbf{q} \in D_{\psi,n} \cap D_{\chi,n}$ and $\psi^*(\mathbf{q}) \mathrel{\mathop{\circ}\limits^{|}} \chi^*(\mathbf{q})$; $\varphi^*(\mathbf{q}) = \psi^*(\mathbf{q}) \otimes \chi^*(\mathbf{q})$.

We say "φ is identically 1 on $\mathfrak{A}$" or "the identity $\varphi = 1$ holds in $\mathfrak{A}$" iff $\varphi^*(\mathbf{q}) = \mathbf{1}$ for all $\mathbf{q} \in D_{\varphi,n}$.

Roughly speaking, the identity $\varphi = 1$ means that the corresponding function on A is $\mathbf{1}$ whenever it is defined.

An identity

$$\varphi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n)$$

can be interpreted in two ways.

1°. Whenever both φ and ψ are defined then they are equal: If $\langle q_1, \ldots, q_n\rangle = \mathbf{q} \in D_{\varphi,n} \cap D_{\psi,n}$ then $\varphi^*(\mathbf{q}) = \psi^*(\mathbf{q})$.
2°. Whenever both φ and ψ are defined and $\varphi^*(\mathbf{q}) \mathrel{\mathop{\circ}\limits^{|}} \psi^*(\mathbf{q})$ then $\varphi^*(\mathbf{q}) = \psi^*(\mathbf{q})$.

The following are examples of identities holding in all partial algebras (in the sense of 1° and therefore also in the sense of 2°):

$$x_1 + x_2 = x_2 + x_1$$

$$x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3.$$

The following identity does not hold in all partial algebras (not even in the sense of 2°):

$$(x_1 + x_2) + (x_3 + x_4) = (x_1 + x_4) + (x_2 + x_3).$$

We construct a partial algebra in which this identity does not hold; the algebra is given by a graph of 11 vertices (v_1 and v_2 are represented twice in the diagram).

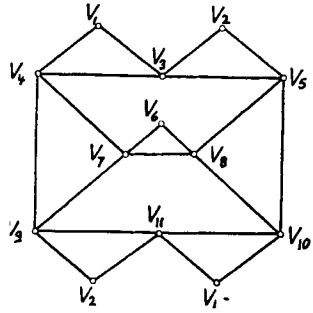


Fig. 1.

In order to define observables we use the following notation: $[i, j, k; a_i, a_j, a_k]$ is the function f whose domain is the set (v_i, v_j, v_k) of vertices and for which we have $f(v_h) = a_h$, $h = i, j, k$. The observables are defined by the functions following on the same line, both functions being equivalent.

$$q_1\colon\ [\ 1,\ 3,\ 4;\ 1,\ 0,\ 0],\quad [\ 1,\ 11,\ 10;\ 1,\ 0,\ 0]$$
$$q_2\colon\ [\ 3,\ 1,\ 4;\ 1,\ 0,\ 0],\quad [\ 3,\ \ 2,\ \ 5;\ 1,\ 0,\ 0]$$
$$q_3\colon\ [\ 2,\ 3,\ 5;\ 1,\ 0,\ 0],\quad [\ 2,\ \ 9,\ 11;\ 1,\ 0,\ 0]$$
$$q_4\colon\ [11,\ 2,\ 9;\ 1,\ 0,\ 0],\quad [11,\ \ 1,\ 10;\ 1,\ 0,\ 0]$$

We then have

$$q_1+q_2\colon\ [1,\ \ 3,\ \ 4;\ 1,\ 1,\ 0],\quad [4,\ 7,\ \ 9;\ 0,\ 1,\ 1]$$
$$q_3+q_4\colon\ [2,\ \ 9,\ 11;\ 1,\ 0,\ 1],\quad [4,\ 7,\ \ 9;\ 1,\ 1,\ 0]$$
$$q_1+q_4\colon\ [1,\ 10,\ 11;\ 1,\ 0,\ 1],\quad [5,\ 8,\ 10;\ 1,\ 1,\ 0]$$
$$q_2+q_3\colon\ [2,\ \ 3,\ \ 5;\ 1,\ 1,\ 0],\quad [5,\ 8,\ 10;\ 0,\ 1,\ 1]$$
$$(q_1+q_2)+(q_3+q_4)\colon\ [4,\ 7,\ \ 9;\ 1,\ 2,\ 1],\quad [6,\ 7,\ 8;\ 1,\ 2,\ 1]$$
$$(q_1+q_4)+(q_2+q_3)\colon\ [6,\ 8,\ 10;\ 1,\ 2,\ 1],\quad [6,\ 7,\ 8;\ 1,\ 1,\ 2]$$

The observables $(q_1+q_2)+(q_3+q_4)$ and $(q_1+q_4)+(q_2+q_3)$ are commeasurable as they are both represented on the triangle (v_6, v_7, v_8); they are different because they are represented there by different functions.

5. As in the case of ordinary commutative algebras, the subset B of idempotent elements of a partial algebra forms a partial Boolean algebra. In detail: Let $\langle A; \mathrel{\circ\!\!-}; +, \cdot, \cdot', \mathbf{1}\rangle$ be a partial algebra. Let B be the subset of elements $a \in A$ such that $a \cdot a = a$. Define $a \mathrel{\circ\!\!-} b$ for $a, b \in B$ iff $a \mathrel{\circ\!\!-} b$ in A; $a \vee b = (a+b) - a \cdot b$ (where $c-d$ is to be understood in the obvious way), $\neg a = \mathbf{1} - a$, $\mathbf{1}$ same element as in A, $\mathbf{0} = 0 \cdot' \mathbf{1}$ (0 number zero). The partial Boolean algebra $\langle B; \mathrel{\circ\!\!-}, \vee, \neg; \mathbf{1}, \mathbf{0}\rangle$ then satisfies the following conditions:

1°. The relation $\mathrel{\circ\!\!-}$ is symmetric and reflexive.
2°. For all $q \in B$: $q \mathrel{\circ\!\!-} \mathbf{1}$ and $q \mathrel{\circ\!\!-} \mathbf{0}$.
3°. The partial function $\vee$ is defined exactly for those pairs $\langle q_1, q_2\rangle$ of $B \times B$ for which $q_1 \mathrel{\circ\!\!-} q_2$.
4°. If any two of q_1, q_2, q_3 are commeasurable then $(q_1 \vee q_2) \mathrel{\circ\!\!-} q_3$ and $\neg q_1 \mathrel{\circ\!\!-} q_2$.
5°. If any two of q_1, q_2, q_3 are commeasurable then the Boolean polynomials in q_1, q_2, q_3 form a Boolean algebra. (As in the case of algebras this condition can be replaced by a more elementary one.)

Properties 1°–5° define the notion of partial Boolean algebra independently of the notion of partial algebra.

Examples of partial Boolean algebras:

(1) Let U^n be the n-dimensional unitary vector space, B the set of linear subspaces of U^n. For $a, b \in B$, $a \mathrel{\circ\!\!-} b$ holds iff there exists a unitary basis of U^n such that some subset of this basis is a basis of a and some subset is a basis of b. $a \vee b$ is the span of a and b; $\neg a$ is the complement of a, $\mathbf{0}$ is the 0-dimensional subspace, $\mathbf{1}$ is the whole space U^n.

(2) Let $\mathfrak{B}_i$, $i \in I$, be a nonempty family of Boolean algebras such that the following conditions are satisfied:

(a) For $i, j \in I$ there exists $k \in I$ such that $B_i \cap B_j = B_k$. (The intersection of two algebras of the family is an algebra of the family; all algebras have therefore the same element $\mathbf{0}$ and the same element $\mathbf{1}$.)

(b) If $B = \bigcup B_i$ (union over I) and if a, b, c are elements of B such that any of them lie in some common algebra B_i, then there exists $k \in I$ such that $a, b, c \in B_k$.

The algebra $\mathfrak{B}$ on the set B is then defined as follows: $a \mathrel{\circ\!\!-} b$ iff there exists $i \in I$ such that $a, b \in B_i$; $a \vee b = c$ in $\mathfrak{B}$ iff there exists

$i \in I$ such that $a \vee b = c$ in $\mathfrak{B}_i$; $\neg a = b$ in $\mathfrak{B}$ iff there exists $i \in I$ such that $\neg a = b$ in $\mathfrak{B}_i$; $\mathbf{1}$ and $\mathbf{0}$ are the common unit and zero elements of the algebras $\mathfrak{B}_i$.

It can be shown that every partial Boolean algebra is isomorphic to an algebra of this type.

6. We now define the "logic" associated with a partial Boolean algebra $\mathfrak{B}$. Let α be a formula of the propositional calculus (in the connectives $\vee$ and $\neg$) and φ the corresponding Boolean polynomial. α is said to be valid in $\mathfrak{B}$ iff $\varphi = 1$ is an identity of $\mathfrak{B}$ in the sense defined in Section 4. A formula is Q-valid if it holds in all partial Boolean algebras, it is C-valid if it holds in all Boolean algebras (i.e., is an identity of the propositional calculus). Clearly, every Q-valid formula is also C-valid.

Theorem. *Let α be a formula in $x_1, \ldots, x_n$ whose only subformulas in x_i alone are x_i or $\neg x_i$ $(i = 1, \ldots, n)$ and such that for all i, j $(1 \leqslant i < j \leqslant n)$ there exists a subformula $\alpha_{i,j}$ in x_i, x_j alone. Then α is Q-valid if it is C-valid.*

Proof. Let $\mathfrak{B}$ be a partial Boolean algebra, let φ be the polynomial corresponding to α and let $\langle q_1, \ldots, q_n \rangle = \mathbf{q}$ be a sequence of elements in $\mathfrak{B}$ belonging to $D_{\varphi, n}$. Let $\varphi_{i,j}$ be a subpolynomial of φ such that no subpolynomial of $\varphi_{i,j}$ is a polynomial in x_i and x_j; $\varphi_{i,j}$ is then one of the four polynomials $x_i \vee x_j$, $\neg x_i \vee x_j$, $x_i \vee \neg x_j$, $\neg x_i \vee \neg x_j$; from this follows $q_i \circ q_j$ and the theorem by Remark (1) in Section 3.

Corollary. *A formula in one or two variables is Q-valid if it is C-valid.*

Examples.

1°. The distributive law

$$[(x_1 \vee x_2) \wedge x_3] \leftrightarrow [(x_1 \wedge x_3) \vee (x_2 \wedge x_3)]$$

satisfies the hypothesis of the theorem and is therefore Q-valid. ($\wedge$, $\rightarrow$, $\leftrightarrow$ are here and in the following understood as being defined in terms of $\vee$ and $\neg$.)

2°. The associative law

$$[(x_1 \vee x_2) \vee x_3] \leftrightarrow [x_1 \vee (x_2 \vee x_3)]$$

does not satisfy the hypothesis of the theorem as there is no subformula containing x_1 and x_3 but not x_2; it is nevertheless Q-valid. For let $\langle q_1, q_2, q_3 \rangle = \mathbf{q}$ be a triple of elements in the partial Boolean

algebra belonging to $D_{\varphi,3}$ (φ being the polynomial corresponding to the formula). We then have $q_1 \mathbin{\overset{|}{\circ}} q_2$, $q_2 \mathbin{\overset{|}{\circ}} q_3$, $(q_1 \vee q_2) \mathbin{\overset{|}{\circ}} q_3$, $q_1 \mathbin{\overset{|}{\circ}} (q_2 \vee q_3)$; from this we obtain $(q_1 \vee q_2) \mathbin{\overset{|}{\circ}} (q_2 \vee q_3)$. Any two of the three elements $q_1, q_2, q_4 = q_2 \vee q_3$ are commeasurable and we have $(q_1 \vee q_2) \vee q_4 = q_1 \vee (q_2 \vee q_4)$. As we also have $q_2 \vee q_4 = q_2 \vee q_3$ we obtain

$$(q_1 \vee q_2) \vee (q_2 \vee q_3) = q_1 \vee (q_2 \vee q_3).$$

In exactly the same way we prove

$$(q_1 \vee q_2) \vee (q_2 \vee q_3) = (q_1 \vee q_2) \vee q_3.$$

The simplest example (known to us) of a C-valid formula which is not Q-valid is

$$[(x_1 \leftrightarrow x_2) \leftrightarrow (x_3 \leftrightarrow x_4)] \leftrightarrow [(x_1 \leftrightarrow x_4) \leftrightarrow (x_2 \leftrightarrow x_3)].$$

The proof is by considering the same algebra and the same observables as in Section 4 for the corresponding formula with $+$ instead of $\leftrightarrow$.

7. We shall now axiomatize the set of Q-valid formulas. Most axiom systems of the classical propositional calculus (e.g., Principia Mathematica) consist of Q-valid formulas. Modus ponens, however, does not hold for the notion of Q-validity: There are Q-valid formulas $\alpha, \alpha \rightarrow \beta$ such that β is not Q-valid. (A refutation of β is given by a sequence of observables for which α cannot be evaluated.)

Let Σ be the set of formulas of the propositional calculus in the variables $x_1, x_2, \ldots$ and the connectives $\vee$ and $\neg$. (As before, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are thought of as being defined.) Let Σ^* be the set of formulas of Σ and the formulas of the type $\mathbin{\overset{|}{\circ}}(\alpha_1, \ldots, \alpha_n)$ where n is a positive integer and $\alpha_1, \ldots, \alpha_n$ are formulas of Σ. We define the following notion: "The sequence $\gamma_1, \ldots, \gamma_m$ of formulas of Σ^* is a Q-proof of the formula α of Σ." The notion of Q-proof is reduced to the following notion: "The sequence $\gamma_1, \ldots, \gamma_m$ of formulas of Σ^* is Φ-admissible." (Φ will be subset of Σ.) A sequence is then a Q-proof if it is (α)-admissible and contains α as one of its formulas. $((\alpha)$ is the unit set of α.)

The notion of admissibility is based on rules of inference.

$$\mathrm{R}_1. \quad \frac{\mathbin{\overset{|}{\circ}}(\alpha_1, \ldots, \alpha_n)}{\mathbin{\overset{|}{\circ}}(\alpha_i, \alpha_j)} \qquad (\text{where } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n).$$

$$R_2.\quad \frac{\mathring{\delta}(\alpha_1, \alpha_1),\ \mathring{\delta}(\alpha_1, \alpha_2),\ \ldots,\ \mathring{\delta}(\alpha_i, \alpha_j)\ \ldots,\ \mathring{\delta}(\alpha_n, \alpha_n)}{\mathring{\delta}(\alpha_1, \ldots, \alpha_n)}.$$

(The premiss consists of the n^2 formulas $\mathring{\delta}(\alpha_i, \alpha_j)$ such that $1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n$.)

$$R_3.\quad \frac{\mathring{\delta}(\alpha_1, \alpha_2),\ \alpha_2 \leftrightarrow \alpha_3}{\mathring{\delta}(\alpha_1, \alpha_3)}.$$

$$R_4.\quad \frac{\mathring{\delta}(\neg \alpha_1, \alpha_2)}{\mathring{\delta}(\alpha_1, \alpha_2)}.$$

$$R_5.\quad \frac{\mathring{\delta}(\alpha_1, \alpha_2, \alpha_3)}{\mathring{\delta}(\alpha_1 \vee \alpha_2, \alpha_3)}.$$

$$S_1.\quad \frac{\mathring{\delta}(\alpha_1, \ldots, \alpha_n)}{\beta(\alpha_1, \ldots, \alpha_n)}.$$

where $\beta(x_1, \ldots, x_n)$ is a C-valid formula.

$$S_2.\quad \frac{\alpha_1,\ \alpha_1 \rightarrow \alpha_2}{\alpha_2}.$$

(This form of modus ponens is of course different from the one mentioned at the beginning of Section 7.) S_1 is a scheme of schemes; it can be replaced by a finite number of ordinary schemes (e.g., schemes corresponding to the axioms of Principia Mathematica).

A sequence $\gamma_1, \ldots, \gamma_n$ of formulas of Σ^* is Φ-admissible iff the following conditions are satisfied:

1°. Φ is a subset of Σ.

2°. For all i, $1 \leqslant i \leqslant n$, γ_i is either of the type $\mathring{\delta}(\beta_1, \beta_1)$ or $\mathring{\delta}(\beta_1, \beta_2)$ (where $\beta_1, \beta_1 \vee \beta_2$ are subformulas of a formula $\alpha \in \Phi$) or there exist indices $i_1, \ldots, i_m$ such that $1 \leqslant i_k < i$ $(k = 1, \ldots, m)$ and γ_i follows from $\gamma_{i_1}, \ldots, \gamma_{i_m}$ by one of the above rules.

Remarks.

(1) The notion $\mathring{\delta}(\alpha_1, \ldots, \alpha_n)$ can be eliminated from our axiom system by replacing the formula $\mathring{\delta}(\alpha_1, \ldots, \alpha_n)$ by a Q-identity in $\alpha_1, \ldots, \alpha_n$ containing all formulas $\alpha_i \vee \alpha_j$ $(1 \leqslant i < j \leqslant n)$ as subformulas.

(2) In the rules of inference and in deductions the "variables" $x_1, x_2, \ldots$ are considered as parameters and hold fixed throughout the argument. So they should perhaps rather be called "constants".

8. Theorem. *There exists a Q-proof for a formula α of Σ if and only if α holds in all partial Boolean algebras.*

1°. Assume that there exists a Q-proof $\gamma_1, \ldots, \gamma_m$ of α (formula in $x_1, \ldots, x_n$), let $\mathfrak{B}$ be a partial Boolean algebra, let $\langle q_1, \ldots, q_n \rangle$ be a sequence of elements in the domain of definition of φ (the polynomial associated with α). We show by induction on i: If γ_i is a formula of Σ, then $\langle q_1, \ldots, q_n \rangle$ is in the domain of definition of the corresponding Boolean polynomial χ_i and $\chi_i^*(q_1, \ldots, q_n) = \mathbf{1}$ (γ_i has therefore no other variables than $x_1, \ldots, x_n$!); if γ_i is a formula $\mathring{\delta}(\beta_1, \ldots, \beta_k)$, then $\langle q_1, \ldots, q_n \rangle$ is in the domain of the corresponding polynomials $\psi_1, \ldots, \psi_k$ and the elements $\psi_i^*(q_1, \ldots, q_n)$ are all in relation $\mathring{\delta}$. Clearly the statement is true if γ_i is $\mathring{\delta}(\alpha_1, \alpha_1)$ or $\mathring{\delta}(\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_1 \vee \alpha_2$ are subformulas of α. If the statement is true for the formulas in the premiss of a rule, it is also true for the conclusion. Let us verify as an example modus ponens (rule S_2): If φ_1, φ_2 are the polynomials associated with the formulas α_1, α_2, the induction hypothesis applied to $\alpha_1, \alpha_1 \to \alpha_2$ gives

$$\varphi_1^*(q_1, \ldots, q_n) = \mathbf{1},$$

$$\neg\, \varphi_1^*(q_1, \ldots, q_n) \cup \varphi_2^*(q_1, \ldots, q_n) = \mathbf{1}.$$

Clearly

$$\neg\, \varphi_1^*(q_1, \ldots q_n) = \mathbf{0},$$

$$\mathbf{0} \cup \varphi_2^*(q_1, \ldots, q_n) = \varphi_2^*(q_1, \ldots, q_n) = \mathbf{1}.$$

2°. Assume that there does not exist a Q-proof of the formula α of Σ. Our aim is to construct a partial Boolean algebra in which α does not hold.

Throughout the remainder of this section α is a fixed formula of Σ; we assume that α contains exactly the variables $x_1, \ldots, x_n$. A formula of Σ^* is called "α-provable" iff there exists an (α)-admissible sequence containing it. We state some simple lemmas on the notion of α-provability.

(a) *If α_1 is α-provable, so is $\mathring{\delta}(\alpha_1, \alpha_1)$.*

(b) *Let Ω be the subset of formulas β of Σ such that $\mathring{\delta}(\beta, \beta)$ is α-provable. Formulas of Ω contain no other variables than $x_1, \ldots, x_n$. $x_1, \ldots, x_n$ and α are formulas of Ω.*

(c) *The relation "$\alpha_1 \leftrightarrow \alpha_2$ is α-provable" is an equivalence relation on the set Ω. This equivalence relation is compatible with the operations $\vee$, $\neg$ and the relation $\mathring{\delta}$.*

(d) *If α_1 and $\alpha_1 \leftrightarrow \alpha_2$ are α-provable so is α_2.*

(e) *If α_1 and α_2 are α-provable, so is $\alpha_1 \leftrightarrow \alpha_2$.*

We now define a partial Boolean algebra $\langle B; \downarrow_{\circ}; \vee, \neg; \mathbf{1}, \mathbf{0}\rangle$ associated with the formula α. Elements of B are the equivalence classes of the relation "$\alpha_1 \leftrightarrow \alpha_2$ is α-provable". The equivalence classes are composed in the obvious way: $[\alpha_1] \vee [\alpha_2]$ is the class of $\alpha_1 \vee \alpha_2$, etc. $[\alpha_1] \downarrow_{\circ} [\alpha_2]$ if and only if $\downarrow_{\circ}(\alpha_1, \alpha_2)$ is α-provable. $\mathbf{1}$ is the class of α-provable formulas. We have $q \downarrow_{\circ} \mathbf{1}$ for all q as for every formula β of Ω the formula $\downarrow_{\circ}(\beta, \beta \leftrightarrow \beta)$ is α-provable. $\mathbf{0}$ is the class $\neg \mathbf{1}$. The axioms of partial Boolean algebras are easily verified.

Let q_i be the class of the formula x_i $(i = 1, \ldots, n)$; if β is any formula of Ω, ψ the Boolean polynomial associated with β, the class of β is the element $\psi^*(q_1, \ldots, q_n)$, i.e., β is α-provable if and only if $\psi^*(q_1, \ldots, q_n) = \mathbf{1}$. The formula α is therefore Q-provable if and only if $\varphi^*(q_1, \ldots, q_n) = \mathbf{1}$ in the partial Boolean algebra just defined.

Remark. The completeness theorem can easily be extended to a theorem on the completeness of the rules $R_1, \ldots, R_5$, S_1, S_2. Let Γ be a subset of Σ^*, $\gamma \in \Sigma^*$; then γ follows from Γ by the rules $R_1, \ldots, R_5$, S_1, S_2 and $\downarrow_{\circ}(x_i, x_i)$ $(i = 1, 2, \ldots)$ $(\Gamma \vdash \gamma)$ iff γ is a consequence of Γ in all partial Boolean algebras $\mathfrak{B}$. (γ is a consequence of Γ in B if γ is defined and true for all sequences $\langle q_1, \ldots, q_n\rangle$ of elements of B for which all the formulas of Γ are defined and true.) A formula $\alpha \in \Sigma$ is therefore Q-provable iff $\Gamma(\alpha) \vdash \alpha$, where $\Gamma(\alpha)$ is the set of commeasurability relations of the subformulas of α.

9. Let S be a set, S_0 a subset of S, let $\vee$ be a binary operation $S^2 \to S$ and let $\neg$ be a unary operation $S \to S$. The algebra $\mathfrak{S} = \langle S, S_0; \vee, \neg\rangle$ is called a "truth-table"; S_0 is the set of designated elements. A formula α in the variables $x_1, \ldots, x_n$ defines a map $\alpha^*: S^n \to S$; α holds in $\mathfrak{S}$ iff α^* maps S^n into S_0.

Theorem. *There exists a truth-table $\mathfrak{S}$ with a two-element S_0 such that a formula is Q-valid iff it holds in $\mathfrak{S}$.*

Proof (in outline). 1°. We construct (e.g., by an infinite direct product) a partial Boolean algebra $\mathfrak{B}$ such that a formula is Q-valid iff it holds in $\mathfrak{B}$. 2°. We define the truth-table $\mathfrak{S}$ as follows: $S = B \cup (u)$ (where $u \notin B$), $S_0 = (1, u)$; the operations $\neg'$ and $\vee'$ in $\mathfrak{S}$ are defined by putting $\neg' q = \neg q$ for $q \in B$, $\neg u = u$; $q_1 \vee' q_2 = q_1 \vee q_2$ for $q_1, q_2 \in B$ and $q_1 \downarrow_{\circ} q_2$, $q_1 \vee' q_2 = u$ otherwise. One verifies that a formula holds in $\mathfrak{S}$ iff it holds in $\mathfrak{B}$.

Remark. There does not exist a truth-table $\mathfrak{S}$ with only one designated element such that a formula is Q-valid iff it holds in $\mathfrak{S}$.

Theorem. *If all Q-valid formulas hold in the truth-table* $\mathfrak{S}$ *having three (or less) elements then all C-valid formulas hold in* $\mathfrak{S}$.

By a theorem in Section 6 the above theorem (suggested by a question of Leon Henkin) is an immediate consequence of the following:

Theorem. *If all C-valid formulas in one or two variables hold in the truth-table* $\mathfrak{S}$ *having three (or less) elements then all C-valid formulas hold in* $\mathfrak{S}$.

Proof. 1°. Assume that the algebra $\mathfrak{S}$ is generated by a two-element subset: There exist elements $a, b \in S$ and a Boolean polynomial φ (corresponding to a formula α) such that $S = (a, b, \varphi^*(a, b))$. Let $\beta(x_1, \ldots, x_n)$ be a formula in the n variables $x_1, \ldots, x_n$ which does not hold in $\mathfrak{S}$. Let $\langle q_1, \ldots, q_n \rangle$ be a sequence such that $\beta^*(q_1, \ldots, q_n) \notin S_0$. Define a formula $\gamma(x_1, x_2)$ as follows: $\gamma(x_1, x_2)$ is $\beta(\xi_1, \ldots, \xi_n)$ where ξ_i is x_1, x_2, or $\alpha(x_1, x_2)$ according to whether q_i is a, b or $\varphi^*(a, b)$. γ does not hold in S because $\gamma^*(a, b) = \beta^*(q_1, \ldots, q_n) \notin S_0$. The formula γ is therefore not C-valid and neither is β.

2°. Assume that the algebra $\mathfrak{S}$ is not generated by a two-element subset of S. Then $\neg x = x$ for all $x \in S$ and $x \vee y = x$ or $x \vee y = y$ for all $x, y \in S$. Every element $x \vee y$ is designated; for if, e.g., $x \vee y = x$, x is a value of $(x_1 \vee x_2) \vee \neg x_2$ which is C-valid. Every C-valid formula is of the form $\neg \ldots \neg (\beta_1 \vee \beta_2)$ and holds therefore in $\mathfrak{S}$.

The above theorem does not contradict Reichenbach [44] which connects "quantum-logic" and three-valued logic because the notions of validity involved are different.

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MODEL-THEORETIC INVARIANTS: APPLICATIONS TO RECURSIVE AND HYPERARITHMETIC OPERATIONS

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Introduction. Both in informal mathematical reasoning and in foundations one sometimes *identifies* objects a and a' in two structures $\mathfrak{A}$, $\mathfrak{A}'$ (of the same similarity type) whatever the 'nature' of the a, a' themselves may be. Thus, a mathematician does not distinguish between the rationals in one commutative field of characteristic 0 and any other. Or, if Φ is an axiom system of arithmetic with the individual constant 0 and the function constant S, and $\Phi \vdash \forall y(Sy \neq 0) \wedge \forall x \forall y(Sx = Sy \rightarrow x = y)$, then $S^n 0$, also denoted by $\overline{n}$, is taken to denote the *same* object in each model $\mathfrak{A} = \langle A, 0, S \rangle$ of Φ. In contrast, the term $\iota_x[(x = 0 \wedge \alpha) \vee (x = S0 \wedge \neg \alpha)]$ would not if α is a formally undecided sentence since it denotes 0 in $\mathfrak{A}$ if $\models_{\mathfrak{A}} \alpha$ and $S0$ if $\models_{\mathfrak{A}} \neg \alpha$ (and $S0 \neq 0$). In terms of the notion: a in A and a' in A' *correspond*, one defines in an obvious way which elements are common to all structures in a class $\mathscr{A}$.

Our main interest is to see what ideas are implicit in this informal notion of correspondence. In Section 1 we give two analyses in model-theoretic terms, applicable to wide classes of structures, and coinciding with the informal notion in familiar cases. The principal general result establishes (for the case of general models) sharp explicit definability for each element of the common part. Since one of the main properties brought out in our analyses is that the common part of $\mathscr{A}$ does not admit automorphisms, we call it the *hard core* of $\mathscr{A}$.

The most interesting applications come about through the following link with recursion theory. The class of recursive sets of natural numbers made its first appearance in the literature in Gödel [31], not in connection with Church's problematic thesis, but under the name of 'entscheidungsdefinite Eigenschaften'. In the spirit of the times this notion was formulated proof-theoretically, but it has a clear model-theoretic content. One considers Φ (containing $\overline{0}, \overline{1}, \ldots$) and calls $\alpha(x)$ *entscheidungsdefinit* provided, for all $\overline{n}$, $\Phi \vdash \alpha(\overline{n})$ or $\Phi \vdash \neg \alpha(\overline{n})$. Thus, granted that $\overline{n}$ denotes

the *same* object in each model $\mathfrak{A}$ of Φ so that $\mathfrak{N}$ is the hard core of $\{\mathfrak{A}: \models_{\mathfrak{A}} \Phi\}$, the set $\mathfrak{N} \cap \{x: \models_{\mathfrak{A}} \alpha(x)\}$ is *independent* of $\mathfrak{A}$, and α is a uniform invariant definition of it; more precisely, invariant *on* $\mathfrak{N}$, since for (absolute) invariance, $\{x: \models_{\mathfrak{A}} \alpha(x)\}$ itself would have to be independent of $\mathfrak{A}$. (It turns out that the latter sets are *finite* sets of natural numbers.)

If $\mathbf{C}$ is the hard core of a class $\mathscr{A}$ of structures there is then an unambiguous extension of the notions of invariance, of invariance *on* $\mathbf{C}$, and of several related notions set out in Section 2. The extensions are meaningful for arbitrary classes of structures, but, of course, dependent on the *language* used in the definitions, e.g., whether finite or infinite formulae α are considered.

The principle involved in the applications is simply that familiar from ordinary mathematical analyses of concrete situations in abstract terms. Thus, we have

(a) Generalization of known results about some familiar cases; given results about general models (e.g., the finiteness theorem), or about recursive and recursively enumerable sets of natural numbers (e.g., maximal simple sets of Friedberg [58]), one reformulates them by replacing some or all occurrences of 'finite' by 'invariantly definable in $\mathscr{A}$', and 'recursive' by 'invariantly definable *on* $\mathbf{C}$ in $\mathscr{A}$', and tries to prove the results for unfamiliar $\mathscr{A}$.

(b) Analyses of the familiar results. Since in any particular application distinct general notions may coincide, one has to analyse which general notions are involved in a special case. [Thus, for systems of arithmetic, $\mathbf{C} = \mathfrak{N}$ and the collection of finite subsets of $\mathbf{C}$ has the following properties: it has cardinal $\aleph_0$ $(= \bar{\bar{\mathbf{C}}})$; it can be mapped into (and onto) $\mathbf{C}$ by a function invariantly definable on $\mathbf{C}$; each element of the collection has cardinal $< \aleph_0$ (and so $< \bar{\bar{\mathbf{C}}}$), is invariantly definable, bounded, i.e., included in an invariantly definable set, *and* conversely.] In particular, different proofs of the same theorem in the familiar case may generalize to proofs of different theorems. This is beautifully illustrated in the case of various maximal simple set constructions; cf. Kreisel-Sacks [a].

Our principal illustrations concern ω-models of countable sets of axioms, i.e., typically, models satisfying the infinite formula $(\forall x \in \omega)\,(x = \bar{0} \vee x = \bar{1} \vee \ldots)$, and recursion theory on the recursive ordinals, though extensions to other segments of the ordinals are unambiguous. Not unexpectedly, the theory becomes particularly neat if one uses languages with suitable infinitely long expressions, *provided one restricts*

the syntax not by simple-minded purely external conditions (e.g., cardinality of the expressions), *but by conditions on* the satisfaction relation $\models_{\mathfrak{A}} \alpha$ *between the pair of objects* $\mathfrak{A}$ *and* α (cf. Discussion, Section 2.2).

While this general model-theoretic approach is implicit in Gödel [31], the discovery of close relations between ω-models and the concept of hyperarithmeticity is due to Mostowski, cf. Grzegorczyk-Mostowski-Ryll-Nardzewski [58] and the polished exposition of Mostowski [62]. Our basic new point is that in the earlier theory one did not analyze just why the numerals played a special role in general models for formal systems of arithmetic, and simply compared subsets of $\mathfrak{N}$ which are invariantly definable on $\mathfrak{N}$ in general and ω-models, respectively. In terms of the present terminology, this neglects the fact that $\mathfrak{N}$ itself is (absolutely) invariantly definable in ω-models. but not in general models; so, sets definable *on* $\mathfrak{N}$ in ω-models are automatically (absolutely) invariantly definable, but not in general models. In short, in the old theory the notions of hard core and absolute invariance were neglected. Some modifications needed for a smooth theory were first introduced in Kreisel [61]. (The partial success of the old theory is explained in 2.1, Lemma 2, below: some elementary results hold for sets definable *on* an *arbitrary part* of the hard core.)

In connection with relative recursiveness and invariant definability from the diagram of Y, $Y \subset \mathbf{C}$ (relative representability in a class of models of Φ, of Mostowski [62]) there is an important distinction between two types of models of Φ which collapses for general models but not for ω-models (Definition 2(c), Lemma 2, Proposition C below). This point enters into our revision of Spector's generalization [55] of Post's problem; cf. Section 3.2.

1. Basic Notions. In 1.1 the notion of common part is formulated abstractly and applied in 1.2 to a series of important classes $\mathscr{A}$. For these also a simpler formulation of the informal notion is meaningful. Suppose we have some kind of 'standard' maximum (concrete) structure $\mathfrak{A}_0$ and consider substructures $\mathfrak{A} \in \mathscr{A}$ ($\mathfrak{A}_0$ need not $\in \mathscr{A}$). Then the common part may literally be defined to be the set-theoretic intersection of all the universes of those $\mathfrak{A}$. This simpler idea is sufficient for many applications, where it coincides with the general notion.

1.1. Let $\mathscr{A}$ be a class of structures satisfying Φ, possibly containing individual constants (which the structure must specify).

Definition. $\mathfrak{A}_1$, *of the similarity type of* $\mathfrak{A}$, *is* rigidly *contained in*

$\mathfrak{A}$, *iff there is a* unique *function* F *from* A_1 *into* A, *such that the restriction of* $\mathfrak{A}$ *to the range* $\tilde{A}$ *of* F *is isomorphic to* $\mathfrak{A}_1$. (*We continue to use* $\tilde{A}$ *in this sense below*; $\tilde{A}$ *is closed under the operations of* $\mathfrak{A}$ *and contains the objects of* A *denoted by the constants of* Φ.)

Corollary. *Let* $\mathbf{C}$ *denote the set of constants of* Φ *and let* $\mathbf{C}_1$ *be an arbitrary labeling of* $A_1 - \mathbf{C}$. *Then* $\mathfrak{A}_1$ *is rigidly contained in all* $\mathfrak{A} \in \mathscr{A}$ *if and only if* (*i*) *in each* $\mathfrak{A}$ *the diagram* Δ_1 *of* $\mathfrak{A}_1$ *is satisfiable,* (*ii*) *if* $\mathbf{C}_1'$ *is a congruent labeling to* $\mathbf{C}_1$, *i.e., obtained from* $\mathbf{C}_1$ *by replacing* $\mathsf{c} \in \mathbf{C}_1$ *by* $\mathsf{c}^* \in \mathbf{C}_1'$ (* *not in the language of* Φ) $\Delta_1, \Delta_1^* \models_{\mathfrak{A}} \mathsf{c} = \mathsf{c}^*$.

(i) expresses that $\mathfrak{A}_1$ is contained in $\mathfrak{A}$; (ii) expresses rigidity.

Theorem. *Let* $\mathscr{A}$ *be the class of general models of* Φ. *If* $\mathfrak{A}_1$ *is rigidly contained in* $\mathscr{A}$, *let* $c_{\mathfrak{A}}$ *be the element of* $\mathfrak{A}$ *corresponding to* c *in* A_1. *Then, for each* $c \in A_1$, *there is a quantifier-free formula* $\Delta_c(x, \vec{y})$ ($\vec{y}$ *standing for a finite sequence*), *such that* (i) $c_{\mathfrak{A}}$ *is the only object* $\models_{\mathfrak{A}} \exists \vec{y} \Delta_c(x, \vec{y})$, *i.e., uniformity,* (ii) $\models_{\mathfrak{A}} (\forall x)[(\exists \vec{y})\Delta_c(x, \vec{y}) \leftrightarrow (\exists \vec{y} \in \tilde{A}^{\omega})\Delta_c(x, \vec{y})]$ (*i.e.,* $\tilde{A}$ *is a* basis *for the existential formula*). (iii) *Further, all basic operations and relations are* decided *by* Φ *for these explicit definitions as arguments.*

Proof by compactness, using rigidity; decidability follows because the full diagram is satisfied in each $\mathfrak{A}$.

Definition. $\mathscr{A}$ *possesses a hard core iff there is a maximal structure rigidly contained in each* $\mathfrak{A} \in \mathscr{A}$.

1.2. To analyse the informal notion of *common part* (c.p.) of $\mathscr{A}$ we use the following evident properties:

A1. The hard core of $\mathscr{A}$ is (isomorphic to) a substructure of c.p.,

A2. C.p. is a substructure of each $\mathfrak{A} \in \mathscr{A}$ (though not necessarily *rigidly* contained),

A3. C.p. is a monotone decreasing function of $\mathscr{A}$.

By (A3), c.p. is a substructure of each $\mathfrak{A}$: $\mathfrak{A} \subset \mathfrak{A}_0 \wedge \mathfrak{A} \in \mathscr{A}$, i.e. of the *set theoretic intersection* considered in 1 above. Below are some examples of intrinsic interest, for which hard cores and intersections are determined for suitable $\mathfrak{A}_0$. In this way it is decided whether A1–A3 determine c.p.

(a) The concrete structure ('general directed magnitude'): the set of *complex numbers* with *addition* and *multiplication*. (i) Commutative fields of characteristic 0; intersection: real rationals. (ii) Real closed fields;

intersection: real algebraic numbers. (iii) Algebraically closed fields of transcendence degree $\geqslant 1$; intersection: algebraic complex numbers. Note that in (i), (ii) the intersections satisfy the axioms mentioned, and are isomorphic to the hard cores of (the general models of) (i), (ii) resp., but not in (iii).

Going over to arbitrary models, the common parts of (i), (ii) are determined by (A1–3) to be isomorphic to the respective hard cores, but not in (iii). Intuitively in the case of two algebraically closed fields, one would not identify $\sqrt{2}+i$ in $\mathfrak{A}$ and $\sqrt{2}+i$ in $\mathfrak{A}'$ since there is nothing to distinguish $\sqrt{2}+i$ and $\pm\sqrt{2}\pm i$: only *sets* of solutions of an irreducible equation are identified. Evidently automorphisms are to be avoided. The hard core of (iii) consists of the real rational numbers and thus agrees with the informal notion of common part.

(b) The concrete structure [set in the sense of the cumulative theory of types built up from the empty set: a model is standard if it consists of (well-founded) sets and is transitive]: *sets with the* $\in$ *relation*, $\emptyset$, and the *operations* $a \cup b$ and $\{a\}$. The following axioms are specially suited to the present problems. (i) Extensionality, the usual 'definitions' of the constants, the (predicatively) restricted replacement axiom, i.e.,

$$(\forall a)[(\forall x \in a)\exists! y \alpha(x, y) \rightarrow (\exists z)(\forall y)[y \in z \leftrightarrow \exists x(x \in a \wedge \alpha(x, y))]],$$

where α may contain parameters, but any quantifiers in $\alpha(x, y)$ are restricted to be elements of a, x, y or the parameters. (ii) In addition to (i), the axiom of union, and of infinity (asserting the existence of the set of all finite von Neumann ordinals): let $\pi(b)$ express closure of b under predecessor, i.e., $\forall x(x \cup \{x\} \in b \rightarrow x \in b)$, then:

$$(\exists \omega)(\emptyset \in \omega \wedge \forall a[a \in \omega \leftrightarrow (\forall b)\{[\pi(b) \wedge a \in b] \rightarrow \emptyset \in b\}]).$$

The intersection of (i) consists of just the hereditarily finite sets (also for the full replacement axiom); if the structure only contained $\in$ ($\emptyset$, $\{\}$, $\cup$ defined) the intersection would be empty because the hereditarily finite sets built up from any set would also be a model. The intersection of (ii) consists of the hereditarily hyperarithmetic sets of rank $< \omega_1$ (the first non-recursive ordinal). (Essentially Theorem 2 (i) of GKT [1] and Kreisel [61], but proved more simply by the methods of GMR [1] or Section 2 below.)

Going over to general models, by extensionality, each model is iso-

[1] GKT = Gandy-Kreisel-Tait [60]; GMR = Grzegorczyk-Mostowski-Ryll-Nardzewski [61].

morphic to a structure of (not necessarily well-founded) sets where $\in$, $\emptyset$, $\cup$, $\{\}$ have standard meaning. Considering only those sets that *are* well-founded, even (ii) has as intersection only the hereditarily finite sets. But without this condition, *each general model of* (ii) *contains a subset which is isomorphic to the union of* ω, *the set* ϱ_1 *of all recursive subsets of* ω, *the set* ϱ_2 *of all subsets of* ϱ_1 *with partial recursive characteristic function, etc., and closed under union* [2]. For $X \in \varrho_1$, if α is any invariant definition of X *on* ω, $\forall y[y \in x \leftrightarrow y \in \omega \wedge \alpha(y)]$ is an explicit definition of an element $X_{\mathfrak{A}}$ of $\mathfrak{A}$ corresponding to X. Note: $X_{\mathfrak{A}}$ contains in general 'non-standard' elements of $\omega_{\mathfrak{A}}$: $\omega \cap X_{\mathfrak{A}} = X$, but $X_{\mathfrak{A}} \neq X$; also, for given X and $\mathfrak{A}$, $X_{\mathfrak{A}}$ is not uniquely determined by X; finally, the definition is not *uniform* because for given α, $X_{\mathfrak{A}} \neq X_{\mathfrak{A}'}$. Thus the hard core is a proper part of this common substructure [cf. the notion of *prime model*; but the structure described is not a model of (ii)].

Dana Scott has shown (by use of Theorem 1.1) for *arbitrary* consistent extensions of (i): if the extension is consistent with Zermelo-Fraenkel's axioms *plus* regularity the hard core consists precisely of the hereditarily finite sets, but not, e.g., if $\exists!x(x=\{x\})$ is a theorem of the extension.

An ω-model of (ii) [or (i)] is, by definition, a model in which the denotation of ω consists precisely of the hereditarily finite sets built up from the empty set [3]. Now the hard core of all ω-models of (ii) consists precisely of the hereditarily hyperarithmetic sets of rank $<\omega_1$, and the same remains true if one adds to (ii) any Π_1^1 set of axioms in the notation (which have an ω-model at all), by Theorem 2 (ii) of GKT, for those elements of the hard core which *are* of rank $< \omega_1$. Furthermore any such element is explicitly definable uniformly for all ω-models.

(c) The concrete structure: *the ordinals, with the ordering relation* $<$, and *binary relations* P_n (defined by transfinite induction), 0 and *successor*

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[2] Compare the notion of GKT: the set $X^2_{\mathfrak{A}}$ represents in $\mathfrak{A}$ the subset X^2 of $\mathfrak{P}(\omega)$ (and for higher types). Here strong uniqueness conditions are required. To each $X' \subseteq \omega$ there corresponds in $\mathfrak{A}$ the (unique) class $\{X: X \in \mathfrak{A} \wedge X \cap \omega = X'\}$; for representable X', and all $\mathfrak{A} \in \mathscr{A}$, this is not empty. Then, for representable X', either $X^2_{\mathfrak{A}}$ contains the whole class or is disjoint from it. These conditions imply continuity of the characteristic functions of representable sets, and thus, e.g., (non-empty) finite sets of finite subsets of ω are not representable; cf. Feferman's review of GKT in J. Symb. Logic.

[3] This does not mean that *all* sets in an ω-model of (ii) are well-founded though, of course, ω is. Models satisfying this stronger condition are called β-models ('β' for bon-ordre) by Mostowski. The Shepherdson-Cohen minimal models make up the hard core of the β-models for ordinary set theory.

S; a model is standard if it is an initial segment of the ordinals. The following axioms are useful. (i) Total order, successor axioms, schema of proof by transfinite induction for arbitrary α: $\forall x[(\forall y<x)\alpha(y)\cdot\rightarrow\alpha(x)]\rightarrow$ $\rightarrow\forall x\alpha(x)$; definition by transfinite recursion: suppose all quantifiers in α_n are bounded by x or y, and α_n does not contain P_n; then $\forall x\forall y[P_n(x,y)\leftrightarrow\alpha_n^{[x]}(P_n,x,y)]$ is an axiom where $\alpha_n^{[x]}$ is obtained from α_n by replacing $P(t,z)$ by $x\leqslant t\vee P(t,z)$, i.e., $\lambda yP_n(x,y)$ is defined from $\lambda z<x\lambda yP_n(z,y)$; the supremum: if all quantifiers in $\alpha(x,y)$ are bounded, $(\forall a)[(\forall x<a)\exists y\alpha(x,y)\cdot\rightarrow\exists z(\forall x<a)(\exists y<z)\alpha(x,y)]$. (ii) In addition to (i) the axiom of infinity $(\exists\omega)(\forall x<\omega)\{Sx\neq\omega\wedge[x=0\vee(\exists y<x)(Sy=x)]\}$.

The intersection of (i) consists of the finite ordinals, and that of (ii) the recursive ordinals.

Going over to general models, every model is isomorphic to one in which each element whose set of predecessors has ordinal σ, *is* the (abstract) ordinal σ. (Other elements can in general not be replaced by the abstract order type of their predecessors because a non-well-ordering may permit automorphisms and so different elements may define the same order type.) Considering only the ordinals, the intersection of (ii) is still only ω because we can consistently add the set of axioms $a_1<\omega$, $a_2<a_1$, ..., $a_{n+1}<a_n$, ...

The *hard core* of all consistent recursively enumerable extensions of (i) is also ω, by use of provably disjoint recursively enumerable sets which are not recursively separable, and decidability of the P_n on the hard core.

An ω-model of (ii) is one in which ω *is* ω. The hard core of all ω-models consists of the recursive ordinals, with P_n determined uniquely. The same remains true on addition of an arbitrary Π_1^1 set of axioms (in the present notation) provided they have an ω-model. Each recursive ordinal α can be explicitly defined, e.g., by defining first an ordering of the finite ordinals with initial segment of type ω_1, as in Gandy [60], and then a functional relation $P_n(x,y)\wedge y<\omega$, $\forall x\exists!yP_n(x,y)$, mapping the ordinals $<\alpha<\omega_1$ into the segment and preserving order [4].

2. Invariants. The theorem of 1.1 gives results on uniform definability of singletons. Now we are concerned with *subsets* of the hard core which can be defined (in one sense or another) in each structure $\mathfrak{A}\in\mathscr{A}$. Let $\mathfrak{C}$

[4] To study the hard core of a (transfinite) type theory in which variables are explicitly supplied with (ordinal) types, one has to combine the considerations of (b) and (c).

be a set of constants intended to define elements of the hard core; but the results obtained will use only the weaker condition: for distinct $c, c' \in \mathbf{C}$, $\Phi \models \mathsf{c} \neq \mathsf{c}'$. One may therefore suppose that the symbols c denote themselves, and so the letters C and $\mathbf{C}$ may be interchanged.

Throughout, $X \subseteq \mathbf{C}$, $C_1 \subseteq \mathbf{C}$, $C_2 \subseteq \mathbf{C}$ and c denotes an element of $\mathbf{C}$, φ a formula with one free variable. (The extension to $X \subseteq \mathbf{C}^n$ is immediate, by use of formulae φ with n free variables. If needed, $\vec{c}$ denotes an element of C^n.)

Definitions. *X is* invariantly *or* uniformly (a) *defined*, (b) *defined* on C_1, *by φ in $\mathscr{A}$ iff*:

(a) $(\forall \mathfrak{A} \in \mathscr{A})[X = \{a : a \models_{\mathfrak{A}} \varphi\}]$, (b) $(\forall \mathfrak{A} \in \mathscr{A})[X = C_1 \cap \{a : a \models_{\mathfrak{A}} \varphi\}]$.

X is (a′) *definable*, (b′) *definable on C_1, in $\mathscr{A}$ iff*:

(a′) $(\forall \mathfrak{A} \in \mathscr{A})\exists \varphi[X = \{a : a \models_{\mathfrak{A}} \varphi\}]$, (b′) $(\forall \mathfrak{A} \in \mathscr{A})(\exists \varphi)[X = C_1 \cap \{a : a \models_{\mathfrak{A}} \varphi\}$, *where, as usual, φ may contain constants for arbitrary elements of A.*

For $Y \subseteq C_2$, X is invariantly defined from Y on C_1 *by φ, containing a predicate letter* Υ *not in the language of Φ, iff*:

(c) $(\forall \mathfrak{A} \in \mathscr{A})(\forall Y' \subseteq A)[Y' \cap C_2 = Y \rightarrow X = C_1 \cap \{a : a, Y' \models_{\mathfrak{A}} \varphi\}]$.

(Evidently, there are notions corresponding to, e.g., (b′), non-uniform definability, etc.)

Remark. If Φ is a set theory, it is *not* assumed that Y' or Y is necessarily an object in the universe of $\mathfrak{A}$. Thus, if Φ contains the comprehension axiom, the letter Υ does not appear in it. In short, Y is regarded as a *property*, not a *set*. More formally: $a, Y' \models_{\mathfrak{A}} \varphi$ should be written: $a, Y' \models_{\mathfrak{A}'} \varphi$ where $\mathfrak{A}'$ is the structure obtained from $\mathfrak{A}$ by adding an additional relation type.

Correspondingly, in 1.2 (b) and (c): if Y is regarded as a set, Υ may appear in α in the replacement and supremum axioms resp., but not if Y is regarded as a property. Note that in the former case the hard core may be altered, but not in the latter.

Some notions of *implicit* definability corresponding to the above explicit ones are these:

(ai) Let the set Ψ contain the predicate symbol $\times$, and possibly others Υ, not contained in Φ. Ψ is a *uniform implicit definition* of X in $\mathscr{A}$ if and only if:

$$(\forall \mathfrak{A} \in \mathscr{A})(\forall X' \subseteq A)[X = X' \leftrightarrow (\exists \vec{Y})(X', \vec{Y} \models_{\mathfrak{A}} \Psi)].$$

(a'i) X is implicitly definable in (each structure of) $\mathscr{A}$ iff:

$$(\forall \mathfrak{A} \in \mathscr{A})(\exists \psi)(\forall X' \subseteq A)[X = X' \leftrightarrow \exists \vec{Y}(X', \vec{Y} \models_{\mathfrak{A}} \psi)]$$

where ψ is a *single* formula.

(bi) ψ is a *uniform implicit definition* of X *on* C_1 iff:

$$(\forall \mathfrak{A} \in \mathscr{A})(\exists X', \vec{Y})(X', \vec{Y} \models_{\mathfrak{A}} \psi) \wedge \forall X'[(\exists \vec{Y}')(X', \vec{Y} \models_{\mathfrak{A}} \psi) \cdot \rightarrow X = X' \cap C_1].$$

Note that in contrast to (ai), X itself need not satisfy $(\exists \vec{Y})(X, \vec{Y} \models_{\mathfrak{A}} \psi)$.

Terminology. For $\mathbf{C} = \omega$, (b) is usually called: reckonable, strongly representable, (strongly) definable, or binumerable. (a) is not generally considered. (bi) is (a slight generalization of) Fraïssé's [61a] relative recursiveness when Φ is the diagram of the basic relations (his: $R_1, \ldots, R_p$ on p. 320) on $\mathbf{C}$ (his: E). In his Section 5, Fraïssé's intended meaning is almost certainly better expressed in terms of models whose universe is E itself (p. 323, l. 5), than by his formal definition.

2.1. *Technical lemmas on general models.* Most of the results below are easy, known, and/or given in Kreisel-Krivine [a], Chapter V. Φ is an arbitrary (possibly uncountable) consistent set of sentences in ordinary first-order language, and $\mathscr{A}$ its class of general models.

Lemma 1. *If X is definable in $\mathscr{A}$, uniformly or not, explicitly or implicitly, X is a* finite *subset of* $\mathbf{C}$.

Lemma 2. For $\mathbf{C}_1 \subset \mathbf{C}$, *if X is definable* on $\mathbf{C}_1$ *in $\mathscr{A}$, there is a φ and a set Φ_1, of cardinality $< \aleph_0 + \Phi$, such that for all c (of course: not all a) and $\mathfrak{A} \in \mathscr{A}$,*

$$c \in X \leftrightarrow [\Phi, \Phi_1 \vdash \varphi(c)] \text{ and } c \notin X \leftrightarrow [\Phi, \Phi_1 \vdash \neg \varphi(c)].$$

Thus, (i) for countable Φ, Φ_1 is finite. (ii) If (the set of Gödel numbers of) Φ is recursively enumerable in some set $\mathfrak{N}_0$ ($\subseteq \mathfrak{N}$), X is recursive in $\mathfrak{N}_0$ *on* $\mathbf{C}_1$, say $\subseteq \mathfrak{N}$; here as usual, X is called: recursive *on* $\mathbf{C}_1$, if there are two recursively enumerable sets Y_1, Y_2 which are not necessarily complementary on $\mathfrak{N}$ itself, such that $\mathbf{C}_1 \cap Y_1 = \mathbf{C}_1 \cap \complement Y_2 = X$. (iii) An example of Keisler shows that, in general, Φ_1 cannot be dropped even for countable Φ, and one of Dana Scott that Φ may have to be infinite for uncountable Φ; but in these examples the hard core is empty. Lemma 2 shows that a definable set is almost invariantly definable, and an analo-

gous argument establishes the analogous result for implicit definability. (iv) If Φ is a Rosser theory, in the sense of Smullyan [61], p. 135, extended to Φ that are not recursively enumerable, a definable set is invariantly definable.

(ii) extends immediately to relative invariant definability from Y. The collapse of the distinction made after Definition 2(c), for the case of *general* models, follows from:

Lemma 3. *Let Φ contain at least second-order type theory in which every finite subset of C exists, i.e., $\Phi \models_{\mathfrak{A}} (\exists S)\forall x[S(x) \leftrightarrow (x = \mathsf{c}_{n_1} \vee \ldots \vee x = \mathsf{c}_{n_p})]$. Suppose Υ does not occur in Φ, $Y \subseteq C$ and $\Psi = \{\Upsilon(\mathsf{c}) : c \in Y\} \cup \{\neg\Upsilon(\mathsf{c}) : c \notin Y\}$. Then the addition of γ: $\exists S\forall x[S(x) \leftrightarrow \Upsilon(x)]$ to $\Phi \cup \Psi$ is conservative.*

If Φ, Ψ, $\gamma \vdash \alpha$ then also Φ, Ψ_1, $\gamma \vdash \alpha$ for some finite subset Ψ_1 of Ψ. Replace $\Upsilon(t)$ throughout by the disjunction of $t = \mathsf{c}$, $\Upsilon(\mathsf{c}) \in \Psi_1$ to form ψ_1', γ'. Then $\Phi \vdash \psi_1' \wedge \gamma'$, and hence, if α does not contain Υ, $\Phi \vdash \alpha$.

2.2. *Technical lemmas on ω-models.* We consider ω-models only for systems of set theory and of ordinals (cf. Section 1.2) and only *countable* Φ, the latter because the ω-rule is not complete for uncountable Φ. For more general definitions and counterexamples, cf. Kreisel-Krivine [a].

We first consider sets of finite formulae and then examine for what infinite formulae the arguments are valid. Since finite ordinals are used in the arithmetization of the syntax of finitary language, we are first concerned with sets of integers: for $X \subseteq \omega$, X *is (uniformly) definable* on ω *if and only if* X *is (uniformly) definable,* since, if $\alpha(x)$ defines X *on* ω, $\alpha(x) \wedge x \in \omega$, resp. $\alpha(x) \wedge x < \omega$ defines X in all ω-models considered.

Lemma 2 of 2.1 holds for ω-models of *countable* Φ by GMR. Consequently, either directly by Kreisel [61], p. 113, or by the completeness of the ω-rule:

If Φ is Π_1^1 in (= inductively defined from) a set $X_0 \subseteq \omega$, and $X \subseteq \omega$ is definable in all ω-models of Φ, then X is hyperarithmetic in X_0.

The following propositions show that, *in contrast to general models, the class of invariantly definable sets* (of integers) *depends on* Φ, and similarly for related notions.

Proposition A. (i) *If $\Phi \in \Pi_1^1$, the invariantly definable sets are hyperarithmetic, but* (ii) *there are $\Phi \in \Sigma_1^1$ such that* all Π_1^1 *subsets of* ω *are invariantly definable (in contrast to Lemma* 1).

Proof: (i) by GMR, (ii), e.g., by Kreisel [61], p. 119 (a).

Proposition B. *Let Φ be ordinary axiomatic second-order arithmetic not containing the letters X, X' and let X be non-hyperarithmetic, Φ_1 the diagram of X for X, Φ_1' its diagram for X'. Then (i) $\Phi \cup \Phi_1 \cup \Phi_1' \cup \cup\{\forall S \exists x[S(x) \leftrightarrow \neg X(x)]\}$ has an ω-model, (ii) its extension by $(\exists S)\forall x [S(x) \leftrightarrow X'(x)]$ does not.*

Proof: (i) by the basis lemma of GKT, (ii) obvious.

This contrasts with Lemma 3; but though of course (i) assures the existence of all *finite* or even *hyperarithmetic* subsets of ω (the comprehension axiom is provable for them), *it does not assure the existence of all sets $\subseteq \omega$ which are invariantly definable in* (i). A simple computation shows this about invariant definability *from* X: let X be O (the set of ordinal notations), and Φ as in Proposition B; then the class of sets invariantly definable in $\Phi \cup \Phi_1$ from (the property) O can be enumerated by a function hyperarithmetic in O, but all functions hyperarithmetic in O are invariantly definable in $\Phi \cup \Phi_1 \cup \{\exists S \forall x[S(x) \leftrightarrow X(x)]\}$ from (the set) O by Mostowski [62; II, 5, 6]. Thus for ω-models, the distinction (cf. the remark in the first part of Section 2) between definability from properties and sets is real.

Proposition C. *If Φ is Π_1^1 in X_0 ($X_0 \subseteq \omega$) then the ω-rule applied to Φ closes off after $\leqslant \omega_1^{X_0}$ steps, where $\omega_1^{X_0}$ is the least ordinal not invariantly definable from X_0 (in all ω-models.)*

Proof as in Theorem 3 of Spector [61], where it is stated for absolutely Π_1^1 sets Φ. It is nearly stated in Mostowski [62], p. 41, II. 5.3 with $\omega_1^{|\Phi|}$ ($|\Phi|$ the characteristic function of Φ) instead of the generally smaller $\omega_1^{X_0}$. The improvement expresses that a rule uses only the occurrence of a formula in a set of axioms, not its absence.

Recall that the ordinary logical rules (for the consequence relation of general models) close off after ω steps, and note that, in the case of general models, for all X_0, ω is the least ordinal not invariantly definable from X_0.

Discussion. To extend, e.g., Lemma 1, whose proof depends on the finiteness theorem for general models, it is necessary to extend the latter to ω-models. Propositions A–C (are set out to) suggest that the relevant property of finiteness is invariant definability in general models and this turns out to be true (cf. next section). Part of Lemma 1 concerns the step from (invariant, possibly implicit) definability to *uniform* explicit definability, and therefore involves essentially a choice of

language. Inspection of, e.g., the proof of Proposition A in Kreisel [61], shows: all that is needed of the formulae Φ is that the satisfaction relation ('the structure described by F satisfies φ') be Σ_1^1 in F and (the Gödel number of) φ and X_0. By use of countable models, F may be taken to be number-theoretic functions. This is satisfied not only by finite formulae, but also infinite formulae φ, e.g., infinite disjunctions and finite strings of quantifiers, provided, regarded as syntactic objects, they are hyperarithmetic in X_0; the collection of such φ is countable, and thus the φ may be given *finite* Gödel numbers. (NB: One cannot rely on simple-minded syntactic conditions, e.g., $\forall x_1 \ldots \forall x_n \ldots [\neg R(x_1, x_2) \vee \neg R(x_2, x_3) \vee \ldots]$ expresses that R is well-founded, hence the satisfaction relation is Π_1^1, but the syntactic object is a simple $\omega + \omega$ sequence. Dana Scott [*], in a related context, introduced the restriction to finite sequences of quantifiers for purely pragmatic reasons; but there too the analysis in terms of the satisfaction relation seems to be the correct theoretical foundation.) Going back to (countable) general models of first-order formulae Φ, a model F (on ω) may be taken as given by a *set* F of number-theoretic functions, namely the characteristic functions of the relations in Φ, and of the Skolem functions of Φ; then the relation 'F satisfies Φ' ($\leftrightarrow$ F satisfies a certain infinite set of propositional formulae) is Π_1^0 in F and Φ. From this point of view, Mostowski's proof of the finiteness theorem in [61] generalizes naturally to ω-models (though in 3.1 we use a different method); only the final step of his proof needs elaboration because, in the case of general models, one can, and he does, take advantage of special compactness properties of continuous mappings from 2^ω into a discrete space.

3. Applications. Only some points of principle are considered.

3.1. *Theory of ω-models.* Theorem. *Suppose Φ is Π_1^1 in X_0 and every $\Phi_1 \subseteq \Phi$, which is hyperarithmetic in X_0, has an ω-model. Then Φ itself has an ω-model.*

Corollary. *If $\Phi \vdash_\omega \varphi$, then there is a $\Phi_1 \subseteq \Phi$, Φ_1 hyperarithmetic in X_0, such that $\Phi_1 \vdash_\omega \varphi$.*

(It seems to be open whether relative hyperarithmeticity, i.e. definability from the *set* X_0, can be replaced by definability from the *property* X_0.)

Proof. Since Φ is Π_1^1 in X_0 there is an arithmetic functional K from $\mathfrak{P}(\omega)$ to $\mathfrak{P}(\omega)$, $\forall X[K(X) \supset X]$ and Φ is the least class $\supset X_0$: $K(\Phi) = \Phi$.

If, for all ordinals α, $X^0 = X_0$, $\alpha \neq 0$: $X^\alpha = \bigcup_{\beta<\alpha} K(X^\beta)$, $X^\alpha = \Phi$ for $\alpha \geqslant \omega_1^{X_0}$ by Spector's [61] Theorem 3. Now let Ω be the (arithmetic) functional mapping sets X of formulae into the set of all logical consequences of X together with a *single* application of the ω-rule. Again $\Omega(X) \supset X$; let the closure be $Cn_\omega(X)$. Consider the sequence of *pairs* of sets of formulae:

$\langle \Phi^0, X^0 \rangle = \langle X_0, X_0 \rangle$; for $\alpha \neq 0$, $\langle \Phi^\alpha, X^\alpha \rangle = \langle X^\alpha \cup \bigcup_{\beta<\alpha} \Omega(\Phi^\beta), \bigcup_{\beta<\alpha} K(X^\beta) \rangle$.

The closure rule is still arithmetic; by above, for $\alpha \geqslant \omega_1^{X_0}$, $X^\alpha = \Phi$ and so $\Phi^\alpha \supset \Phi$. By completeness of the ω-rule, $\Phi^\alpha \supset Cn_\omega(\Phi)$, for all $\alpha \geqslant \omega_1^{X_0}$. So, if $\Phi \vdash_\omega \varphi$, $\varphi \in \Phi^\alpha$ for some $\alpha < \omega_1^{X_0}$. Since, for $\alpha < \omega_1^{X_0}$, $\Phi^\alpha \subset Cn_\omega(X^\alpha)$, and Φ^α is hyperarithmetic in X_0, the theorem is proved.

In view of this generalization we prefer to speak of the finiteness theorem for general models, and not of: compactness theorem. For, though compactness is used in most proofs (and shortens them), a generalization to non-compact cases is possible, as above.

The use of the theorem simplifies and improves a good deal of GKT. Also it brings out more clearly the relation between (i) my construction, in Kreisel [61], p. 120, (ii), of a Π_1^1 set Φ (extension of ordinary second-order arithmetic) which is not ω-equivalent to any $\Pi_1^1 \cap \Sigma_1^1$ (sub)set and (ii) non-finite axiomatizability of ordinary arithmetic. (Of course, if no special conditions are imposed on Φ, one can take any non-hyperarithmetic Π_1^1 set X and put $\Phi = \{\mathsf{P}(\bar{n}) : n \in X\}$ imitating the construction of infinite sets not equivalent to finite ones for general models.)

3.2. *Recursion theory on the recursive ordinals.* The following two languages are suitable for isolating important areas of classical recursion theory.

(a) *Elementary theory of recursively enumerable sets.* The only constants are $\emptyset$, $=$, $\subseteq$ and one kind of variable. In the interpretations, $\emptyset$, $=$, $\subseteq$ have the usual meaning. We consider three interpretations (for the range of the variables):

(i) Recursively enumerable subsets of ω. For this interpretation, $R_0(x) \underset{\text{def}}{=} (\exists y)(x = \mathbf{C}y)$ defines the recursive sets, and (Lacombe): $\mathrm{Fin}(x) \underset{\text{def}}{=} (\forall y)[y \subseteq x \to R_0(y)]$ the finite ones. Non-trivial parts of recursion theory can be formulated in this language such as decomposition of recursively enumerable sets or existence of a maximal simple set, Friedberg [58], but cf. (b) below.

In terms of model-theoretic invariants (for *general* models of the theory of ordinals), recursive functions are those invariantly definable on ω; a recursively enumerable set is defined as the *range* of such a function. We consider ranges included (ii) in the hard core ($=\omega$ for general models), (iii) in ω, and go over to ω-models (of Π_1^1 sets of axioms).

(ii) Π_1^1 subsets of ω_1 [5]. Now $R_0(x)$ defines the sets hyperarithmetic *on* ω_1, and $\mathrm{Fin}(x)$ the finite ones.

(iii) Π_1^1 subsets of ω. Now, $R_0(x)$ defines the hyperarithmetic subsets of ω, and $\mathrm{Fin}(x)$ the finite ones. This coincides with Mostowski [62].

It is not impossible that all three interpretations are equivalent for this language. H. Rogers suggested during the symposium that the three classes of sets may be isomorphic with respect to inclusion, and R. L. Vaught that the theory might be decidable.

(b) *Elementary theory of monadic recursive functions.* The language contains two kinds of variables ($x, y, \ldots$ for individuals and $f, g, \ldots$ for monadic functions), constants 0, $=$ and application $[f(x)]$.

(i) Universe of individuals $=\omega$, of functions: number-theoretic recursive functions. (a) (i) above can be interpreted here by defining $f \subseteq g: \forall x \exists y[f(x)=g(y)]$, and so the definitions of R_0 and Fin carry over. Alternative definitions here are: $R_0{}^+(f) \underset{\text{def}}{\Leftrightarrow} (\exists g)(\forall x)\{g(x)=0 \leftrightarrow (\exists y)[f(y)=x]\}$, $F(f) \underset{\text{def}}{\Leftrightarrow} (\forall g)$ [the range of the function g restricted to the range of f is $R_0{}^+$]. $B(f) \underset{\text{def}}{\Leftrightarrow} \exists g[f \subseteq g \wedge F(g)]$ here also defines the finite ($=$bounded recursively enumerable) sets.

By use of inverse pairing functions the theory of n-ary recursive functions can be interpreted in the present language.

Model-theoretically, we again regard the universe of individuals as (ii) the hard core, (iii) ω, the functions as those (ii) invariantly definable on the hard core, (iii) on ω, and go over to ω-models (of Π_1^1 sets of axioms).

(ii) Universe of individuals $=\omega_1$, of functions: those invariantly definable *on* $\omega_1 \cdot \forall f[R_0{}^+(f) \leftrightarrow R_0(f)]$, but $\neg \forall f[F(f) \leftrightarrow \mathrm{Fin}(f)]$ since $F(f)$ defines the absolutely hyperarithmetic subsets of ω_1, and $\neg \forall f[B(f) \rightarrow F(f)]$.

[5] What is meant is: subsets of ω_1, which are the range of some function which is invariantly definable *on* ω_1, in the sense of the present paper. The notation is chosen because any such set X can also be defined as follows: If W is the initial segment of a recursive ordering of ω of type ω_1, e.g., as in Gandy [60], and $|n|$ the ordinal of the initial segment determined by n (in the field of W), then $X=\{|n|: n \alpha X'\}$ where X' is a Π_1^1 subset of ω.

Thus by using F, Fin, B, respectively, in the three places where 'finite' occurs in the maximal simple set theorem, one gets 27 formally distinct generalizations! For details, cf. Kreisel-Sacks [a].

(iii) Universe of individuals $=\omega$, of functions: those (whose graph is) invariantly definable *on* ω, i.e., the (absolutely) hyperarithmetic number-theoretic functions. The range of every such function is hyperarithmetic and thus $\forall f F(f)$ (and not all Π_1^1 subsets of ω can be enumerated). So the theory of Π_1^1 sets cannot be interpreted here by the definition a(iii) above.

Relative recursiveness (between recursively enumerable sets X and Y). For general models of recursively enumerable axiom systems Φ, containing a minimum Φ_0 of arithmetic, the following relations are equivalent: (1) X is invariantly definable from Y *on* the common part of all models of Φ, Y being regarded as a set, (2) with Y being regarded as a property, (3) if German lower case letters denote invariantly definable subsets of of the common part [i.e. enumerated by f: $F(f)$], there is g: $R_0(g)$, $n \in X \leftrightarrow (\exists \mathfrak{a} \subset Y)(\exists \mathfrak{a}' \subset \mathbf{C}Y)[g(n, \mathfrak{a}, \mathfrak{a}') = 1]$, $n \notin X \leftrightarrow (\exists \mathfrak{a} \subset Y)(\exists \mathfrak{a}' \subset \mathbf{C}Y)$ $[g(n, \mathfrak{a}, \mathfrak{a}') = 2]$, (4) as (3) with $n \in X$ replaced by $\mathfrak{n} \subset X$, $n \notin X$ by $\mathfrak{n} \subset \mathbf{C}X$, $g(n, \mathfrak{a}, \mathfrak{a}')$ by $\mathfrak{g}(\mathfrak{n}, \mathfrak{a}, \mathfrak{a}')$.

Note that for recursively enumerable sets X, Y the relations (3), (4) can be expressed in the language (b) above for the interpretations b(i), b(iii). – The existence of Φ_0 shows that the relations (1) and (2) are invariant invariants! i.e., the same relation is defined for each $\Phi \supset \Phi_0$ considered. – The common part of all models of Φ *together* with the diagram of Y (on ω) is still ω, even if Y is considered as a set.

For ω-models and Π_1^1 sets Φ. By Mostowski [62], (1) is an invariant invariant, but the argument does not apply to (2) for all Y (on ω_1). Also, the use of infinite formulae does not alter (1), but, in general, increases (2). By definition the common part of all ω-models of Φ is not altered if Y is regarded as a property, but in general is increased if Y is regarded as a set. – Trivially (1) $\supset$ (2) $\supset$ (3) $\supset$ (4); the inclusions are proper. (1) $\supset$ (2) by p. 187, (2) $\supset$ (3) by recent work of Driscoll since (2) is transitive and (3) is not, (3) $\supset$ (4) because a simple computation shows that (4) is transitive.

For each of the relations (1)–(4) Post's problem generalizes to this [for the interpretation b(ii)]. To find Π_1^1 sets of recursive ordinals X, Y which are incomparable for the relation considered. – Friedberg's proof [58] generalises directly for (3) and hence for (4), Kreisel-Sacks [a]. It

is open for (2). A satisfactory solution for (1) requires X, Y with the property that the common part of all ω-models of Φ_0 *together* with the diagrams on ω_1 of X, Y resp., be still ω_1. (Cf. *generic* sets of Cohen [63, 64].) (*Added in proof.* The existence of such X and Y has been established by Sacks, in his *Metarecursively enumerable sets and admissible ordinals*, forthcoming in Bull. Amer. Math. Soc.)

The generalization above conflicts with Spector's [55] on two counts. First he uses Π_1^1 subsets of ω, instead of ω_1, second the relation (1), which is equivalent to relative hyperarithmeticity for all Π_1^1 systems Φ containing Φ_0.

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SEMANTICAL ANALYSIS OF MODAL LOGIC II. NON-NORMAL MODAL PROPOSITIONAL CALCULI

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This paper continues the investigations of Kripke [63]. Not only is Kripke [63] presupposed; the reader is advised to have it on hand for ready reference. The notations and terminology of that paper are used freely; in particular, $P, Q, R, \ldots$ are atomic formulae, and $A, B, C, \ldots$ are arbitrary formulae built from them using the connectives $\wedge$, $\sim$, $\Box$. All propositional calculi in this paper have the same formation rules as those of Kripke [63]. However, they in general will lack the rule of necessitation of Kripke [63] and thus will not be normal. Hence (in our opinion), they are intuitively somewhat unnatural; but nevertheless they have an elegant model theory. Among these systems are, notably, Lewis's S2 and S3; this paper extends the results of Kripke [63] to these and other systems. The results of this paper were announced in Kripke [63, abstract].

All systems considered here will be formulated with axiom schemata; substitution is a derived rule.

1. Generalities; Halldén's property. By R1, or modus ponens, we mean the rule: $A, A \supset B / B$.

Theorem 1. *Let* PL *be any propositional calculus containing the theorem schema* $A \supset (A \wedge \ldots A)$ *and its converse* $(A \wedge \ldots A) \supset A$. *If modus ponens is an admissible (derivable) rule of* PL, PL *can be recursively axiomatized with modus ponens as sole rule of inference.*

Proof. (Essentially given in Craig [53].) Form PL′ as follows: Axioms: $(A \wedge \ldots A) \supset A$ (any number of conjuncts A), and $B \wedge \ldots B$ (m conjuncts), where m is the Gödel number of a proof of B in PL. R1 is the only rule. Then clearly PL contains PL′, and PL′ contains PL because if B is any theorem of PL and m is the Gödel number of a proof thereof, $B \wedge \ldots B$ and $B \wedge \ldots B \cdot \supset \cdot B$ (m conjuncts) are both provable in PL′, whence B is by R1. Q.E.D.

S2, S3, and most systems considered below have an "incompleteness" property of Halldén [51]: some formula $A \vee B$ is derivable, where A and B have no propositional variables in common and neither A nor B is derivable. Halldén essentially showed that no system with his property can possess a normal characteristic matrix.

Theorem 2. *Let* PL *be a propositional calculus containing all classical tautologies and with* R1 *as sole postulated rule. Let A and B be formulae with no propositional variables in common. Suppose $A \vee B$ is provable, and let* PL′ (PL″) *be formed by adding all substitution instances of A (of B) as a new axiom schema to* PL. *Then* $\mathrm{PL} = \mathrm{PL}' \cap \mathrm{PL}''$.

Proof. Let C be a theorem of PL′ and of PL″. We need to derive it in PL. Since $\vdash C$ in PL′, there are finitely many instances $A_1, \ldots, A_m$ of A from which C is derivable in PL′. Hence, by the deduction theorem, $\vdash A_1 \wedge \ldots A_m \cdot \supset \cdot C$ in PL. Similarly, $\vdash B_1 \wedge \ldots B_n \cdot \supset \cdot C$ in PL for certain substitution instances $B_1, \ldots, B_n$ of B. So we need only derive $A_1 \wedge \ldots A_m \cdot \vee \cdot B_1 \wedge \ldots B_n$ in PL. By relettering B if necessary, we can assume that $A_1, \ldots, A_m$, as well as A, contain no variables in common with B; no matter how B is thus relettered, we can still derive $A \vee B$ in PL. Since A and B have no propositional variables in common, by substitution, we get $\vdash A_i \vee B$ in PL $(i = 1, \ldots, m)$; hence $\vdash A_1 \wedge \ldots A_m \cdot \vee \cdot B$. Since in this latter formula the disjuncts still have no variable in common, substituting again we get $\vdash (A_1 \wedge \ldots A_m) \vee B_j$ $(j = 1, \ldots, n)$, and hence $\vdash (A_1 \wedge \ldots A_m) \vee (B_1 \wedge \ldots B_n)$. Q.E.D.

Theorem 3. *If* PL *is a propositional calculus, it possesses a normal characteristic matrix if and only if the following four conditions hold:*

(*a*) PL *is consistent (i.e.,* $\dashv A \wedge \sim A$).
(*b*) R1 *is an admissible rule of* PL.
(*c*) PL *contains all classical tautologies.*
(*d*) PL *does not possess Halldén's property.*

The necessity of the conditions is clear. The proof of sufficiency, which we omit, proceeds by constructing a normal characteristic matrix by Lindenbaum's method.

Although we are assuming that all propositional calculi have the formation rules of Kripke [63], it is clear that the theorems above hold for a wider class of formation rules. Further, Theorem 2 applies, e.g., to systems containing the Hilbert-Bernays positive logic (in place of classical logic).

2. Propositional calculi considered. All systems considered contain all tautologies of classical propositional logic as axioms; thus these axioms will not be listed explicitly [1].

The system E2 of Lemmon [57] is characterized by the axioms:

A1. $\Box A \supset A$,

A3. $\Box (A \supset B) \cdot \supset \cdot \Box A \supset \Box B$,

and the rules R1 and

(Eb) If $\vdash A \supset B$, then $\vdash \Box A \supset \Box B$.

We get a system E3 if we replace A3 in E2 by the stronger axiom:

(1) $\Box (A \supset B) \supset \Box (\Box A \supset \Box B)$.

It is easy to show that (Eb) is equivalent to $A \supset B / \Diamond A \supset \Diamond B$.

Now by Theorem 1, E2 and E3 can be reformulated with R1 as sole rule of inference. Alternatively, we can get more natural reaxiomatizations of these two systems by dropping the rule (Eb), and replacing it by the clause: if B is an axiom and A is any formula, $\Box A \supset \Box B$ is an axiom. Then, with R1 as sole postulated rule, we can obtain $B / \Box A \supset \Box B$ as a derived rule, and thence (Eb) (by the inference: $A \supset B / \Box A \supset \Box (A \supset B)$, hence $\Box A \supset \Box B$ by A3 and truth-functional logic) [2]. Given a formulation of E2 (E3) with R1 as sole (primitive) rule, define S2 (S3) as the result of adding to this formulation the schema $\Box (A \supset A)$. Further, for a formulation of E2 with R1 as sole rule, define $S2^n$ by adding thereto:

(2) $\Box^n (A \supset A) \qquad (0 \leqslant n < \infty)$.

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[1] This proviso was inadvertently neglected in Kripke [63]. In connection with Kripke [63], Footnote 2 on p. 69, we should mention that an outline of Hintikka's work has since appeared as Hintikka [61]; and that the "pupils" of T. J. Smiley mentioned in that footnote can now be more accurately specified as Francis Drake.

[2] We can prove that if B is derivable, so is $\Box A \supset \Box B$, by induction on the length of a proof of B in the reaxiomatized version of E2 (E3). If B is an axiom, the desired result is clear. If B is derived from C and $C \supset B$ by R1, we can assume by the inductive hypothesis that $\Box A \supset \Box C$ and $\Box A \supset \Box (C \supset B)$ are derivable. By axiom A3 and truth-functional logic, we obtain $\vdash \Box A \supset (\Box C \supset \Box B)$ from the second of these; and from this together with the first we get the desired $\vdash \Box A \supset \Box B$.

Note that the rule $B / \Box A \supset \Box B$ is a trivial consequence of (Eb) (since $A \supset B$ is derivable from B). So the reaxiomatized version of E2 (E3) is clearly *contained* in E2 (E3). Above we indicated how to show that it *contains* E2 (E3).

Here $\Box^n$ indicates a sequence of n necessity signs. Thus $S2^0$ is E2; $S2^1$ is S2.

Now if A is a theorem of E2 (E3), so is $A \supset A \cdot \supset \cdot A$, hence by (Eb) $\vdash \Box (A \supset A) \supset \Box A$. Hence $\vdash \Box A$ in S2 (S3). Similarly (using n applications of (Eb)), $\Box^n(A \supset A) \supset \Box^n A$ is a theorem of E2, so by R1 $\Box^n A$ is a theorem of $S2^n$. In Theorem 6 below we will show conversely that if $\vdash \Box^n A$ in $S2^n$, then $\vdash A$ in E2.

If we define $S2^\infty$ by adding (2) for all n as schemata to a formulation of E2 with R1 as sole rule, it is easily shown that if B is an axiom of $S2^\infty$, $\vdash \Box B$ in $S2^\infty$ (for every axiom is either $\Box^n(A \supset A)$ or a theorem of E2). Hence by induction on the length of a proof of B in $S2^\infty$,[3] $\vdash B$ in $S2^\infty$ implies $\vdash \Box B$; and thus $S2^\infty$ is equivalent to Feys-von Wright's M (cf. Kripke [63]).

Notice that in E3, since $\vdash \Box B \supset \Box (A \supset B)$, by two applications of (1) to the consequent, we get $\vdash \Box B \supset \Box (\Box \Box A \supset \Box \Box B)$, and hence $\vdash \Box \Box A \cdot \supset \cdot \Box B \supset \Box \Box B$. Hence it can be shown that, if we defined $S3^2$ by adding (1) to $S2^2$, $S3^2$ would collapse into S4. (Some intuitive motivation for the facts that $S3^2 = S4$, $S2^\infty = M$ will become apparent from the model theory below.)

The systems $S2^n$, E3, and S3 will be discussed, model-theoretically, below. In the rest of this section, we mention other systems, and their relations to the previous systems and to each other. The Falsum system is the system with R1 as sole rule and the following single axiom:

(3) $\sim \Box A$.

Axiom (3) reduces "$\Box$" to a *falsum* operator. It can easily be verified that E2 and E3 admit an interpretation with "$\Box$" as the *falsum* operator. So does Lemmon's E4, whose rules are R1 and (Eb) and whose axioms are A1, A3, and

A4. $\Box A \supset \Box \Box A$.

Lemmon's E5 collapses into S5; the present writer has proposed instead a system E5 defined by A1, A3,

A2. $\Box A \cdot \supset \cdot \sim \Box B \supset \Box \sim \Box B$,

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[3] We have just handled the case where B is an axiom. If B is derived by R1 from C and $C \supset B$, we can assume by inductive hypothesis that $\Box C$ and $\Box (C \supset B)$ are derivable. From the second of these we get by A3 $\vdash \Box C \supset \Box B$, hence using $\Box C$ and R1, we obtain the desired $\vdash \Box B$.

and R1 and (Eb). This system is still contained in the Falsum system.

Now since $\vdash B \cdot \supset \cdot A \supset A$, by (Eb) $\vdash \Box B \supset \Box (A \supset A)$ in E2; *i.e.*, $\vdash \Box (A \supset A) \vee \sim \Box B$. Hence, applying Theorem 2 to E2 (E3) reformulated with R1 as sole rule, E2 (E3) is the intersection of S2 (S3) with the Falsum system. Theorem 2 further implies that E4 (E5) is analogously related to S4 (S5).

We remark that all systems so far considered (except the Falsum) are contained in the Trivial system, which is based on the axioms $\Box A \supset A$ and $A \supset \Box A$. R1 is the only rule.

The Łukasiewicz Ł-modal system is axiomatized by the axioms A1 and

(4) $A \supset B \cdot \supset \cdot \Box A \supset \Box B$.

R1 is the only rule (cf. Łukasiewicz [53]). Curry's Gentzen rules for possibility in Curry [52] turn out to be equivalent to the Ł-modal system, although by the time Curry [52] was published, he no longer advocated this system as natural. Another formulation of the system has been rediscovered by Jean Porte. The Ł-modal system is contained in both the Trivial and the Falsum systems. In fact, we easily derive, from $\vdash B \supset \cdot A \supset B$ and (4), $\vdash B \cdot \supset \cdot \Box A \supset \Box B$, hence $\vdash \sim \Box A \cdot \vee \cdot B \supset \Box B$. So by Theorem 2, the Ł-modal system is the intersection of the Trivial and the Falsum systems, a fact first proved by Prior [57]. Thus the Ł-modal system contains all the E-systems.

3. **Models.** Any attempt to develop a model theory for E2 and E3 must face the problem that no formula of the form $\Box B$ is provable in E2 or E3, not even $\Box (A \supset A)$. If, as in Kripke [63], however, we were to define a formula $\Box B$ as true in a "world" $\mathbf{H}_1$ iff B is true in every $\mathbf{H}_2$ "possible relative to" $\mathbf{H}_1$, and evaluated the truth-functions by truth-tables, then since $A \supset A$ would clearly be true in any world $\mathbf{H}_2$, $\Box (A \supset A)$ would have to be true in any world $\mathbf{H}_1$. Similar difficulties arise in S2 and S3 (in $S2^n$), owing to the fact that $\Box^2(A \supset A)(\Box^n(A \supset A))$ is unprovable.

We observe, however, that $\vdash \Box B \supset \Box (A \supset A)$ in E2, and hence $\vdash \sim \Box (A \supset A) \supset \sim \Box B$; if $A \supset A$ is not necessary, then nothing is. This leads us to divide the "possible worlds" into two classes; the "normal worlds", in which necessity is evaluated according to the Leibnitzian prescription of Kripke [63]; and the non-normal worlds, in which $\Box B$ is always false.

We define, then, a *model structure* (m.s.) to be an ordered triple $(\mathbf{G}, \mathbf{K}, \mathbf{R})$,

where $\mathbf{K}$ is a set, $\mathbf{G} \in \mathbf{K}$, and $\boldsymbol{R}$ is a relation defined on $\mathbf{K}$ with the following *quasi-reflexivity* property: If $\mathbf{H}_1 \boldsymbol{R} \mathbf{H}_2$ ($\mathbf{H}_1, \mathbf{H}_2 \in \mathbf{K}$), then $\mathbf{H}_1 \boldsymbol{R} \mathbf{H}_1$. An element of $\mathbf{K}$ (a "world") $\mathbf{H}$ such that $\mathbf{H} \boldsymbol{R} \mathbf{H}$ is called *normal*.

(Alternatively, we could define a model structure as a quadruple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R}, \mathbf{N})$, where $\mathbf{K}$ is a set, $\mathbf{G} \in \mathbf{K}$, $\mathbf{N} \subseteq \mathbf{K}$, and $\boldsymbol{R}$ is a relation on $\mathbf{K}$ which is reflexive on $\mathbf{N}$. The normal worlds are the elements of $\mathbf{N}$. This definition leads to a slightly more general class of models at the expense of one more primitive; its advantages will become apparent in Section 7.)

A m.s. is also called an E2 m.s. If $\boldsymbol{R}$ is transitive, a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an E3 m.s. An E2 (E3) m.s. in which $\mathbf{G}$ is normal is called an S2 (S3) m.s.

Temporarily let Q be the following relation between elements of $\mathbf{K}$ and non-negative integers: $\mathbf{H} Q n$ iff either $n = 0$ and $\mathbf{H} = \mathbf{G}$ or $n \geqslant 1$ and there exist $\mathbf{H}_0, \ldots, \mathbf{H}_n \in \mathbf{K}$ such that $\mathbf{H}_0 = \mathbf{G}$, $\mathbf{H}_n = \mathbf{H}$, and $\mathbf{H}_i \boldsymbol{R} \mathbf{H}_{i+1}$ $(0 \leqslant i < n)$. Define the *depth* n of an element $\mathbf{H}$ of $\mathbf{K}$ to be the least n such that $\mathbf{H} Q n$; if no such n exists, we say the depth is *infinite*.

We define an E2 m.s. to be an $\mathrm{S2}^n$ m.s. iff every $\mathbf{H} \in \mathbf{K}$ of depth $< n$ is normal. For $n = 0$ $(n = 1)$, this coincides with the previous notion of E2 (S2) m.s.

An E2 (E3, $\mathrm{S2}^n$, S3) *model* φ for a wff A of E2 (E3, $\mathrm{S2}^n$, S3) is a binary function $\varphi(P, \mathbf{H})$ associated with a given E2 (E3, $\mathrm{S2}^n$, S3) m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$. As in Kripke [63], the first variable P ranges over atomic subformulae of A, while the second variable $\mathbf{H}$ ranges over elements of $\mathbf{K}$. The range of φ is the set $\{\mathbf{T}, \mathbf{F}\}$.

Given a model φ associated with a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, we define for any subformula B of A a value $\varphi(B, \mathbf{H})$. If B is atomic, the value has already been given. If $\varphi(B, \mathbf{H})$ and $\varphi(C, \mathbf{H})$ have been defined, for each $\mathbf{H}$, then define $\varphi(B \wedge C, \mathbf{H})$, $\varphi(\sim B, \mathbf{H})$ as in Kripke [63]. We define $\varphi(\Box B, \mathbf{H}) = \mathbf{T}$ iff $\mathbf{H}$ is normal and $\varphi(B, \mathbf{H}') = \mathbf{T}$ for all $\mathbf{H}'$ such that $\mathbf{HRH}$; otherwise, $\varphi(\Box B, \mathbf{H}) = \mathbf{F}$. (Hence, in particular, $\varphi(\Box B, \mathbf{H}) = \mathbf{F}$ automatically, if $\mathbf{H}$ is not normal.)

As in Kripke [63], a formula A is true in a model φ associated with a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ iff $\varphi(A, \mathbf{G}) = \mathbf{T}$. A is valid (for $\mathrm{S2}^n$, E3, S3) iff it is true in every ($\mathrm{S2}^n$, E3, S3) model. Notice that here, unlike the situation in Kripke [63], the distinguished element $\mathbf{G}$ in a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ plays an essential role in the definitions. For example, in an S2 model φ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, $\varphi(\Box(A \supset A), \mathbf{G})$ must equal $\mathbf{T}$, since $\mathbf{G}$ is normal in an S2 m.s.; hence $\Box(A \supset A)$ is valid in S2.

We can define *connected* models and *tree* models as in Kripke [63]. As in Kripke [63], we could confine ourselves to connected models

without essentially changing the theory. Notice that, in a connected m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, every $\mathbf{H} \in \mathbf{K}$ has a finite depth; and that, in a connected m.s. where $\boldsymbol{R}$ is transitive, every $\mathbf{H} \in \mathbf{K}$ has depth $\leqslant 1$. Hence, if we define a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ to be an $S2^{\infty}$ m.s. if every $\mathbf{H} \in \mathbf{K}$ of finite depth is normal, then a connected $S2^{\infty}$ m.s. is an M m.s. (cf. Kripke [63]). If we define a $S3^2$ m.s. as an E3 m.s. where every $\mathbf{H} \in \mathbf{K}$ of depth <2 is normal, then a connected $S3^2$ m.s. is an S4 m.s. Conversely, under these definitions, an M (S4) m.s. is an $S2^{\infty}$ ($S3^2$) m.s. These facts provide the model-theoretic motivation for the results in Section 2 that $S2^{\infty} = M$, $S3^2 = S4$.

4. Semantic tableaux. We can adopt almost all of the notions of Kripke [63] for tableau constructions without change. Again, we have, at a given stage of the construction, a system of alternative sets of tableaux. Each alternative set $\mathscr{S}$ is ordered by a relation R which now need be only quasi-reflexive. The ordered set $\mathscr{S}$ can, in a natural way, be given the form of a tree; the origin $\mathbf{t}_0$ of the tree is the (descendant in $\mathscr{S}$ of) the main tableau. Replacing $\mathbf{G}$ by $\mathbf{t}_0$, we can define the depth n of a tableau $\mathbf{t}$ in $\mathscr{S}$, just as it was previously defined for a world: the depth n of $\mathbf{t}$ is the least n such that either $n = 0$ and $\mathbf{t} = \mathbf{t}_0$, or $n \geqslant 1$ and there exist $\mathbf{t}_1, \ldots, \mathbf{t}_n$ such that $\mathbf{t}_n = \mathbf{t}$, $\mathbf{t}_i \, R \, \mathbf{t}_{i+1} (0 \leqslant i < n)$.

Now a tableau $\mathbf{t}$ of an alternative set $\mathscr{S}$ is called *normal* iff $\mathbf{t} \, R \, \mathbf{t}$. We require R to be quasi-reflexive: $\mathbf{t}_1 \, R \, \mathbf{t}_2$ implies $\mathbf{t}_1 \, R \, \mathbf{t}_1$. In E3 and S3 constructions, we further require that R be transitive. In S2 and S3 constructions, the main tableau $\mathbf{t}_0$ must be normal. In $S2^n$ constructions any tableau of depth $<n$ is normal. Further, for all systems we require that a tableau with a formula $\Box$ B on the left be normal (since, correspondingly, $\varphi(\Box B, \mathbf{H}) = \mathbf{T}$ implies that $\mathbf{H}$ is normal). In short, as in Kripke [63], the properties assumed for R parallel those of $\boldsymbol{R}$; and, as in Kripke [63], the relation R is to hold in a set only if the preceding requires it to hold.

All rules except Yr can be adopted without change [4] from Kripke [63]. Yr now reads:

Yr. If $\Box A$ appears on the right of a normal tableau $\mathbf{t}$, we start out a new tableau $\mathbf{t}'$, with A on the right, and such that $\mathbf{t} \, R \, \mathbf{t}'$.

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[4] For *Ar*, the italicized condition in Kripke [63], p. 73, included there to ensure reflexivity of R, should be dropped in favor of: "*Further if* $\mathbf{t} \, R \, \mathbf{t}$, *we stipulate that* $\mathbf{t}_1 \, R \, \mathbf{t}_1$."

Notice that Yr, as stated, actually renders the "requirement" of quasi-reflexivity superfluous. For if $\mathbf{t}\ R\ \mathbf{t}$ is not required for some reason other than quasi-reflexivity (e.g., occurrence of some formula $\Box B$ on the left, or being of depth $<n$ in an $S2^n$ construction), then Yr will forever be inapplicable to $\mathbf{t}$, so that no tableau $\mathbf{t}'$ with $\mathbf{t}\ R\ \mathbf{t}'$ will ever be introduced, and thus quasi-reflexivity will hold vacuously. Hence, in an E2 or E3 (an S3, an $S2^n$) construction, a tableau $\mathbf{t}$ will be normal iff it has some formula $\Box B$ on the left (either has some formula $\Box B$ on the left or is the main tableau, either has some formula $\Box B$ on the left or is of depth $<n$).

Theorem 4. *A is valid in* E2 (E3, S3, $S2^n$) *iff the construction for A is closed (in the appropriate system).*

The proof of this theorem is exactly like that of the corresponding theorem in Kripke [63], Section 3.2.

4.1. Trees and a reformulation of the rules. As in Kripke [63], each of the alternative sets of a stage of a construction can be given the form of a tree. In fact, if, following Kripke [63], we define $\mathbf{t}_1\ S\ \mathbf{t}_2$ to hold iff $\mathbf{t}_1$ and $\mathbf{t}_2$ are descendants in $\mathscr{S}$ of tableaux $\mathbf{t}_1'$ and $\mathbf{t}_2'$, such that $\mathbf{t}_2'$ was introduced by an application of Yr to $\mathbf{t}_1'$, then S is a tree relation on $\mathscr{S}$. Following Kripke [63], Section 3.3., we raise the question whether the rules can be reformulated in terms of S, in such a way that a rule applied to a tableau $\mathbf{t}$ can affect at most $\mathbf{t}$ and tableaux contiguous to $\mathbf{t}$ (Kripke [63], p. 81). Notice that, if we replace R by S, we can define the depth of a tableau $\mathbf{t}$ in terms of S, rather than R; for $S2^n$ constructions this notion of depth will coincide with the original one (but for E3 and S3, where R is transitive, it will not). Now, in terms of S, we can define a tableau $\mathbf{t}$ of an $S2^n$ construction to be normal iff it contains a formula $\Box A$ on the left or is of depth $<n$, for an E3 construction iff it contains a formula $\Box A$ on the left, and for an S3 construction iff it contains a formula $\Box A$ on the left or is the main tableau of its alternative set. We can then reformulate all rules except Y1 simply by replacing R by S (for Ar we drop the condition intended to ensure quasi-reflexivity; cf. Kripke [63], p. 80). For $S2^n$ Y1 now becomes like Y1 for M in Kripke [63], p. 81:

Y1. Let $\Box A$ appear on the left of $\mathbf{t}$. Then put A on the left of $\mathbf{t}$ and of every tableau $\mathbf{t}'$ such that $\mathbf{t}\ S\ \mathbf{t}'$.

For S3 and E3, we need to reformulate Y1 so as to obtain a surrogate for the transitivity property of R. We base our reformulation on the following fact, easily verified for a model φ on an E3 m.s. $(\mathbf{G}, \mathbf{K}, \mathbf{R})$: If $\varphi(\Box A, \mathbf{H}_1) = \mathbf{T}$, $\mathbf{H}_1 \in \mathbf{K}$, $\mathbf{H}_2$ is a normal element of $\mathbf{K}$, and $\mathbf{H}_1 R \mathbf{H}_2$, then $\varphi(\Box A, \mathbf{H}_2) = \mathbf{T}$. So Y1 becomes:

Y1. Let $\Box A$ appear on the left of a tableau $\mathbf{t}$. Then: (1) put A on the left of $\mathbf{t}$; (2) put A on the left of every tableau $\mathbf{t}'$ such that $\mathbf{t} S \mathbf{t}'$; (3) put $\Box A$ on the left of every *normal* tableau $\mathbf{t}'$ such that $\mathbf{t} S \mathbf{t}'$.

Notice that if $\Box A$ appears on the left of $\mathbf{t}$ and $\mathbf{t} S \mathbf{t}'$, $\mathbf{t}' S \mathbf{t}''$, then $\mathbf{t}'$ must be normal (otherwise it could not be a descendant of a tableau to which Yr was applied, so that $\mathbf{t}' S \mathbf{t}''$ would be impossible). Hence by Y1, we put $\Box A$ on the left of $\mathbf{t}'$, and thus we put A on the left of $\mathbf{t}''$. This process can be iterated an arbitrary number of times, so we get the *effect* of transitivity (cf. the analogous discussion for S4, Kripke [63], p. 81).

The equivalence of the "S-formulation" to the "R-formulation" can be proved without much trouble. The "S-formulation" will be used throughout Section 5.

5. Completeness theorem.

5.1. Consistency property. We prove the completeness theorem for all systems $S2^n$ $(0 \leqslant n < \infty)$ and for E3 and S3. We first need to show that every provable formula of a given system is valid (in the appropriate model theory). For E2 and E3, we can verify the validity of the axiom schemata directly, and then verify the rules R1 and (Eb). R1 is trivial. To verify (Eb), suppose we are given an E2 (E3) m.s. $(\mathbf{G}, \mathbf{K}, \mathbf{R})$, and an associated model φ such that $\varphi(\Box A \supset \Box B, \mathbf{G}) = \mathbf{F}$, so that $\Box A \supset \Box B$ is not valid in E2 (E3); we need to verify that $A \supset B$ is not valid in E2 (E3). Since $\varphi(\Box A \supset \Box B, \mathbf{G}) = \mathbf{F}$, $\varphi(\Box A, \mathbf{G}) = \mathbf{T}$, and hence $\mathbf{G}$ is normal. Hence, since $\varphi(\Box B, \mathbf{G}) = \mathbf{F}$, there exists $\mathbf{H}_0 \in \mathbf{K}$ such that $\varphi(B, \mathbf{H}_0) = \mathbf{F}$ and $\mathbf{G} \mathbf{R} \mathbf{H}_0$. Since $\mathbf{G} \mathbf{R} \mathbf{H}_0$ and $\varphi(\Box A, \mathbf{G}) = \mathbf{T}$, $\varphi(A, \mathbf{H}_0) = \mathbf{T}$. So $\varphi(A \supset B, \mathbf{H}_0) = \mathbf{F}$. Now $(\mathbf{H}_0, \mathbf{K}, \mathbf{R})$ is an E2 (E3) m.s., and we define a model $\varphi'(P, \mathbf{H})$ thereon by stipulating that $\varphi'(P, \mathbf{H}) = \varphi(P, \mathbf{H})$ for all $\mathbf{H} \in \mathbf{K}$. That is, φ and φ' are the same binary *functions*, but one is associated with $(\mathbf{G}, \mathbf{K}, \mathbf{R})$ and the other with $(\mathbf{H}_0, \mathbf{K}, \mathbf{R})$. Hence, since $\varphi(A \supset B, \mathbf{H}_0) = \mathbf{F}$, $\varphi'(A \supset B, \mathbf{H}_0) = \mathbf{F}$, and φ' falsifies $A \supset B$, Q.E.D. So we have now verified that all theorems of E2 (E3) are valid in E2 (E3). In $S2^n$, all axioms are theorems of E2 (hence valid in E2 and *a fortiori*

in S2^n) or are of the form $\Box^n(A \supset A)$. The latter are easily verified in S2^n to be valid on account of the restriction (in an S2^n m.s. $(\mathbf{G}, \mathbf{K}, \mathbf{R})$) that every $\mathbf{H} \in \mathbf{K}$ of depth $< n$ be normal. The rule R1, which is the only rule of S2^n, is verified trivially. Similarly, for S3, $\Box(A \supset A)$ is obviously valid, since $\mathbf{G}$ must be normal in an S3 m.s. $(\mathbf{G}, \mathbf{K}, \mathbf{R})$; and the other verifications are trivial.

5.2. Completeness property. As in Kripke [63], we prove that if the construction based on S for A is closed, A is valid. The rank of a tableau, the associated and the characteristic formula, are as in Kripke [63]. As in Kripke [63], we need to prove the following:

Lemma. *If A_0 is the characteristic formula of the initial stage of an* S2^n (E3 or S3) *construction, and B_0 is the characteristic formula of any stage, then* $\vdash A_0 \supset B_0$ in E2 (E3).

Proof. As in Kripke [63], we prove that the characteristic formula of any stage A_m implies the characteristic formula A_{m+1} of the next stage. In fact, we prove that, in any S2^n construction, $\vdash A_m \supset A_{m+1}$ is already provable in E2; and similarly that, for an E3 or S3 construction, $\vdash A_m \supset A_{m+1}$ is already provable in E3. We can repeat all the "preliminary observations" in the proof of the corresponding lemma in Kripke [63], noting that the inference from $\vdash X \supset Y$ to $\vdash \Diamond X \supset \Diamond Y$ (Kripke [63], p. 84, lines 13–16), which was justified in Kripke [63] by an appeal to the rule R2, is justified in E2 and E3 by (Eb). And in fact, all the cases considered in Kripke [63] except Y1 can be repeated here without change. (It is notable that the revision of Yr makes no difference to the verification of the case Yr—in fact, the case would go through even if the applicability of Yr were not restricted to normal tableaux.) So we verify only Y1:

Case Y1. In S2^n, we have $\Box A$ on the left of a tableau $\mathbf{t}$, and we are required to put A on the left of $\mathbf{t}$ and of every tableau $\mathbf{t}'$ such that $\mathbf{t} S \mathbf{t}'$. As in Kripke [63], putting A on the left of $\mathbf{t}$ itself is simply justified by the axiom $\Box A \supset A$. To justify putting A on the left of $\mathbf{t}'$, where $\mathbf{t} S \mathbf{t}'$, observe (as in Kripke [63]) that if the characteristic formula of $\mathbf{t}$ is $X \wedge \Box A \wedge \Diamond B$, where B is the characteristic formula of $\mathbf{t}'$, putting A on the left of $\mathbf{t}'$ changes this characteristic formula of $\mathbf{t}$ to $X \wedge \Box A \wedge \Diamond (B \wedge A)$. We can justify this transformation (as in Kripke [63]) if we can prove $\Box A \wedge \Diamond B \cdot \supset \cdot \Diamond (B \wedge A)$ in E2. Well, we certainly

have $\vdash A \supset (B \cdot \supset \cdot B \wedge A)$, so by (Eb) we get $\vdash \Box A \supset \Box (B \cdot \supset \cdot B \wedge A)$. But by the axiom A3, restated in terms of possibility, $\vdash \Box (B \cdot \supset \cdot B \wedge A) \cdot \supset \cdot \Diamond B \supset \Diamond (B \wedge A)$, so hence $\vdash \Box A \cdot \supset \cdot \Diamond B \supset \Diamond (B \wedge A)$, which reduces to the desired $\vdash \Box A \wedge \Diamond B \cdot \supset \cdot \Diamond (B \wedge A)$. Now for E3 and S3, we need to verify further that it is legitimate to put $\Box A$ on the left of every normal tableau $\mathbf{t}'$ such that $\mathbf{t} S \mathbf{t}'$. Now if $\mathbf{t} S \mathbf{t}'$, $\mathbf{t}'$ can be normal only if it has some formula $\Box C$ on the left. (In S3, the main tableau $\mathbf{t}_0$ of a set is stipulated to be normal even if it lacks such a formula; but the condition $\mathbf{t} S \mathbf{t}'$ clearly rules out the possibility that $\mathbf{t}'$ can be the main tableau, i.e., the origin of the tree structure.) So we can assume that the characteristic formula B of $\mathbf{t}'$ has the form $B' \wedge \Box C$. Thus we can justify putting $\Box A$ on the left of $\mathbf{t}'$ if we can prove $\Box A \wedge \Diamond (B' \wedge \Box C) \cdot \cdot \supset \cdot \Diamond (B' \wedge \Box C \wedge \Box A)$ in E3. Now we have $\vdash A \cdot \supset \cdot C \supset A$, and hence by (Eb), $\vdash \Box A \supset \Box (C \supset A)$ in E3. But, by the characteristic axiom of E3, $\vdash \Box (C \supset A) \supset \Box (\Box C \supset \Box A)$. Further, it is easy to prove that $\vdash \Box (\Box C \supset \Box A) \cdot \supset \cdot \Diamond (B' \wedge \Box C) \supset \Diamond (B' \wedge \Box C \wedge \Box A)$. Combining these results, by the transitivity of $\supset$ we get $\vdash \Box A \cdot \supset \cdot \Diamond (B' \wedge \Box C) \supset \supset \Diamond (B' \wedge \Box C \wedge \Box A)$, which is easily transformed into the desired result. Q.E.D.

Theorem 5. *If A is valid, then A is provable.*

Proof. The theorem, of course, is asserted relative to a particular system. We prove it first for E2 and E3. If A is valid, then the construction for A is closed. Hence there is a stage of the construction such that each alternative set at this stage contains a closed tableau. Let $D_1 \vee \ldots D_m$ be the characteristic formula of the stage, where the D_j $(j = 1, \ldots, m)$ are the characteristic formulae of the alternative sets of the stage. We prove $\vdash \sim D_j$ for each j. Now D_j is the characteristic formula of an alternative set $\mathscr{S}$ which contains a closed tableau $\mathbf{t}$. Now let $\mathbf{t}_0$ be the main tableau of the alternative set $\mathscr{S}$. Since $\mathscr{S}$ has the structure of a tree, there is a unique branch of the tree going from $\mathbf{t}_0$ to $\mathbf{t}$. That is to say, there are tableaux $\mathbf{t}_1, \ldots, \mathbf{t}_p$ such that $\mathbf{t}_p = \mathbf{t}$ and $\mathbf{t}_i S \mathbf{t}_{i+1}$ $(i = 0, \ldots, p-1)$. (We intend this assertion to allow the possibility $\mathbf{t} = \mathbf{t}_p = \mathbf{t}_0$, $p = 0$.) We assert that, if X_i is the characteristic formula of $\mathbf{t}_i$, then $\vdash \sim X_i$. Since the characteristic formula of a set is defined as identical with the characteristic formula of its main tableau, X_0 is identical with D_j, and the assertion implies $\vdash \sim D_j$. To prove the assertion, we show that $\vdash \sim X_p$, and that for $i \geqslant 1$, $\vdash \sim X_i$ implies $\vdash \sim X_{i-1}$. Now X_p, being the characteristic formula of $\mathbf{t}$, has the form

$Y \wedge B \wedge \sim B$, where B occurs on both the left and the right of $\mathbf{t}\,(=\mathbf{t}_p)$, so clearly $\vdash \sim X_p$. Assuming $\vdash \sim X_i$, if $i \geqslant 1$, we have $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$. Since $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$, the tableau $\mathbf{t}_{i-1}$ must be normal; for otherwise the rule Yr would never have been applied to it, contrary to the fact that $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$. So (remembering that we are dealing with E2 or E3), $\mathbf{t}_{i-1}$ has some formula $\Box B$ on the left. Hence its characteristic formula X_{i-1} has the form $Y \wedge \Box B \wedge \Diamond X_i$. But since $\vdash \sim X_i$, *a fortiori* $\vdash B \supset \sim X_i$, hence by (Eb), $\vdash \Box B \supset \Box \sim X_i$, and hence $\vdash \sim (\Box B \wedge \Diamond X_i)$. This implies $\vdash \sim X_{i-1}$, as required.

So we have proved $\vdash \sim D_j$ for each j. Hence we obtain $\vdash \sim (D_1 \vee \ldots D_n)$. But, by the lemma just proved, since the characteristic formula of the initial stage is $\sim A$, $\vdash \sim A \supset (D_1 \vee \ldots D_m)$. Hence, combining these results, $\vdash A$.

This proof worked for E2 and E3. What about $S2^n$ and S3? We treat S2, S3, and $S2^2$; the treatment of $S2^n$ for arbitrary n is similar. In S2 and S3, we can no longer assert that a normal tableau must have some formula $\Box B$ on the left; the main tableau $\mathbf{t}_0$ is an exception. But the axiom $\Box (C \supset C)$ of S2 and S3 shows that, for any formula Y, $\vdash Y \cdot \supset \cdot \Box (C \supset C) \wedge Y$. By contraposition, $\vdash \sim (Y \wedge \Box (C \supset C)) \supset \sim Y$ in S2 and S3. Thus if Y is X_0, the preceding proof certainly shows that $\vdash \sim (\Box (C \supset C) \wedge X_0)$ even in E2 (E3); but then, by modus ponens, we can derive $\vdash \sim X_0$ in S2 (S3). Similarly in $S2^2$, we can no longer be sure that $\mathbf{t}_0$ and $\mathbf{t}_1$ contain a formula $\Box B$ on the left, even if both are normal, although we can be sure of this for any tableau of depth $\geqslant 2$. But the characteristic formula X_0 of $\mathbf{t}_0$ has the form $Y \wedge \Diamond X_1$. Using the axiom $\Box\Box(C \supset C)$ of $S2^2$, we get $\vdash Y \wedge \Diamond X_1 \cdot \supset \cdot Y \wedge \Box (C \supset C) \wedge \Diamond (\Box (C \supset C) \wedge X_1)$. (If $p = 0$, X_1 drops out, and we simply use $\vdash Y \cdot \supset \cdot Y \wedge \Box (C \supset C)$.) Now, we can infer (by the preceding reasoning), successively in E2, $\sim (\Box (C \supset C) \wedge X_1)$, and then $\sim (Y \wedge \Box (C \supset C) \wedge \Diamond (\Box (C \supset C) \wedge X_1))$. Combining this with the previous result, we get $\vdash \sim (Y \wedge \Diamond X_1)$, i.e., $\vdash \sim X_0$. Similarly for $S2^n$; and then the reasoning goes through as before. Q.E.D.

6. Applications.

Theorem 6. $\vdash A$ *in* E2 (E3) *if and only if* $\vdash \Box^n A$ ($\vdash \Box A$) *in* $S2^n$ (S3).

Proof. The "only if" part has been proved above. For the "if" part, observe, e.g., for S2, that if A is not derivable in E2, then there

is an E2 countermodel φ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ for it. Define a m.s. $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$, where $\mathbf{K}' = \mathbf{K} \cup \{\mathbf{G}'\}$, $\mathbf{G}' \notin \mathbf{K}'$, and, for $\mathbf{H}_1'$ and $\mathbf{H}_2' \in \mathbf{K}'$, $\mathbf{H}_1' \boldsymbol{R} \mathbf{H}_2'$ iff either $\mathbf{H}_1' = \mathbf{G}'$ and $\mathbf{H}_2' = \mathbf{G}$ or $\mathbf{G}'$, or $\mathbf{H}_1' \in \mathbf{K}$, $\mathbf{H}_2' \in \mathbf{K}$ and $\mathbf{H}_1' \boldsymbol{R} \mathbf{H}_2'$. Let φ' be an extension of φ to $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$; i.e., $\varphi'(P, \mathbf{H}) = \varphi(P, \mathbf{H})$ for $\mathbf{H} \in \mathbf{K}$. Then it is easily proved by induction that $\varphi'(B, \mathbf{H}) = \varphi(B, \mathbf{H})$ for $\mathbf{H} \in \mathbf{K}$ and any subformula B of A. So $\varphi'(A, \mathbf{G}) = \varphi(A, \mathbf{G}) = \mathbf{F}$. Since $\mathbf{G}' \boldsymbol{R} \mathbf{G}$, $\varphi'(\Box A, \mathbf{G}') = \mathbf{F}$. Since $\mathbf{G}' \boldsymbol{R} \mathbf{G}'$, $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$ is an S2 m.s., hence φ' is an S2 model, so that $\Box A$ is not valid, hence not provable, in S2. Similarly for $S2^n$ and S3. (In the case of $S2^n$, $n > 1$, instead of adjoining a single element $\mathbf{G}'$ to $\mathbf{K}$, we form $\mathbf{K}'$ by adjoining a string of new elements $\mathbf{G}_1, \ldots, \mathbf{G}_n$, and stipulating that $\mathbf{G}_i \boldsymbol{R}' \mathbf{G}_i$ $(i = 1, \ldots, n)$, $\mathbf{G}_i \boldsymbol{R}' \mathbf{G}_{i+1}$ $(i = 1, \ldots, n-1)$, and $\mathbf{G}_n \boldsymbol{R}' \mathbf{G}$. Further for $\mathbf{H}, \mathbf{H}' \in \mathbf{K}$, $\mathbf{H} \boldsymbol{R}' \mathbf{H}'$ iff $\mathbf{H} \boldsymbol{R} \mathbf{H}'$. Form $(\mathbf{G}_1, \mathbf{K}', \boldsymbol{R}')$, where $\boldsymbol{R}'$ holds only as required above; this is clearly an $S2^n$ m.s. If φ' extends φ to $(\mathbf{G}_1, \mathbf{K}', \boldsymbol{R}')$, then $\varphi(\Box^n A, \mathbf{G}_1) = \mathbf{F}$ if $\varphi(A, \mathbf{G}) = \mathbf{F}$.) Q.E.D.

We can now, incidentally, show that the axiomatizations of S2 and S3 given here are in fact equivalent to the original systems of Lewis (Lewis and Langford [59]). Actually, we prove their equivalence to Lemmon's P2 and P3, respectively; a proof that Lemmon's systems are equivalent to Lewis's forms of S2 and S3 has been given in Lemmon [57].

The *axioms* of P2 (P3) are all axioms of E2 (E3). In addition to modus ponens, Lemmon's rules for P2 (P3) are a rule stipulating that if A is a tautology or an axiom, $\vdash \Box A$; and a rule $\Box(A \supset B)/\Box(\Box A \supset \Box B)$. To derive the first rule in S2 (S3), as it has been formulated here, observe that if A is a tautology or an axiom of P2 (P3), it is a theorem of E2 (E3); so, by Theorem 6, $\Box A$ is a theorem of S2 (S3). For the second rule, observe that if $\vdash \Box (A \supset B)$ in S2 (S3), by Theorem 6 $\vdash A \supset B$ in E2 (E3), hence by (Eb) $\vdash \Box A \supset \Box B$ in E2 (E3), and hence by Theorem 6 $\vdash \Box (\Box A \supset \Box B)$ in S2 (S3). So S2 (S3) contains P2 (P3); the converse inclusion is trivial.

We notice that, as in Kripke [63], the tableaux lead to a decision procedure for the propositional calculi considered. In fact, in $S2^n$, a formula A of degree m either is valid or has a finite tree countermodel, in which each branch of the tree is of length $\leqslant m$. The proof is like that for M in Kripke [63]. In E3 and S3, we prove, similarly to the proof for S4 in Kripke [63], that a formula A either is valid or has both a finite countermodel and a (possibly infinite) denumerable tree countermodel.

Theorem 7. *If A is a formula of degree $\leqslant m$, then $\vdash A$ in $S2^{m+1}$ if and only if $\vdash A$ in* M.

Proof. The "only if" part is trivial. If A is not provable in $S2^{m+1}$, then it has, being of degree $\leqslant m$, a finite tree countermodel φ in which each branch is of length $\leqslant m$, and hence if $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is the associated model structure, every $\mathbf{H} \in \mathbf{K}$ has depth $\leqslant m$. Since $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an $S2^{m+1}$ model structure, this shows that every $\mathbf{H} \in \mathbf{K}$ is normal, and hence that $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an M m.s. But then φ is a countermodel to A in M.

Remark. By use of semantic tableaux and a modified proof, we can improve $S2^{m+1}$ in this result to $S2^m$. The latter is a "best possible" result; for no formula $\Box^m(A \supset A)$ $(m \geqslant 1)$ can be provable in $S2^{m+1}$ (otherwise, by Theorem 6, $\vdash \Box (A \supset A)$ in E2).

Corollary. M *has no axiomatization with only finitely many axiom schemata, and with modus ponens as sole rule of inference.*

Proof. If it had such an axiomatization, let m be the maximum modal degree of the schemata [5]. By Theorem 7 [5], every axiom would be provable in $S2^{m+1}$, and since modus ponens is a rule of $S2^{m+1}$, M would be contained in $S2^{m+1}$. But $\Box^{m+2}(A \supset A)$, though a theorem of M, is not a theorem of $S2^{m+1}$, as shown in the remark just made; so the *reductio* is complete.

This corollary has long been a widespread unproved conjecture.

Finally, notice that, as in Kripke [63], given a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ we can define a *proposition* ϱ as a mapping of elements of $\mathbf{K}$ into $\{\mathbf{T}, \mathbf{F}\}$, or, equivalently, as a subset of $\mathbf{K}$. Given propositions ϱ and σ, define $\varrho \wedge \sigma$, $\sim \varrho$ as in Kripke [63]; $\Box \varrho$ is now defined by $\Box \varrho\,(\mathbf{H}) = \mathbf{T}$ iff $\mathbf{H}$ is normal and $\varrho(\mathbf{H}') = \mathbf{T}$ for all $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}\,\boldsymbol{R}\,\mathbf{H}'$; otherwise $\Box \varrho(\mathbf{H}) = \mathbf{F}$.

As in Kripke [63], we get matrices for $S2^n$, E3, S3 from model structures for these systems. Let $\mathbf{K} = \{\mathbf{G}, \mathbf{H}\}$, and let $1 = \{\mathbf{G}, \mathbf{H}\}$, $2 = \{\mathbf{G}\}$, $3 = \{\mathbf{H}\}$, $4 =$ empty set, taking propositions as subsets of $\mathbf{K}$, as in Kripke

[5] Strictly speaking, a single schema like $\Box (A \supset A)$ can have an arbitrarily large modal degree, depending on the modal degree of A. We make the convention that the degree of a schema is to be computed as if the schematic letters therein are atomic formulae; thus $\Box (A \supset A)$ has degree 1. Note that if a schema has degree m in this sense, we can replace distinct schematic letters by distinct propositional variables to obtain a formula B of degree m in the ordinary sense. Hence, by Theorem 7, $\vdash B$ in $S2^{m+1}$. But since $S2^{m+1}$ contains the rule of substitution as a derived rule, the original schema thus is derivable in $S2^{m+1}$.

[63], p. 93. If $\mathbf{G}\,\boldsymbol{R}\,\mathbf{G}$, $\mathbf{G}\,\boldsymbol{R}\,\mathbf{H}$, but neither $\mathbf{H}\,\boldsymbol{R}\,\mathbf{G}$ nor $\mathbf{H}\,\boldsymbol{R}\,\mathbf{H}$, $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an S3 m.s., and the matrix in terms of 1, 2, 3, 4, again with 1 and 2 as designated values, coincides with Group I of Lewis-Langford [59].

Notice that it is impossible to obtain infinite characteristic matrices for $S2^n$, E3, and S3 of the types obtained in Kripke [63] for the systems considered there; for such characteristic matrices would be normal, contrary to Halldén's results.

It should be mentioned that, by a proof like that in Kripke [63], Section 5.3, we can prove the following theorem on S2 and S3: If $\vdash \Box A \vee \Box B$, then $\vdash \Box A$ or $\vdash \Box B$. Similarly in $S2^n$, $n \geqslant 1$, if $\vdash \Box^n A \vee \Box^n B$, then $\vdash \Box^n A$ or $\vdash \Box^n B$. For all the E-systems, the property is trivial: since "$\Box$" is interpretable as a *falsum* operator, no formula $\Box A \vee \Box B$ can be derivable.

7. Other systems. The literature contains systems S6 ($S2 + \Diamond \Diamond A$ [6]), S7 ($= S3 + \Diamond \Diamond A$[6]), S8 ($= S3 + \Box \Diamond \Diamond A$[6]). Theorem 2 implies [6] that $S3 = S4 \cap S7$, $S2 = S2^2 \cap S6$, since $\vdash \Diamond \Diamond A \vee (\Box B \supset \Box \Box B)$ in S3 and $\vdash \Diamond \Diamond A \vee \Box \Box (B \supset B)$ in S2. If we define an S6 (S7) m.s. to be an S2 (S3) m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ with at least one $\mathbf{H} \in \mathbf{K}$ of depth 1 which is not normal (*i.e.*, to be an S2 (S3) m.s. which is not an $S2^2$ ($S3^2$) m.s.), and an S8 m.s. to be an S3 m.s. in which for every normal $\mathbf{H} \in \mathbf{K}$ there is an $\mathbf{H}' \in \mathbf{K}$, such that $\mathbf{H}\,\boldsymbol{R}\,\mathbf{H}'$, and which is *not* normal, we can derive completeness theorems for the resulting systems.

Finally, if we drop quasi-reflexivity from the requirements on $\boldsymbol{R}$, and simply define a model structure to be a quadruple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R}, \mathbf{N})$, where $\mathbf{N}$ is the set of normal elements, we get a model theory for Lemmon's D2. If $\boldsymbol{R}$ is required to be transitive, the theory works for D3.

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[6] Actually, S6 is usually defined as the result of adding $\Diamond \Diamond A$ to *Lewis's* formulation of S2 (Lewis and Langford [59]), which (unlike our formulation) contains rules of inference other than R1; similarly for S7 and S8. We can show, however, that if $\Diamond \Diamond A$ is added to S2 as formulated above (with R1 as sole rule), Lewis's rules are derivable; and similarly for S7 and S8. Thus there is no problem about applying Theorem 2, which assumes that R1 is the only primitive rule in the systems considered.

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THE FRAENKEL-MOSTOWSKI METHOD FOR INDEPENDENCE PROOFS IN SET THEORY *

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1. Introduction. The Fraenkel–Mostowski (FM) method as introduced by Fraenkel [22, 28, 28a, 32, 35, 37] and Mostowski [38, 38a, 39] turned out to be very useful in proving the independence of the axiom of choice and related statements. Until the recent appearance of Cohen [63, 64] it was the only method available for obtaining such results. The major deficiency of the FM method is that it does not apply to "ordinary" set theories, i.e., to set theories in which all the elements are sets and in which the axiom of foundation holds. The FM method is applied to the following two types of set theories. The first type, the one which Fraenkel and Mostowski originally dealt with, are set theories which admit the existence of urelements, i.e., elements which are not sets. These set theories are obtained from the "ordinary" set theories by what essentially amounts to a weakening of the axiom of extensionality. The second type, to which this method was applied by Mendelson [56], Shoenfield [55] and Specker [57], are set theories which permit the existence of unfounded sets, i.e., they permit the existence of sequences of the type $\ldots \in x_n \in x_{n-1} \in \ldots \in x_2 \in x_1$ or other similar phenomena. These set theories are obtained by weakening the axiom of foundation. The same proofs of independence were obtained in both types of set theories, except for a proof of the independence of the axiom of foundation from some other statements in Mendelson [58]. Since all other known proofs are carried out in, practically, the same way in both types of set theories we shall henceforth mention only the first type, tacitly assuming that everything applies to the second type as well.

The new method of Cohen [63, 64] for proving the independence of the axiom of choice and related statements is stronger than the FM method in that it applies also to "ordinary" set theories. However, independence

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proofs by the FM method are still interesting for the following reasons. First, ideas developed in those proofs turned out to be useful in proving the corresponding results by the Cohen method. Second, there are results of the FM method for which the corresponding results of the Cohen method are not yet known and may not be known for a long time. Moreover, there are independence results which hold for set theories which admit urelements, but do not hold for "ordinary" theories, e.g., the result that the axiom of choice is independent of the statement "the power set of a well-ordered set can be well-ordered". Third, independence proofs by the FM method resulted in some interesting theorems of an algebraic-combinatorial nature as in Halpern [a] and Läuchli [a].

All the articles known to the author which deal with applications or results of the FM method are mentioned in the present paper. As a rule we shall not give references for published mathematical results in set theory which we shall mention. For results obtained by the FM method we shall refer only to papers in which the final results are presented; the reader can consult these papers for further references.

The results of the FM method apply, in one form or another, to many different set theories. However, to fix ideas we shall deal with a set theory of the Gödel–Bernays type, i.e., with classes, which admits urelements (and which does not contain the axiom of choice), e.g., the set theory $\mathfrak{S}$ of Mostowski [39]. We shall denote such a set theory by $\mathscr{S}$. Henceforth, whenever we shall say that a statement is provable or refutable we shall mean provable or refutable, respectively, in $\mathscr{S}$. All the independence results which will be mentioned are based on the tacit assumption that $\mathscr{S}$ is consistent. We shall mention only those results of the FM method which concern sets exclusively, since the corresponding results which concern classes are obtained in the same way.

2. One body of results. We shall denote by $|x|$ the cardinality of the set x. We shall consider statements in which symbols of cardinals occur as abbreviations of longer sentences in which such symbols do not occur. If u is a set of sets we shall say that f is a *choice function* for u if and only if $(\forall x)\,(x \in u \wedge x \neq 0 \rightarrow f(x) \in x)$. A very common family of consequences of the axiom of choice is the following, where α and β stand for cardinal numbers (i.e., cardinals of well-ordered sets). C^{β}_{α} ($C^{\beta}_{<\alpha}$, $C^{\beta}_{\leqslant\alpha}$, C^{β}_{wo}). For every set u of sets such that $|u| = \beta$ and $(\forall x)\,(x \in u \rightarrow |x| = \alpha)$ (resp., $(\forall x)\,(x \in u \rightarrow |x| < \alpha)$, $(\forall x)\,(x \in u \rightarrow |x| \leqslant \alpha)$,

$(\forall x)\,(x \in u \rightarrow x$ can be well-ordered)) there exists a choice function.

C_α ($\mathsf{C}_{<\alpha}$, $\mathsf{C}_{\leqslant\alpha}$, C_{wo}). For every set u of sets such that $(\forall x)\,(x \in u \rightarrow |x| = \alpha)$ (resp., $(\forall x)\,(x \in u \rightarrow |x| < \alpha)$, $(\forall x)\,(x \in u \rightarrow |x| \leqslant \alpha)$, $(\forall x)\,(x \in u \rightarrow x$ can be well-ordered)) there exists a choice function.

C^β ($\mathsf{C}^{<\beta}$). For every set u of sets such that $|u| = \beta$ (resp., $|u| < \beta$) there exists a choice function.

For a binary relation R let $\mathfrak{F}(R) = \{x \mid (\exists y)\,(\langle xy\rangle \in R \vee \langle yx\rangle \in R)\}$. By a graph we mean a symmetric irreflexive binary relation. We say that a graph R is *n-colorable* if there are n pairwise disjoint sets $A_1, \ldots, A_n$ such that $\mathfrak{F}(R) = \bigcup_{i=1}^n A_i$ and $A_i \times A_i \cap R = 0$ for $1 \leqslant i \leqslant n$.

P_n. If every finite subgraph of the graph R is n-colorable then R is n-colorable.

Ordering principle. For every set x there exists a relation R which orders it.

Order extension principle. For every set x and relation S which partially orders x there exists a relation $R \supseteq S$ which (totally) orders x.

Prime ideal theorem. In every Boolean algebra there is a prime ideal.

Positive results concerning the consequences of the axiom of choice which we just considered are given in Figure 1 by arrows. A positive result not shown there is $m < n \rightarrow (\mathsf{P}_n \rightarrow \mathsf{P}_m)$. Negative results are given by crossed arrows $\vdash\!\!\vdash\!\!\vdash\!\!\rightarrow$. When we write $\mathsf{A} \vdash\!\!\vdash\!\!\vdash\!\!\rightarrow \mathsf{B}$ and near the arrow we write C, $[i]$ we mean that A does not imply B, it does not even imply C and the proof is given in $[i]$ (e.g., C_2 does not imply C_3^ω, as proved by Mostowski [45]). The major open problems are indicated in Figure 1 by broken arrows $\dashrightarrow$. Two independence results will be proved in Section 4, where a finer structure of the corresponding part of Figure 1 will be considered. Unlike the relations between the statements $\mathsf{C}_{\leqslant n}$, the relations between the statements C_n are very complicated. Mostowski [45] obtained a necessary condition and a sufficient condition for a conjunction of C_n's to imply a C_n. Mostowski conjectured that his necessary condition is also sufficient but no proof of it is known yet. It is even unknown whether $\mathsf{C}_3 \wedge \mathsf{C}_5 \wedge \mathsf{C}_{13}$ implies C_{15}.

For results related to the prime ideal theorem but not mentioned here see Halpern [62].

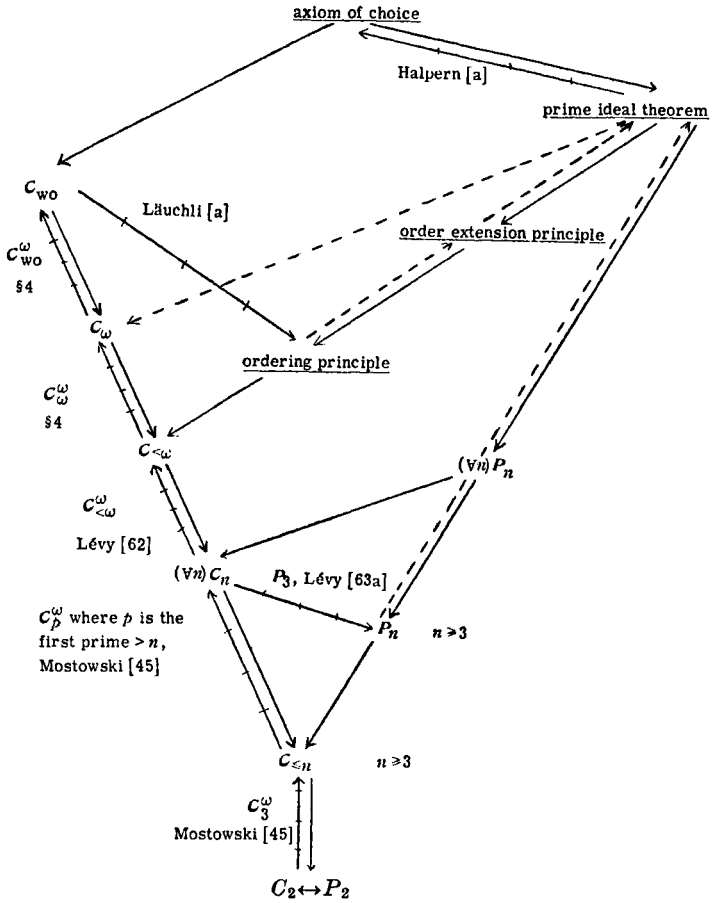


Fig. 1

3. Survey of additional results and problems. Mostowski [48] proves that the axiom of dependent choices does not imply the axiom of choice. He also hints at a proof that for regular α $C^{<\alpha}$ does not imply C^{α}_2 (see the first part of Section 4 for remarks on the role of α in such a statement). The interdependence of generalized forms of the axiom of dependent choices, the C^{α}'s and related statements is investigated in Lévy [a]. The main open problem in this area is whether C^{ω} implies the axiom of dependent choices.

There are several articles dealing with definitions of finiteness (Lindenbaum–Mostowski [38], Mostowski [38, 38a], Doss [45], Fraïssé [58] and Lévy [58]). The two basic metamathematical results (from which all

other ones can be derived by mathematical means) are the result of Lindenbaum and Mostowski that one cannot prove that every infinite set is the union of two disjoint infinite sets (see proofs in Fraïssé [58], Lévy [58]) and the result that the ordering principle does not imply that the power set of every infinite set is reflexive (Lévy [58]).

Let r, s stand for cardinals. Let $r \leqslant^\star s$ denote the statement "$r=0$ or there exist sets u, v and a function f such that $|u|=r$, $|v|=s$ and f maps v on u". We shall consider the following statements.

B_1. $(\forall r, s)\,(r \leqslant^\star s \rightarrow s \not< r)$.

B_2. $(\forall r, s)\,(r \leqslant^\star s \wedge s \leqslant^\star r \rightarrow r = s)$.

B_3. $(\forall r, s)\,(r \leqslant^\star s \rightarrow r \leqslant s)$.

Obviously axiom of choice $\rightarrow \mathsf{B}_3 \rightarrow \mathsf{B}_2 \rightarrow \mathsf{B}_1$. It is unknown to the author whether any of the reverse implications is provable. It is worth mentioning that two of the independence results which appear in Läuchli [62] and Lévy [60b] follow mathematically from the irrefutability of the existence of infinite irreflexive sets. Tarski proved (oral communication) that B_1 implies that every irreflexive set is finite as follows. Let A be an irreflexive infinite set. Let s be the cardinality of the set of all finite sequences without repetitions of members of A. One can easily see that $s \not\geqslant \aleph_0$ hence $s < s+1$. On the other hand it is easy to verify that $s+1 \leqslant^\star s$, which contradicts B_1. A second oral remark of Tarski is that $(\forall r, s)\,(2^r = 2^s \rightarrow r = s)$ implies B_2, since, as easily seen, $r \leqslant^\star s$ and $s \leqslant^\star r$ imply $2^r \leqslant 2^s$ and $2^s \leqslant 2^r$, hence $2^r = 2^s$ and $r = s$.

One of the most interesting open problems in this field is whether $(\forall r)\,(r \geqslant \aleph_0 \rightarrow 2r = r)$ implies the axiom of choice. For independence results additional to the above see Esenin-Volpin [54] (Suslin's hypothesis), Fraïssé [58], Läuchli [62] (various theorems of algebra and topology), Mostowski [58] ("every set is equinumerous to a subset of the power set of an ordinal" is not implied by the ordering principle) and Specker [57] (for every infinite cardinal r $r^2 \leqslant 2^r$). For other applications of the FM method see Ellentuck [62] and Lévy [60b, 62, 63a]. The FM method has been applied also by R. E. Bradford to obtain results concerning decidability in the additive theory of cardinals which have not been published yet.

4. A few detailed results.

Theorem 1. *If it is consistent with* $\mathscr{S}$ + axiom of choice *to assume that there exists a group of order* $\aleph_\alpha$ *which has no proper subgroups of*

index $<\aleph_\beta$ *then* $C_{<\beta}$ *does not imply* C^{ω}_{α} *in* $\mathcal{S}$ (*the* index *of a subgroup is the number of* (*left*) *cosets of the group with respect to that subgroup*).

The meaning of a statement like "$C_{<\beta}$ does not imply C^{ω}_{α}" is clear if α and β are certain particular ordinals like ω, ω_1, ω_ω, etc., but the general statement uses free mathematical variables in a metamathematical statement, and hence has no meaning without further explanation. For such explanation see Lévy [a, Section 3].

In the proof of Theorem 1 we shall follow closely Mostowski [45].

Lemma 2. *Let* G *be a group of order* $\aleph_\alpha$ *which has no proper subgroup of index* $<\aleph_\beta$, *and let* $H=\prod_{i<\omega}G$, *where* $\prod$ *denotes weak direct product. For every proper subgroup* L *of* H $\mathrm{Ind}\,(H/L)\geqslant\aleph_\beta$, *where* $\mathrm{Ind}\,(H/L)$ *is the index of* L *in* H.

Proof. Let G_i be the subgroup of $\prod_{i<\omega}G$ consisting of the sequences all of whose components, except possibly the i-th, are units. The union of the G_i's for $i<\omega$ obviously generates H. Since L is a proper subgroup of H there is an i such that $G_i\nsubseteq L$. $\mathrm{Ind}\,(H/L)\geqslant\mathrm{Ind}\,(G_i/G_i\cap L)$ since members of G_i in different left cosets with respect to $G_i\cap L$ are in different left cosets with respect to L. Since $G_i\nsubseteq L$ and since G_i is isomorphic to G, we have, by our assumption *on* G, $\mathrm{Ind}\,(G_i/G_i\cap L)\geqslant\aleph_\beta$, *hence* $\mathrm{Ind}\,(H/L)\geqslant\aleph_\beta$.

Proof of Theorem 1. If $\mathcal{S}$+axiom of choice is consistent with the existence of a group as mentioned in the theorem then so is the theory obtained from $\mathcal{S}$+axiom of choice by adding to it an axiom asserting that there are at least $\aleph_\alpha$ urelements (see, e.g., Lévy [a]). Let G be a group as mentioned in the theorem. By Cayley's theorem we can consider G as a group of permutations on a set A of cardinality $\aleph_\alpha$ such that for every $x\in A$ there is a $\varphi\in G$ with $\varphi(x)\neq x$.

Let K_i, $i<\omega$, be sets of urelements of cardinality $\aleph_\alpha$ and let G_i be a group of permutations of K_i which is isomorphic (as a permutation group) to G, which acts on A. Let $K=\bigcup_{i<\omega}K_i$ and let F be the weak direct product of the permutation groups G_i, $i<\omega$, hence a group of permutations of K. Let b range over finite sets of natural numbers. A permutation $\varphi\in F$ is said to be *b-identical* iff it is the identity on each K_i for which $i\in b$. The subgroup of all b-identical members of F is denoted with F^b. We shall say that an element x is a K-*element* iff all the urelements in its transitive closure are in K, where the transitive closure of a set x is the least set u such that $x\in u$ and for every y, z

$y \in z \in u \rightarrow y \in u$. We extend every permutation φ of K to an $\in$-automorphism of the class of all K-elements, the extended permutation is denoted by φ too. A K-element x is said to be *b-symmetric* iff $\varphi(x) = x$ for each $\varphi \in F^b$. An element x is said to be an M-*element* iff for each y in its transitive closure there is a b such that y is b-symmetric. A class X is said to be an M-*class* iff all its members are M-elements and the class is closed under F^b, for some b. As is done in all applications of the FM method (see, e.g., Mostowski [39]) one verifies that the M-elements, M-classes and the $\in$ relation constitute a model for $\mathscr{S}$ (or, rather, replacing the notions of element and class by M-element and M-class, respectively, gives an interpretation of $\mathscr{S}$ in $\mathscr{S}$). Furthermore, one can easily verify that each K_i has the cardinality $\aleph_\alpha$ in the model and that the set $\{\langle i, K_i\rangle | i < \omega\}$ is an M-element but it has no choice function which is an M-element, i.e., $C^\omega_{\omega_\alpha}$ fails in the model.

To see that $C_{<\omega_\beta}$ holds in the model we shall show that for every M-element x which is a set of sets which have $<\aleph_\beta$ members each there exists an M-element which is a function such that for every $y \in x$ $f(y) \in y$. Let x be as above. Since x is an M-element x is b-symmetric for some b. We define for M-elements u, v $u \sim_b v$ if for some $\varphi \in F^b$ $\varphi(u) = v$. It is easily seen that $\sim_b$ is an equivalence relation such that for a b-symmetric x $u \in x \wedge u \sim_b v \rightarrow v \in x$ and hence x is the union of $\sim_b$-equivalence classes. By the axiom of choice there is a subset z of x which contains exactly one representative of each equivalence class. Since $z \subseteq x$ each member y of z has $<\aleph_\beta$ members. By the axiom of choice there are functions a and g on z such that for every $y \in z$ y is $a(y)$-symmetric and $g(y) \in y$. Let $H_y = F^{a(y) \cup b}$, $L_y = \{\varphi | \varphi \in H_y \wedge \varphi(g(y)) = g(y)\}$. H_y is easily seen to be isomorphic to $\prod_{i<\omega} G$, hence by Lemma 2 $L_y = H_y$ or $\mathrm{Ind}\,(H_y/L_y) \geqslant \aleph_\beta$. As in the proof in Lévy [62, p. 482] y has at least $\mathrm{Ind}\,(H_y/L_y)$ members, hence $\mathrm{Ind}\,(H_y/L_y) \geqslant \aleph_\beta$ is false and $L_y = H_y$ for every y. From this it follows that there is an M-element f which is a choice function for x as in Lévy [62a, p. 482].

Theorem 3. *For a limit number* α $C_{<\omega_\alpha}$ *does not imply* C_{ω_α}; *if* α *is not a limit number then* $C_{<\omega_{\alpha-1}}$ *does not imply* C_{ω_α}.

Proof. If $\mathscr{S}$ is consistent then by the method of Gödel [58] and additional standard methods (see Lévy [a, Section 3]) one can prove that $\mathscr{S}$ + axiom of choice + generalized continuum hypothesis is consistent. Within the latter theory let G be a simple group of cardinality $\aleph_\alpha$ (e.g., the alternating group of permutations of a set of cardinality $\aleph_\alpha$). As

shown by W. R. Scott [55, Corollary 2 ff.] if α is a limit number G has no subgroups of index $<\aleph_\alpha$ and if α is not a limit number then G has no subgroups of index $<\aleph_{\alpha-1}$. Theorem 3 follows now from Theorem 1.

It still remains an open problem if for a non-limit ordinal α $C_{<\omega_\alpha}$ (which is equivalent to $C_{\omega_{\alpha-1}}$) implies C_{ω_α}. The negative answer will be known if one proves, even with the aid of the axiom of choice and the generalized continuum hypothesis, the existence of a group G of order $\aleph_\alpha$ which has no subgroups of index $<\aleph_\alpha$.

In analogy to the proof of Lévy [62, Theorem 6] one can prove the following theorem.

Theorem 4. *If ω_δ is the least regular cardinal confinal with ω_α then $(\forall\gamma<\alpha)\, C_{\omega_\gamma}$ does not imply $C^{\omega_\delta}_{\omega_\alpha}$.*

One can easily prove that for $\lambda<\delta$ $(\forall\gamma<\alpha)\, C_{\omega_\gamma} \rightarrow C^{\omega_\lambda}_{<\omega_\alpha}$. A similar proof yields $(\forall\gamma)\, C_{\omega_\gamma} \rightarrow C_{wo}$.

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FREE PRODUCT IN GENERAL ALGEBRAS

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There are two important notions in modern algebra which have the label "free" attached to them. These are free algebra and free product. Let us recall their definitions.

Suppose $\mathfrak{A}_0$ is a class of algebras of a given similarity type. An algebra A, of the same type, is called $\mathfrak{A}_0$-*free*, or (more precisely) $\mathfrak{A}_0$-*freely generated by* the set X iff (1) A is generated by the set X and (2) every mapping $\varphi : X \to D$, where D is an arbitrary algebra in $\mathfrak{A}_0$, can be extended to a homomorphism $h : A \to D$ ("extended" means $h|X = \varphi$).

For the definition of the free product, let B and C be subalgebras of A; A is an $\mathfrak{A}_0$-*free product* of B and C iff (1) B and C together generate A and (2) every pair of homomorphisms $h_1 : B \to D$, $h_2 : C \to D$ into an algebra D in $\mathfrak{A}_0$ may be simultaneously extended to a homomorphism $h : A \to D$.

Note that the definitions above refer to the concepts of "free for" and not "free in": we did not assume that A is in $\mathfrak{A}_0$.

The similarity of these notions is evident, but there are also considerable differences between them.

We have another definition of free algebra, which constitutes a "logical" counterpart for the "algebraic" definition, but there is no such second definition for the free product.

The "logical" definition of free algebra uses the notion of satisfiability of equations by the elements of an algebra, and runs as follows: A is $\mathfrak{A}_0$-freely generated by the set X iff (1) (as before) A is generated by the set X and (2) for every equation $\tau(x_1, \ldots, x_k) = \vartheta(x_1, \ldots, x_k)$ (the x_i are variables and it is assumed that no variables other than $x_1, \ldots, x_k$ occur either in the term τ or in the term ϑ), if this equation is satisfied by distinct elements $a_1, \ldots, a_k$ belonging to X (i.e., $\tau(a_1, \ldots, a_k) = \vartheta(a_1, \ldots, a_k)$ holds), then $\tau(x_1, \ldots, x_k) = \vartheta(x_1, \ldots, x_k)$ is a true equation (holds tautologically) in every algebra in $\mathfrak{A}_0$.

Both definitions of free algebra are equivalent but the latter one, as is known, has some important direct consequences.

The aim of this paper is to give a "logical" definition for the concept of free product. The main idea lies in the intuitive meaning of an algebraic connection. If we agree to say that elements $a_1, \ldots, a_k$ are algebraically connected when some "non-trivial" equation holds for them, then we may say that the "logical" definition of free algebra states that there are no algebraic connections between elements of the set which generates freely a free algebra, except those tautologically true in $\mathfrak{A}_0$.

When we say: "A is an $\mathfrak{A}_0$-free product of B and C", the word "free" again means that there are no algebraic connections between B and C, i.e., no equation $\tau(b_1, \ldots, b_k, c_1, \ldots, c_l) = \vartheta(b_1, \ldots, b_k, c_1, \ldots, c_l)$ with b_i in B and c_i in C holds, and again we have to make exceptions for the "trivial" cases and allow them to hold. By these we should understand all such connections which can be "justified" by general rules (tautologies) of the class $\mathfrak{A}_0$ and by connections holding between elements of the factors. To make this idea clearer let us give some examples from group theory.

Example 1. In the free product $G_1 \star G_2$ of two groups ($\mathfrak{A}_0$ is now the class of all groups) the square g_1^2 of an element g_1 in G_1 does not, in general, commute with an element in G_2. This means that the algebraic connection $x^2y = yx^2$ (or $x^2yx^{-2}y^{-1} = e$) does not hold, in general, between G_1 and G_2. It may happen, however, that it holds for some elements g_1 in G_1 and g_2 in G_2, (for example, when g_1 is of order 2, i.e., $g_1^2 = e$). There is nothing peculiar in this since we have as a general rule in group theory: $x^2 = e \rightarrow x^2y = yx^2$. This rule and the fact that $g_1^2 = e$ justify the algebraic connection $g_1^2 g_2 = g_2 g_1^2$.

Example 2. Let us consider the free product with respect to the class of all groups in which $x^2y = yx^2$ is true for every pair of elements g, g'. In this case, when G is a free product of G_1, G_2 we can find many pairs g_1, g_2 from G_1 and G_2 respectively which commute. If, e.g., G_1 has the property that every element has a square root, then the direct sum of G_1 and G_2 is their free product (with respect to the named class) and so G_1 and G_2 commute. This is possible because $x_1 = x_2^2 \rightarrow x_1y = yx_1$ holds as a general rule in the class under consideration. If $x_1 = x_2^2$ is satisfied, then $x_1y = yx_1$ is satisfied too.

Let us note that the rules in Example 1 and Example 2 which explain the algebraic connections between the factors of the free products have

a common form: they are both open conditional formulas. There is, however, a great difference between Example 1 and Example 2. In the first case, all variables (actually only one) occurring in the antecedent occur in the consequent; in the second case the variable x_2 occurs only in the antecedent. To this quite "logical" (indeed syntactical) difference corresponds still another difference of a purely algebraic character. As is known from the theory of groups, if the group G is the free product of G_1 and G_2, and H_1 and H_2 are subgroups of G_1 and G_2, respectively, then the subgroup of G generated by H_1 and H_2 is a free product of H_1 and H_2. This statement is true when we consider the free product with respect to the class of all groups, but fails in the case when the class of groups is restricted, as in Example 2, to groups in which $x^2y=yx^2$ holds. If G belongs to this class and is a free product with respect to this class of two infinite cyclic groups $\{g_1\}$ and $\{g_2\}$, then g_1 and g_2 do not commute, but g_1^2 and g_2 do. Therefore, the subgroup generated by $\{g_1^2\}$ and $\{g_2\}$ is not a free product of these groups with respect to the same class [1]. The role of the variable occurring only in the antecedent is clear.

Example 3. To show that to "justify" an algebraic connection between two factors of a free product we may have to use algebraic connections in both factors of the free product, let us consider such products with respect to the class of groups in which $x^2y^2=y^2x^2$ is true. Now, for g_1, g_2 in factors G_1, G_2, respectively, $g_1g_2=g_2g_1$ may hold when both g_1 and g_2 are square. Thus the "justifying" rule is $x_1=x_2^2 \cdot y_1=y_2^2 \rightarrow$ $\rightarrow x_1y_1=y_1x_1$. The equations appearing in the antecedent refer to different factors; therefore they are written in different variables.

Example 4. This example is of a more artificial nature then the previous three. It shows that if the class $\mathfrak{A}_0$ is not as regular as, for example, the class of all groups or of groups with some equational conditions, then it is possible to have very peculiar products and it is not so easy to find rules to "justify" algebraic connections between the factors.

Let G be a simple group without any proper subgroup G_0 isomorphic to the whole group G (in general G may be any algebra having only one proper endomorphism) and let $\mathfrak{A}_0$ be the class of all proper subgroups of G. If G_0 is in $\mathfrak{A}_0$ and $h_1 : G \rightarrow G_0$, $h_2 : G \rightarrow G_0$ are homomorphisms,

[1] I am indebted to Bjarni Jónsson for this example.

then both are zero-homomorphisms, i.e., $h_i : G \to \{e\}$, $i = 1, 2$. This shows that G is an $\mathfrak{A}_0$-free product of $G_1 = G$ and $G_2 = G$. Let us note that in this case for g_1 in G_1 and g_2 in G_2 the algebraic connection $g_1 = g_2$ may hold.

Now we will propose a generalization of the concept of free product. First, however, we have to introduce some conventions concerning symbolism.

Equations will be denoted by Greek letters $\omega, \varepsilon, \eta, \varrho$, with superscripts when necessary. We shall denote finite collections of elements of algebras B and C, respectively, by b and c, also with superscripts when necessary. For example, ε may be an equation of the form $\tau(x_1, \ldots, x_k) = \vartheta(x_1, \ldots, x_k)$ and b may be the collection of $b_1, \ldots, b_k$; in this case $\varepsilon(b)$ denotes simply $\tau(b_1, \ldots, b_k) = \vartheta(b_1, \ldots, b_k)$. Obviously the number of elements in b must be suitably chosen to fit the variables in ε.

Let B and C be subalgebras of A, let $\{\omega^t\}$, $t \in T$, be a set of equations, let $\{b^t\}$, $t \in T$, and $\{c^t\}$, $t \in T$, be sets of collections of elements of B and C, respectively, such that $\omega^t(b^t, c^t)$ makes sense.

By definition A is an *$\mathfrak{A}_0$-free product with conditions* $\{\omega^t(b^t, c^t)\}$, $t \in T$, (briefly: *conditional free product*) of B and C iff: (1) A is generated by B and C; (2) every pair of homomorphisms $h_1 : B \to D$, $h_2 : C \to D$ into an algebra D in $\mathfrak{A}_0$, such that for every $t \in T$, $\omega^t(h_1 b^t, h_2 c^t)$ holds in D, can be simultaneously extended to a homomorphism $h : A \to D$.

It is important to observe that in the definition of the conditional free product we do not assume that the algebraic connections $\omega^t(b^t, c^t)$ hold in A. They may, but it is not necessary that they do. If they do, then there exist algebraic connections between the factors B and C of the product A, which perhaps, cannot be "justified" by the rules of the class $\mathfrak{A}_0$ only, and then the conditional product will not be the free product in the former sense.

Example 5. Let G_1 and G_2 be subgroups generating the whole group G and let G_0 be the common part of G_1 and G_2. Then G is a free product of G_1 and G_2 with amalgamated subgroup G_0 iff G is a conditional free product with conditions $g = g$ where g is in G_0.

In this example g on the left side of the equation is to be considered as an element of G_1 and g on the right side as an element of G_2. Applying the definition we see that the requirement is: if $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$, and if for g in G_0, $h_1(g) = h_2(g)$, then there is a homomorphism $h : G \to H$ which is a common extension of h_1 and h_2.

Example 6. For two groups G_1 and G_2 let us consider the free product with respect to the class of all groups and with conditions: $g_1g_2=g_2g_1$ for all g_1 in G_1 and g_2 in G_2. Now the homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ must have a common extension only if the images of G_1 and G_2 in H commute. It follows that if G is a direct product of G_1 and G_2, then G is the conditional free product with the above conditions.

Now let A be an algebra generated by its subalgebras B and C. Further, let X and Y be two disjoint sets of variables, e.g., $X=(x_1, x_2, \ldots)$ and $Y=(y_1, y_2, \ldots)$, and let $f: X \to B$, $g: Y \to C$ be fixed one-to-one functions from X and Y onto B and C. By these assumptions for every equation $\omega(\mathfrak{x}, \mathfrak{y})$ written in some x's and y's, $\omega(f\mathfrak{x}, g\mathfrak{y})$ makes sense. It is simply the result of substituting the corresponding elements of B and C for x's and y's in ω. Reciprocally, if $\omega(b, c)$ is an equation in elements from B and C, then $\omega(f^{-1}b, g^{-1}c)$ makes sense and is an equation in x's and y's.

Suppose we are interested in finding out if A is an $\mathfrak{A}_0$-free product of its subalgebras B and C with conditions $\{\omega^t(b^t, c^t)\}$, $t \in T$. Let us consider the following condition:

(C) For every connection $\varrho(b, c)$ which holds between B and C, there exist: a subset T_1 of T, equations ε^i and η^j and collections b^i, c^j of elements in B and C, respectively, such that $\varepsilon^i(b^i)$ and $\eta^j(c^j)$ hold for every i in I and j in J, and moreover the following formula

$$(\star) \qquad \prod_{t \in T_0} \omega^t(f^{-1}b^t, g^{-1}c^t) \cdot \prod_{i \in I} \varepsilon^i(f^{-1}b^i) \cdot \prod_{j \in J} \eta^j(g^{-1}c^j) \to \varrho(f^{-1}b, g^{-1}c)$$

is true in $\mathfrak{A}_0$.

The formula $(\star)$ is a conditional formula, but the condition of $(\star)$ may be a conjunction of infinitely many equations. If we restrict the formula to be finite, i.e., if we restrict the sets T_0, I, and J to be finite, then we shall call the restricted condition (C_0). Condition (C_0) is obviously stronger than (C). Let us note that neither (C) nor (C_0) depends on f and g.

Theorem 1. *For an algebra A generated by B and C, condition (C) holds iff A is an $\mathfrak{A}_0$-free product of B and C with conditions $\{\omega^t(b^t, c^t)\}$, $t \in T$.*

The proof is relatively simple. Let $h_1 : B \to D$ and $h_2 : C \to D$, with D in $\mathfrak{A}_0$, be two homomorphisms, such that $\omega^t(h_1b, h_2c)$, $t \in T$, hold. Since every element a in A may be represented in the form $a=\tau(b', c')$, where

elements b' are in B, c' in C and τ is a term, hence we may define the common extension $h : A \to D$ of h_1 and h_2 simply by accepting $h(a) = \tau(h_1 b', h_2 c')$. This defines a homomorphism iff $h(a)$ does not depend on b', c' and the term τ. Suppose $a = \tau(b', c') = \vartheta(b'', c'')$; we shall write the equation $\tau(b', c') = \vartheta(b'', c'')$ in a shorter form as $\varrho(b, c)$. We must prove $\varrho(h_1 b, h_2 c)$ holds in D.

Now let us turn to condition (C). Since $\varrho(b, c)$ is a connection between B and C, a formula of the form $(\star)$ is true in $\mathfrak{A}_0$, and hence in D. Moreover, $\varepsilon^i(b^i)$, $\eta^j(c^j)$ hold in B and C, respectively. It follows that

$$\prod_{t \in T} \omega^t(h_1 b^t, h_2 c^t) \cdot \prod_{i \in I} \varepsilon^i(h_1 b^i) \cdot \prod_{j \in J} \eta^j(h_2 c^j) \to \varrho(h_1 b, h_2 c)$$

holds in D as a substitution of $(\star)$ and $\varepsilon^i(h_1 b^i)$, $\eta^j(h_2 c^j)$ hold in D, because equations are preserved by homomorphisms. Finally, $\omega^t(h_1 b^t, h_2 c^t)$ hold in D by the assumptions on h_1, h_2. Since all conditions hold in D, $\varrho(h_1 b, h_2 c)$ holds in D also.

To prove that condition (C) is necessary, let us assume that A is an $\mathfrak{A}_0$-free product of B and C with conditions $\{\omega^t(b^t, c^t)\}$, $t \in T$, and let $\{\varepsilon^i(b^i)\}$, $i \in I$, and $\{\eta^j(c^j)\}$, $j \in J$, be the sets of all equations holding in B and C, respectively. These sets are actually what we usually call "positive diagrams" of B and C and it is known that:

If $\varphi : B \to D$ is an arbitrary mapping, then φ is a homomorphism iff $\varepsilon^i(\varphi b^i)$, $i \in I$, are true in D.

Of course, a similar statement is true for $\eta^j(c^j)$, $j \in J$, and C. Now let us assume that $\varrho(b, c)$ holds in A. We shall prove that

$$(\star\star) \quad \prod_{t \in T} \omega^t(f^{-1} b^t, g^{-1} c^t) \cdot \prod_{i \in I} \varepsilon^i(f^{-1} b^i) \cdot \prod_{j \in J} \eta^j(g^{-1} c^j) \to \varrho(f^{-1} b, g^{-1} c)$$

is true in $\mathfrak{A}_0$.

Assume the contrary; then there exist an algebra D in $\mathfrak{A}_0$ and a substitution of elements from D for x's and y's in $(\star\star)$ such that $(\star\star)$ becomes false. That substitution may be described by two functions $k : X \to D$ and $l : Y \to D$, giving elements $k(x)$ and $l(y)$ in D which are to be substituted for x in X and y in Y. Therefore

$$\prod_{t \in T} \omega^t(kf^{-1} b^t, lg^{-1} c^t) \cdot \prod_{i \in I} \varepsilon^i(kf^{-1} b^i) \cdot \prod_{j \in J} \eta^j(lg^{-1} c^j) \cdot \neg\, \varrho(kf^{-1} b, lg^{-1} c).$$

If we put $h_1 = kf^{-1}$ and $h_2 = lg^{-1}$ then we see that $\varepsilon^i(h_1 b^i)$, $i \in I$, is true, so $h_1 : B \to D$ is a homomorphism; $\eta^j(h_2 c^j)$, $j \in J$ is true, hence $h_2 : C \to D$ is a homomorphism; $\omega^t(h_1 b^t, h_2 c^t)$, $t \in T$ is true, so h_1 and h_2 have a

common extension; $\varrho(h_1 b, h_2 c)$ is not true. But all the above are contradictory with the assumption that $\varrho(b, c)$ is true. If $h : A \to D$ is a common extension of h_1 and h_2, then $h_1 b = hb$, $h_2 c = hc$ and $\varrho(h_1 b, h_2 c)$ must be true because equations are preserved by homomorphisms.

We have proved that $(\star\star)$ which is of the form $(\star)$ is true in $\mathfrak{A}_0$. From the previous assumptions: $\varepsilon^i(b^i)$, $i \in I$, and $\eta^j(c^j)$, $j \in J$, hold in B and C. Therefore condition (C) is satisfied.

If $\mathfrak{A}_0$ is an elementary class, i.e., a set of elementary formulas exists such that an algebra A belongs to $\mathfrak{A}_0$ iff every formula in this set is true in A, then, as is well known, if an infinite formula of the form $\prod_{s \in S} \xi^s(\mathfrak{x}^s) \to \xi(\mathfrak{x})$ is true in $\mathfrak{A}_0$, then for a finite subset S_0 of S the formula $\prod_{s \in S_0} \xi^s(\mathfrak{x}^s) \to \xi(\mathfrak{x})$ is true in $\mathfrak{A}_0$. This shows that in the case of an elementary class condition (C_0) is necessary. Since it is certainly sufficient, too, we have the following:

Theorem 2. *If $\mathfrak{A}_0$ is an elementary class, then for an algebra A generated by B and C, condition (C_0) holds iff A is an $\mathfrak{A}_0$-free product of B and C with conditions $\{\omega^t(b^t, c^t)\}$, $t \in T$.*

The conditions $\{\omega^t(b^t, c^t)\}$, $t \in T$, for an $\mathfrak{A}_0$-free product are called *analytic* iff for every t in T, every substitution of elements in B for x's and elements of C for y's in $\omega^t(f^{-1}b^t, g^{-1}c^t)$ is also included among them. A conditional $\mathfrak{A}_0$-free product with analytic conditions is called an *analytic* $\mathfrak{A}_0$-free product.

It is easy to see that an analytic $\mathfrak{A}_0$-free product depends on a set of equations $\{\omega^t(\mathfrak{x}^t, \mathfrak{y}^t)\}$, $t \in T$, in variables x's and y's. Its definition may be expressed as follows: A is an analytic $\mathfrak{A}_0$-free product with condition $\{\omega^t(\mathfrak{x}^t, \mathfrak{y}^t)\}$, $t \in T$, of B and C iff (1) B and C generate A and (2) for every D in $\mathfrak{A}_0$ and every two homomorphisms $h_1 : B \to D$, $h_2 : C \to D$, h_1 and h_2 have a common extension $h : A \to D$ provided that for every t in T and all collections b^t in B, c^t in C, (such that $\omega^t(b^t, c^t)$ makes sense) $\omega^t(h_1 b^t, h_2 c^t)$ holds in D.

Let us now consider analytic $\mathfrak{A}_0$-free products with a set of conditions $\zeta = \{\omega^t(\mathfrak{x}^t, \mathfrak{y}^t)\}$, $t \in T$, and let A', B, C be arbitrary algebras.

If there exist two subalgebras B' and C' of A', isomorphic to B and C, respectively, such that A' is an analytic $\mathfrak{A}_0$-free product of B' and C' with conditions ζ, then $\langle A', B', C' \rangle$ is called a $\zeta - \mathfrak{A}_0$-*free product* of B and C. A $\zeta - \mathfrak{A}_0$-free product $\langle A', B', C' \rangle$ of B and C is called *minimal*

iff for every $\zeta-\mathfrak{A}_0$-free product $\langle A'', B'', C''\rangle$ and for every pair of isomorphisms $\varphi : B'' \to B'$, $\psi : C'' \to C'$ there is a homomorphism $h : A'' \to A'$ which is a common extension of φ and ψ.

Obviously if a minimal $\zeta-\mathfrak{A}_0$-free product of B and C exists, then it is unique up to isomorphism. This means that if $\langle A', B', C'\rangle$ and $\langle A'', B'', C''\rangle$ are two minimal $\zeta-\mathfrak{A}_0$-free products of B and C, then there exists an isomorphism $\varphi : A' \to A''$ which maps B' onto B'' and C' onto C''. Unfortunately, such a minimal product does not always exist, and the problem of finding conditions for classes of algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$ and for the set of analytic conditions ζ under which a minimal $\zeta-\mathfrak{A}_0$-free product of every pair of algebras B and C in $\mathfrak{A}_1$ exists is open.

Example 7. Let B be an algebra with two one-element subalgebras $\{b_1\}$ and $\{b_2\}$ which cannot be mapped one on another by means of an automorphism of B, and let $C=\{c\}$ be a one-element algebra. For ζ we shall take the set containing only one condition: $x=y$. It is clear that $\langle B, B, \{b_1\}\rangle$ and $\langle B, B, \{b_2\}\rangle$ are $\zeta-\mathfrak{A}_0$-free products of B and C. Suppose that $\langle A', B', \{b'\}\rangle$ is a minimal $\zeta-\mathfrak{A}_0$-free product of B and C; then there are homomorphisms $\varphi : B \to A'$, $\psi : B \to A'$ which map B isomorphically onto B' and $\varphi(b_1)=b'$, $\psi(b_2)=b'$. But in such a case both must be isomorphisms transforming B onto $A'=B'$, and consequently $\varphi^{-1}\psi$ is an automorphism of B with $\varphi^{-1}\psi(b_2)=b_1$, which contradicts the assumption.

In most cases a minimal analytic free product does exist. For instance, as shown in Example 6, a direct product of two groups is a conditional free product. Actually it is a minimal analytic free product, so it always exists.

There are many problems related to free products of all kinds. Let me mention one which seems to be of a different sort even for the ordinary unrestricted $\mathfrak{A}_0$-free product with an equational class $\mathfrak{A}_0$.

As shown in Examples 1 and 2 there exist classes $\mathfrak{A}_0$ with the property: (P_1) If A is an $\mathfrak{A}_0$-free product of B and C and B_0 and C_0 are subalgebras of B and C, respectively, then the algebra generated by B_0 and C_0 is an $\mathfrak{A}_0$-free product of B_0 and C_0.

But there also exist equational classes in which this property fails.

It may be easily shown that a class has property (P_1) iff for every product condition (C) is satisfied with a formula ($\star$) (with empty T_0) in which every variable occurring in conditions $\varepsilon^i(f^{-1}b^i)$, $\eta^j(g^{-1}c^j)$ occurs also in $\varrho(f^{-1}b, g^{-1}c)$.

But this is not a "logical" characterization of the class $\mathfrak{A}_0$. What we want to have is a condition on the set of equations true in $\mathfrak{A}_0$ which is necessary and sufficient for an equational class $\mathfrak{A}_0$ to have property (P_1).

This problem recalls another one concerning free algebras. It is well known that some equational classes $\mathfrak{A}_0$ have the following property:

(P_2) Every subalgebra of an $\mathfrak{A}_0$-free algebra is itself an $\mathfrak{A}_0$-free algebra.

For example, groups, abelian groups, and even abelian groups with property $px=0$ where p is prime have this property. But abelian groups with properties $nx=0$ where n is a composite integer do not have it. We do not know any characterization of the equational classes $\mathfrak{A}_0$ with property (P_2) which refers to equations true in $\mathfrak{A}_0$. We do not even know if every equational class with property (P_2) has property (P_1).

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MODEL-THEORETIC METHODS AND RESULTS IN THE THEORY OF CYLINDRIC ALGEBRAS *

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We shall deal in this paper with the cylindric algebras of Tarski, in particular with the class of representable cylindric algebras. The principal model-theoretic method exploited here is that of ultraproducts. The model-theoretic results are connected with the equational character of the class of representable cylindric algebras and with the problem of finite axiomatizability of this class.

The intent of Section 1 is mainly expository, and consists of a simplified development of some general theorems about representable algebras. Theorems 1.3, 1.4, 1.5, 1.8 and 1.13 seem to be new results. Section 2 contains the main new result of the paper. There we show that the class of representable three-dimensional cylindric algebras is not finitely axiomatizable. The methods – apart from the use of an ultraproduct construction – are similar to methods introduced by Jónsson and Lyndon in the study of relation algebras. The fairly "deep" theorem of Bruck and Ryser concerning orders of finite projective planes is used in an essential way.

1. Representable cylindric algebras. We assume as known the basic definitions in the theory of cylindric algebras. We use the notation of Henkin and Tarski [61]. Proofs of some elementary facts about cylindric algebras may be found in Henkin [56]; in one essential theorem below we use results of Halmos [62] and Galler [57]. Most of the results of this section are due to Henkin and Tarski. We recall that $\mathscr{RCA}_\alpha$ and $\mathscr{CS}_\alpha$ are the classes of all representable cylindric algebras and of all cylindric set algebras respectively, of dimension α.

Theorem 1.1. *Every $\beta\gamma$-reduct of an $\mathscr{RCA}_\alpha$ is an $\mathscr{RCA}_\alpha$.*

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Proof. It suffices to prove that, given a $\mathscr{CS}_\alpha$ $\mathfrak{A}$ with base U and a non-zero element x of A, there is a homomorphism f of the $\beta\gamma$-reduct $\mathfrak{B}$ of $\mathfrak{A}$ into a $\mathscr{CS}_\beta$ such that $f(x) \neq 0$. Choose $t \in x$. If $s \in U^\beta$, define $s^+ \in U^\alpha$ as follows:

$$s^+(\varkappa) = \begin{cases} t(\varkappa) & \text{if } \varkappa \notin \text{range of } \gamma, \\ s(\gamma^{-1}(\varkappa)) & \text{if } \varkappa \in \text{range of } \gamma. \end{cases}$$

The required function f is defined as follows: $f(y) = \{s \in U^\beta : s^+ \in y\}$, for all $y \in A$.

Theorem 1.2. *If $\alpha \leqslant \beta$, then every $\mathscr{RCA}_\alpha$ can be neatly embedded in an $\mathscr{RCA}_\beta$.*

Proof. It suffices to prove the theorem for a $\mathscr{CS}_\alpha$ $\mathfrak{A}$ with base U. For each $x \in A$ let $f(x) = \{t \in U^\beta : t \restriction \alpha \in x\}$. Then f neatly embeds $\mathfrak{A}$ in a $\mathscr{CS}_\beta$.

These two simple theorems are quite important for the present development. The following theorem is a simple generalization of Corollary 1.16 of Frayne–Morel–Scott [62]. For this theorem we suppose that L is a first-order language with relation symbols, operation symbols and individual constants, and we suppose that N is the set of non-logical constants of L.

Theorem 1.3. *Suppose $\mathscr{K}$ is a class of L-structures and $\mathfrak{A}$ is an L-structure. Let $\mathscr{F} = \{\langle F, K\rangle : F$ is a finite subset of A and K is a finite subset of $N\}$. Suppose $\mathscr{M}$ is a subset of $\mathscr{F}$ such that for every $\langle F, K\rangle \in \mathscr{F}$ there exists a pair $\langle G, H\rangle \in \mathscr{M}$ such that $F \subseteq G$ and $K \subseteq H$. Suppose that for every $\langle F, K\rangle \in \mathscr{M}$ the F-generated substructure of the K-reduct of $\mathfrak{A}$ is a substructure of the K-reduct of some member of $\mathscr{K}$. Then $\mathfrak{A}$ can be isomorphically embedded in an ultraproduct of members of $\mathscr{K}$.*

Proof. For each $\langle F, K\rangle \in \mathscr{M}$ let $M_{FK} = \{\langle G, H\rangle \in \mathscr{M} : F \subseteq G$ and $K \subseteq H\}$. Then $\{M_{FK} : \langle F, K\rangle \in \mathscr{M}\}$ has the finite intersection property and so is included in an ultrafilter D on $\mathscr{M}$. For every $\langle F, K\rangle \in \mathscr{M}$ choose $\mathfrak{B}_{FK} \in \mathscr{K}$ such that the F-generated substructure of the K-reduct of $\mathfrak{A}$ is a substructure of the K-reduct of $\mathfrak{B}_{FK}$. Then there is a function f on A into $\prod_{\langle F,K\rangle \in \mathscr{M}} B_{FK}/D$ such that $f(a) = g/D$ with $g(F, K) = a$ whenever $a \in F$ and $\langle F, K\rangle \in \mathscr{M}$, and f is the required isomorphism.

From Theorems 1.1 through 1.3 we obtain the following two results

at once. Recall that $\mathscr{LCA}_\alpha$ is the class of all locally finite dimensional cylindric algebras of dimension α.

Lemma 1.4. *If $\mathfrak{A}$ is a $\mathscr{CA}_\alpha$ such that every finitely generated subalgebra of every finite reduct of $\mathfrak{A}$ is representable, then $\mathfrak{A}$ is isomorphically embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s.*

Theorem 1.5. *Every $\mathscr{RCA}_\alpha$ is isomorphically embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s.*

Problem 1. Is every $\mathscr{RCA}_\alpha$ elementarily embeddable in an ultraproduct of representable $\mathscr{LCA}_\alpha$'s?

Using a modified form of Theorem 1.3 it can be seen that appropriate generalizations of Theorems 1.1 and 1.2 would serve to answer this question positively.

Lemma 1.6. *An ultraproduct of $\mathscr{RCA}_\alpha$'s is an $\mathscr{RCA}_\alpha$.*

Proof. Suppose that for each $t \in T$ $\mathfrak{A}_t$ is a $\mathscr{CS}_\alpha$ with base U_t and x is in the universe B of $\prod_{t\in T} \mathfrak{A}_t$, D is an ultrafilter over T, and $x/D \neq 0$. It suffices to find a homomorphism f of $\prod_{t\in T} \mathfrak{A}_t/D$ into a $\mathscr{CS}_\alpha$ such that $f(x/D) \neq 0$. Now there is an $s \in \prod_{t\in T} U_t^\alpha$ such that $s_t \in x_t$ whenever $t \in T$ and $x_t \neq 0$. Let $s'(\varkappa)_t = s_t(\varkappa)$ for all $\varkappa < \alpha$ and $t \in T$, and let $s''(\varkappa) = s'(\varkappa)/D$ for all $\varkappa < \alpha$. Let $\mathscr{C}$ be a function on $\prod_{t\in T} U_t/D$ into $\prod_{t\in T} U_t$ such that $\mathscr{C}(y/D)/D = y/D$ for all $y \in \prod_{t\in T} U_t$. Now for each $w \in (\prod_{t\in T} U_t/D)^\alpha$ define π_w as follows:

$$\pi_w(\varkappa) = \begin{cases} s'(\varkappa) \text{ if } s''(\varkappa) = w(\varkappa), \\ \mathscr{C}(w(\varkappa)) \text{ if } s''(\varkappa) \neq w(\varkappa), \end{cases}$$

for all $\varkappa < \alpha$. The desired function f is defined as follows:

$$f(y/D) = \{w \in (\prod_{t\in T} U_t/D)^\alpha : \{t : pr_t \circ \pi_w \in y_t\} \in D\}$$

for all $y \in B$, where for each $t \in T$ pr_t is the function mapping B into A_t such that $pr_t(z) = z_t$ for all $z \in B$.

The problem arises whether an ultraproduct of $\mathscr{CS}_\alpha$'s is isomorphic to a $\mathscr{CS}_\alpha$. For α finite the preceding proof can easily be modified to give a positive answer. The answer is negative for α infinite, e.g., for $\alpha = \omega$. For, let T be a set of cardinality 2^{2^ω} and for each $t \in T$ let $\mathfrak{A}_t$ be the $\mathscr{CS}_\omega$ of all subsets of 2^ω; thus $U_t = 2$ for all $t \in T$. By Theorem 1.25 of Frayne–Morel–Scott [62] choose D such that $\prod_{t\in T} \mathfrak{A}_t/D$ has

cardinality $2^{2^{\omega}}$. Now $\prod_{t\in T} U_t/D$ has just two elements, and a $\mathscr{CS}_{\omega}$ with base 2 has $\leqslant 2^{2^{\omega}}$ elements. The equation $c_0c_1c_2(-d_{01}\cdot -d_{02}\cdot -d_{12})=0$ holds in $\mathfrak{A}_t$ for each $t\in T$ and hence also in $\prod_{t\in T} A_t/D$. It follows that $\prod_{t\in T}\mathfrak{A}_t/D$ is not isomorphic to a $\mathscr{CS}_{\omega}$.

Theorem 1.7. *Suppose $\mathfrak{A}\in\mathscr{CA}_{\alpha}$. Then the following conditions are equivalent:*

(i) $\mathfrak{A}\in\mathscr{RCA}_{\alpha}$;
(ii) *every finite reduct of $\mathfrak{A}$ is representable;*
(iii) *every finitely generated subalgebra of $\mathfrak{A}$ is representable;*
(iv) *every finitely generated subalgebra of every finite reduct of $\mathfrak{A}$ is representable.*

This theorem can be easily derived from the preceding results.

For the next theorem we suppose that we are given first-order languages $L_0, L_1, \ldots, L_{\omega}$ with sets of non-logical constants $N_0, N_1, \ldots, N_{\omega}$ respectively, such that $N_0\subseteq N_1\subseteq\ldots\subseteq N_{\omega}=\bigcup_{\varkappa<\omega} N_{\varkappa}$. Also suppose that for each $\varkappa\leqslant\omega$ $\Delta_{\varkappa}$ (resp., $\Gamma_{\varkappa}$) is a set of $L_{\varkappa}$-sentences (resp., $L_{\varkappa}$-formulas with sole free variable v_0) such that $\Delta_0\subseteq\Delta_1\subseteq\ldots\subseteq\Delta_{\omega}=\bigcup_{\varkappa<\omega}\Delta_{\varkappa}$ (resp., $\Gamma_0\subseteq\Gamma_1\subseteq\ldots\subseteq\Gamma_{\omega}=\bigcup_{\varkappa<\omega}\Gamma_{\varkappa}$).

Theorem 1.8. *Suppose that $\mathfrak{A}$ is an L_0-structure and that for every $\varkappa<\omega$ $\mathfrak{A}$ is a substructure of the N_0-reduct of an $L_{\varkappa}$-structure $\mathfrak{B}_{\varkappa}$ such that $\mathfrak{B}_{\varkappa}$ is a model of $\Delta_{\varkappa}$ and x satisfies φ in $\mathfrak{B}_{\varkappa}$ for each $x\in A$ and each φ in $\Gamma_{\varkappa}$.*

Then $\mathfrak{A}$ is a substructure of the N_0-reduct of an L_{ω}-structure $\mathfrak{C}$ such that $\mathfrak{C}$ is a model of Δ_{ω} and x satisfies φ in $\mathfrak{C}$ for each $x\in A$ and each φ in Γ_{ω}.

Proof. For each $\varkappa<\omega$, let $M_{\varkappa}=\{\lambda\in\omega : \varkappa\leqslant\lambda\}$. Then $\{M_{\varkappa} : \varkappa<\omega\}$ is included in an ultrafilter D on ω. The set $\prod_{\varkappa<\omega} B_{\varkappa}/D$ can be given a natural L_{ω}-structure, and one can easily define an isomorphic embedding of $\mathfrak{A}$ into the N_0-reduct of this structure so that the conclusion of the theorem can be satisfied.

Theorems 1.1–1.8 are all of a rather elementary nature. The more profound results of the representation theory stem from the following theorem, which will not be proved here. A proof can be found by consulting Halmos [1] and Galler [1][1].

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[1] Several simpler but unpublished proofs, due to Henkin and Tarski, are known. Perhaps the simplest was used in a modified form in Daigneault–Monk [63].

Theorem 1.9 (Tarski). *For $\alpha \geqslant \omega$, $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha \subseteq \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$.*

Theorem 1.10 [2]. *Suppose $\mathfrak{A} \in \mathscr{C}\mathscr{A}_\alpha$. Then the following conditions are equivalent:*

(i) $\mathfrak{A} \in \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$;
(ii) *for every $\varkappa < \omega$, $\mathfrak{A}$ can be neatly embedded in a $\mathscr{C}\mathscr{A}_{\alpha+\varkappa}$;*
(iii) *$\mathfrak{A}$ can be neatly embedded in a $\mathscr{C}\mathscr{A}_{\alpha+\omega}$.*

Proof. That (i) implies (ii) follows from Theorem 1.2; that (ii) implies (iii) is seen by Theorem 1.8. Now assume (iii) holds. Then each finite reduct of $\mathfrak{A}$ is a subalgebra of a reduct of an $\mathscr{L}\mathscr{C}\mathscr{A}_\omega$, and so by Theorem 1.9 is representable. Hence by Theorem 1.7 $\mathfrak{A}$ is representable.

We are now in a position to describe the class $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ in model-theoretic terms.

Theorem 1.11. *$\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is an equational class (in the wider sense); if $\alpha \geqslant \omega$, then $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is the least $\mathbf{UC}_\delta$ including $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$.*

Proof. The equivalence—1.10 (i) equivalent to 1.10 (iii)—can be used to show that $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is closed under the operation of taking homomorphic images, so $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$ is equational [3]. If $\mathscr{K}$ is a $\mathbf{UC}_\delta$ including $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$ and if $\mathfrak{A} \in \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$, then by Theorem 1.5 $\mathfrak{A}$ is a subalgebra of a member of $\mathscr{K}$, so $\mathfrak{A} \in \mathscr{K}$.

Problem 3. For $\alpha \geqslant \omega$: is there an $\mathbf{EC}_\delta$ $\mathscr{K}$ such that $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha \subseteq \subseteq \mathscr{K} \subset \mathscr{R}\mathscr{C}\mathscr{A}_\alpha$?

Actually we can give a more detailed description of the equational character of the class $\mathscr{R}\mathscr{C}\mathscr{A}_\alpha$, a description which shows the universal character of $\mathscr{L}\mathscr{C}\mathscr{A}_\omega$ for equations. To this end we introduce the following notation: if ε is an equation, Γ is the set of $\varkappa < \alpha$ such that a symbol $c_\varkappa$ or $d_{\varkappa\lambda}$ occurs in ε, and δ is a one-to-one function mapping Γ into ω, then by $\delta^*(\varepsilon)$ we mean the equation obtained from ε by replacing each index $\varkappa$ of symbols occurring in ε by $\delta(\varkappa)$; δ is called an *ω-mapping for ε*.

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[2] From this theorem one can derive many generalizations of Theorem 1.9 in a rather rapid fashion; see Monk [61]. Using the notation of that paper, the following new result follows from the present Theorem 1.5: $\mathscr{G}\mathscr{C}\mathscr{A}_\alpha$, $\mathscr{W}\mathscr{D}\mathscr{C}\mathscr{A}_\alpha$, and the class of semi-simple $\mathscr{C}\mathscr{A}_\alpha$'s ($\alpha \geqslant \omega$) are not $\mathbf{EC}_\delta$'s; and in the same way one can prove that $\mathscr{L}\mathscr{C}\mathscr{A}_\alpha$ ($\alpha \geqslant \omega$) is not an $\mathbf{EC}_\delta$ (this was known previously because of Feferman–Vaught [59], Corollary 6.7.2).

[3] For further details, see Monk [61].

Theorem 1.12. *For each equation ε in the elementary theory of $\mathscr{CA}_\alpha$'s let δ_ε be an ω-mapping for ε. Then $\mathscr{RCA}_\alpha$ is the class of all models of all equations ε such that $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega$.*

Proof. Let $\Sigma = \{\varepsilon : \varepsilon$ is an equation identically satisfied in every $\mathscr{RCA}_\alpha\}$ and let $\Gamma = \{\varepsilon : \delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega\}$. It suffices to show that $\Sigma = \Gamma$. First we take the case $\alpha \geqslant \omega$. Note first that if $\delta_\varepsilon \subseteq \gamma$ and γ is a permutation of α, then ε holds in a $\mathscr{CA}_\alpha$ $\mathfrak{A}$ iff $\delta_\varepsilon^*(\varepsilon)$ holds in the $\alpha\gamma$-reduct $\mathfrak{A}'$ of $\mathfrak{A}$; moreover $\mathfrak{A}$ is representable iff $\mathfrak{A}'$ is, and $\mathfrak{A}$ is locally finite iff $\mathfrak{A}'$ is. Hence $\varepsilon \in \Sigma$ iff $\delta_\varepsilon^*(\varepsilon) \in \Sigma$. Now suppose $\varepsilon \in \Sigma$. Then $\delta_\varepsilon^*(\varepsilon) \in \Sigma$, so in particular $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\alpha$ and hence by 1.9 and 1.2 in every $\mathscr{LCA}_\omega$; thus $\varepsilon \in \Gamma$. Suppose conversely that $\varepsilon \in \Gamma$. Then $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{LCA}_\omega$ and hence in every $\mathscr{LCA}_\alpha$, since the ω-reduct of an $\mathscr{LCA}_\alpha$ is an $\mathscr{LCA}_\omega$. By Theorem 1.5 $\delta_\varepsilon^*(\varepsilon) \in \Sigma$, so $\varepsilon \in \Sigma$.

Now we take the case $\alpha < \omega$. If $\varepsilon \in \Sigma$, then ε holds in every $\mathscr{RCA}_\omega$ by 1.1, so that $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{RCA}_\omega$ and in particular in every $\mathscr{LCA}_\omega$, and $\varepsilon \in \Gamma$. Assume $\varepsilon \in \Gamma$. Then by 1.5 $\delta_\varepsilon^*(\varepsilon)$ holds in every $\mathscr{RCA}_\omega$ and so ε holds in every $\mathscr{RCA}_\omega$. Hence by 1.2 $\varepsilon \in \Sigma$.

The well known argument of Craig [53] can be modified to prove the following theorem.

Theorem 1.13. *For $\alpha < \omega$ there is a primitive recursive set Σ of equations such that $\mathscr{RCA}_\alpha$ is the set of all models of Σ.*

Proof [4]. Let $\Gamma = \{\varepsilon : \varepsilon$ holds in every $\mathscr{RCA}_\alpha\}$. Then by Theorem 2.11 of Henkin–Tarski [61], Γ is recursively enumerable. Let f be a primitive recursive function enumerating Γ, and let $\Delta = \{\varepsilon :$ there is a $p \leqslant \varepsilon$ such that ε has the form $(\varphi - \psi) + \ldots + (\varphi - \psi) = 0$, where $f(p) = \varphi \doteq \psi$ and $-$ is symmetric difference$\}$. Clearly Δ is primitive recursive and $\mathscr{RCA}_\alpha$ is the set of all models of Δ.

The problem still remains whether or not $\mathscr{RCA}_\alpha$ is finitely axiomatizable for $\alpha < \omega$ [5]. We treat this problem in the next section.

2. Non-finite axiomatizability of $\mathscr{RCA}_3$. Since $\mathscr{RCA}_0 = \mathscr{CA}_0$ and $\mathscr{RCA}_1 = \mathscr{CA}_1$, these two classes are finitely axiomatizable. Also, $\mathscr{RCA}_2$ is

[4] For the purposes of this proof we shall identify formulas with natural numbers.

[5] If is also of interest to consider the case $\alpha \geqslant \omega$, where of course finite axiomatizability fails but where the problem of a kind of uniform infinite axiomatizability is still open.

finitely axiomatizable, since $\mathscr{RCA}_2$ is the class of all models of the equations defining $\mathscr{CA}_2$'s together with two further equations (see Henkin and Tarski [61], Theorem 2.18). The main purpose of this section, on the other hand, is to show that $\mathscr{RCA}_3$ is not finitely axiomatizable. The method used involves two steps. First, non-representable $\mathscr{CA}_3$'s are associated with numbers m for which there is no projective plane of order m, following the pattern of Lyndon [61]. Second, an ultraproduct of these non-representable algebras is shown to be representable.

If R is an equivalence relation on 3, let $\binom{R}{2} = \{\{\alpha\beta\} : \alpha \text{ and } \beta \text{ are distinct } R\text{-equivalence classes}\}$. Let G be any set with at least three elements. Let G' be the set of all pairs $\langle R, f\rangle$ such that R is an equivalence relation on 3, f maps $\binom{R}{2}$ into G, and the range of f does not consist of exactly two elements. Let $\langle A_G, \cup, \cap, \sim\rangle$ be the Boolean algebra formed from all subsets of G'. For $\varkappa, \lambda < 3$ define

$$d_{\varkappa\lambda} = \{\langle R, f\rangle : \langle \varkappa\lambda\rangle \in R\};$$

$$c_\varkappa\{\langle R, f\rangle\} = \{\langle S, g\rangle : S \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2 \text{ and } f\{\alpha\beta\} = g\{\gamma\delta\} \text{ whenever } 0 \neq \alpha \cap \gamma \neq \{\varkappa\} \text{ and } 0 \neq \beta \cap \delta \neq \{\varkappa\}\};$$

$$c_\varkappa a = \bigcup_{t \in a} c_\varkappa\{t\} \text{ if } a \in A_G.$$

Finally, let $\mathfrak{A}_G = \langle A_G, \cup, \cap, \sim, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda<3}$.

Lemma 2.1. *$\mathfrak{A}_G$ is a simple $\mathscr{CA}_3$, in which the condition*

$$c_\varkappa c_\lambda a = 1 \tag{*}$$

holds for $a \neq 0$ and $\varkappa \neq \lambda$.

Proof. We shall verify postulates P1–P8 of Henkin–Tarski [61], simplicity, and (*). Of these, P1, P2, P3 and P6 obviously hold for $\mathfrak{A}_G$. Since $c_\varkappa$ is clearly completely additive, it suffices for P4 to assume that x and y are atoms (singletons). Note also that P5 and simplicity are implied by (*), and that (*) only needs to be proved for a a singleton, in virtue of P3.

Verification of P4. We wish to show that $c_\varkappa(\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}) = c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. First suppose that $\langle T, h\rangle$ is a member of the left-hand side of this equation. Then $\langle R, f\rangle \in c_\varkappa\{\langle S, g\rangle\}$ and $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\}$. Thus $T \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2 = S \cap (3 \sim \{\varkappa\})^2$. If

$0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}$ with α and β T-equivalence classes while γ and δ are S-equivalence classes, let ε and ξ be R-equivalence classes such that $0 \neq \alpha \cap \gamma \cap \varepsilon \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \cap \xi \neq \{\varkappa\}$; then $h\{\alpha\beta\} = f\{\varepsilon\xi\} = g\{\gamma\delta\}$. Thus $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. Assume conversely that this membership relation holds. Then $R \cap (3 \sim \{\varkappa\})^2 = T \cap (3 \sim \{\varkappa\})^2 = S \cap (3 \sim \{\varkappa\})^2$. Suppose α and β are R-equivalence classes, γ and δ are S-equivalence classes, and $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$, $0 \neq \beta \cap \delta \neq \{\varkappa\}$. Then there are T-equivalence classes ε and ξ such that $0 \neq \alpha \cap \gamma \cap \varepsilon \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \cap \xi \neq \{\varkappa\}$. Hence $f\{\alpha\beta\} = h\{\varepsilon\xi\} = g\{\gamma\delta\}$. Thus $\langle R, f\rangle \in c_\varkappa\{\langle S, g\rangle\}$, and $\langle T, h\rangle \in c_\varkappa(\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\})$.

Verification of (*). We wish to show that $\langle S, g\rangle \in c_\varkappa c_\lambda\{\langle R, f\rangle\}$, with $\varkappa \neq \lambda$. Let $3 = \{\varkappa, \lambda, \mu\}$. Let $T = [R \cap (3 \sim \{\lambda\})^2] \cup [S \cap (3 \sim \{\varkappa\})^2] \cup \cup \{\langle \varkappa\lambda\rangle, \langle \lambda\varkappa\rangle : \langle \varkappa\mu\rangle \in R$ and $\langle \lambda\mu\rangle \in S\}$. Clearly T is an equivalence relation on 3. Let $\mathscr{H}$ be the set of functions on $\binom{T}{2}$ into G whose range has just one, or exactly three, elements. If $\langle \mu\lambda\rangle \in T$, choose $h \in \mathscr{H}$ so that $h\{[\mu]_T, [\varkappa]_T\} = f\{[\mu]_R, [\varkappa]_R\}$ if $\langle \mu\varkappa\rangle \notin R$ ($[\mu]_T$ is the T-equivalence class of which μ is a member). If $\langle \mu\varkappa\rangle \in T$, choose $h \in \mathscr{H}$ so that $h\{[\mu]_T, [\lambda]_T\} = g\{[\mu]_S, [\lambda]_S\}$ if $\langle \mu\lambda\rangle \notin S$. Finally, if $\langle \mu\lambda\rangle, \langle \mu\varkappa\rangle \notin T$, choose $h \in \mathscr{H}$ such that $h\{[\mu]_T, [\varkappa]_T\} = f\{[\mu]_R, [\varkappa]_R\}$ if $\langle \mu\varkappa\rangle \notin R$ and $h\{[\mu]_T, [\lambda]_T\} = g\{[\mu]_S, [\lambda]_S\}$ if $\langle \mu\lambda\rangle \notin S$. Then in any case $\langle T, h\rangle \in c_\lambda\{\langle R, f\rangle\}$ and $\langle S, g\rangle \in c_\varkappa\{\langle T, h\rangle\}$, so $\langle S, g\rangle \in c_\varkappa c_\lambda\{\langle R, f\rangle\}$.

Verification of P7. We wish to show that $d_{\varkappa\lambda} = c_\mu(d_{\varkappa\mu} \cdot d_{\mu\lambda})$, where $\varkappa, \lambda \neq \mu$. First suppose $\varkappa \neq \lambda$. Clearly $d_{\varkappa\mu} \cdot d_{\mu\lambda} = \{\langle 3^2, 0\rangle\}$ and hence the desired result follows. Second, suppose $\varkappa = \lambda$. Let $\langle R, F\rangle \in G'$. Let $3 = \{\varkappa\mu\nu\}$, and let

$$S = \begin{cases} 3^2 \text{ if } \langle \varkappa\nu\rangle \in R \\ \{\langle \varkappa\mu\rangle, \langle \varkappa\varkappa\rangle, \langle \mu\varkappa\rangle, \langle \mu\mu\rangle, \langle \nu\nu\rangle\} \text{ if } \langle \varkappa\nu\rangle \notin R. \end{cases}$$

Clearly S is an equivalence relation on 3.

Let $g = 0$ if $\langle \varkappa\nu\rangle \in R$, and let g be such that $g([\varkappa]_S, [\nu]_S) = f([\varkappa]_R, [\nu]_R)$ if $\langle \varkappa\nu\rangle \notin R$. Then $\langle S, g\rangle \in d_{\varkappa\mu}$ and $\langle R, f\rangle \in c_\mu\{\langle S, g\rangle\}$, so $\langle R, f\rangle \in c_\mu d_{\varkappa\mu}$.

Verification of P8. By the complete additivity of $c_\varkappa$ it suffices to show that $c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\} = 0$, where $\langle \varkappa\lambda\rangle \in R \cap S$, $\langle R, f\rangle \neq \langle S, g\rangle$, and $\varkappa \neq \lambda$. We cannot have $R = S = 3^2$, since then $f = g = 0$ and $\langle R, f\rangle = \langle S, g\rangle$. Say $\{\varkappa\lambda\mu\} = 3$. Suppose $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$. If $R = S$, then since $R \neq 3^2$ and $\langle \varkappa\lambda\rangle \in R$, we have $\langle \lambda\mu\rangle \notin R$, and so $\langle \lambda\mu\rangle \notin S$; hence $\langle \lambda\mu\rangle \notin T$ and $f([\lambda]_R, [\mu]_R) = h([\lambda]_T, [\mu]_T) = g([\lambda]_S, [\mu]_S)$, so that $f = g$ and

$\langle R, f\rangle = \langle S, g\rangle$, which contradicts an earlier assumption. Thus $R \neq S$. By symmetry we may assume that $R = 3^2$, while $S \neq 3^2$. In particular, $\langle \lambda\mu\rangle \in R \sim S$, which contradicts $\langle T, h\rangle \in c_\varkappa\{\langle R, f\rangle\} \cap c_\varkappa\{\langle S, g\rangle\}$.

Lemma 2.2. *If G has at least four elements and if G is a line in a projective plane P, then $\mathfrak{A}_G \in \mathscr{RCA}_3$* [6].

Proof. Let $U = P \sim G$. Define

$$F\{\langle R, f\rangle\} = \{x \in U^3 : x_\varkappa = x_\lambda \text{ iff } \langle \varkappa\lambda\rangle \in R, \text{ and } f\{\alpha\beta\} \in \overline{x_\varkappa x_\lambda} \text{ whenever } \varkappa \in \alpha, \lambda \in \beta, \text{ and } \alpha \neq \beta\};$$

$$F(a) = \bigcup_{t \in a} F\{t\}$$

for $a \in A_G$. Clearly F preserves $\bigcup$. If $x \in F\{\langle R, f\rangle\} \cap F\{\langle S, g\rangle\}$, then $\langle \varkappa\lambda\rangle \in R$ iff $x_\varkappa = x_\lambda$ iff $\langle \varkappa\lambda\rangle \in S$, and so $R = S$; since furthermore two distinct lines intersect in exactly one point, we have $f = g$. It follows that $F(a) \cap F(-a) = 0$ for all $a \in A_G$. If $x \in U^3$, let $R = \{\langle \varkappa\lambda\rangle : x_\varkappa = x_\lambda\}$; thus R is an equivalence relation on 3. Let $f\{\alpha\beta\} = \overline{x_\varkappa x_\lambda} \cdot G$ whenever $\varkappa \in \alpha$, $\lambda \in \beta$, and $\alpha \neq \beta$. Then f maps $\binom{R}{2}$ into G, and the range of f does not consist of exactly two elements. Clearly $x \in F\{\langle R, f\rangle\}$. Hence $F(a) \cup F(-a) = 1$ for all $a \in A_G$, and F preserves $-$. Let $\langle R, f\rangle \in G'$ be given. If $R = 3^2$, let $x \in U^3$ be such that $x_0 = x_1 = x_2$. If R has exactly two equivalence classes $\alpha = \{\varkappa\lambda\}$ and $\beta = \{\mu\}$, let x_μ be a point not on G and let $x_\varkappa = x_\lambda$ be a further point on the line $\overline{x_\mu f\{\alpha\beta\}}$. If R has three equivalence classes but the range of f is a singleton $\{u\}$, let x_0 be a point not on G and let x_1 and x_2 be two further points on the line $\overline{x_0 u}$. Finally, if R has three equivalence classes and the range of f consists of three distinct elements, let x_0 be a point not on G, let x_1 be a further point on the line $\overline{x_0 f\{\{0\}\{1\}\}}$, and let $x_2 = \overline{x_0 f\{\{0\}\{2\}\}} \cdot \overline{x_1 f\{\{1\}\{2\}\}}$. In any of these cases, $x \in F\{\langle R, f\rangle\}$. Hence $F(a) \neq 0$ whenever $a \neq 0$, and so F is one-to-one. Clearly $F(d_{\varkappa\lambda}) = D_{\varkappa\lambda}$ whenever $\varkappa, \lambda < 3$. Finally, suppose $x \in F(c_\varkappa\{\langle R, f\rangle\})$. Say $x \in F\{\langle S, g\rangle\}$, where $S \cap (3 \sim \{\varkappa\})^2 = R \cap (3 \sim \{\varkappa\})^2$ and $f\{\alpha\beta\} = g\{\gamma\delta\}$ whenever $0 \neq \alpha \cap \gamma \neq \{\varkappa\}$ and $0 \neq \beta \cap \delta \neq \{\varkappa\}$. Let $y_\lambda = x_\lambda$ for every $\lambda \in 3 \sim \{\varkappa\}$. If $\{\varkappa\}$ is not an R-equivalence class, say

[6] Throughout the remainder of the paper we follow the notation of Pickert [55] for projective planes, except that: a line is considered to be the set of all points lying on it, $\overline{PQ}$ is the line joining P and Q, and $l \cdot m$ is the intersection point of l and m.

$\langle \varkappa\lambda\rangle \in R$ and let $y_\varkappa = x_\lambda$. If the R-equivalence classes are $\{\varkappa\}$ and $\{\lambda\mu\}$, let $y_\varkappa$ be a third point on the line $\overline{x_\lambda f\{\{\varkappa\}\{\lambda\mu\}\}}$. If the R-equivalence classes are $\{\varkappa\}$, $\{\lambda\}$ and $\{\mu\}$ and if the range of f is a singleton, let $y_\varkappa$ be a fourth point on the line through $f\{\{\lambda\}\{\mu\}\}$, x_λ and x_μ. Lastly if the R-equivalence classes are $\{\varkappa\}$, $\{\lambda\}$ and $\{\mu\}$ and if the range of f has exactly three elements, let $y_\varkappa = \overline{x_\lambda f\{\{\varkappa\}\{\lambda\}\}} \cdot \overline{x_\mu f\{\{\varkappa\}\{\mu\}\}}$. In any of these cases it is easily seen that $y \in F\{\langle R, f\rangle\}$, so that $x \in C_\varkappa F\{\langle R, f\rangle\}$. Thus $F(c_\varkappa\{\langle R, f\rangle\}) \subseteq C_\varkappa F\{\langle R, f\rangle\}$. The converse is verified in a similar manner.

Lemma 2.3. *If G has at least four elements but is finite, and if $\mathfrak{A}_G$ is representable, then G is a line in some projective plane.*

Proof. Since $\mathfrak{A}_G$ is simple, the hypothesis implies the existence of an isomorphism F of $\mathfrak{A}_G$ onto a $\mathscr{CS}_3$ with base U. We may assume that $U \cap G = 0$. Let $P = G \cup U$; members of P are the points of our projective plane. For the lines of P we take G together with all sets

$$L(x, u) = \{x, u\} \cup \{v : \langle vuu\rangle \in F\{\langle R, f\rangle\}\},$$

where $x \in G$, $u \in U$, R has equivalence classes $\{0\}$ and $\{12\}$, and $f\{\{0\}\{12\}\} = x$. To show that we have a projective plane, we shall verify the conditions (1.1), (1.2), (1.16), (1.17) and (1.18) of Pickert [55]. To do this we first need two auxiliary results:

(a) If $u, v \in U$, $x \in G$, and $v \in L(x, u)$, then $u \in L(x, v)$.

To prove (a), we may assume that $u \neq v$. Let S have equivalence classes $\{01\}$, $\{2\}$, and let g be such that $g\{\{01\}\{2\}\} = x$. Then with R and f as above we have $d_{01} \cdot c_1\{\langle R, f\rangle\} = \{\langle S, g\rangle\}$. Hence $D_{01} \cap C_1 F\{\langle R, f\rangle\} = F\{\langle S, g\rangle\}$. By assumption $\langle vuu\rangle \in F\{\langle R, f\rangle\}$, so $\langle vvu\rangle \in F\{\langle S, g\rangle\}$. Now let T have equivalence classes $\{02\}$, $\{1\}$, and let h be such that $h\{\{02\}\{1\}\} = x$. Then $d_{02} \cdot c_0\{\langle S, g\rangle\} = \{\langle T, h\rangle\}$, so $\langle uvu\rangle \in F\{\langle T, h\rangle\}$. Also, $d_{12} \cdot c_2\{\langle T, h\rangle\} = \{\langle R, f\rangle\}$, so $\langle uvv\rangle \in F\{\langle R, f\rangle\}$ and $u \in L(x, v)$.

(b) If $u, v \in U$, $x \in G$, and $v \in L(x, u)$, then $L(x, u) = L(x, v)$.

To prove (b), by (a) and symmetry it suffices to show that $L(x,u) \subseteq L(x,v)$. Suppose $w \in L(x, u)$ and $u \neq v \neq w \neq u$. We are given that $\langle wuu\rangle \in \in F\{\langle R, f\rangle\}$ and $\langle vuu\rangle \in F\{\langle R, f\rangle\}$. As in the proof of (a) we see that $\langle vvu\rangle \in F\{\langle S, g\rangle\}$. Let V have equivalence classes $\{0\}$, $\{1\}$, $\{2\}$, with k mapping $\binom{V}{2}$ onto $\{x\}$. Then $c_1\{\langle R, f\rangle\} \cap c_0\{\langle S, g\rangle\} = \{\langle S, g\rangle, \langle V, k\rangle\}$.

Hence $\langle wvu\rangle \in F\{\langle S, g\rangle\} \cup F\{\langle V, k\rangle\}$; since clearly $\langle wvu\rangle \notin F\{\langle S, g\rangle\}$, we have $\langle wvu\rangle \in F\{\langle V, k\rangle\}$. Now $d_{12} \cdot c_2\{\langle V, k\rangle\} = \{\langle R, f\rangle\}$, so $\langle wvv\rangle \in F\{\langle R, f\rangle\}$ and hence $w \in L(x, v)$.

Now to verify the axioms for a projective plane, we note that (1.1) and (1.16) are obvious. To verify (1.2), suppose that P_i lies on l_k, $i, k = 1, 2$, while $l_1 \neq l_2$. If one of l_1, l_2 is G, then obviously $P_1 = P_2$. Suppose $l_1 = L(x, u)$ and $l_2 = L(y, v)$. If $P_1 \in G$, then $P_1 = x = y$. If $P_2 \in G$, then $P_1 = P_2$. If $P_2 \notin G$, then $L(x, u) = L(x, P_2) = L(y, v)$, which contradicts the assumption $l_1 \neq l_2$. One argues similarly if $P_2 \in G$. Suppose $P_1, P_2 \notin G$. Then $L(x, u) = L(x, P_1)$ and $L(y, v) = L(y, P_1)$; since $P_2 \in L(x, P_1) \cap L(y, P_1)$ we have $\langle P_2 P_1 P_1\rangle \in F\{\langle R, f\rangle\} \cap F\{\langle R, g\rangle\}$, where the range of f is $\{x\}$ and the range of g is $\{y\}$. Hence $x = y$ since F preserves $-$, and $L(x, u) = L(y, v)$, again a contradiction. Thus (1.2) holds.

To verify (1.17), note that G intersects each line $L(x, u)$. Now suppose two distinct lines $L(x, u)$ and $L(y, v)$ are given. We may assume that $x \neq y$ and $u \neq v$. Let R have equivalence classes $\{0\}$ and $\{12\}$ and let f and g map $\binom{R}{2}$ onto $\{x\}$ and $\{y\}$ respectively. Let S have equivalence classes $\{01\}$ and $\{2\}$. Then it is easily seen that

(c) $d_{01} \cdot c_0(c_1\{\langle R, f\rangle\} \cap c_2\{\langle R, g\rangle\}) = \{\langle S, h\rangle : h \text{ maps } \binom{S}{2} \text{ onto } \{z\}, \text{ with } z \neq x, y\}$.

Now since $\mathfrak{A}_G$ is finite, there is an h such that $\langle vvu\rangle \in F\{\langle S, h\rangle\}$ [7]. Say h maps $\binom{S}{2}$ onto $\{z\}$. If $z = x$, then $\langle vuu\rangle \in F\{\langle R, f\rangle\}$ so that $v \in L(x, u) \cap L(y, v)$. If $z = y$, then $\langle vuu\rangle \in F\{\langle R, g\rangle\}$ so that $u \in L(y, v) \cap L(x, u)$. Assume $z \neq x, y$. Then by (c) choose w such that $\langle wvu\rangle \in F(c_1\{\langle R, f\rangle\} \cap c_2\{\langle R, g\rangle\})$. Then $\langle wuu\rangle \in F\{\langle R, f\rangle\}$ and $\langle wvv\rangle \in F\{\langle R, g\rangle\}$, so $w \in L(x, u) \cap L(y, v)$. Thus (1.17) holds.

To prove (1.18), let x, y, z be distinct elements of G, and let $u \in U$. If R has equivalence classes $\{0\}$ and $\{12\}$ and f maps $\binom{R}{2}$ onto $\{z\}$, then $c_0\{\langle R, f\rangle\} = d_{12}$. Hence $\langle uuu\rangle \in C_0 F\{\langle R, f\rangle\}$, so choose v such that $\langle vuu\rangle \in F\{\langle R, f\rangle\}$. It is easily seen that x, y, u, v are four points no three of which are collinear.

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[7] It is only here that the assumption that G is finite is used. The necessity for some assumption to assure the existence of h was first noticed in another context by Tarski; see Lyndon [56].

The above three lemmas were suggested by the work of Lyndon [61] on relation algebras. Now we can prove the main result of this section.

Theorem 2.4. *$\mathscr{RCA}_3$ is not finitely axiomatizable* [8].

Proof. Let $M = \{m \in \omega : 3 \leqslant m$ and there is no projective plane of order $m\}$. By the Bruck–Ryser theorem [49], M is infinite. For each $m \in M$ let $K_m = \{n \in M : m \leqslant n\}$. Then $\{K_m : m \in M\}$ has the finite intersection property and so there is an ultrafilter D on the set of all subsets of M such that $\{K_m : m \in M\} \subseteq D$. For each $m \in M$ let G_m be a set with $m+1$ elements. Then by Lemma 2.3, $\mathfrak{A}_{G_m}$ is non-representable for each $m \in M$. The proof will be complete when we show that $\mathfrak{B} = \prod_{m \in M} \mathfrak{A}_{G_m}/D$ is representable.

Let $H = \prod_{m \in M} G_m/D$. By Theorem 6.5 of Kochen [61], H has $2^{\aleph_0}$ elements. Hence by Lemma 2.2, $\mathfrak{A}_H$ is representable. We shall show that $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}_H$. Using Theorem 2.2 of Frayne–Morel–Scott [62], we easily see that $\mathfrak{B}$ is atomic and that for each atom b of $\mathfrak{B}$ there is an equivalence relation R on 3 and a function

$$g \in \prod_{m \in M} G_m{}^{\binom{R}{2}}$$

such that $b = h/D$, where $h_m = \{\langle R, g_m \rangle\}$ for each $m \in M$. We let $h = \mathscr{A}(R, g)$, and we define g' mapping $\binom{R}{2}$ into $\prod_{m \in M} G_m$ as follows:

$$(g'\{\alpha\beta\})_m = g_m\{\alpha\beta\}$$

for all $\{\alpha\beta\} \in \binom{R}{2}$ and all $m \in M$. If k maps $\binom{R}{2}$ into $\prod_{m \in M} G_m$, define k^* as follows:

$$(k^*_m)\ \{\alpha\beta\} = (k\{\alpha\beta\})_m;$$

thus $k^* \in \prod_{m \in M} G_m{}^{\binom{R}{2}}$.

Let $\mathscr{C}$ be a choice function for H and let π be the natural mapping of $\prod_{m \in M} G_m$ onto H. The isomorphism F embedding $\mathfrak{B}$ in $\mathfrak{A}_H$ is defined as follows:

$$F(b) = \{\langle R, g \rangle : \mathscr{A}(R, (\mathscr{C} \circ g)^*)/D \leqslant b\}$$

for all $b \in B$. Clearly $F(\sum X) = \bigcup_{x \in X} F(x)$ whenever $\sum X$ exists; F preserves $-$; and F is one-to-one. If $\varkappa, \lambda < 3$, then

[8] Building upon Lyndon [61] the corresponding result for relation algebras can also be proved in a fashion analogous to the proof below. Theorem 2.4 is not surprising then, in view of Monk [61a].

$$\begin{aligned}\langle R, g\rangle \in F(d_{\kappa\lambda}) \quad &\text{iff } \{m : \langle R, (\mathscr{C} \circ g)^*_m\rangle \in d_{\kappa\lambda}\} \in D\\ &\text{iff } \{m : \langle \kappa\lambda\rangle \in R\} \in D\\ &\text{iff } \langle \kappa\lambda\rangle \in R\\ &\text{iff } \langle R, g\rangle \in d_{\kappa\lambda}.\end{aligned}$$

To show that $F(c_\kappa b) = c_\kappa F(b)$ for all $b \in B$ it suffices to take the case in which b is an atom $\mathscr{A}(R, f)/D$. Assume $\langle S, g\rangle \in F(c_\kappa \mathscr{A}(R, f)/D)$. Then $\mathscr{A}(S, (\mathscr{C} \circ g)^*)/D \leqslant c_\kappa \mathscr{A}(R, f)/D$ and hence $\{m : \langle S, (\mathscr{C} \circ g)^*_m\rangle \in c_\kappa\{\langle R, f_m\rangle\}\} \in D$. Thus $S \cap (3 \sim \{\kappa\})^2 = R \cap (3 \sim \{\kappa\})^2$ and $\{m : (\mathscr{C} \circ g)^*_m\{\alpha\beta\} = f_m\{\gamma\delta\}$ whenever $0 \neq \alpha \cap \gamma \neq \{\kappa\}$ and $0 \neq \beta \cap \delta \neq \{\kappa\}\} \in D$. It follows that $(\mathscr{C} \circ g)\{\alpha\beta\}/D = f'\{\gamma\delta\}/D$ whenever $0 \neq \alpha \cap \gamma \neq \{\kappa\}$ and $0 \neq \beta \cap \delta \neq \{\kappa\}$; hence $g\{\alpha\beta\} = (\pi \circ f')\{\gamma\delta\}$ under these circumstances, and thus $\langle S, g\rangle \in$ $\in c_\kappa\{\langle R, \pi \circ f'\rangle\}$. Clearly $\{\langle R, \pi \circ f'\rangle\} = F(\mathscr{A}(R, f)/D)$, so $\langle S, g\rangle \in$ $\in c_\kappa F(\mathscr{A}(R, f)/D)$. Hence $F(c_\kappa \mathscr{A}(R, f)/D) \subseteq c_\kappa F(\mathscr{A}(R, f)/D)$, and the converse is proved in a similar fashion. This completes the proof.

The above method does not immediately extend to higher dimensions, so the following problem is open.

Problem 4. For $3 < \alpha < \omega$, is $\mathscr{RCA}_\alpha$ finitely axiomatizable?

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REDUCTIONS OF HIGHER-ORDER LOGIC *

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A higher-order logic is considered which contains variables of all finite and transfinite types, together with a more restricted logic in which the types of variables are all in a sense describable ordinals. It is shown that several important problems connected with the restricted higher-order logic, pertaining, for instance, to the determination of spectra and the characterization of logical truth, are reducible to the corresponding problems for second-order logic; the principal results of this sort extend earlier results of Zykov and Hintikka. It is obtained as a corollary that the set of logical truths of second-order logic does not appear in any reasonable extension of the Kleene hierarchy.

1. Preliminaries. As ingredients of *second-order formulas* we assume the following disjoint categories of *symbols* to be available: (1) the usual sentential connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and brackets, (2) a denumerable set of *individual variables*, (3) for each positive integer n, a denumerable set of n-place *predicate variables*, (4) the two quantifiers $\wedge$ and $\vee$ (respectively universal and existential), (5) for each natural number n, a proper class [1] of n-place *predicate constants*.

Each symbol is assumed to be a 1-place sequence. By a *second-order expression* is understood a finite sequence each of whose 1-place subse-

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[1] As a basis for the metatheory is assumed a system of set theory involving both sets (which are members of something) and proper classes (which are members of nothing); a class is either a set or a proper class. To be explicit, I have in mind the axomiatic theory of Bernays–Morse, including the Axiom of Choice, as presented in the Appendix of Kelley [55]. The *ordinals* are defined in such a way that each ordinal is the set of all smaller ordinals, and the *cardinal* of a set A is the least ordinal with the same cardinality as A. Thus the cardinals form a subclass of the ordinals.

quences is a symbol of one of the sorts listed above. The class of *second-order formulas* is the least class Φ such that (1) Φ contains all second-order expressions $\mathsf{P}\mathsf{u}_0 \ldots \mathsf{u}_{n-1}$, where n is a positive integer, P is an n-place predicate constant or predicate variable, and $\mathsf{u}_0, \ldots, \mathsf{u}_{n-1}$ are individual variables, (2) Φ is closed under the application of sentential connectives, and (3) $\wedge \mathsf{u}\phi$ and $\vee \mathsf{u}\phi$ are in Φ whenever ϕ is in Φ and u is either an individual variable or a predicate variable.

We shall employ the usual notion of a model with one modification: models with an empty universe will be admitted. Thus a *model* is to be a pair $\langle A, F \rangle$ such that A is a set (possibly empty), F is a function whose domain is a set of predicate constants, and whenever P is an n-place predicate constant in the domain of F, $F(\mathsf{P})$ is an n-place relation on A (that is, a set of ordered n-tuples of members of A).

If $\mathfrak{A} = \langle A, F \rangle$, then we understand by the *universe* of $\mathfrak{A}$, or $U_{\mathfrak{A}}$, the set A, and by the *language* of $\mathfrak{A}$ the domain of F; if P is in the language of $\mathfrak{A}$, then $\mathsf{P}_{\mathfrak{A}}$ is to be $F(\mathsf{P})$. If $\mathfrak{A}, \mathfrak{B}$ are models, then $\mathfrak{A}$ is said to be *isomorphic* to $\mathfrak{B}$ or a *submodel* of $\mathfrak{B}$ under the usual conditions (including the requirement that $\mathfrak{A}$ and $\mathfrak{B}$ have the same language). By an *L-model* is understood a model whose language is L. With each model $\mathfrak{A}$ we can associate an *isomorphism type* $\tau\mathfrak{A}$ in such a way that (1) $\tau\mathfrak{A}$ is always a set, and (2) for any models $\mathfrak{A}$ and $\mathfrak{B}$ having the same language, $\tau\mathfrak{A} = \tau\mathfrak{B}$ if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ [2]. If L is a set of predicate constants, then by an *L-type* we understand an isomorphism type α such that $\alpha = \tau\mathfrak{A}$ for some L-model $\mathfrak{A}$. If α, β are isomorphism types, we say that $\alpha \leqslant \beta$ if and only if there are models $\mathfrak{A}, \mathfrak{B}$ such that $\tau\mathfrak{A} = \alpha$, $\tau\mathfrak{B} = \beta$, and $\mathfrak{A}$ is a submodel of $\mathfrak{B}$.

Definition 1. *If ϕ is a second-order formula whose free variables are all individual variables and in the standard ordering are $\mathsf{v}_0, \ldots, \mathsf{v}_{n-1}$, then ϕ is said to be* satisfied *in a model $\mathfrak{A}$ by a sequence $\langle a_0, \ldots, a_{n-1} \rangle$ of elements of $U_{\mathfrak{A}}$ if* (1) *the predicate constants occurring in ϕ are all in the language of $\mathfrak{A}$, and* (2) *ϕ is true when* (i) *each predicate constant P occurring in ϕ is regarded as denoting $\mathsf{P}_{\mathfrak{A}}$,* (ii) *the variables $\mathsf{v}_0, \ldots, \mathsf{v}_{n-1}$ are regarded (when occurring freely) as denoting $a_0, \ldots, a_{n-1}$ respectively,* (iii) *the bound*

[2] We could for instance define $\tau\mathfrak{A}$ as the set of all models $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ and $U_{\mathfrak{B}}$ is the cardinal of $U_{\mathfrak{A}}$. I prefer not to select at this point any particular definition of isomorphism type but to employ only the properties (1) and (2). This will permit the later convenience of identifying particular kinds of isomorphism types with particular set-theoretical objects in an *ad hoc* manner.

individual variables of ϕ are regarded as ranging over $U_{\mathfrak{A}}$, and (iv) *for each positive integer n, the bound n-place predicate variables of ϕ are regarded as ranging over the set of all n-place relations on $U_{\mathfrak{A}}$.* (In this definition, as in others below, an intuitive notion of truth is used, but one which can be exactly defined by well-known methods, like those in Tarski and Vaught [57]. Slight modifications, required to accommodate empty models, are spelled out in Kalish and Montague [a].) *We then say that ϕ is* true *in $\mathfrak{A}$ if ϕ contains no free variables and ϕ is satisfied in $\mathfrak{A}$ by the empty sequence Λ.*

As ingredients of *higher-order formulas* we assume the following disjoint categories of *symbols* to be available: (1) the sentential connectives, brackets, quantifiers, and predicate constants introduced above, (2) for each ordinal α, a denumerable set of *variables of type α* (the variables of types 0 and 1 coinciding respectively with our former individual variables and 1-place predicate variables), and (3) the symbol η, regarded as indicating membership, and the identity symbol $=$.

As before, each of these symbols is assumed to be a 1-place sequence, and a *higher-order expression* is to be a finite sequence each of whose 1-place subsequences is one of these symbols. The class of *higher-order formulas* is the least class Φ such that (1) Φ contains all higher-order expressions $\mathsf{u}\eta\mathsf{v}$, $\mathsf{u}=\mathsf{v}$, $\mathsf{P}\mathsf{u}_0 \ldots \mathsf{u}_{n-1}$, where u, v are variables of arbitrary type, n is a positive integer, P is an n-place predicate constant, and $\mathsf{u}_0, \ldots, \mathsf{u}_{n-1}$ are variables of type 0, (2) Φ is closed under the application of sentential connectives, and (3) $\wedge\mathsf{u}\phi$ and $\vee\mathsf{u}\phi$ are in Φ whenever ϕ is in Φ and u is a variable of any type.

We shall now extend the notions of satisfaction and truth to arbitrary higher-order formulas. A consideration of transfinite types suggests departing from a direct generalization of Definition 1; contrary to the course exemplified in clause (iv) of that definition, we shall regard the types as cumulative. There are several reasons for considering the notion introduced below to be the correct notion of truth for higher-order formulas; but we shall proceed rather arbitrarily here, leaving a detailed justification of the definition to the paper Montague [a].

Let ε be a 2-place predicate constant and Σ a 1-place predicate constant; we regard these symbols as expressing membership and the property of being a class (or non-individual), respectively. Let $\mathsf{t}, \ldots, \mathsf{z}$ be distinct individual variables and P a 1-place predicate variable. We understand by $\mathrm{RFS}_{\varepsilon, \Sigma, \mathsf{t}, \ldots, \mathsf{z}, \mathsf{P}}$ the universal closure of the conjunction

of the following formulas (which are given in Montague–Scott–Tarski [a] as the axioms for *second-order rank-free set theory with individuals*; both here and below ε is for perspicuity written between its two arguments rather than before them, as strict conformity to our definitions would require):

$$\Lambda x\,[x\varepsilon u \leftrightarrow x\varepsilon v] \wedge \Sigma u \wedge \Sigma v \rightarrow \Lambda P\,[Pu \leftrightarrow Pv],$$

$$\Sigma u \rightarrow \Lambda P \vee v\,[\Sigma v \wedge \Lambda x\,[x\varepsilon v \leftrightarrow x\varepsilon u \wedge Px]],$$

$$\Sigma u \rightarrow \vee w \Lambda y\,[y\varepsilon w \leftrightarrow \neg\, \Sigma y \vee \vee v(v\varepsilon u \wedge \Lambda t\,[t\varepsilon y \rightarrow t\varepsilon v])],$$

$$\Lambda x \vee z \Lambda u \Lambda w\,[\Lambda y\,[y\varepsilon u \rightarrow y\varepsilon z] \wedge \Lambda y\,[y\varepsilon w \leftrightarrow \neg\, \Sigma y \vee \vee v(v\varepsilon u \wedge \Lambda t\,[t\varepsilon y \rightarrow t\varepsilon v])] \rightarrow w\varepsilon z \vee \Lambda y\,[y\varepsilon x \rightarrow y\varepsilon w]],$$

$$y\varepsilon x \rightarrow \Sigma x.$$

Let $\mathfrak{A}$ be an $\{\varepsilon, \Sigma\}$-model in which the second-order formula $\mathrm{RFS}_{\varepsilon, \Sigma, t, \ldots, z, P}$ is true. Then $T_{\mathfrak{A}}(0)$ is to be the set of individuals of $\mathfrak{A}$; more generally, if α is any ordinal, $T_{\mathfrak{A}}(\alpha)$ is to be the αth cumulative type in $\mathfrak{A}$, in accordance with the following recursive definition. (1) $T_{\mathfrak{A}}(0)$ is the set of objects x in $U_{\mathfrak{A}}$ such that $\langle x \rangle$ is not in $\Sigma_{\mathfrak{A}}$. (2) $T_{\mathfrak{A}}(\alpha+1)$ is the set of objects x in $U_{\mathfrak{A}}$ such that for all y, if $\langle y, x \rangle$ is in $\varepsilon_{\mathfrak{A}}$, then y is in $T_{\mathfrak{A}}(\alpha)$. (3) if λ is a limit number, then $T_{\mathfrak{A}}(\lambda)$ is the union of the sets $T_{\mathfrak{A}}(\alpha)$ for $\alpha < \lambda$. It turns out that $U_{\mathfrak{A}}$ is $T_{\mathfrak{A}}(\alpha)$ for some ordinal α; we call the least such α the *rank* of $\mathfrak{A}$.

Definition 2. *If ϕ is a higher-order formula whose free variables are all individual variables and are in the standard ordering* $v_0, \ldots, v_{n-1}$, *then we say that ϕ is* satisfied *in a model $\mathfrak{A}$ by a sequence $\langle a_0, \ldots, a_{n-1} \rangle$ of elements of $U_{\mathfrak{A}}$ if* (1) *the predicate constants occurring in ϕ are all in the language of $\mathfrak{A}$, and there exist ε, Σ,* t, …, z, P, $\mathfrak{B}$ *such that* (2) *ε, Σ are respectively a 2-place and a 1-place predicate constant,* (3) t, …, z *are distinct individual variables,* (4) P *is a 1-place predicate variable,* (5) $\mathfrak{B}$ *is an $\{\varepsilon, \Sigma\}$-model in which* $\mathrm{RFS}_{\varepsilon, \Sigma, t, \ldots, z, P}$ *is true,* (6) $T_{\mathfrak{B}}(0)$ *is* $U_{\mathfrak{A}}$, (7) *all variables occurring in ϕ have types which are at most the rank of $\mathfrak{B}$, and* (8) *ϕ is true when* (i) *each predicate constant* P *occurring in ϕ is regarded as denoting* $P_{\mathfrak{A}}$, (ii) *η is interpreted as $\varepsilon_{\mathfrak{B}}$ and $=$ as the identity relation,* (iii) *the variables* $v_0, \ldots, v_{n-1}$ *are regarded (when occurring freely) as denoting $a_0, \ldots, a_{n-1}$ respectively, and* (iv) *for each ordinal α, the bound variables of ϕ of type α are regarded as ranging over the set $T_{\mathfrak{B}}(\alpha)$. As before, ϕ is said to be* true *in $\mathfrak{A}$ if ϕ contains no free variables and ϕ is satisfied in $\mathfrak{A}$ by Λ.*

Definitions 1 and 2 overlap in their range of applicability, but there is no danger of ambiguity. If ϕ is both a second-order formula and a higher-order formula, then all variables occurring in ϕ are of type 0, and ϕ is satisfied (or true) in the sense of Definition 1 if and only if ϕ is satisfied (or true) in the sense of Definition 2.

By a *formula* is understood either a second-order formula or a higher-order formula, and by a *sentence* a formula in which no variable is free. Two formulas ϕ and ψ with the same free variables are said to be *equivalent* if, for any model $\mathfrak{A}$ such that the language of $\mathfrak{A}$ contains all predicate constants occurring in ϕ or ψ, and any finite sequence a of elements of $U_{\mathfrak{A}}$ having as many places as ϕ and ψ have free variables, ϕ is satisfied in $\mathfrak{A}$ by a if and only if ψ is satisfied in $\mathfrak{A}$ by a.

To each second-order formula ϕ containing no predicate variables of more than one place, there corresponds in a natural way a higher-order formula ϕ'; we obtain ϕ' by replacing each subformula Pu, where P is a 1-place predicate variable and u an individual variable, by $\mathsf{u}\eta\mathsf{P}$. It can then be shown that ϕ is equivalent to ϕ'. Thus the decision to employ cumulative types has no effect on the truth of second-order formulas.

Lemma 1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic models and ϕ is a sentence containing no predicate constants beyond those in the language of $\mathfrak{A}$, then ϕ is true in $\mathfrak{A}$ if and only if ϕ is true in $\mathfrak{B}$.*

A direct generalization of second-order logic would require the introduction of predicate variables not only of various types but also of various numbers of places. The complications of such a course do not, however, seem worthwhile. What can be expressed with predicate variables of several places can be equivalently expressed with 1-place predicate variables, provided that one is willing to ascend in type.

By LT, or the class of *logical truths*, is understood the class of sentences ϕ such that ϕ is true in every model whose language contains all predicate constants occurring in ϕ.

The higher-order formulas clearly form a proper class; to change this we must restrict both the predicate constants and the types of variables that may occur. Let L be a set of predicate constants and Γ a set of ordinals. (In particular, Γ may be an ordinal.) Then by Γ-Fm_L, or the set of *Γ-order formulas* of L, is understood the set of higher-order formulas ϕ such that all variables occurring in ϕ have types in Γ and all predicate

constants occurring in ϕ are in L; Γ-St_L, or the set of *Γ-order sentences* of L, is the set of Γ-order formulas of L having no free variables.

By II-Fm_L and II-St_L we understand respectively the set of second-order formulas all of whose predicate constants are in L and the set of ϕ in II-Fm_L such that ϕ has no free variables. By ESF_L, or the set of *existential second-order formulas* of L, is understood the set of ϕ in II-Fm_L such that ϕ has the form $\mathsf{V}\mathsf{P}\psi$, where P is a 1-place predicate variable and no predicate variable is bound in ψ; USF_L is the set of negations of formulas in ESF_L.

Let ϕ be a sentence and L the set of predicate constants occurring in ϕ. By the *spectrum* of ϕ we understand the class of isomorphism types α such that ϕ is true in some L-model $\mathfrak{A}$ for which $\tau\mathfrak{A}=\alpha$. By $\inf(\phi)$ and $\sup(\phi)$ we understand the greatest lower bound and least upper bound respectively of the spectrum of ϕ with respect to $\leqslant$, provided that such bounds exist and are unique; otherwise $\inf(\phi)$ and $\sup(\phi)$ are the ordinal 0.

Let Φ be a class of sentences. A class Γ of isomorphism types is said to be *Φ-describable* if there is a member ϕ of Φ such that Γ is the spectrum of ϕ [3].

We shall be interested in two specializations of these notions. First, let L be the empty set Λ. Then we can identify the isomorphism type of an L-model $\mathfrak{A}$ with the *cardinal* of $U_{\mathfrak{A}}$; the class of all L-types will then coincide with the class of cardinal numbers [4]. If ϕ is in II-St_L, then the spectrum of ϕ coincides with what Zykov calls the spectrum of ϕ in Zykov [53]. The relation $\leqslant$ in this case amounts to the usual less-than-or-equal relation between cardinals, and is therefore a reflexive well-ordering. Consequently, if ϕ is a sentence true in at least one L-model, then $\inf(\phi)$ exists and is a member of the spectrum of ϕ; and if ϕ is a sentence containing no predicate constants and such that the spectrum of ϕ has an upper bound, then $\sup(\phi)$ exists. If Φ is any *set* of sentences containing no predicate constants, then the least upper bound of the set of cardinals $\inf(\phi)$ for $\phi\varepsilon\Phi$ exists and is called the *index* of Φ, and the least upper bound of the set of cardinals $\sup(\phi)$ for $\phi\varepsilon\Phi$ also exists and is called the *Hanf number* of Φ. (These two notions were introduced by William Hanf in [62c].) If Γ is a set of ordinals, then a *Γ-describable*

[3] The notions introduced above would also have interest if reference to an individual sentence were replaced by reference to a set of sentences.

[4] Notice that if we had not decided to admit empty models, there would be no L-type corresponding to the cardinal 0.

class (or set) *of cardinals* is a class (or set) of cardinals which is (Γ-St_A)-describable, and a *Γ-describable cardinal* is a cardinal α such that $\{\alpha\}$ is a Γ-describable set of cardinals. Similarly, a II-*describable class* (or set) *of cardinals* is one which is (II-St_A)-describable, and a II-*describable cardinal* is a cardinal α such that $\{\alpha\}$ is a II-describable set of cardinals.

For a second specialization we may assume that $L=\{\mathsf{P}\}$ and that P is a 2-place predicate constant. Then we can identify the L-types with *relation types*. In particular, we may consider those L-models $\mathfrak{A}$ for which $\mathsf{P}_{\mathfrak{A}}$ is a strict (or irreflexive) well-ordering (a condition equivalent to the truth in $\mathfrak{A}$ of the sentence $\wedge\mathsf{x} \neg \mathsf{Pxx} \wedge \wedge\mathsf{x}\wedge\mathsf{y}\wedge\mathsf{z}\,[\mathsf{Pxy} \wedge \mathsf{Pyz} \rightarrow \mathsf{Pxz}] \wedge \wedge\mathsf{x}\wedge\mathsf{y}\,[\mathsf{Pxy} \vee \mathsf{x}=\mathsf{y} \vee \mathsf{Pyx}] \wedge \wedge\mathsf{Q}\,[\vee\mathsf{x}\mathsf{x}\eta\mathsf{Q} \rightarrow \vee\mathsf{x}(\mathsf{x}\eta\mathsf{Q} \wedge \neg \vee\mathsf{y}\,[\mathsf{y}\eta\mathsf{Q} \wedge \mathsf{Pyx}])]$, where x, y, z are distinct individual variables and Q a variable of type 1); the type $\tau\mathfrak{A}$ of such a model may be identified with the *ordinal number* of the relational system $\langle U_{\mathfrak{A}}, \mathsf{P}_{\mathfrak{A}}\rangle$. The relation $\leqslant$ between types then amounts to the usual less-than-or-equal relation between ordinals, and is of course again a reflexive well-ordering. If Γ is a set of ordinals, then a *Γ-describable class* (or set) *of ordinals* is a class (or set) of ordinals which is (Γ-$\mathrm{St}_{\{\mathsf{P}\}}$)-describable for some 2-place predicate constant P, and a *Γ-describable ordinal* is an ordinal α such that $\{\alpha\}$ is a Γ-describable set of ordinals. We can characterize in an analogous way a II-*describable class of ordinals* and a II-*describable ordinal*.

The following lemma is due to David Kaplan.

Lemma 2. *If L is a set of predicate constants and ϕ is in* II-St_L, *then there exists ψ in* II-St_L *such that* (1) *ψ contains no predicate variables of more than two places and* (2) *for every model $\mathfrak{A}$, ψ is true in $\mathfrak{A}$ if and only if $U_{\mathfrak{A}}$ is infinite and ϕ is true in $\mathfrak{A}$.*

Most of our results will have to do with Γ-order logic, where Γ is a set of ω-describable ordinals. Various wider notions of a describable ordinal could replace ω-describability, but at the expense of complicating the constructions. The following lemma, which follows easily from Lemma 2, shows that ω-describability is at least sufficient to comprehend II-describability.

Lemma 3. (a) *Every* II-*describable cardinal or ordinal is a* 4-*describable cardinal or ordinal respectively.* (b) *Every* II-*describable class of infinite cardinals or ordinals is a* 4-*describable class of cardinals or ordinals respectively.*

2. Definition of the reduction. If L is a set of predicate constants and Γ a set of ordinals, then the set of *expressions corresponding to L and Γ* consists of (1) those second-order expressions in which no predicate constants occur beyond those in L, together with (2) those higher-order expressions in which no predicate constants occur beyond those in L and no variables occur beyond those having types in Γ. We say that v is a *new-variable function* for L and Γ if (1) v is a 3-place function, (2) v is defined for all triples $\langle k, n, \phi \rangle$ such that k, n are finite ordinals and ϕ is an expression corresponding to L and Γ, (3) for every finite ordinal n, every expression ϕ corresponding to L and Γ, and every u, u is a variable of type n not occurring in ϕ if and only if u is $v(k, n, \phi)$ for some finite ordinal k, and (4) whenever $\langle k, n, \phi \rangle$, $\langle k', n, \phi \rangle$ are triples for which v is defined, if $v(k, n, \phi)$ is $v(k', n, \phi)$, then k is k'. We may regard $v(k, n, \phi)$ as the kth variable of type n not occurring in ϕ (in the order determined by v).

Suppose that ψ is a function from variables to formulas. Then the *relativization* of ϕ by ψ, or $\phi^{(\psi)}$, is defined by the following recursion for any higher-order formula ϕ whose variables are all in the domain of ψ. (1) If P is an $(n+1)$-place predicate constant, $\mathsf{u}_0, \ldots, \mathsf{u}_n$ are individual variables, and u, v are variables of arbitrary types, then $(\mathsf{P}\mathsf{u}_0 \ldots \mathsf{u}_n)^{(\psi)}$ is $\mathsf{P}\mathsf{u}_0 \ldots \mathsf{u}_n$, $(\mathsf{u}\eta\mathsf{v})^{(\psi)}$ is $\mathsf{u}\eta\mathsf{v}$, and $(\mathsf{u}=\mathsf{v})^{(\psi)}$ is $\mathsf{u}=\mathsf{v}$. (2) $(\neg\, \phi)^{(\psi)}$ is $\neg\, \phi^{(\psi)}$; $(\phi \to \chi)^{(\psi)}$ is $(\phi^{(\psi)} \to \chi^{(\psi)})$; and similarly for the other sentential connectives. (3) $(\wedge \mathsf{u} \phi)^{(\psi)}$ is $\wedge \mathsf{u}(\psi(\mathsf{u}) \to \phi^{(\psi)})$, and $(\vee \mathsf{u} \phi)^{(\psi)}$ is $\vee \mathsf{u}(\psi(\mathsf{u}) \wedge \phi^{(\psi)})$.

For the remainder of this section assume that ε is a 2-place predicate constant, Σ a 1-place predicate constant, L a set of predicate constants, Γ a set of ordinals, and v a new-variable function for $L \cup \{\varepsilon, \Sigma\}$ and $\Gamma \cup \omega$.

If g is a biunique function from variables to individual variables and ϕ is a higher-order formula all of whose variables are in the domain of g, then the *replacement* in ϕ based on g and ε, or $\phi_{g,\varepsilon}$, is defined by the following recursion. (1) If P is an $(n+1)$-place predicate constant, $\mathsf{u}_0, \ldots, \mathsf{u}_n$ are individual variables, and u, v are variables of arbitrary type, then $(\mathsf{P}\mathsf{u}_0 \ldots \mathsf{u}_n)_{g,\varepsilon}$ is $\mathsf{P}g(\mathsf{u}_0) \ldots g(\mathsf{u}_n)$, $(\mathsf{u}=\mathsf{v})_{g,\varepsilon}$ is $g(\mathsf{u})=g(\mathsf{v})$, and $(\mathsf{u}\eta\mathsf{v})_{g,\varepsilon}$ is $g(\mathsf{u})\, \varepsilon\, g(\mathsf{v})$. (2) $(\neg\, \phi)_{g,\varepsilon}$ is $\neg\, (\phi_{g,\varepsilon})$; $(\phi \to \psi)_{g,\varepsilon}$ is $(\phi_{g,\varepsilon} \to \psi_{g,\varepsilon})$; and similarly for the other sentential connectives. (3) $(\wedge \mathsf{u} \phi)_{g,\varepsilon}$ is $\wedge g(\mathsf{u}) \phi_{g,\varepsilon}$, and $(\vee \mathsf{u} \phi)_{g,\varepsilon}$ is $\vee g(\mathsf{u}) \phi_{g,\varepsilon}$.

We understand J to be the *Cantor pairing function*, which in a biunique and recursive way assigns to every pair of finite ordinals a finite ordinal.

If ϕ is in ω-$\mathrm{St}_{\{\varepsilon\}}$ and a is an individual variable, then $\mathrm{OT}_{\phi,\varepsilon,v}(\mathsf{a})$ is to be the formula $(\phi_{g,\varepsilon})^{(\psi)}$, where g is that function whose domain is the set of variables of finite type and such that for all finite ordinals k and n, $g(v(k, n, \Lambda))$ is $v(J(k, n), 0, \mathsf{a})$, and ψ is that function whose domain is the set of variables of type 0 and such that whenever u is a variable of type 0, $\psi(g(\mathsf{u}))$ is the formula $g(\mathsf{u})\varepsilon\mathsf{a}$, and whenever n is a finite ordinal and u is a variable of type $n+1$, $\psi(g(\mathsf{u}))$ is the formula $\forall g(\mathsf{w})\ [\wedge g(\mathsf{t})\ [g(\mathsf{t})\varepsilon g(\mathsf{w}) \leftrightarrow \psi(g(\mathsf{t}))] \wedge \wedge g(\mathsf{t})\ [g(\mathsf{t})\varepsilon g(\mathsf{u}) \rightarrow g(\mathsf{t})\varepsilon g(\mathsf{w})]]$ (where t is $v(0, n, \Lambda)$ and w is $v(0, n+1, \mathsf{u})$). We shall be interested in $\mathrm{OT}_{\phi,\varepsilon,v}(\mathsf{a})$ only when ϕ is a sentence whose spectrum is $\{\alpha\}$, for some ordinal α; and in this case $\mathrm{OT}_{\phi,\varepsilon,v}(\mathsf{a})$ may be read 'a is a set having α as its order type.'

If again ϕ is in ω-$\mathrm{St}_{\{\varepsilon\}}$ and a is an individual variable, then $\mathrm{T}_{\phi,\varepsilon,\Sigma,v}(\mathsf{a})$ is to be the formula $\forall \mathsf{b}\ [\Sigma\mathsf{b} \wedge \wedge\mathsf{x}(\mathsf{x}\varepsilon\mathsf{b} \leftrightarrow \wedge\mathsf{k}\ [\wedge\mathsf{m}\wedge\mathsf{n}(\wedge\mathsf{y}\ [\mathsf{y}\varepsilon\mathsf{m} \rightarrow \mathsf{y}\varepsilon\mathsf{k}] \wedge \wedge\mathsf{y}\ [\mathsf{y}\varepsilon\mathsf{n} \leftrightarrow \neg\ \Sigma\mathsf{y} \vee \forall\mathsf{x}(\mathsf{x}\varepsilon\mathsf{m} \wedge \wedge\mathsf{z}\ [\mathsf{z}\varepsilon\mathsf{y} \rightarrow \mathsf{z}\varepsilon\mathsf{x}])] \rightarrow \mathsf{n}\varepsilon\mathsf{k} \vee \wedge\mathsf{y}\ [\mathsf{y}\varepsilon\mathsf{b} \rightarrow \mathsf{y}\varepsilon\mathsf{n}]) \rightarrow \rightarrow \mathsf{x}\varepsilon\mathsf{k}]) \wedge \mathrm{OT}_{\phi,\varepsilon,v}(\mathsf{b}) \wedge \Sigma\mathsf{a} \wedge \wedge\mathsf{y}\ [\mathsf{y}\varepsilon\mathsf{a} \leftrightarrow \neg\ \Sigma\mathsf{y} \vee \forall\mathsf{x}(\mathsf{x}\varepsilon\mathsf{b} \wedge \wedge\mathsf{z}\ [\mathsf{z}\varepsilon\mathsf{y} \rightarrow \mathsf{z}\varepsilon\mathsf{x}])]]$, where b, k, m, n, x, y, z are respectively $v(0, 0, \mathsf{a}), \ldots, v(6, 0, \mathsf{a})$. If the spectrum of ϕ is $\{\alpha\}$ and α is an ordinal, then $\mathrm{T}_{\phi,\varepsilon,\Sigma,v}(\mathsf{a})$ may be read 'a is the αth cumulative type'.

If a is an individual variable and n a finite ordinal, then $\mathrm{Dom}_{n,\varepsilon,v}(\mathsf{a})$ is to be the formula $\mathsf{a}=\mathsf{a}$ if n is 0, and the formula $\forall\mathsf{b}_1 \ldots \forall\mathsf{b}_n\ [\mathsf{a}\varepsilon\mathsf{b}_1 \wedge \ldots \ldots \wedge \mathsf{b}_{n-1}\varepsilon\mathsf{b}_n]$, where $\mathsf{b}_1, \ldots, \mathsf{b}_n$ are respectively $v(0, 0, \mathsf{a}), \ldots, v(n-1, 0, \mathsf{a})$, if $n>0$. We may read $\mathrm{Dom}_{n,\varepsilon,v}(\mathsf{a})$ as 'a is n-dominated'.

Suppose that g is a biunique function from variables to individual variables, ϕ is a sentence in Γ-St_L all of whose variables are in the domain of g, n is a finite ordinal, $\mathsf{u}_0, \ldots, \mathsf{u}_k$ are all the variables occurring in ϕ, listed in order of first occurrence, and d is a function whose domain contains $\mathsf{u}_0, \ldots, \mathsf{u}_k$ and such that $d(\mathsf{u}_0), \ldots, d(\mathsf{u}_k)$ are in $(n+2)$-$\mathrm{St}_{\{\varepsilon\}}$. Then the corresponding *second-order reduction* of ϕ, or $\mathrm{Red}_{g,d,\varepsilon,\Sigma,n,v}(\phi)$, is the sentence $\mathrm{RFS}_{\varepsilon,\Sigma,\mathsf{b}_0,\ldots,\mathsf{b}_6,\mathsf{P}} \wedge \forall\mathsf{a}_0 \ldots \forall\mathsf{a}_k[\mathrm{T}_{d(\mathsf{u}_0),\varepsilon,\Sigma,v}(\mathsf{a}_0) \wedge \ldots \wedge \mathrm{T}_{d(\mathsf{u}_k),\varepsilon,\Sigma,v}(\mathsf{a}_k) \wedge \mathrm{Dom}_{n,\varepsilon,v}(\mathsf{a}_0) \wedge \ldots \wedge \mathrm{Dom}_{n,\varepsilon,v}(\mathsf{a}_k) \wedge [\neg\ \mathrm{Dom}_{n+1,\varepsilon,v}(\mathsf{a}_0) \vee \ldots \vee \neg\ \mathrm{Dom}_{n+1,\varepsilon,v}(\mathsf{a}_k)] \wedge (\phi_{g,\varepsilon})^{(\psi)}]$, where $\mathsf{a}_0, \ldots, \mathsf{a}_k, \mathsf{b}_0, \ldots, \mathsf{b}_6, \mathsf{P}$ are respectively $v(0, 0, \phi_{g,\varepsilon}), \ldots$ $\ldots, v(k, 0, \phi_{g,\varepsilon}), v(0, 0, \Lambda), \ldots, v(6, 0, \Lambda), v(0, 1, \Lambda)$, and ψ is that function whose domain is $\{g(\mathsf{u}_0), \ldots, g(\mathsf{u}_k)\}$ and such that, for $i \leqslant k$, $f(g(\mathsf{u}_i))$ is the formula $g(\mathsf{u}_i)\varepsilon\mathsf{a}_i$.

3. Spectrum problems.

Definition 3. *If α is a cardinal and β an ordinal, then $\beth_{\alpha,\beta}$ is to be a cardinal determined in accordance with the following recursive clauses:*

$\beth_{\alpha,0}$ is α; $\beth_{\alpha,\beta+1}$ is $\alpha+2^{\beth_{\alpha,\beta}}$; if β is a limit ordinal, then $\beth_{\alpha,\beta}$ is the least upper bound of the cardinals $\beth_{\alpha,\gamma}$ for $\gamma<\beta$.

Theorem 1. *Let n be a finite ordinal, Γ be any set of $(n+2)$-describable ordinals, ϕ be in Γ-St_A, and β be the maximum of the types of variables occurring in ϕ. Then there exists a sentence ψ in II-St_A such that the spectrum of ψ is the class of cardinals $\beth_{\alpha,\beta+1+n}$ for which α is in the spectrum of ϕ; ψ may, moreover, be chosen so as to have the form $\forall\mathsf{P}\forall\mathsf{Q}\delta$, where P is a 2-place predicate variable, Q is a 1-place predicate variable, and δ is in USF_A.*

Proof. Let ε be a 2-place predicate constant and Σ a 1-place predicate constant. Let g be a biunique function whose domain is the set of variables occurring in ϕ and whose values are individual variables. Clearly there is a function d with the same domain such that whenever u is a variable of type α occurring in ϕ, $d(\mathsf{u})$ is in $(n+2)$-$\mathrm{St}_{\{\varepsilon\}}$ and the spectrum of $d(\mathsf{u})$ is $\{\alpha\}$. Let v be a new-variable function for $\{\varepsilon, \Sigma\}$ and $\Gamma \cup \omega$, let P be a 2-place predicate variable, let Q be a 1-place predicate variable not occurring in $\mathrm{Red}_{g,d,\varepsilon,\Sigma,n,v}(\phi)$, and let χ be the result of replacing ε by P and Σ by Q in $\mathrm{Red}_{g,d,\varepsilon,\Sigma,n,v}(\phi)$. It is easily seen that $\forall\mathsf{P}\forall\mathsf{Q}\chi$ is equivalent to some sentence of the form $\forall\mathsf{P}\forall\mathsf{Q}\delta$, where δ is in USF_A. We may then take ψ to be $\forall\mathsf{P}\forall\mathsf{Q}\delta$.

It is clear that if ϕ is in II-St_L (where L is any set of predicate constants), then there exists a sentence ϕ' in ω-St_L such that ϕ is equivalent to ϕ'. Thus, as a special case of Theorem 1, we may let n be 0 and replace the assumption that ϕ is in Γ-St_A by the assumption that ϕ is in II-St_A; the resulting assertion corresponds closely to Theorem III of Zykov [53], which, however, contains slight advantages that could not be expected in a general theorem.

If Γ is any set of cardinals, understand by $<[\Gamma]$ the restriction of $<$ to Γ, that is, the set of ordered pairs $\langle\alpha, \beta\rangle$ such that α, β are in Γ and $\alpha<\beta$. The following is a simple consequence of Lemma 3 and Theorem 1.

Theorem 2. *Assume the Generalized Continuum Hypothesis. If n is a finite ordinal, $4\leqslant n$, Γ is the set of n-describable cardinals, and Δ is the set of II-describable cardinals, then $<[\Gamma]$ and $<[\Delta]$ have the same order type.*

This consequence was pointed out by Barry Kurtzman, who has obtained an independent proof not involving the Continuum Hypothesis.

Theorem 3. *Suppose that Γ is a set of ω-describable ordinals, $4 \subseteq \Gamma$, α and α' are respectively the index and the Hanf number of Γ-St_A, β and β' are respectively the index and the Hanf number of II-St_A, and γ and γ' are respectively the index and the Hanf number of the set of sentences in II-St_A having the form $\forall \mathsf{P} \forall \mathsf{Q} \phi$ (where P is a 2-place predicate variable, Q is a 1-place predicate variable, and ϕ is in USF_A). Then*

$$\alpha = \beta = \gamma < \alpha' = \beta' = \gamma'.$$

The equalities in this theorem follow immediately from Lemma 2 and Theorem 1. The fact that $\gamma < \alpha'$ requires a separate proof, which will not, however, be given here.

If Γ satisfies the hypothesis of Theorem 3, then in view of Theorems 2 and 3 and Lemma 3(a) we might suppose that the Γ-describable cardinals and the II-describable cardinals coincide. It follows from the next theorem, which will not be proved here, that this supposition would be false.

Theorem 4. *Let n be a finite ordinal which is at least 3, and let α be the first cardinal which is not n-describable. Then α is $(n+1)$-describable. If, moreover, $n \geqslant 4$, then α is not II-describable.*

4. Logical truth. If L is a countable (that is, at most denumerable) set of predicate constants and Γ a countable set of ordinals, then nr is said to be a *Gödel-numbering corresponding to L and Γ* if (1) nr is a biunique function whose values are finite ordinals, (2) the domain of nr is the set of expressions corresponding to L and Γ, and (3) the following sets and relations are recursive: the set of numbers $nr(\mathsf{P})$ such that P is a predicate constant in L, the set of numbers $nr(\mathsf{P})$ such that P is a predicate variable, the set of numbers $nr(\mathsf{u})$ such that u is a variable having type in Γ, the range of nr, the set of pairs $\langle nr(\mathsf{P}), n\rangle$ such that n is in ω and P is an n-place predicate constant in L, the set of pairs $\langle nr(\mathsf{P}), n\rangle$ such that n is in ω and P is an n-place predicate variable, the set of pairs $\langle nr(\mathsf{u}), n\rangle$ such that n is in $\omega \cap \Gamma$ and u is a variable of type n, and the set of triples $\langle nr(\phi), nr(\psi), nr(\chi)\rangle$ such that ϕ, ψ are in the domain of nr and χ is the concatenation of ϕ and ψ.

Assume in what follows that L is a countable set of predicate constants, Γ is a countable set of ordinals, ε is a 2-place predicate constant not in L, Σ is a 1-place predicate constant not in L, nr is a Gödel-numbering corresponding to $L \cup \{\varepsilon, \Sigma\}$ and $\Gamma \cup \omega$, and S, P are distinct 3-place predicate constants.

If R is a set of, or relation between, expressions corresponding to $L \cup \{\varepsilon, \Sigma\}$ and $\Gamma \cup \omega$, then R is said to be *nr-recursive* if the image of R under nr (which will be a set of, or relation between, finite ordinals) is recursive. An n-place function is regarded as a special case of an $(n+1)$-place relation, and therefore no special definition of nr-recursivity for functions is required.

Theorem 5. *Suppose that an nr-recursive function d exists whose domain is the set of variables having types in Γ, and such that whenever u is a variable of type α and α is in Γ, $d(u)$ is in ω-$\mathrm{St}_{\{\varepsilon\}}$ and the spectrum of $d(\mathsf{u})$ is $\{\alpha\}$. Then $\mathrm{LT} \cap \Gamma\text{-}\mathrm{St}_L$ is recursively reducible to $\mathrm{LT} \cap \mathrm{ESF}_{L\cup\{\varepsilon,\Sigma\}}$; indeed, there exists an nr-recursive function f with domain Γ-St_L, with range included in $\mathrm{ESF}_{L\cup\{\varepsilon,\Sigma\}}$, and such that whenever ϕ is in Γ-St_L, we have: ϕ is in $\mathrm{LT} \cap \Gamma\text{-}\mathrm{St}_L$ if and only if $f(\phi)$ is in $\mathrm{LT} \cap \mathrm{ESF}_{L\cup\{\varepsilon,\Sigma\}}$.*

Proof. Clearly there is an nr-recursive new-variable function v for $L \cup \{\varepsilon, \Sigma\}$ and $\Gamma \cup \omega$, and there is a biunique nr-recursive function g whose domain is the set of variables having types in Γ and whose values are individual variables. Let h be that function with domain Γ-St_L such that for each ϕ in Γ-St_L, $h(\phi)$ is the least finite ordinal n such that whenever u is a variable having type in Γ and occurring in ϕ, $d(\mathsf{u})$ is in $(n+2)$-$\mathrm{St}_{\{\varepsilon\}}$. Then clearly h is nr-recursive. Let t be that function with domain Γ-St_L such that for each ϕ in Γ-St_L, $t(\phi)$ is the sentence $\neg\, \mathrm{Red}_{g,d,\varepsilon,\Sigma,h(\Phi),v}(\neg\, \phi)$. It is easily seen that t is recursive. The desired function f can now be obtained from t; intuitively speaking, $f(\phi)$ is to be the result of moving the second-order quantifier to the front in $t(\phi)$.

The special case of Theorem 5 in which Γ is ω is essentially Theorem III of Hintikka [55b]. As an immediate consequence of Theorem 5 we have the following.

Theorem 6. *If Γ is a finite set of ω-describable ordinals, then the conclusion of Theorem 5 holds.*

By $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$, or the *numerical model* corresponding to S and P, is understood that $\{\mathsf{S}, \mathsf{P}\}$-model $\mathfrak{A}$ such that (1) $U_{\mathfrak{A}}$ is ω, (2) $\mathsf{S}_{\mathfrak{A}}$ is the set of triples $\langle m, n, p\rangle$ for which m, n are finite ordinals and p is $m+n$, and (3) $\mathsf{P}_{\mathfrak{A}}$ is the set of triples $\langle m, n, p\rangle$ for which m, n are finite ordinals and p is $m \cdot n$. If $\mathfrak{A}$ is any model and B a subset of $\mathfrak{A}$, then ϕ is said to *define* B in $\mathfrak{A}$ if (1) ϕ is a formula with exactly one free individual variable and no free variables which are not individual variables, (2) all predicate

constants occurring in ϕ are in the language of $\mathfrak{A}$, and (3) for all x in $U_{\mathfrak{A}}$, x is in B if and only if the one-place sequence $\langle x \rangle$ satisfies ϕ in $\mathfrak{A}$. If in addition Δ is any set of ordinals, then B is said to be *Δ-definable* in $\mathfrak{A}$ if there exists ϕ in Δ-Fm_L, where L is the language of $\mathfrak{A}$, such that ϕ defines B in $\mathfrak{A}$. By $f\star(A)$ is understood the set of objects $f(x)$ such that x is in A.

It is a consequence of Gödel's completeness theorem that $nr\star(\mathrm{LT} \cap \cap\ 1\text{-}\mathrm{Fm}_L)$ and $nr\star(\mathrm{LT} \cap \mathrm{USF}_L)$ are 1-definable in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$. It is rather easily seen that the same cannot be said for the set of Gödel numbers of second-order logical truths. It is natural, however, to ask whether this set occurs in some natural extension of the Kleene arithmetical hierarchy, for instance, whether it is 2-definable or ω-definable in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$. The next theorem, which was derived from Theorem 6 by Vaught and me, gives a negative answer; indeed, we have the stronger result that in a 'natural' higher-order language not even the set of Gödel numbers of *existential* second-order logical truths is definable. First a lemma is required which asserts the existence of self-referential sentences.

Lemma 4. *If* 0 *is in* Γ, f *is an nr-recursive function whose domain includes* Γ-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$, ϕ *is in* Γ-$\mathrm{Fm}_{\{\mathsf{S},\mathsf{P}\}}$, ϕ *has exactly one free variable, and that is an individual variable, then there exists* ψ *in* Γ-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$ *such that* ψ *is true in* $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$ *if and only if* $\langle nr(f(\psi)) \rangle$ *satisfies* ϕ *in* $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$.

Theorem 7. *If all members of* Γ *are* ω*-describable ordinals and* L *contains* S *and* P, *then* $nr\star(\mathrm{LT} \cap \mathrm{ESF}_{L \cup \{\varepsilon, \Sigma\}})$ *is not* Γ*-definable in* $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$.

Proof. Assume the hypothesis and that ϕ is a formula in Γ-$\mathrm{Fm}_{\{\mathsf{S},\mathsf{P}\}}$ which defines $nr\star(\mathrm{LT} \cap \mathrm{ESF}_{L \cup \{\varepsilon, \Sigma\}})$ in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$. Let Δ be the set consisting of the ordinals 0 and 1, together with the types of variables occurring in ϕ. By Theorem 6, there exists an nr-recursive function f with domain Δ-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$ such that (1) the range of f is included in $\mathrm{ESF}_{\{\mathsf{S},\mathsf{P},\varepsilon,\Sigma\}}$ and (2) for all ψ in Δ-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$, ψ is in $\mathrm{LT} \cap \Delta$-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$ if and only if $f(\psi)$ is in $\mathrm{LT} \cap \mathrm{ESF}_{\{\mathsf{S},\mathsf{P},\varepsilon,\Sigma\}}$. It is well known that there exists a sentence χ in 2-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$ such that for every $\{\mathsf{S}, \mathsf{P}\}$-model $\mathfrak{A}$, χ is true in $\mathfrak{A}$ if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$. We now employ the Paradox of the Liar. By Lemma 4, there exists ψ in Δ-$\mathrm{St}_{\{\mathsf{S},\mathsf{P}\}}$ such that ψ is true in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$ if and only if $\langle nr\,(f(\chi \to \psi)) \rangle$ satisfies $\neg\, \phi$ in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$. It follows by (1), (2), and Lemma 1 that ψ is true in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$ if and only if ψ is not true in $\mathfrak{N}_{\mathsf{S},\mathsf{P}}$, and the theorem is proved.

In contrast with Theorem 7 we have the following positive definability result, whose proof, though simple, will not be given here.

Theorem 8. *If* α *is the index of* II-St_A, *then* $nr\star(\text{LT} \cap \text{II-St}_L)$ *is* $(\alpha+1)$*-definable in* $\mathfrak{N}_{S,P}$.

I do not know whether $\alpha+1$ is the least ordinal satisfying the conclusion of Theorem 8.

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OMITTING CLASSES OF ELEMENTS

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By a *class* of elements we mean a class defined by a set of formulas. More precisely, suppose L is some countable, first-order language with equality, Σ is a set of formulas in L having a common single free variable, and $\mathfrak{A}$ is an L-structure. Then the elements of class Σ in $\mathfrak{A}$ are the elements of $\mathfrak{A}$ which satisfy all the formulas of Σ in $\mathfrak{A}$. We say $\mathfrak{A}$ omits Σ if there are no elements of class Σ in $\mathfrak{A}$.

Suppose T is a theory in L. It is an easy consequence of the Löwenheim-Skolem-Tarski theorem that if T has an infinite model containing an element of class Σ, then it has such models in each infinite power. On the other hand, the existence of an infinite model of T omitting Σ may not imply the existence of models omitting Σ in each infinite power. For example, suppose T is the theory of ordered fields and Σ is the class of non-Archimedean elements. Then the models of T which omit Σ are the Archimedean-ordered fields and, hence, there are none of power greater than $2^{\aleph_0}$.

It is fairly easy to show that there must be some cardinal, $\varkappa$, such that for any theory T and any class Σ the existence of a model of T of power $\varkappa$ which omits Σ implies the existence of such models in each infinite power. The principal result of this paper is the determination of this cardinal. The proof depends upon a partition theorem of Erdös and Rado.

For references to related earlier results see Section 3 and Vaught [63b].

1. We denote the cardinality of a set A by $\bar{\bar{A}}$. The letters $\varkappa$ and λ will always denote infinite cardinals; α, β, and γ ordinal numbers; and δ a limit ordinal number. The letters m and n will denote finite ordinals. The least cardinal greater than $\varkappa$ is $\varkappa^+$. The notation $2^{\varkappa}_{\alpha}$ is defined inductively by

$$2^{\varkappa}_{0}=\varkappa \text{ and } 2^{\varkappa}_{\alpha}=\sum 2^{\lambda} \ (\lambda=2^{\varkappa}_{\beta},\ \beta<\alpha).$$

In particular, the beth sequence is defined by $\beth_\alpha=2^{\aleph_0}_{\alpha}$. Hence, the generalized continuum hypothesis may be stated as $\beth_\alpha=\aleph_\alpha$ for all α.

The letter T will always denote a theory in a countable first-order language L, and Σ will denote a set of formulas in L having a common single free variable. We shall also use the letter Σ to denote the class of elements defined by Σ.

2. Theorem 2.1. *If $\mathfrak{A}$ is an infinite model of T which omits Σ then there are models of T which omit Σ in each power $\lambda \leqslant \overline{\mathfrak{A}}$.*

Proof. Consider an elementary subsystem of $\mathfrak{A}$ of power λ.

For a given T and Σ it follows from 2.1 that there are only two possibilities: (1) there are models of T in every infinite power omitting Σ, or (2) there is some cardinal $\varkappa$ such that there are models of T omitting Σ in just those infinite powers which are less than $\varkappa$. In the latter case we say $\varkappa$ is *characterized* by T and Σ ($\varkappa = \mathscr{C}(T, \Sigma)$). We denote by $\mathscr{C}$ the class of cardinals which may be so characterized. Since we have restricted ourselves to a countable language we have $\overline{\overline{\mathscr{C}}} \leqslant 2^{\aleph_0}$.

Theorem 2.2. (1) $\aleph_0 \in \mathscr{C}$; $\aleph_1 \in \mathscr{C}$. (2) *If $\varkappa^+ \in \mathscr{C}$ then $(2^\varkappa)^+ \in \mathscr{C}$ and $\varkappa^{++} \in \mathscr{C}$.* (3) *If $\varkappa = \Sigma \varkappa_n$ $(n \in \omega)$, $\varkappa > \varkappa_n$ (all $n \in \omega$), and $\varkappa_n \in \mathscr{C}$ (all $n \in \omega$), then $\varkappa \in \mathscr{C}$ and $\varkappa^+ \in \mathscr{C}$.* (4) *$\aleph_\alpha \in \mathscr{C}$ and $(\beth_\alpha)^+ \in \mathscr{C}$* (for $\alpha < \omega_1$).

Proof. (1) The case of $\aleph_0$ is trivial; take for Σ a class which cannot be omitted in any model of T.

To characterize $\aleph_1$ let T be some first-order theory of the integers and Σ the class of non-standard integers.

(2) Suppose T and Σ characterize $\varkappa^+$. To characterize $(2^\varkappa)^+$ we use a construction of R. Robinson. We add to our language L a new unary relation symbol R and a binary relation symbol E. The theory T' is the set of statements which assert that in any model $\mathfrak{A}$ of T': (i) the elements of R form a model of T; (ii) if $b, c \in A$ and $b \neq c$ then $\{a \in A;\ R(a) \text{ and } E(a, b)\} \neq \{a \in A;\ R(a) \text{ and } E(a, c)\}$. The class Σ' is defined as the elements of R which are of class Σ (when the formulas of Σ are relativized to R). Noting that (ii) implies that distinct elements of A correspond to distinct subsets of R, one may verify that T' and Σ' characterize $(2^\varkappa)^+$.

In a similar fashion, to characterize $\varkappa^{++}$ we add to our language L a new unary relation symbol R, a new binary relation symbol $<$, and a ternary relation symbol F. The theory T'' is the set of statements which assert that in any model $\mathfrak{A}$ of T'': (i) the elements of R form a model of T; (ii) $<$ is a linear ordering of A; and (iii) for all $a \in A$, $F(a, \ , \ )$

is a one-one map of the elements less than a into R. Σ'' is defined similarly to Σ'. One may verify that T'' and Σ'' characterize $\varkappa^{++}$.

(3) Suppose for each $n \in \omega$ T_n and Σ_n characterize $\varkappa_n$. For each $n \in \omega$ we add a new unary relation symbol R_n to the language. The theory T is the theory which asserts (i) $m \neq n$ implies $R_m \cap R_n = \emptyset$ $(m, n \in \omega)$; and (ii) for each $n \in \omega$ the elements of R_n form a model of T_n. The elements of class Σ are those which are either (a) of class Σ_n in some R_n (with Σ_n relativized to R_n), or (b) not in any $R_n (n \in \omega)$. It may be verified that T and Σ characterize $\varkappa^+$.

To characterize $\varkappa$ we add to the above two new binary relation symbols P and Q. The theory T' is the theory T of the last paragraph together with statements asserting that: (iii) Q is an equivalence relation; (iv) for each $n \in \omega$, R_n is an equivalence class of Q; and (v) P is a one-one map of the entire model into a single equivalence class of Q. Then T' and Σ characterize $\varkappa$ since any model omitting Σ must now have power $\leqslant \varkappa_n$ for some n, i.e., have power $< \varkappa$.

Part (4) of the theorem follows by induction from parts (1), (2) and (3).

We shall see in Theorem 3.1 that every element of $\mathscr{C}$ is $< \beth_{\omega_1}$. So, assuming the generalized continuum hypothesis, Theorem 2.2 gives all the members of $\mathscr{C}$. An open question is as follows. Call a class Σ an *elementary type* with respect to T if Σ is maximal consistent with respect to T. What cardinals can be characterized by T and Σ with Σ an elementary type with respect to T? The only cardinals known to be so characterizable are $\aleph_0$, $\aleph_1$, and $(2^{\aleph_0})^+$.

3. The principal result of this paper is:

Theorem 3.1. *If for every $\alpha < \omega_1$ there is a model of T of power $\geqslant \beth_\alpha$ which omits Σ, then there is in every infinite power a model of T which omits Σ.*

We defer the proof of this theorem until the next section in order to first discuss some of the methods which will be used in the proof and some earlier, related results.

To prove Theorem 3.1 one must show the existence of a model of T of prescribed cardinality and having a certain property, namely, that of omitting Σ. If this property were one which could be asserted by a first-order sentence, or even an infinite set of such sentences, one could apply the Löwenheim-Skolem theorem. Indeed, this can be done for the property of *not* omitting Σ. The property of omitting Σ cannot, in general,

be asserted by a set of first-order sentences. What we will do is add sentences that assert that Σ is, in a certain sense, omitted "locally".

Consider the Henkin proof of the completeness theorem (Henkin [49]). For every sentence of the form "$(\exists v)\varphi(v)$" in the theory T a new individual constant a is added to the language and the sentence "$\varphi(a)$" is added to the theory. It can be shown that after iterating this process sufficiently the enlarged theory will have a model in which every element is represented by one of the individual constants.

Suppose that at each step, when one added a constant a, it was possible to find a formula $\sigma \in \Sigma$ such that it was consistent to add the sentence "$\neg\, \sigma(a)$" to the theory T and all the previously added sentences. Notice that these new sentences do not assert that a model of the theory contains no elements of class Σ, but only that none of the elements represented by individual constants are of class Σ. Still, they do imply that there is at least one model which omits Σ, namely, the one whose every element is represented by a constant.

This process was carried out, in one form or another by Ryll-Nardzewski, Ehrenfeucht, Engeler, Svenonius, and Vaught (for references see Vaught [63b]). The result is the following theorem.

Theorem 3.2. *Suppose T is a complete theory. A necessary and sufficient condition that there be a model of T omitting Σ is that there be no formula $\varphi(v)$ such that both:* (i) $\models_T (\exists v)\varphi(v)$ *and* (ii) *for all elements, a, of models of T, $\varphi(a)$ implies that a is of class Σ.*

The next theorem is from Ehrenfeucht and Mostowski [56] but we have reformulated it to emphasize its "local" character.

Theorem 3.3 (*Ehrenfeucht and Mostowski*). *Suppose T is a theory with an infinite model and $(X, <)$ a linearly ordered set. Then there is a model $\mathfrak{A}$ of T with $X \subseteq A$ such that for all $n \in \omega$ and sequences $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ in X, $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ satisfy the same formulas in $\mathfrak{A}$.*

This remarkable result has many interesting consequences. In particular, Ehrenfeucht used it to show that for any theory T (having an infinite model) there will be some countable set of types of elements such that T has arbitrarily large models containing only elements of those types.

The proof of the theorem of Ehrenfeucht and Mostowski depends on

a combinatorial theorem of Ramsey. The proof of Theorem 3.1 will depend on an extension of Ramsey's theorem (Theorem 39 of Erdös and Rado [56]).

For any set X let $X^{(n)}$ denote the set of all subsets of X having exactly n elements.

Theorem 3.4 (*Erdös and Rado*). *If* $\overline{\overline{Y}} > 2^{\kappa}_{n-1}$, $Y^{(n)} = \bigcup C_i (i \in I)$ and $\overline{\overline{I}} < \kappa$, *then there is an* $X \subseteq Y$ *and a* $j \in I$ *such that* $X > \kappa$ *and* $X^{(n)} \subseteq C_j$.

4. We return to the proof of Theorem 3.1.

It is well known that one may replace existential quantifiers by the use of new individual constant symbols and function symbols. In this way one may transform any theory into a theory having essentially the same models, but in which all the sentences are universal sentences, i.e., contain no existential quantifiers in prenex form. Without loss of generality, we will henceforth assume that all theories considered have been so transformed. If $\mathfrak{A}$ is a model of such a transformed theory and $X \subseteq A$, then there is a submodel of $\mathfrak{A}$ *generated* by X and denoted by $\mathfrak{M}(X, \mathfrak{A})$. Indeed, the elements of $\mathfrak{M}(X, \mathfrak{A})$ are just the values taken on by the various terms of the language when the elements of X are substituted in them.

Let T be a theory and Σ a set of formulas defining a class of elements. Let $\sigma_0, \sigma_1, \ldots$ be an enumeration of Σ and $\tau_0, \tau_1, \ldots$ an enumeration of the terms of the language. Define $h(n)$ to be the least number such that the free variables of τ_n are among $v_0, \ldots, v_{h(n)-1}$. The crux of the proof of Theorem 3.1 is contained in the next lemma.

Lemma 4.1. *Suppose* T *and* Σ *satisfy the hypothesis of* 3.1. *Then there is a function* $F : \omega \to \omega$ *such that for any linearly ordered set* $(X, <)$ *there is a model* $\mathfrak{A}$ *of* T *with* $X \subseteq A$ *such that*

$$\text{(I)} \qquad \neg\, \sigma_{F(m)}(\tau_m(x_0, \ldots, x_{h(m)-1}))$$

holds in $\mathfrak{A}$ *whenever* $m \in \omega$ *and* $x_0 < \ldots < x_{h(m)-1}$.

Proof of lemma. By the completeness theorem it is sufficient to show that there is a function F such that for every linearly ordered set T is consistent with all the required sentences of type (I). By the compactness theorem we need only show the consistency for one infinite linearly ordered set to establish the consistency for all. We shall define F inductively by: $F(n)$ is the least number such that for all $\alpha < \omega_1$:

(*) there is a linearly ordered set $(X, <)$ of power $> \beth_\alpha$ and a model $\mathfrak{A}$ of T such that $X \subseteq A$, $\mathfrak{A}$ omit Σ and all formulas of type (I) are satisfied in $\mathfrak{A}$ for $m \leqslant n$ and $x_0 < \ldots x_{h(m)-1}$ in X.

We must, of course, show that there is at least one such number. Suppose $F(0), \ldots, F(n-1)$ have been defined such that (*) holds for them for all $\alpha < \omega_1$. We first show that for a fixed value of $\alpha < \omega_1$ there is a value of $F(n)$ for which (*) holds. By the induction hypothesis there is a linearly ordered set $(X, <)$ of power $> \beth_{\alpha + h(n)}$ and a model $\mathfrak{A}$ of T omitting Σ with $X \subseteq A$ and such that all formulas of type (I) are satisfied in $\mathfrak{A}$ for $m < n$ and $x_0 < \ldots < x_{h(m)-1}$ in X. (Notice that for $n = 0$ this reduces to the hypothesis of the theorem.) We define $C_i \subseteq X^{(h(n))}$ by $\{x_0, \ldots, x_{h(n)-1}\} \in C_i$ if $x_0 < \ldots < x_{h(n)-1}$ and $\models_{\mathfrak{A}} \neg \sigma_i(\tau_n(x_0, \ldots, x_{h(n)-1}))$. Since $\mathfrak{A}$ omits Σ, $X^{(h(n))} = \bigcup C_i$ $(i \in \omega)$. By Theorem 3.4 there is a $j \in \omega$ and an $X_0 \subseteq X$ such that $\overline{\overline{X_0}} > \beth_\alpha$ and $X_0{}^{(h(n))} \subseteq C_j$. So if we let $F(n) = j$ we can satisfy (*) for the fixed value of α. Of course, for a different value of α we may have to take a different value for $F(n)$. But since there are only ω possible values for $F(n)$ there must be one for which (*) holds for a set of values of α cofinal with ω_1. That value for $F(n)$ will then satisfy (*) for all $\alpha < \omega_1$. This completes the inductive step and proves Lemma 4.1.

To prove Theorem 3.1 from Lemma 4.1 requires only a couple of observations.

First, the conclusion of 4.1 concerns only substitutions of properly ordered sequences from X into the terms of the language, but these are sufficient to give all the elements of $\mathfrak{M}(X, \mathfrak{A})$. The reason is that instead of permuting the order of the x's one may permute the variables in the term and substitute into the new term given thereby.

Second, by suitably modifying the theory T (e.g., by adding a unary relation to correspond to each $\sigma \in \Sigma$) we may replace "holds in $\mathfrak{A}$" by "holds in $\mathfrak{M}(X, \mathfrak{A})$" in the conclusion of Lemma 4.1. Then $\mathfrak{M}(X, \mathfrak{A})$ will be a model of T omitting Σ and since (for infinite X) $\mathfrak{M}(X, \mathfrak{A})$ has the same power as X, Theorem 3.1 is established.

5. In this section we consider some "two-cardinal" theorems. We say T and Σ admit $(\varkappa, \lambda)$ if there is a model $\mathfrak{A}$ of T of power $\varkappa$ such that the set of elements of $\mathfrak{A}$ of class Σ has power λ. This notion was introduced by Vaught for the case where Σ contained only a single formula. He obtained several results for that case, notably a "Löwenheim-Skolem theorem" for two cardinals (see Morley and Vaught [62]) and more

recently Theorem 5.2 below. My result (Theorem 5.1) was suggested by Vaught's results. However, the method of proof is quite different and, indeed, provides an alternate proof of Theorem 5.2 (but not of some stronger forms of 5.2 which Vaught's method gives). It should be understood that Vaught's results preceded mine.

Theorem 5.1. *If for every* $\alpha<\omega_1$ T *and* Σ *admit a pair* $(\varkappa, \lambda)$ *with* $\varkappa>2^{\lambda}_{\alpha}$ *then* T *and* Σ *admit* $(\varkappa, \aleph_0)$ *for all* $\varkappa$.

Theorem 5.2 (*Vaught*). *If* Σ *consists of a single formula then* 5.1 *remains true if* ω_1 *is replaced by* ω.

The proof of Theorem 5.1 is parallel to that of Theorem 3.1. Instead of using Lemma 4.1 we use the following lemma.

Lemma 5.3. *Suppose* T *and* Σ *satisfy the hypothesis of Theorem* 5.1. *Then there is a function* $F: \omega\to\omega+1$ *such that for any linearly ordered set* $(X,<)$ *there is a model* $\mathfrak{A}$ *of* T *with* $X\subseteq A$ *such that for each* $n\in\omega$: *if* $F(n)$ *is finite then*

$$\text{(I)}\qquad \neg\,\sigma_{F(n)}(\tau_n(x_0, \ldots, x_{h(n)-1}))$$

holds in $\mathfrak{A}$ *for all* $x_0<\ldots<x_{h(n)-1}$ *in* X.
If $F(n)=\omega$ *then*

$$\text{(II)}\qquad \tau_n(x_0, \ldots, x_{h(n)-1})=\tau_n(y_0, \ldots, y_{h(n)-1})$$

holds for all $x_0<\ldots<x_{h(n)-1}$ *and* $y_0<\ldots<y_{h(n)-1}$ *in* X.

The proof of Lemma 5.3 is similar to the proof of 4.1. We define $F(n)$ by induction to be the least ordinal such that for all $\alpha<\omega_1$:

(**) there is a linearly ordered set $(X,<)$ and a model $\mathfrak{A}$ of T with $X\subseteq A$ such that: (i) $\overline{\overline{X}}>2^{\lambda}_{\alpha}$ where λ is the power of the class Σ in $\mathfrak{A}$, and (ii) for all $m\leqslant n$ and all $x_0<\ldots<x_{h(m)-1}$ and $y_0<\ldots<y_{h(m)-1}$ in X, if $F(m)$ is finite (I) holds in $\mathfrak{A}$, and if $F(m)=\omega$ (II) holds in $\mathfrak{A}$.

We proceed (in a fashion similar to that used in proving Lemma 4.1) to assume $F(0), \ldots, F(n-1)$ to be defined satisfying (**) for each $\alpha<\omega_1$. Thus for such $\alpha<\omega_1$ there is a linear ordered set $(X,<)$ and a model $\mathfrak{A}$ of T with $X\subseteq A$ and $\overline{\overline{X}}>2^{\lambda}_{\alpha+h(n)}$ which satisfies all the proper formulas of type (I) and (II) for $m<n$. Suppose $x_0<\ldots<x_{h(n)-1}$; we define $\{x_0, \ldots, x_{h(n)-1}\}\in C_i$ if $\neg\,\sigma_i(\tau_n(x_0, \ldots, x_{h(n)-1}))$ and $\{x_0, \ldots, x_{h(n)-1}\}\in D_a$ if $\tau_n(x_0, \ldots, x_{h(n)-1})=a$ and a is of class Σ in $\mathfrak{A}$. Then $X^{(h(n))}=\bigcup C_i\cup\bigcup D_a$

($i \in \omega$, $a \in$ class Σ in $\mathfrak{A}$). By Theorem 3.4 there is an $X_0 \subseteq X$ with $\overline{\overline{X_0} > 2^{\lambda}_{\alpha}}$ and there is either a $j \in \omega$ with $X_0{}^{(h(n))} \subseteq C_j$ or there is an $a \in A$ with $x_0{}^{(h(n))} \subseteq D_a$. In the first case $F(n) = j$, and in the second case $F(n) = \omega$ will satisfy (**) for the fixed value of α. The rest of the proof is the same as that of Lemma 4.1. The deduction of Theorem 5.1 from Lemma 5.3 is essentially the same as the deduction of Theorem 3.1 from Lemma 4.1.

If Σ consists of a single formula, say σ_0, then $F(n)$ can take on only two values, 0 and ω, so the whole proof can be carried out with ω, substituted for ω_1 in the hypothesis. This establishes Theorem 5.2.

In the case where Σ consists of a single formula, Chang and Keisler [62] show (assuming the generalized continuum hypothesis) that if T and Σ admit $(\varkappa, \lambda)$ then they admit $(\varkappa, \lambda')$ for all $\varkappa \geqslant \lambda' \geqslant \lambda$. Thus, in the case where Σ is a single formula the conclusion of the theorem may be strengthened to say that T and Σ admit all pairs $(\varkappa, \lambda)$ with $\varkappa \geqslant \lambda$.

It is somewhat surprising that the theorem cannot be so strengthened in the general case. Keisler has devised an example of a theory T and a class Σ which admit $(\varkappa, \varkappa)$ and $(\varkappa, \aleph_0)$ for all $\varkappa$, but do not admit any other pairs!

6. Suppose now we remove the restriction that our language have only a countable number of symbols and consider languages having, say, $\aleph_\beta$ symbols. It is natural to conjecture that 2.2 and 3.1 remain true if ω_1 is replaced by $\omega_{\beta+1}$. The best result I have been able to establish is the following which I state without proof.

Theorem 6.1. *In languages having at most $\aleph_\beta$ symbols:* (1) *Theorem* 2.2 *remains true if ω_1 is replaced by $\omega_{\beta+1}$, and* (2) *Theorem* 3.1 *remains true if ω_1 is replaced by $\omega_{\gamma+1}$ where $2^{\aleph_\beta} = \aleph_\gamma$.*

Although our results concern first-order languages they have applications to some non-elementary languages. Consider a theory T which asserts (among possibly many other things) that a certain definable subset Q of each model satisfies some first-order theory of the integers. If Σ is the class of non-standard integers, then the models of T which omit Σ are those models of T in which Q is actually the integers. Martin Helling observed this, and that, therefore, Theorem 3.1 implies that $\beth_{\omega_1}$ is the Hanf number of the weak-second-order logic of Tarski. (The Hanf number of a language is the least cardinal $\varkappa$ such that any theory in the language which has a model of power $\varkappa$ has a model in every

infinite power.) Edgar Lopez-Escobar used Helling's result to show that $\beth_{\omega_1}$ is the Hanf number for a certain language containing infinitely long formulas with only a finite number of quantifiers (see Scott [*]).

7. We conclude with a few unsolved problems. The first of these is to strengthen Theorem 6.1.

The second, and most interesting one, is to characterize those classes which can be omitted in models of a given power. Theorem 3.2 gives a nice characterization of those classes which cannot be omitted in any power. But we have no nice property which distinguishes between those classes which can be omitted: (1) only in countable models, (2) in some uncountable models but not arbitrarily large ones, and (3) in models of all infinite powers.

The third problem is a much more definite one. If one examines the proof of Theorem 2.2 one sees that all the examples ultimately depend on the one characterizing $\aleph_1$, and in that one Σ is the class of non-standard integers. Is there any essentially different way to characterize uncountable cardinals? We make this question precise as follows. We say a formula, $\varphi(v)$ is *algebraic* in a theory T if in every model of T there are only a finite number of elements satisfying $\varphi(v)$. By the compactness theorem there will then be a finite bound on the number of elements in any model which satisfy $\varphi(v)$. An element of a model of T is *algebraic* if it satisfies a formula algebraic in T. Clearly, the number of algebraic elements in any model is countable.

Conjecture. If T is complete and T and Σ characterize $\varkappa > \aleph_0$, then every model of T contains an infinite number of algebraic elements.

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UNIVERSAL GROUPS OF AUTOMORPHISMS OF MODELS

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Introduction. Let $\mathfrak{A}=\langle A, R\rangle$ be a first-order structure. By $Aut\ \mathfrak{A}$ we shall denote the group of all automorphisms of $\mathfrak{A}$, i.e., the group of all one-to-one mappings of A onto itself which preserve the relation R. We shall say that a group G has a *faithful representation* by automorphisms of $\mathfrak{A}$ iff for some subgroup $G' \subseteq Aut\ \mathfrak{A}$, G is isomorphic to G'. This usage of the term "representation" is analogous to the usage by algebraists who study representations of groups by groups of linear transformations, i.e., by subgroups of the group of all non-singular linear mappings of a vector space onto itself.

In this paper we introduce two concepts. For a set Σ of first-order sentences we define $\mathscr{G}(\Sigma)$ to be the class of all groups G which have a faithful representation by automorphisms of some model $\mathfrak{A}$ of Σ. We also introduce the class $\mathscr{K}_{\text{un}}$ of universal groups G of automorphisms which have the property that for *every* set Σ which has infinite models, G has a faithful representation by automorphisms of a model of Σ.

The classes $\mathscr{G}(\Sigma)$ and $\mathscr{K}_{\text{un}}$ turn out to have interesting properties. In particular they are elementary (in the wider sense), i.e., the classes of models of sets of first-order sentences of the language of group theory. For the class $\mathscr{K}_{\text{un}}$ we are able to construct a very simple set Ω_{un} such that $\mathscr{K}_{\text{un}} = Mod\ \Omega_{\text{un}}$.

These results make the classes $\mathscr{G}(\Sigma)$ and $\mathscr{K}_{\text{un}}$ a natural object of study by the logician. We have a whole class of new problems of the following form. Given a set Σ of first-order axioms such as the axioms for algebraically closed fields, to find explicitly "natural" axioms Γ_Σ for the class $\mathscr{G}(\Sigma)$ of groups of automorphisms of models of Σ. In this paper we solve this problem for the set Σ_{ord} of axioms for simple ordering.

Notations. As a rule we use the notations and terminology summarized by the editors of this volume. We shall mention a few notations and concepts which although taken from the literature may be unfamiliar to some readers.

If Σ is a set of sentences then $Mod\ \Sigma$ will denote the class of all models of Σ. A class $\mathscr{K}$ of structures will be called *elementary* (in the wider sense) iff $\mathscr{K} = Mod\ \Sigma$ for some set Σ of first-order sentences. We shall often drop the adjective "in the wider sense" and simply say "elementary" (though "elementary" is used elsewhere in a more restricted sense). $\mathscr{K}$ is called *axiomatizable* iff for some recursive Σ, $\mathscr{K} = Mod\ \Sigma$.

$\bigwedge_i \sigma_i$, where i ranges over a *finite* set I is used to abbreviate the logical conjunction of the formulas σ_i, $i \in I$.

We shall view each integer as the set of all smaller integers. Thus $n = \{0, 1, \ldots, n-1\}$. The power set of a set S will be denoted by 2^S.

Unless specifically stated to the contrary, we assume that the structures under discussion are of the type $\langle A, R\rangle$ where R is a binary relation on A. This convention will simplify notations without restricting the generality of our results.

The notation $Aut\ \mathfrak{A}$ for the group of all automorphisms of a structure $\mathfrak{A}$ was explained in the Introduction.

1. The class $\mathscr{G}(\Sigma)$ and its properties.

Definition 1. *Let $\mathscr{K}$ be a class of similar structures. The class $\mathscr{G}(\mathscr{K})$ will consist of all groups G for which there exists a structure $\mathfrak{A} \in \mathscr{K}$ such that G is isomorphic to a subgroup of $Aut\ \mathfrak{A}$.*

If $G \in \mathscr{G}(\mathscr{K})$ we shall say that G has a *faithful representation* by a group of automorphisms of a structure in $\mathscr{K}$. We shall also say that G *acts faithfully* on a structure in $\mathscr{K}$.

The most interesting case of this concept is where $\mathscr{K}$ is an elementary class.

Definition 2. *Let Σ be a consistent set of a first-order language. The class $\mathscr{G}(\Sigma)$ dual to Σ is defined to be $\mathscr{G}(Mod\ \Sigma)$.*

The class $\mathscr{G}(\Sigma)$ of groups turns out to be interesting from the logician's point of view because of the following:

Theorem 1. *Let Σ be a consistent set of sentences of a first-order language. The class $\mathscr{G}(\Sigma)$ is an elementary universal class in the wider sense, $\mathscr{G}(\Sigma) \in \mathrm{UC}_\Delta$. Furthermore, if Σ is recursive then there exists a recursive set Γ_Σ of sentences (of the language of group theory) such that $\mathscr{G}(\Sigma) = Mod\ \Gamma_\Sigma$.*

Remark. It should be noted that when we say that a class $\mathscr{G}$ of groups is elementary we mean that $\mathscr{G}$ is the class of models of a set of sentences written only by means of the non-logical constant $\cdot$ to denote multiplication, and a singulary function constant to denote the group-inverse function $Inv(x)=x^{-1}$ which is also taken as a primitive.

We shall give two proofs. The first will derive the theorem from a general result of Łoś [55a] and Tarski [54, II]. The second shorter proof uses the theory of ultraproducts and is more algebraic in nature.

Proof I. We shall use the following result due to Łoś and Tarski. Let Σ_1 be a consistent set of sentences of a first-order language and let $\mathscr{K}_1=Mod\,\Sigma_1$. If the typical structure $\mathfrak{A}\in\mathscr{K}_1$ has the form $\mathfrak{A}=\langle A, R_1, \ldots, R_n\rangle$ and k is a fixed integer $1\leqslant k\leqslant n$, then the structure $\mathfrak{A}_k=\langle A, R_1, \ldots, R_k\rangle$ will be called the *k-reduct* of $\mathfrak{A}$. The class $\mathscr{K}$ of all substructures of k-reducts of structures in $\mathscr{K}_1$ is a universal class (in the wider sense). Furthermore, if Σ_1 is recursive then there exists a recursive set Ω of universal sentences such that $\mathscr{K}=Mod\,\Omega$.

We assume as usual that the sentences in Σ contain just one binary non-logical constant so that models of Σ are structures of the type $\langle A, R\rangle$ where $R\subseteq A\times A$. It is convenient to view groups as structures of the type $\langle G, \cdot, Inv\rangle$ where $\cdot$ is the binary group operation and Inv is the singulary group-inverse operation $Inv\,(x)=x^{-1}$. In this formulation, a substructure $\langle G', \cdot, Inv\rangle$ of the above group is automatically a subgroup. As is customary in algebra, we refer to a group by mentioning just its universe G.

A necessary and sufficient condition on a group G' to be in $\mathscr{G}(\Sigma)$ is that there exist a structure $B=\langle B, G, \cdot, Inv, A, R, \star\rangle$ where G and A are sets (monadic relations), R is a binary relation, $\cdot$ and $\star$ are binary operations and Inv a singulary operation, such that the following holds. The group G' is isomorphic to a substructure of $\langle G, \cdot|G, Inv|G\rangle$. The structure B is a model of the set Σ_1 of sentences (of an appropriate first-order language) expressing the following conditions.

(1) $\langle G, \cdot|G, Inv|G\rangle$ is a group.

(2) $\langle A, R|A\rangle$ is a model of Σ.

(3) $B=G\cup A$, $G\cap A=\emptyset$.

(4) $x\in G$ and $y\in A$ imply $x\star y\in A$; otherwise $x\star y=e_G$ (e will always denote the unit of a group).

(5) For every $g \in G$ the mapping $a \to g \star a$ (a ranges over A) is an automorphism of $\langle A, R|A\rangle$.

(6) $g_1, g_2 \in G$ and $a \in G$ imply $(g_1 \cdot g_2) \star a = g_1 \star (g_2 \star a)$.

(7) $g \star a = a$ for all $a \in A$ implies $g = e_G$.

The construction of the set Σ_1 is straightforward. The crucial device is to use the predicate symbols of the formal language which correspond to G and A to relativize quantifiers in the appropriate way. Note that if Σ is recursive then Σ_1 is recursive.

The class $\mathscr{K}$ of all substructures $\mathfrak{B}' = \langle B', G', \cdot, Inv\rangle$ of the 3-reducts $\mathfrak{B}_3 = \langle B, G, \cdot, Inv\rangle$ where $\mathfrak{B} \in Mod\ \Sigma_1$ is, by the theorem cited above, a universal class. $\mathscr{K}$ is, of course, closed with respect to isomorphisms. Assume $\mathscr{K} = \mathrm{Mod}\ \Omega$ where Ω is a set of universal sentences. Also, if Σ_1 is recursive then Ω may be chosen to be recursive.

Add now to Ω a sentence σ expressing the condition (i)$B' = G'$. It is clear that $\langle B', G', \cdot, Inv\rangle \in Mod\,\Omega \cup \{\sigma\}$ if and only if $\langle G', \cdot, Inv\rangle \in \mathscr{G}(\Sigma)$. Because of ($i$) one can easily obtain from $\Omega \cup \{\sigma\}$ a set Γ_Σ of sentences in the language of group theory such that $\mathscr{G}(\Sigma) = Mod\ \Gamma_\Sigma$. Finally, if Σ, and hence Ω, is recursive then Γ_Σ is recursive.

Proof II. The class $\mathscr{G}(\Sigma)$ is closed with respect to substructures. We shall prove that $\mathscr{G}(\Sigma) \in \mathrm{UC}_\Delta$ by showing that $\mathscr{G}(\Sigma)$ is closed with respect to ultraproducts; see Łoś [55b] and Kochen [61] for the characterization of universal classes used here.

Let $G_i \in \mathscr{G}(\Sigma)$ for $i \in I$. For every $i \in I$ there exists a structure $\mathfrak{A}_i = \langle A_i, R_i\rangle \in Mod\ \Sigma$ such that G_i is a group of automorphisms of $\mathfrak{A}_i$. Let D be an ultrafilter on I. By $\prod_i S_i$ we shall denote the cartesian product of a collection of sets or structures indexed by I and by $\prod_i S_i/D$ we shall denote the product reduced by D (the ultraproduct). Also if $s \in \prod_i S_i$ then $[s]_D$ will denote the equivalence class of s in the reduced product $\prod_i S_i/D$. Let

$$G = \prod_i G_i/D, \quad \mathfrak{A} = \prod_i \mathfrak{A}_i/D.$$

For $g \in \prod_i G_i$ and $a \in \prod_i A_i$ define $g(a) = a' \in \prod_i A_i$ by $a'(i) = g(i)(a(i))$ for all $i \in I$, where on the right hand we have the result of application of the automorphism $g(i) \in G_i$ to the element $a(i) \in A_i$. Define an operation

of $[g]_D$ on $\mathfrak{A}$ by $[g]_D([a]_D)=[g(a)]_D$ for all $[a]_D \in \mathfrak{A}$. It is readily verifiable that if $[g]_D=[g_1]_D$ and $[a]_D=[a_1]_D$ then $[g(a)]_D=[g_1(a_1)]_D$. The above definition is, therefore, independent of the representatives g and a. Calculating coordinatewise with the representatives g and a of $[g]_D$ and $[a]_D$ and using the fact that $g(i)$ is an automorphism of $\mathfrak{A}_i$ we see that $[g]_D$ is an automorphism of $\mathfrak{A}$. Furthermore,

$$[g_1]_D([g_2]_D([a]_D))=[g_1g_2]_D([a]_D)$$

where the product $g_1g_2 \in \prod_i G_i$ is defined coordinatewise, and $[g_1g_2]_D=$ $=[g_1]_D[g_2]_D$. Also, if $[g]_D \neq [e]_D$, then for some $[a]_D$ we have $[g]_D([a]_D) \neq [a]_D$. Thus G acts as a group of automorphisms of $\mathfrak{A}$. Now $\mathfrak{A} \in Mod\,(\Sigma)$, hence $G \in \mathscr{G}(\Sigma)$. This completes the proof that $\mathscr{G}(\Sigma) \in \mathrm{UC}_\Delta$. The ultraproduct method does not yield the assertion concerning axiomatizability of $\mathscr{G}(\Sigma)$.

Remark. We proved Theorem 1 for the case that $Inv\,(x)=x^{-1}$ is taken as a primitive. If we want to drop the definable Inv and view groups as structures of the type $\langle G, \cdot \rangle$, then $\mathscr{G}(\Sigma)$ is no longer a universal class because the group axioms are no longer universal sentences. Using Theorem 1 we can easily see, however, that $\mathscr{G}(\Sigma)=Mod\,\Gamma'_\Sigma$ where Γ'_Σ consists of the group axioms and universal sentences.

2. Algebraic characterization of universal groups.

Definition 3. *A group G will be called a* universal group *(of automorphisms) iff for every set Σ of sentences (of any language) which has infinite models, $G \in \mathscr{G}(\Sigma)$,* i.e., *$G$ acts faithfully on a model of Σ. The class of all universal groups will be denoted by $\mathscr{K}_{\mathrm{un}}$.*

Remark. It is not hard to deduce from Theorem 1 that $\mathscr{K}_{\mathrm{un}}$ is an elementary and even axiomatizable class. The first assertion is immediate because $\mathscr{K}_{\mathrm{un}}=\bigcap_\Sigma \mathscr{G}(\Sigma)$, where Σ ranges over sets with infinite models, and each $\mathscr{G}(\Sigma)$ is elementary. We shall not prove here the assertion concerning axiomatizability because in Section 3 we shall construct explicitly a very simple set Ω of sentences such that $\mathscr{K}_{\mathrm{un}}=Mod\,\Omega$.

The exact determination of the class $\mathscr{K}_{\mathrm{un}}$ depends on a fundamental result of Ehrenfeucht and Mostowski [56, Theorem 5.7]. Let $\bar{\Sigma}$ be a set of sentences which has infinite models. Let $\langle B, \leqslant \rangle$ be an ordered

set and G a group of order-preserving permutations of B. There exists a model $\mathfrak{A}=\langle A, R\rangle$ of $\bar{\Sigma}$ such that $B\subseteq A$ and for every $g\in G$ there exists an automorphism f of $\mathfrak{A}$ such that $g=f|B$.

Following Ehrenfeucht and Mostowski [56] we say, in the above situation, that $F=Aut\,\mathfrak{A}$ *strongly contains* G. Note that if F strongly contains G it is not necessarily true that G is isomorphic to a subgroup of F. If F strongly contains G then G is a homomorphic image of a subgroup of F.

Definition 4. *A group G is* order-preserving *iff there exists an ordered set $\mathfrak{B}=\langle B, \leqslant\rangle$ such that G is isomorphic to a subgroup of $Aut\,\mathfrak{B}$.*

If Σ_{ord} is the set of the usual axioms of ordering then the class of order preserving groups is precisely $\mathscr{G}(\Sigma_{\mathrm{ord}})$ (Definition 2).

Theorem 2. *A group G is a universal group of automorphisms if and only if it is order-preserving. Thus $\mathscr{K}_{\mathrm{un}}=\mathscr{G}(\Sigma_{\mathrm{ord}})$.*

Proof. That $\mathscr{K}_{\mathrm{un}}\subseteq\mathscr{G}(\Sigma_{\mathrm{ord}})$ is trivial so that all we have to show is $\mathscr{G}(\Sigma_{\mathrm{ord}})\subseteq\mathscr{K}_{\mathrm{un}}$. Let Σ be a set of sentences having infinite models. To construct a model of Σ having G as a group of automorphisms we apply the Ehrenfeucht–Mostowski theorem to a set of sentences $\bar{\Sigma}$ obtained from Σ by introduction of Skolem functions.

We may assume that every sentence $\sigma\in\Sigma$ is in prenex form

$$\sigma=\forall x_1\ldots\forall x_n\exists y_1\ldots\exists y_m\,\forall x_{n+1}\ldots\exists y_{m+1}\ldots M(x_1,\ldots,x_k,y_1,y_2,\ldots,y_{p(\sigma)}).$$

Augment the first-order language of Σ by introducing for each existentially quantified variable y_i of σ a function constant f_i^{σ}, $1\leqslant i\leqslant p(\sigma)$. The order of f_i^{σ} will be the number of universal quantifiers preceding $\exists y_i$. Let $\bar{\sigma}$ be

$$\bar{\sigma}=\forall x_1\forall x_2\ldots\forall x_k M(x_1,\ldots,x_k,f_1^{\sigma}(x_1,\ldots,x_n),\ldots,f_{p(\sigma)}^{\sigma}(x_1,\ldots,x_k)),$$

and let $\bar{\Sigma}=\{\bar{\sigma}|\sigma\in\Sigma\}$.

It is well known that: (i) Every model $\mathfrak{A}=\langle A, R\rangle$ of Σ is a reduct of some model $\bar{\mathfrak{A}}=\langle A, R, f_i^{\sigma}\rangle_{\sigma\in\Sigma,\,1\leqslant i\leqslant p(\sigma)}$ of $\bar{\Sigma}$. (ii) If $\bar{\mathfrak{A}}$ is a model of $\bar{\Sigma}$ then $\mathfrak{A}$ is a model of Σ. (iii) If $\bar{\mathfrak{A}}$ is a model of $\bar{\Sigma}$, $B\subseteq A$ (A is the universe of $\bar{\mathfrak{A}}$ as above), and $\bar{B}$ is the closure of B under the functions f_i^{σ}, $\sigma\in\Sigma$, $1\leqslant i\leqslant p(\sigma)$, then $\langle\bar{B}, R|\bar{B}\rangle$ is a model of Σ. This last statement follows from (ii) and the fact that $\langle\bar{B}, R|\bar{B}, f_i^{\sigma}|\bar{B}\rangle_{\sigma\in\Sigma,\,1\leqslant i\leqslant p(\sigma)}$ is a model of $\bar{\Sigma}$.

The set $\bar{\Sigma}$ has, by (i), infinite models. Let $\bar{\mathfrak{A}}=\langle A, R, f_i^{\sigma}\rangle_{\sigma\in\Sigma,\,1\leqslant i\leqslant p(\sigma)}$

be a model of $\bar{\Sigma}$ such that $B \subseteq A$, and $Aut(\bar{\mathfrak{A}})$ strongly contains G where G is a group of automorphisms of the ordered set $\langle B, \leqslant \rangle$. Let $\bar{B}$ be the closure of B with respect to the functions f_i^σ.

Let $g \in G$ and let $\varphi \in Aut\, \bar{\mathfrak{A}}$ be such that $g = \varphi | B$. The mapping $\bar{g} = \varphi | \bar{B}$ is a mapping of $\bar{B}$ onto itself and hence, being a restriction of an automorphism of $\bar{\mathfrak{A}}$, is an automorphism of $\bar{\mathfrak{B}} = \langle \bar{B}, R|\bar{B}, f_i^\sigma|\bar{B}\rangle_{\sigma \in \Sigma, 1 \leqslant i \leqslant p(\sigma)}$. Indeed, if $b \in \bar{B}$ then $b = t(b_1, \ldots, b_n)$ where t is a term built out of a finite number of the f_i^σ and $b_i \in B$, $1 \leqslant i \leqslant n$. Thus $\bar{g}b = \varphi b = t(\varphi b_1, \ldots, \varphi b_n) = t(gb_1, \ldots, gb_n)$; since $gb_i \in B$ and $\bar{B}$ is closed with respect to the f_i^σ, we have $\bar{g}b \in \bar{B}$. Furthermore $c = t(\varphi^{-1}b_1, \ldots, \varphi^{-1}b_n) = t(g^{-1}b_1, \ldots, g^{-1}b_n)$ is an element of $\bar{B}$ and $gc = \varphi c = b$. Thus g is a mapping of $\bar{B}$ onto $\bar{B}$.

Assume that $\bar{g}$ and $\bar{g}_1$ are two automorphisms of $\bar{\mathfrak{B}}$ such that $\bar{g}|B = \bar{g}_1|B = g \in G$. With $b \in \bar{B}$ as above we have

$$\bar{g}b = t(\bar{g}b_1, \ldots, \bar{g}b_n) = t(gb_1, \ldots, gb_n) = t(\bar{g}_1 b_1, \ldots, \bar{g}_1 b_n) = \bar{g}_1 b,$$

hence $\bar{g} = \bar{g}_1$.

Thus every $g \in G$ is the restriction to B of *precisely one* automorphism $\bar{g}$ of $\bar{\mathfrak{B}}$. Denote the set $\{\bar{g} | g \in Aut\, \bar{\mathfrak{B}}, \bar{g}|B = g \in G\}$ by $\bar{G}$. The mapping $g \to \bar{g}$, where $g \in G$, $\bar{g} \in \bar{G}$, and $\bar{g}|B = g$, establishes an isomorphism between the group G and the set $\bar{G}$ with the composition operation. Thus $\bar{G}$ is a *group* of automorphisms of $\bar{\mathfrak{B}}$ and $G \approx \bar{G}$. The reduct $\langle \bar{B}, R|\bar{B}\rangle$ of $\bar{\mathfrak{B}}$ is, by (iii), a model of Σ. Hence $G \in \mathscr{G}(\Sigma)$. Since G was any order-preserving group and Σ was any set of sentences with infinite models, we have $\mathscr{G}(\Sigma_{\text{ord}}) \subseteq \mathscr{K}_{\text{un}}$.

3. Elementary characterization of universal groups. We know now that $\mathscr{K}_{\text{un}} = \mathscr{G}(\Sigma_{\text{ord}})$. It follows from Theorem 1 that there exists a recursive set Ω such that $\mathscr{G}(\Sigma_{\text{ord}}) = Mod\, \Omega$. Theorem 1, however, will lead to a rather complicated set Ω. In this chapter we shall treat directly the problem of axiomatizing $\mathscr{K}_{\text{un}} = \mathscr{G}(\Sigma_{\text{ord}})$ and obtain a simple solution.

Definition 5. *A group G is* left orderable *if there exists a total ordering $\leqslant$ of G such that for every $g_1, g_2, g \in G$, $g_1 \leqslant g_2$ implies $gg_1 \leqslant gg_2$.*

The following result is due to P. M. Cohn [57]. *A group is order-preserving if and only if it is left orderable.*

Example. Let $\langle R, \leqslant \rangle$ be the real line with the usual ordering. The order-preserving mappings of R onto R are the continuous strictly

increasing functions f such that $f(\infty)=\infty$ and $f(-\infty)=-\infty$. Let G be the group of these functions under composition. The group G is, by the result cited above, left orderable. We claim that *G is not orderable* (a group is called *orderable* iff for some ordering $\leqslant$, from $g_1 \leqslant g_2$ both $gg_1 \leqslant gg_2$ and $g_1 g \leqslant g_2 g$ follow).

To prove this, note that in an orderable group if $xy^2=y^2x$ then $xy=yx$. For otherwise $xyx^{-1} \neq y$ and in the ordering of the group we have, say, $y<xyx^{-1}$. This implies, in an orderable group, that $y^2<xy^2x^{-1}$ and hence $xy^2 \neq y^2x$.

Let $\varphi(x)$, $-\infty<x<\infty$ be a strictly positive function of period 2, i.e., $\varphi(x+2)=\varphi(x)$, such that $\int_x^{x+2} \varphi(t)dt=2$, assume also that some number c, $\int_c^{c+1} \varphi(t)dt \neq 1$. Define

$$f(x)=\int_0^x \varphi(t)dt.$$

The function f is in G. Also $f(x+2)=f(x)+2$ for all $-\infty<x<\infty$, and $f(c+1) \neq f(c)+1$.

Let $g(x)=x+1$, thus $g \in G$. Now $(fg)(x)=f(x+1)$, $(gf)(x)=f(x)+1$. Thus $fg \neq gf$. On the other hand,

$$(fg^2)(x)=f(x+2)=f(x)+2=(g^2f)(x),$$

hence $g^2f=fg^2$ in the group G. By the previous remark the group G is *not* orderable.

To give an elementary characterization of $\mathscr{K}_{\text{un}}=\mathscr{G}(\Sigma_{\text{ord}})$ we need two lemmata on left orderable groups.

Lemma 3. *A necessary and sufficient condition on a group G to be left orderable is that there exist a subset $P \subset G$ such that*

(i) *P is closed with respect to the group multiplication.*

(ii) *$e \notin P$ where e is the unit of G.*

(iii) *For every $g \in G$, $g \neq e$, either $g \in P$ or $g^{-1} \in P$.*

Proof. Assume that G is left orderable and that $\leqslant$ is a left ordering of G. Define

$$P=\{g \mid g \in G,\ g<e\}.$$

Clearly P satisfies (i)–(iii).

Let $P \subset G$ be such that (i)–(iii) hold. Define a relation $\leqslant$ on G by

$g_1 \leqslant g_2$ if and only if $g_1^{-1}g_2 \in P$ or $g_1 = g_2$. The relation $\leqslant$ is a left ordering of G.

Lemma 4. *A group G is left orderable if and only if there exist two subsets $P \subset G$ and $P' \subset G$ such that*

(1) *For every $g_1, g_2, g_3 \in P$, $g_1g_2 \neq e$ and $g_1g_2g_3 \neq e$.*

(2) *For every $g_1, g_2, g_3 \in P'$, $g_1g_2 \neq e$ and $g_1g_2g_3 \neq e$.*

(3) $P \cup P' = G - \{e\}$.

Proof. Assume that G is left orderable by $\leqslant$ and define P as in the proof of Lemma 3. Define

$$P' = \{g | g \in G, e < g\}.$$

Conditions (1)–(3) are satisfied by P and P'.

Let the pair P and P' of subsets of G satisfy (1)–(3). We shall show that P satisfies (i)–(iii) of Lemma 3 and hence G is left orderable. If $g \neq e$ then, by (3), $\{g, g^{-1}\} \subseteq P \cup P'$. Now $\{g, g^{-1}\} \subseteq P$ is impossible since $gg^{-1} = e$ would contradict (1). Similarly $\{g, g^{-1}\} \not\subseteq P'$. Thus for $g \neq e$, $g \in P$ and $g^{-1} \in P'$, or vice versa.

To show that P is closed with respect to multiplication assume that $g_1, g_2 \in P$ but $g_1g_2 \notin P$. Now $g_1g_2 = g \neq e$ by (1). Thus $g \in P'$ and hence $g^{-1} \in P$. But $g_1g_2g^{-1} = e$ contradicting (1). Finally $e \notin P$ by (3).

Theorem 5. *A group G is left orderable if and only if it is a model of the following axioms.*

Let n be a positive integer and let ψ be a function assigning to every subset $s \subseteq n = \{0, 1, \ldots, n-1\}$ a pair (i, j) or a triple (i, j, k) such that $\{i, j\} \subseteq s$ ($\{i, j, k\} \subseteq s$) or $\{i, j\} \subseteq n - s$ (or $\{i, j, k\} \subseteq n - s$). Denote by Ψ the set of all such functions ψ for all n.

Let $w_{\psi(S)}$ be the term x_ix_j if $\psi(s) = (i, j)$ and $w_{\psi(s)} = x_ix_jx_k$ if $\psi(s) = (i, j, k)$.

For $0 < n$ and $\psi \in \Psi$, ψ a function on 2^n, define

$$\alpha_{n,\psi} = \forall x_0 \ldots \forall x_{n-1} \left[\bigwedge_{s \in 2^n} w_{\psi(s)} = e \rightarrow x_0 = e \vee \ldots \vee x_{n-1} = e\right].$$

A group is G left orderable if and only if $G \in Mod\, \Omega_{\text{un}}$ where

$$\Omega_{\text{un}} = \{\alpha_{n,\psi} | n = 1, 2, \ldots, \psi \in \Psi\}.$$

Proof. Assume that G is a left orderable group and let $g_0, \ldots, g_{n-1}$ be elements of G such that $g_i \neq e$, $0 \leqslant i \leqslant n-1$. Let $\psi \in \Psi$ and $\alpha_{n,\psi}$ be

as above. If $s=\{i|i<n,\ e<g_i\}$, where $\leqslant$ is a left ordering of G, then the product obtained from $w_{\psi(s)}$ by substituting g_i for x_i, $i<n$, is either a product of two or three elements of G which are greater than e, or the product of two or three elements smaller then e. In any event, the product does not equal e. This shows that $\alpha_{n,\psi}$ is true in G and hence $G \in Mod\, \Omega_{\text{un}}$.

To prove sufficiency we observe, first of all, that a group is left orderable if and only if every finitely generated subgroup is left orderable. This follows, for example, from the compactness theorem. Thus it suffices to prove the assertion for finitely generated and hence countable groups.

Let $G \in Mod\, \Omega_{\text{un}}$ be a countable group and let $g_0, g_1, \ldots$ be an enumeration of $G-\{e\}$. For $1 \leqslant n$ define $S_n=\{g_0, \ldots, g_{n-1}\}$. We claim that there exists a partition $S_n=P_n \cup P_n'$, $P_n \cap P_n' \neq \emptyset$ such that for all $g_i, g_j, g_k \in P_n$, $g_i g_j \neq e$ and $g_i g_j g_k \neq e$, and similarly for P_n'. Otherwise there would exist for every partition $S_n=P_n \cup P_n'$ a pair (i, j) or a triple (i, j, k) such that $\{g_i, g_j\}$ ($\{g_i, g_j, g_k\}$) is a subset of P_n or P_n' and $g_i g_j=e$ ($g_i g_j g_k=e$). For $s=\{i|g_i \in P_n\}$, define $\psi(s)$ to be the above pair or triple. Now $\psi \in \Psi$ and $g_0, \ldots, g_{n-1}$ satisfy the conjunction

$$\bigwedge_{s \in 2^n} w_{\psi(s)}=e \wedge x_0 \neq e \wedge \ldots \wedge x_{n-1} \neq e,$$

in violation of $G \in Mod\, \Omega_{\text{un}}$.

Thus for every $1 \leqslant n$ there exists a partition $S_n=P_n \cup P_n'$ with the desired properties. Note that if $k<n$ then $P_n \cap S_k$ and $P_n' \cap S_k$ yield a partition of S_k with the desired properties. Using König's infinity lemma we can now derive the existence of a partition of $\{g_0, g_1, \ldots\}=$ $=G-\{e\}=P \cup P'$ satisfying conditions (1)–(3) of Lemma 4. Thus G is left orderable.

Corollary 6. $\mathscr{K}_{\text{un}}=Mod\, \Omega_{\text{un}}=\mathscr{G}(\Sigma_{\text{ord}})$.

The set Ω_{un} is infinite. This is necessarily so in view of the following:

Theorem 7. *The class $\mathscr{K}_{\text{un}}=\mathscr{G}(\Sigma_{\text{ord}})$ is not finitely axiomatizable.*

Proof. It is well known that every torsion-free abelian group is orderable and hence in $\mathscr{K}_{\text{un}}$. This can also be proved from Theorem 5. Let Z_p be the cyclic group of order p where p is a prime. Z_p is a model of the sentence

$$\alpha_p=\forall x\, [x \neq e \rightarrow \bigwedge_{0<n<p} x^n \neq e].$$

Let I be the set of all primes and D be a non-principal ultrafilter on I. Consider the ultraproduct $\prod_p Z_p/D=G$. G is an abelian group. Since for $p<q$, $\alpha_q \rightarrow \alpha_p$, G is a model of all sentences α_p, $p \in I$, and hence a torsion-free group. Thus $G \in \mathscr{K}_{\text{un}}$. But $Z_p \notin \mathscr{K}_{\text{un}}$ for all p because Z_p is not orderable. Hence $\mathscr{K}_{\text{un}}$ is not finitely axiomatizable; see Kochen [61] and Keisler [61] for the characterization of finitely axiomatizable classes used here.

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TOPICS IN NON-ARCHIMEDEAN MATHEMATICS

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1. Introduction. Students of the history of mathematics know that Leibniz wished to base the differential and integral calculus on a number system that includes infinitely small and infinitely large quantities. More precisely, he regarded the new numbers as ideal elements which were supposed to have the same properties as the familiar real numbers and stated that their introduction was useful for the *art of invention.*

Neither Leibniz nor his immediate successors were able to establish a rational framework within which this claim could be substantiated and (with rare exceptions) the attempt was abandoned by later generations, who followed the lines laid down by Cauchy and Weierstrass. Even so, there were individual mathematicians such as du Bois–Reymond, Levi–Civita, and quite recently, C. Schmieden and D. Laugwitz, who tried to revive Leibniz' ideas and who obtained certain interesting though partial results in this direction.

In the fall of 1960, it occurred to the present speaker that suitable extensions of the real number system—including infinitely small and infinitely large quantities as required by Leibniz, and yet possessing in a definite sense *the same properties* as the real numbers—can be obtained without difficulty by model-theoretic arguments. Such extensions (which are by no means unique) are analogous and related to the non-standard models of arithmetic first envisaged by Skolem. Accordingly, they have been called non-standard models of analysis. Some early results in this field were given in A. Robinson [61a] and developed in greater detail in A. Robinson [63]. Since then the scope of the theory has been enlarged in successive stages. In the present paper I propose to show by means of a number of suitably chosen examples how this has been done and how it has resulted in developments which should be of interest to mathematicians and logicians alike. On the mathematical side, I shall touch upon the theory of functions of a complex variable, topological spaces, and linear operators in Hilbert space.

2. Non-standard analysis of real numbers. Let ${}^\circ\mathfrak{R}$ be the field of real numbers. In a first-order language which contains distinct symbols for *all* individuals (numbers) *and* relations of ${}^\circ\mathfrak{R}$ (including the relations $E(x, y)$, $Q(x, y)$, $S(x, y, z)$, $P(x, y, z)$ for $x=y$, $x<y$, $x+y=z$ and $xy=z$ respectively), let K_0 be the set of all sentences which hold in ${}^\circ\mathfrak{R}$. By the usual finiteness (or compactness) argument, K_0 is consistent with the set of sentences $H=\{Q(0, a) \wedge Q(a, b_\nu)\}$ where 0 denotes the zero of ${}^\circ\mathfrak{R}$, as usual, b_ν varies over (symbols for) all positive elements of ${}^\circ\mathfrak{R}$, and a is an additional individual constant. Let ${}^*\mathfrak{R}$ be a model of $K_0 \cup H$. In this model the element (denoted by) a is positive and *infinitesimal*, i.e., such that the repeated addition of $|a|$ to itself any finite number of times yields a result smaller than 1. ${}^*\mathfrak{R}$ is thus a non-archimedean field, as we may conclude also from the fact that it is a proper extension of ${}^\circ\mathfrak{R}$. More particularly, ${}^*\mathfrak{R}$ is a proper elementary extension of ${}^\circ\mathfrak{R}$. It will be said to be a non-standard model of analysis.

Let $N(x)$ be the one-place predicate (one-place relation) which characterizes the set of natural numbers ${}^\circ\mathfrak{N}$ in ${}^\circ\mathfrak{R}$. Then the subset ${}^*\mathfrak{N}$ of ${}^*\mathfrak{R}$ which consists of all elements ${}^*\mathfrak{R}$ that satisfy $N(x)$ constitutes a proper elementary extension of ${}^\circ\mathfrak{N}$, and hence a non-standard model of arithmetic.

The infinitesimal elements a of ${}^*\mathfrak{R}$ can be characterized also by the property that $|a|<b$ for all positive elements b of ${}^\circ\mathfrak{R}$. If $|a|<b$ for *some* positive element of ${}^\circ\mathfrak{R}$ then a is said to be *finite*. For any finite element a of ${}^*\mathfrak{R}$, there exists a uniquely determined element of ${}^\circ\mathfrak{R}$, denoted by ${}^\circ a$ or $\mathrm{st}(a)$ and called the *standard part* of a, such that the difference ${}^\circ a - a$ is infinitesimal.

Within this framework we may reconstruct the elementary parts of the differential and integral calculus, as shown in some detail in the closing sections of A. Robinson [63]. A typical example is as follows.

Any infinite sequence of real numbers $\{s_n\}$ is represented in the language of K_0 by a two-place relation $F(x, y)$ whose domain of the first place coincides with ${}^\circ\mathfrak{N}$, and such that the argument of the first place determines the argument of the second place uniquely in ${}^\circ\mathfrak{R}$. These facts about F in relation to ${}^\circ\mathfrak{R}$ can be stated as sentences of K_0, and, accordingly, hold also in ${}^*\mathfrak{R}$. In ordinary language, $F(x, y)$ represents in ${}^*\mathfrak{R}$ a sequence $\{{}^*s_n\}$, where ${}^*s_n$ belongs to ${}^*\mathfrak{R}$ for all n in ${}^*\mathfrak{N}$, and ${}^*s_n = s_n$ for finite n.

Suppose $\{s_n\}$ is bounded in ${}^\circ\mathfrak{R}$. Thus, for some m in ${}^\circ\mathfrak{R}$ and for all n in ${}^\circ\mathfrak{N}$, $|s_n|<m$. Since this is a fact which can be expressed as a sentence

of ${}^\circ K$, $|{}^*s_n| < m$ also in ${}^*\mathfrak{R}$, i.e., also for arbitrary *infinite* $n = \omega$, where an element ω of ${}^*\mathfrak{N}$ is *infinite* if it belongs to ${}^*\mathfrak{N} - {}^\circ\mathfrak{N}$. Since $|{}^*s_\omega| < m$ it is finite and accordingly (see above) possesses a standard part $\sigma = \mathrm{st}({}^*s_\omega)$. We claim that σ is a limit point (accumulation point) of $\{s_n\}$, according to the standard definition in ${}^\circ\mathfrak{R}$. In order to verify this claim we have to show that for any given $\varepsilon > 0$ in ${}^\circ\mathfrak{R}$ and for any $n \in {}^\circ\mathfrak{N}$, there exists an $P \in {}^\circ\mathfrak{N}$ which is greater than n such that $|s_P - \sigma| < \varepsilon$. Now the property of x, "x is a natural number greater then n such that $|s_x - \sigma| < \varepsilon$", can be formulated as a predicate $Q(x)$ in the language of K_0, and we have to show that the sentence $Y = (\exists x)Q(x)$ holds in ${}^\circ\mathfrak{R}$, i.e., that it belongs to K_0. For this purpose, we only have to establish that Y holds in ${}^*\mathfrak{R}$ (for either Y or $\sim Y$ holds in ${}^\circ\mathfrak{R}$, and if $\sim Y$ holds in ${}^\circ\mathfrak{R}$ then it belongs to K_0 and hence holds also in ${}^*\mathfrak{R}$). But $Q(\omega)$ holds in ${}^*\mathfrak{R}$ since $|{}^*s_\omega - \sigma|$ is infinitesimal, and so Y holds in ${}^*\mathfrak{R}$, as required. We have proved:

Every bounded sequence (in ${}^\circ\mathfrak{R}$) possesses an accumulation point; in other words, we have proved the theorem of Bolzano and Weierstrass.

3. Higher-order non-standard analysis. Within the framework described so far, we can prove theorems on *individual* functions and relations, i.e., theorems which do not involve any higher order quantification. A survey shows that this suffices for the more elementary parts of the theories of real and complex variables. Moreover, while we do not enlarge the domain of functions and other higher order concepts directly on passing from ${}^\circ\mathfrak{R}$ to ${}^*\mathfrak{R}$, this can be done in a roundabout way. For example, if $g(x, y)$ is a function of two variables in ${}^\circ\mathfrak{R}$, we may consider the function $f(x) = g(x, \omega)$ for some value of ω in ${}^*\mathfrak{R} - {}^\circ\mathfrak{R}$, and we may then obtain a function of one variable in ${}^*\mathfrak{R}$ which is not obtained by the extension of any function of one variable in ${}^\circ\mathfrak{R}$. Thus, taking

$$g(x, y) = (y/\pi)^{1/2} \exp(-yx^2)$$

and putting $y = \omega$, an infinite natural number, we obtain a function $f(x) = g(x, \omega)$ which is infinitesimal for all standard x (i.e., all x in ${}^\circ\mathfrak{R}$) except $x = 0$. Now consider the function

$$J(y) = \int_{-\infty}^{\infty} g(x, y)dx$$

which is defined for all y in ${}^\circ\mathfrak{R}$ (and is, in the present case, known to

be equal to 1 for all such y). Then we may *regard* $J(y)$, for all y in ${}^*\mathfrak{R}$ also, as the integral $\int_{-\infty}^{\infty} g(x, y)dx$. In this sense

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

where $f(x) = (\omega/\pi)^{1/2} \exp(-\omega x^2)$ as described above. This shows that $f(x)$ may be thought of as a realization of Dirac's delta function.

However, if one wishes to handle problems involving higher order concepts more freely, it is appropriate to extend the general framework of the analysis. There are various ways for doing this, either along set-theoretical or along type-theoretical lines. The author has carried out the procedure in detail for a particular version of type theory (A. Robinson [62]). It is based on the assumption that our formal language contains, in addition to symbols for all individuals in the set of real numbers (or equivalently, in any given set of cardinal $\mathfrak{c}$), also symbols for all sets, relations, sets of sets, sets of relations, etc., and that quantification is permitted with respect to all of these. By ${}^\circ\mathfrak{R}$ we shall now mean the higher-order structure which can be described in this language, and by K_0 the set of all sentences which hold in this structure. Following Henkin [50a], we know that the finiteness principle of the lower predicate calculus can be extended to the kind of language now under consideration provided we reinterpret the notion of the set of all subsets, or of all relations, sets of relations, etc., in a given structure so as to include (possibly) only a subset of the totality of all entities of the given type. In this sense, the set of sentences $K_0 \cup H$, with $H = \{Q(0, a) \wedge Q(a, b_\nu)\}$ as before, and with the new definition of K_0, possesses a model ${}^*\mathfrak{R}$ which is a proper extension of ${}^\circ\mathfrak{R}$. The reinterpretation of higher-order concepts which is necessary if we wish to insure that ${}^*\mathfrak{R}$ satisfies the sentences of K_0 is illustrated by the following example. Let α be the symbol which denotes the set ${}^\circ A$ of all subsets of ${}^\circ\mathfrak{R}$ when the sentences of K_0 are interpreted in ${}^\circ\mathfrak{R}$ and consider the interpretation of α with reference to ${}^*\mathfrak{R}$. It is not difficult to see that α still denotes a set ${}^*A$ of sets of individuals of ${}^*\mathfrak{R}$. However, we are going to show that ${}^*A$ cannot now include *all* sets of individuals of ${}^*\mathfrak{R}$. More particularly, let ${}^\circ\mathfrak{N}$ and ${}^*\mathfrak{N}$ be the set of natural numbers in ${}^\circ\mathfrak{R}$ and the corresponding set in ${}^*\mathfrak{R}$, as before. Then ${}^\circ\mathfrak{N}$ cannot be included in A. For ${}^*\mathfrak{N}$ is included in ${}^*A$ since ${}^\circ\mathfrak{N}$ is included in ${}^\circ A$. Also, it is a property of the set ${}^\circ A$ which can be expressed as a sentence of K_0 and, accordingly, applies also to ${}^*A$, that the difference of any two elements of the set in question (${}^\circ A$ or ${}^*A$) also belongs to the set. Thus, if ${}^\circ\mathfrak{N}$ belongs to ${}^*A$ then

${}^*\mathfrak{N} - {}^\circ\mathfrak{N}$ also belongs to ${}^*A$. On the other hand, every non-empty set of natural numbers in ${}^\circ\mathfrak{N}$ possesses a smallest element and this must be true also in ${}^*\mathfrak{N}$. But ${}^*\mathfrak{N} - {}^\circ\mathfrak{N}$ is a non-empty set of natural numbers which does not possess a smallest element for if $\omega \in {}^*\mathfrak{N} - {}^\circ\mathfrak{N}$ (i.e., if ω is an infinite natural number) then $\omega - 1 \in {}^*\mathfrak{N} - {}^\circ\mathfrak{N}$ as well. This proves our assertion.

The theory of functions of a complex variable can be brought within the scope of the theory of ${}^\circ\mathfrak{N}$ by regarding complex numbers as pairs of real numbers. A complex function of a complex variable $z = f(w)$, with $z = x + iy$, $w = u + iv$, can then be expressed by a four-place relation $F(x, y, u, v)$ such that $F(a, b, c, d)$ holds for real numbers a, b, c, d if and only if $a + ib = f(c + id)$. Thus, if $\Phi(t)$ is a one-place relation which characterizes a family of complex functions of a complex variable – i.e., which is satisfied by certain four-place relations – in ${}^\circ\mathfrak{N}$, then $\Phi(t)$ will define a corresponding family of functions in ${}^*\mathfrak{N}$. These "generalized" functions can be used in order to investigate the properties of the "standard" complex functions in ${}^\circ\mathfrak{N}$, as shown in detail in A. Robinson [62]. Thus, the employment of functions of this kind can replace the familiar normal families method in the proof of Riemann's mapping theorem and in the theory of exceptional values of entire functions (Picard's theorem, Julia's directions).

I shall now give a concrete illustration of the methods which become available in this way. It is chosen from the so-called analytic theory of polynomials which is concerned with the location of the zeros of complex polynomials in the complex plane.

Let ${}^\circ Z$ be the complex plane in the ordinary sense. We may regard it as a set of pairs of elements of ${}^\circ\mathfrak{N}$. Let ${}^\circ\Pi$ be the set of all complex polynomials with complex coefficients. As we pass from ${}^\circ\mathfrak{N}$ to ${}^*\mathfrak{N}$, ${}^\circ Z$ is converted into an extension ${}^*Z$ of ${}^\circ Z$, and ${}^\circ\Pi$ is converted into a set of functions ${}^*\Pi$ which have the properties of the elements of ${}^\circ\Pi$ to the extent in which they can be expressed as sentences of K_0. In particular, ${}^*\Pi$ contains the extensions to ${}^*Z$ of all the standard polynomials $p(z)$ with coefficients in ${}^\circ Z$.

With every polynomial $p(z)$ in ${}^\circ\Pi$ and with every natural number $k \in {}^\circ\mathfrak{N}$, there is associated a complex number $a_k \in {}^\circ\mathfrak{Z}$ which is the coefficient of z^k in the MacLaurin expression for $p(z)$. For $p(z) \neq 0$, the least k for which $a_k \neq 0$ is the *degree* of $p(z)$. Also, the number of non-zero coefficients of any $p(z) \in {}^\circ\Pi$ will be called the *rank* of $p(z)$. Finally, given any positive number ϱ in ${}^\circ\mathfrak{N}$ we define $\Omega[p(z), \varrho]$ as the

number of zeros z of $p(z) \neq 0$, such that $|z| < \varrho$, taking into account multiplicities.

All these functionals extend to ${}^*\Pi$ within ${}^*\mathfrak{R}$. However, it must be borne in mind that the degree and rank of a polynomial (element of ${}^*\Pi$) may now be infinite natural numbers. Some thought is required in order to see how $\nu = \Omega[p(z), \varrho]$ can be expressed within our formal language, but this can be done by formalizing

"there exists a finite set $\{z_1, \ldots, z_\nu\}$ such that $|z_i| < \varrho$ for $i = 1, 2, \ldots, \nu$ and $p(z) = (z - z_1) \ldots (z - z_\nu) q(z)$ where $q(z)$ is a polynomial which does not vanish for any z, $|z| < \varrho$."

Here again, ν may be infinite for $p(z)$ and ϱ in ${}^*\mathfrak{R}$.

A *monomial* in ${}^\circ\Pi$ or ${}^*\Pi$ is, as usual, a polynomial of rank 1. The general form of a monomial in ${}^*\Pi$ is az^n where $a \in {}^*Z$ and $n \in {}^*\mathfrak{N}$, and where a and n may or may not belong to ${}^\circ Z$ and ${}^\circ\mathfrak{N}$, respectively. Any polynomial of *finite* positive rank may be written as a *finite* sum (*finite* in the ordinary sense, in both cases) of monomials of different degrees, so

$$p(z) = c_1 z^{n_1} + c_2 z^{n_2} + \ldots + c_r z^{n_r}, \quad r \geqslant 1 \tag{3.1}$$

where $n_1 < n_2 < \ldots < n_r$ and $c_i \neq 0$, $1 \leqslant i \leqslant r$, and where the n_i may be finite or infinite.

Let az^m and bz^n be two monomials, of different degrees, finite or infinite. We say that bz^n *dominates* az^m, $bz^n \gg az^m$, if there exists a *finite* positive number ϱ in ${}^\circ\mathfrak{R}$ and that $|bz^n| > |az^m|$ for all z such that $|z|$ is *finite* and greater than ϱ. In order to appreciate the significance of this relation notice that for a, b, n, m in ${}^\circ\mathfrak{R}$, the monomial of higher degree dominates the other. This is not necessarily true in ${}^*\mathfrak{R}$ but it can still be shown that either $az^m \gg bz^n$ or $bz^n \gg az^m$ (and the two cases are mutually exclusive). It can also be shown that if $bz^n \gg az^m$ and $cz^k \gg bz^n$, and $k \neq m$, then $cz^k \gg az^m$. This shows that among the monomials which constitute a polynomial of finite rank as written out in 3.1 there is one that dominates all the others. The degree of this monomial will be called the *order* of the polynomial, ord $p(z)$. It turns out that, for sufficiently large finite $|z|$ the modulus of the dominant monomial becomes larger than the sum of the remaining monomials of $p(z)$. On this basis, an adaptation of classical arguments (use of Rouché's theorem) provides the following result.

Theorem 3.2. *Let $p(z)$ be a non-zero polynomial of finite rank in*

*II. *If* $\operatorname{ord} p(z)=\mu$ *is finite,* $\mu \in {}^{\circ}\Re$, *then the number of finite zeros of* $p(z)$ *(i.e., with finite* $|z|$*) is* μ. *If* $\operatorname{ord} p(z)$ *is infinite then the number of finite zeros of* $p(z)$ *is infinite.*

This theorem leads to a far-reaching generalization of the following classical result which is due to Kakeya.

Let $p(z)$ be a polynomial of degree not exceeding n, $n \geqslant 1$, such that at least $k \geqslant 1$ roots of $p(z)$ are included in the circle $|z| \leqslant \varrho$. There exists a positive number $\theta=\theta(n, k)$ – which does not depend on the particular coefficients of $p(z)$ – such that $p'(z)$ has at least $k-1$ roots in the circle $|z| \leqslant \varrho\theta$.

Equivalently, we may state in the conclusion that $zp'(z)$ has at least k roots in the circle $|z| \leqslant \varrho\theta$. Thus, Kakeya's theorem is a special case of the following:

Theorem 3.3. *There exists a function* $\phi(r, k)$ *whose arguments range over the positive integers (classically, i.e., in* ${}^{\circ}\Re$*) and which takes real positive values (i.e., in* ${}^{\circ}\Re$*) and which possesses the following property: For any positive real* ϱ *and for any polynomial of rank not exceeding* r,

$$p(z)=c_1z^{n_1}+c_2z^{n_2}+ \ \dots \ c_rz^{n_r},\ 0 \leqslant n_1 < n_2 < \ \dots \ < n_r,$$

and for any set of complex numbers λ_i *such that* $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \ \dots \ \leqslant |\lambda_r|$ *– if* $p(z)$ *possesses at least* k *roots in the circle* $|z| \leqslant \varrho$ *then the polynomial*

$$g(z)=\lambda_1c_1z^{n_1}+\lambda_2c_2z^{n_2}+ \ \dots \ +\lambda_rc_rz^{n_r}$$

possesses at least k *roots in the circle* $|z| \leqslant \varrho\phi(r, k)$.

Note that this is a classical theorem, without reference to any non-standard extension of the real or complex numbers. The substitution $z=\varrho z'$ shows that, for the proof, it is sufficient to suppose $\varrho=1$.

Given any positive integers r and k in ${}^{\circ}\Re$, suppose that no $\phi(r, k)$ as asserted by the theorem exists. Then the following statement holds in ${}^{\circ}\Re$.

"For every $\sigma>0$, there exist a polynomial $p(z)$ and constants λ_i that satisfy the assumptions of the theorem, with $\varrho=1$, such that the corresponding $g(z)$ possesses not more than $k-1$ roots in the circle $|z| \leqslant \sigma$."

The statement in quotation marks can be expressed as a sentence X which then belongs to K_0, and hence holds also in ${}^*\Re$. Choosing for σ an *infinite* positive number (an element of ${}^*\Re$ greater than any

number of ${}^\circ\mathfrak{R}$), we then obtain $p(z)$ and λ_i in ${}^*\mathfrak{R}$ which satisfy the assumptions of the theorem, with $\varrho = 1$, while $g(z)$ has less then k zeros in the circle $|z| \leqslant \sigma$ and, a fortiori, has less than k finite zeros (and hence does not vanish identically). However, bearing in mind Theorem 3.2 above, we shall refute this possibility by showing that $g(z)$ must be of order k at least.

Indeed, since $g(z)$ does not vanish identically, it includes a non-vanishing monomial, $\lambda_j c_j z^{n_j}$ say, which dominates all the other monomials of $g(z)$, so that $\operatorname{ord} g(z) = n_j$. Similarly, $p(z)$ includes a dominant polynomial, $c_i z^{n_i}$ say, and since $p(z)$ has at least k finite zeros we have $n_i = \operatorname{ord} p(z) \geqslant k$. Now suppose that $j < i$, $n_j < n_i$. At the same time, $c_i z^{n_i}$ dominates $c_j z^{n_j}$ so that

$$|c_i z^{n_i} / c_j z^{n_j}| > 1$$

for sufficiently large finite $|z|$. But $|\lambda_i| \geqslant |\lambda_j|$, by assumption, and so, for such $|z|$,

$$|\lambda_i c_i z^{n_i} / \lambda_j c_j z^{n_j}| = |\lambda_i / \lambda_j| \, |c_i z^{n_i} / c_j z^{n_j}| > 1,$$

which contradicts the fact that $\lambda_j c_j z^{n_j}$ is the dominant monomial in $g(z)$. We conclude that $j \geqslant i$, $n_j \geqslant n_i \geqslant k$, so that $g(z)$ possesses at least k finite zeros. This contradiction proves 3.3.

Kakeya's theorem is obtained from 3.3 by taking $\lambda_i = n_i = i - 1$. Other applications of our method to the analytic theory of polynomials are given in A. Robinson [62].

4. Application to topological spaces. As we pass from classical analysis to the consideration of other structures, e.g., of topological spaces, we have to decide how to select the appropriate extensions that correspond to the non-archimedean fields used in the preceding sections. We are going to describe a scheme which is adequate for this purpose and which is suitable for both first- and higher-order languages.

Let ${}^\circ\mathfrak{M}$ be a structure which corresponds either to a first-order language, as in Section 2, or to a higher-order language, as in Section 3. Let K_0 be the set of all sentences which hold in ${}^\circ\mathfrak{M}$, within the language under consideration, in terms of a given vocabulary. For the purpose of the following definition it is not necessary that this vocabulary include symbols for *all* individuals and relations (of first and higher order, as the case may be) in ${}^\circ\mathfrak{M}$.

Let $Q(x, y)$ be any two-place predicate (well-formed formula with

two free variables) which is definable in the vocabulary of K_0. We shall say that $Q(x, y)$ is *concurrent* in ${}^\circ\mathfrak{M}$ if for every finite set of entities (individuals or relations) $a_1, \ldots, a_n$, $n \geqslant 1$, which belong to the domain of the first place of Q in ${}^\circ\mathfrak{M}$, there exists an entity b in ${}^\circ\mathfrak{M}$ such that $Q(a_1, b), \ldots, Q(a_n, b)$ all hold in ${}^\circ\mathfrak{M}$. In this connection, we say that a_i belongs to the domain of the first place of Q if $Q(a_i, b)$ holds in ${}^\circ\mathfrak{M}$ for *some* b.

For every concurrent predicate Q we have introduced a distinct symbol b_Q, and we define K_Q as the set of sentences $\{Q(a_\nu, b_Q)\}$ where a_ν varies over all (symbols for) elements of the domain of the first place of Q. Let

$$H = K_0 \cup \bigcup_Q K_Q$$

where Q varies over all concurrent predicates in ${}^\circ\mathfrak{M}$ which are defined in the given vocabulary. A familiar argument shows that H is consistent. The models ${}^*\mathfrak{M}$ of H will take over the part played previously by the non-standard models of analysis ${}^*\mathfrak{R}$. At the same time we may verify without difficulty that the more general procedure sketched above also yields non-standard models of analysis, as in Sections 2 and 3, for the appropriate vocabularies. Thus, taking the first-order case considered in Section 2, with a vocabulary which includes (symbols for) all individuals and relations in the field of real numbers, ${}^\circ\mathfrak{M} = {}^\circ\mathfrak{R}$, we claim that ${}^*\mathfrak{M}$ as introduced above is a non-standard model of analysis in the sense of Section 2. Indeed, consider the relation which is in ordinary language

"y is positive and is smaller than x."

Evidently, this can be expressed as a predicate in the language of K_0, more particularly, like all other predicates that range over individuals it is actually given by an atomic relation in the language of K_0, $Q(x, y)$ say. The domain of the first place of Q consists of all positive real numbers. It follows that in any model ${}^*\mathfrak{M}$ of H, b_Q denotes an element which is positive and smaller than any ordinary (standard) real number. This shows that ${}^*\mathfrak{M}$ is a non-standard model of analysis. Similar arguments apply to the higher order theory.

Adopting the higher order approach, let ${}^\circ\mathfrak{M}$ be an arbitrary topological space, and suppose that the vocabulary of the language includes (symbols for) all individuals, sets, relations, sets of relations, etc. Let ${}^*\mathfrak{M}$ be an extension of ${}^\circ\mathfrak{M}$ as introduced above. The points (individuals) of ${}^*\mathfrak{M}$ include, in particular, the points of ${}^\circ\mathfrak{M}$, which will be called *standard*

points. Similarly, if ${}^\circ U$ is the set of open sets in ${}^\circ\mathfrak{M}$, and ${}^* U$ is the corresponding set in ${}^*\mathfrak{M}$, then ${}^* U$ contains in particular the extensions of the open sets of ${}^\circ\mathfrak{M}$. These extensions will be called the *standard open sets* of ${}^*\mathfrak{M}$. Notice that in general, a standard open set is a proper extension of the corresponding set in ${}^\circ\mathfrak{M}$. For example, if ${}^\circ\mathfrak{M} = {}^\circ\mathfrak{R}$ as in Section 3, with the usual topology and ${}^\circ A$ is the open interval between 0 and 1 then the corresponding ${}^* A$ contains all elements of ${}^*\mathfrak{R}$ between 0 and 1, standard or not, but at the same time is a *standard* open set according to our definition.

For any point a in ${}^*\mathfrak{M}$, we define the *monad* of a as the intersection of all standard open sets which include a. If ${}^\circ\mathfrak{M} = {}^\circ\mathfrak{R}$ and ${}^*\mathfrak{M} = {}^*\mathfrak{R}$, and a is a standard point (ordinary real number) then the monad of a consists of all points b in ${}^*\mathfrak{R}$ such that $b - a$ is infinitesimal. In general, the monad of a point a need not be a set within the theory of ${}^*\mathfrak{M}$ (just as ${}^\circ\mathfrak{N}$ is not a set within the theory of ${}^*\mathfrak{R}$). A point of ${}^*\mathfrak{M}$ which is included in the monad of a standard point is called *near-standard*. We are going to prove

Theorem 4.1. *A space* ${}^\circ\mathfrak{M}$ *is compact if and only if all points of* ${}^*\mathfrak{M}$ *are near-standard.*

In this theorem, ${}^*\mathfrak{M}$ is *any* extension of ${}^\circ\mathfrak{M}$ of the kind introduced above. A space is called compact if it satisfies the Heine–Borel condition, without reference to any separation axiom.

Proof of 4.1. Suppose that ${}^\circ\mathfrak{M}$ is not compact. Then there exists an open covering of ${}^\circ\mathfrak{M}$, ${}^\circ B = \{{}^\circ A_\nu\}$ such that for every finite subset $\{{}^\circ A_1, \ldots, {}^\circ A_n\}$ of ${}^\circ B$ there is a point in ${}^\circ\mathfrak{M}$ which does not belong to ${}^\circ A_1 \cup {}^\circ A_2 \cup \ldots \cup {}^\circ A_n$. Consider the relation $Q(x, y)$ which is expressed informally by

"The complement of x belongs to ${}^\circ B$ and x contains y."

Then $Q(x, y)$ is concurrent, according to the definition given above. It follows that ${}^*\mathfrak{M}$ contains a point b_Q such that b_Q is contained in the complements of all ${}^* A_\nu$ (which are the extensions of the corresponding ${}^\circ A_\nu$ from ${}^\circ\mathfrak{M}$ to ${}^*\mathfrak{M}$). Thus, for every standard point in ${}^*\mathfrak{M}$, b_Q belongs to the complement of some standard open set that contains the point in question, and hence does not belong to its monad. This shows that b_Q is not near-standard and establishes the sufficiency of the condition.

Conversely, suppose that the point b in ${}^*\mathfrak{M}$ does not belong to the

monad of any standard point. It is then possible to choose for every standard point a_ν a standard open set ${}^*A_\nu$ which includes a_ν and which does not include b. The set of corresponding open sets in ${}^\circ\mathfrak{M}$, ${}^\circ B = \{{}^\circ A_\nu\}$, is then an open covering of ${}^\circ\mathfrak{M}$. We claim that no finite subset of ${}^\circ B$ covers ${}^\circ\mathfrak{M}$. For if the subset $\{{}^\circ A_1, \ldots, {}^\circ A_k\}$ of ${}^\circ B$ were a covering of ${}^\circ\mathfrak{M}$, then the formal equivalent of the statement

"For all points x, $x \in {}^\circ A_1 \cup {}^\circ A_2 \cup \ldots \cup {}^\circ A_k$"

would hold in ${}^\circ\mathfrak{M}$ and so

"For all points x, $x \in {}^*A_1 \cup {}^*A_2 \cup \ldots \cup {}^*A_k$"

would hold in ${}^\circ\mathfrak{M}$. In particular, then $b \in {}^*A_1 \cup {}^*A_2 \cup \ldots \cup {}^*A_k$, which is contrary to the definition of the ${}^*A_\nu$. This completes the proof of 4.1.

Theorem 4.1 provides a simple proof of the theorem of Heine–Borel that a closed and bounded set ${}^\circ A$ in (ordinary) n-dimensional Euclidean space ${}^\circ\mathfrak{R}^n$, $n \geqslant 1$, is compact. Indeed, by 4.1 we only have to show that all points of ${}^*A$ are near-standard if ${}^\circ A$ is a closed and bounded set. Let $a = (a_1, \ldots a_n)$ be any point in ${}^*A$. Then the coordinates $a_1, \ldots, a_n$ of a are all finite, and hence possess standard parts, ${}^\circ a_1, \ldots, {}^\circ a_n$. Then it can be shown without difficulty that the point ${}^\circ a = ({}^\circ a_1, \ldots, {}^\circ a_n)$ belongs to ${}^\circ A$ and that a belongs to the monad of ${}^\circ a$.

5. Application to normed linear spaces. Let ${}^\circ\mathfrak{M}$ be a metric space. In the usual topology induced by the metric of ${}^\circ\mathfrak{M}$, a point a in ${}^*\mathfrak{M}$ is near-standard if and only if the distance of a from some standard point in ${}^*\mathfrak{M}$ is infinitesimal. This is true in particular if ${}^\circ\mathfrak{M}$ is a normed linear space. In that case, we call a point a in ${}^*\mathfrak{M}$ *norm-finite* if the norm of a is a finite real number as defined in Section 2 above.

Now let ${}^\circ\mathfrak{M}$ be a Banach space, and let ${}^\circ T$ be a linear operator on ${}^\circ\mathfrak{M}$ into ${}^\circ\mathfrak{M}$. ${}^\circ T$ is said to be compact if it transforms every bounded set in ${}^\circ\mathfrak{M}$ into a relatively compact set, i.e., into a set which is compact in ${}^\circ\mathfrak{M}$. This important class of operators can be characterized very simply in terms of the corresponding ${}^*T$ in ${}^*\mathfrak{M}$. It turns out that ${}^\circ T$ is compact if and only if ${}^*T$ maps every norm-finite point on a near-standard point. This definition can be used among other things in order to give a direct proof of the fact that an integral operator with continuous kernel is compact in the space of continuous functions.

Finally, let us consider the application of our method to (infinite-dimensional) separable Hilbert space. Such a space can be represented by the infinite sequences $\{s_n\}$ such that $\sum s_n^2$ is finite or such that $\sum s_n \bar{s}_n$ is finite (for real or complex Hilbert space, as the case may be). Accordingly we may consider its theory already within the framework of Section 3, although the more general approach of Section 4 appears more natural. Within this framework let ${}^\circ\mathfrak{M}$ be the given Hilbert space, together with its sets, relations, sets of relations, etc., as before, and together with its ordinary field of coefficients, ${}^\circ\mathfrak{R}$ say (for the real case), and let ${}^*\mathfrak{M}$ be an extension, as introduced in Section 4, incorporating a non-standard model of analysis, ${}^*\mathfrak{R}$, as the appropriate extension of ${}^\circ\mathfrak{R}$.

Let $\{e_1, e_2, e_3, \ldots\}$ be an orthogonal basis in ${}^\circ\mathfrak{M}$ so that an arbitrary point of the space can be represented by a sequence of coordinates $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ with respect to this basis. On passing to ${}^*\mathfrak{M}$, $\{e_i\}$ is extended to a set $\{{}^*e_i\}$, i.e., a set of points in the extended space, with subscripts ranging over the appropriate extension of the natural numbers, ${}^*\mathfrak{N}$, such that $e_i = {}^*e_i$ for all finite i. Any point in the extended space is given by a sequence $\{\lambda_i\}$ with i ranging over ${}^*\mathfrak{N}$ and λ_i in ${}^*\mathfrak{R}$, such that the infinite sum $\sum_{i=1}^{\infty} \lambda_i^2$ converges in the sense of ${}^*\mathfrak{M}$. That is to say, $\sum_{i=1}^{\infty} \lambda_i^2$ may be a finite real number in ${}^*\mathfrak{R}$, or an infinite real number in ${}^*\mathfrak{R}$, provided it exists at all. It can be shown that the point represented by $\{\lambda_i\}$ is *norm-finite* in the sense obtained above precisely when $\sum_{i=1}^{\infty} \lambda_i^2$ is finite (i.e., not greater than some element of ${}^\circ\mathfrak{R}$) and that it is *near-standard* if and only if $\sum_{i=\omega}^{\infty} \lambda_i^2$ is infinitesimal for all infinite natural numbers ω.

We shall now indicate how this set-up can be used for the development of spectral theory. Let ${}^\circ T$ be a linear operator which is defined and hence bounded, on the space ${}^\circ\mathfrak{M}$. Let ${}^*T$ be the extension of ${}^\circ T$ to the space ${}^*\mathfrak{M}$. Now choose an infinite natural number $\omega \in {}^*\mathfrak{N}$ and let ${}^\omega\mathfrak{M}$ be the subspace of ${}^*\mathfrak{M}$ which consists of the point of ${}^*\mathfrak{M}$ whose coordinates λ_i vanish for $i > \omega$. Then ${}^\omega\mathfrak{M}$ is a "finite-dimensional" space in a non-standard sense, i.e., its dimension is the "natural" number ω.

Let ${}^\omega P$ be the projection operator from ${}^*\mathfrak{M}$ onto ${}^\omega\mathfrak{M}$, so

$$ {}^\omega P\colon \{\lambda_1, \lambda_2, \ldots, \lambda_\omega, \lambda_{\omega+1}, \lambda_{\omega+2}, \ldots\} \to \{\lambda_1, \lambda_2, \ldots, \lambda_\omega, 0, 0, \ldots\}. $$

Define the operator ${}^\omega T$ as the restriction of the product ${}^\omega P T {}^\omega P$ to

${}^{\omega}\mathfrak{M}$. Then ${}^{\omega}T$ is a linear operator on the "finite-dimensional" space ${}^{\omega}\mathfrak{M}$ and, accordingly, can be represented by an $\omega \times \omega$ matrix in the usual way. In particular, if ${}^{\circ}T$ is self-adjoint, so are ${}^{*}T$ and ${}^{\omega}T$. Applying the standard theory of matrices in finite-dimensional spaces, we find that ${}^{\omega}T$ possesses ω eigenvalues, equal or different, and to each eigenvalue there corresponds an eigenspace such that ${}^{\omega}\mathfrak{M}$ is the direct sum of these spaces. The question now arises how to reduce these results about ${}^{\omega}T$ to results about ${}^{\circ}T$ in ${}^{\circ}\mathfrak{M}$. This reduction is not trivial, but we have carried it out in detail for compact operators ${}^{\circ}T$. In that case the eigenvalues λ_i of ${}^{\omega}T$ are finite and hence, possess standard parts ${}^{\circ}\lambda_i$. It can then be shown that any ${}^{\circ}\lambda_i$ which is different from 0 is an eigenvalue of ${}^{\circ}T$. Moreover, for λ_i for which ${}^{\circ}\lambda_i \neq 0$, i.e., which are not infinitesimal, any corresponding eigenvector turns out to be near-standard, owing to the compactness of ${}^{\circ}T$.

If ${}^{\circ}T$ is not compact, we can at least appreciate in a general way how it may happen that certain elements of the spectrum are left without eigenvectors. Such elements may be expected to correspond to eigenvalues λ_i of ${}^{\omega}T$ in ${}^{\omega}\mathfrak{M}$ whose eigenvectors, though chosen to be of finite norm, are not near-standard, and hence do not correspond to any eigenvectors in ${}^{\circ}\mathfrak{M}$.

6. Methodological considerations. As mentioned in A. Robinson [61a] and [62], certain non-standard models of analysis can be realized as ultraproducts. This has the advantage of making the theory more concrete in appearance. Luxemburg (in his lecture notes, California Institute of Technology, 1962) and Takeuti [62] have in fact based their contributions to non-standard analysis on structures of this type. Using a theorem due to Frayne one can show that the models ${}^{*}\mathfrak{M}$ introduced in Section 4 above can be embedded in ultrapowers of ${}^{\circ}\mathfrak{M}$ and hence, that for any given ${}^{\circ}\mathfrak{M}$, ${}^{*}\mathfrak{M}$ may be taken as an ultrapower of ${}^{\circ}\mathfrak{M}$. There are other structures of special types which would serve our purpose. However, for all the applications of the theory made so far, any one of the structures ${}^{*}\mathfrak{M}$ introduced in Section 4 is as good as another. Accordingly, we preferred to base our considerations on the formal properties of the models ${}^{*}\mathfrak{M}$ rather than on their mode of construction.

It may be said that in spite of the model-theoretic garb of our method, it really represents an extension of the *deductive procedures* used in various branches of mathematics. And, if we wish to pass to a more

philosophical level of discussion, we may well argue that the operation of the existence of standard models as opposed to non-standard models, e.g., of arithmetic, has a meaning only inasmuch as it implies the adoption of certain rules of deduction which distinguish the standard model from any non-standard model that is under consideration at the same time.

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THE DECISION PROBLEM FOR FIELDS

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1. Introduction. One aspect of the theory of models which has been developing in the last 30 years is the decision problem for various mathematical theories and the related problems of definability. The decision problem for fields furnishes an excellent illustration of the methods and results.

The decision problem for a given field $\mathfrak{F}$ is the problem of determining whether or not the set Σ of true (first-order) sentences of $\mathfrak{F}$ is recursive. If Σ is recursive, we say $\mathfrak{F}$ is *decidable*; if not, we say $\mathfrak{F}$ is *undecidable*. More generally, the decision problem for a class $\mathscr{C}$ of fields is the problem of determining whether or not the set Γ of sentences true in all fields of $\mathscr{C}$ is recursive. The known decidable fields are:

D1. any finite field,
D2. any real closed field,
D3. any algebraically closed field.

The known undecidable fields are:

U1. the rational field,
U2. any finite extension of the rational field,
U3. any pure transcendental extension of a formally real field [1],
U4. certain fields of formal power series over an undecidable field (in particular the field of formal power series over the rational field),
U5. certain fields whose definitions depend on non-recursive sets,
U6. fields elementarily equivalent to any of the above fields.

We will discuss these results and single out some of the open problems which are either of particular interest or appear likely of solution. We consider formulas in the first-order language L with equality whose mathematical symbols are $+$, $\cdot$, 0, and 1. Thus a field $\mathfrak{F}$ is a structure

[1] In general, we will omit the words 'pure' and 'formally'. By a *transcendental extension*, we will always mean a *pure* transcendental extension and by a *real field*, we will always mean a *formally* real field.

$<F, +, \cdot, 0, 1>$ which satisfies the usual finite set of axioms. We say a set S is *definable* in $\mathfrak{F}$ if there is a formula φ with one free variable such that a satisfies φ in $\mathfrak{F}$ if and only if $a \in S$. Any formula ψ with free variables $x_1, \ldots, x_n$ is equivalent to a formula of the form

$$\underset{u_1}{X_1} \ldots \underset{u_k}{X_k} \; P(x_1, \ldots, x_n, u_1, \ldots, u_k) = Q(x_1, \ldots, x_n, u_1, \ldots, u_k)$$

where $X_1, \ldots, X_k$ are quantifiers, P and Q are polynomials with natural number coefficients (i.e., sums of 1's). We can reduce ψ to this form by first putting ψ in prenex form, then applying the following equivalences as needed, and finally transposing the terms so that they have positive coefficients:

$$u = 0 \vee w = 0 \leftrightarrow u \cdot w = 0,$$

$$u = 0 \wedge w = 0 \leftrightarrow \underset{x,y}{\wedge} u \cdot x = w \cdot y,$$

$$u \neq 0 \leftrightarrow \underset{x}{\vee} u \cdot x = 1.$$

Hence any set definable in $\mathfrak{F}$ can be obtained from an algebraic surface in an appropriate space over $\mathfrak{F}$ by taking projections and complements.

2. Decidable fields. A finite field $\mathfrak{F}$ is clearly decidable since any sentence can be systematically reduced to a Boolean combination of sentences of the diagram of $\mathfrak{F}$, which is finite.

A field is *formally real* if and only if -1 is not the sum of squares of elements of the field. A formally real field is *real closed* if and only if it has no proper algebraic extension which is formally real. Examples are the field of real numbers and the field of real algebraic numbers. This definition of a real closed field does not appear to be a first-order property of fields. However Artin and Schreier [27], who introduced the concept, also showed that a real closed field $\mathfrak{F}$ could be characterized by the conditions:

1. $\mathfrak{F}$ is a formally real field;
2. every element of $\mathfrak{F}$ is either a square or the negative of a square;
3. every polynomial equation in one variable over $\mathfrak{F}$ of odd degree has a solution in $\mathfrak{F}$.

These conditions can be written without difficulty as an infinite set of sentences of L.

A real closed field can be ordered in one and only one way. For, in a real closed ordered field, we must have true

$$(1) \qquad \bigwedge_{x,y} (x<y \leftrightarrow \bigvee_{z} (y=x+z^2 \wedge z \neq 0)).$$

Let L' be the language obtained from L by adding the relation symbol "$<$". Let P be the set of axioms for a real closed field together with (1), i.e., P is the set of axioms for a real closed ordered field. In 1930 [2], Tarski found a decision method for determining whether or not an arbitrary sentence of L' is derivable from P. His method of systematically eliminating the quantifiers of a given formula culminates in the theorem:

Every formula φ of L' is P*-equivalent to a formula θ of L' with no free variables other than those of φ and no bound variables.*

Thus θ is a Boolean combination of formulas of the form $\alpha>\beta$ and $\alpha=\beta$ where each α and β are obtained from the free variables of φ, 0, and 1 by additions and multiplications. Tarski gave explicit rules for obtaining θ from a formula φ by using a generalization of Sturm's theorem. In case φ is a sentence, θ is a combination of equalities and inequalities among natural numbers. Hence θ can easily be verified or disproved as the case may be on the basis of P. Thus P is complete.

We also obtain as a corollary to Tarski's theorem the following characterization of sets definable in the field of real numbers:

A set S is definable in the field of real numbers if and only if S is a finite union of (open, closed, half-open) intervals with algebraic or infinite endpoints.

This decision method, or algorithm, of Tarski is on the edge of practicality. With continued improvement in machines and programming techniques, we can expect someday to have a machine carry out this algorithm (or variants of it[3]) to solve, for example, certain packing and covering problems of elementary geometry which are now beyond the reach of mathematicians.

Since the axioms for real closed fields are also complete, as we see by using (1) to eliminate inequalities, any pair of real closed fields are

[2] This work was not published in full until 1948, when the monograph (see Tarski [51]) was prepared for publication by J. C. C. McKinsey.

[3] Seidenberg [54] has given another algorithm based on a single application of Sturm's theorem which should shorten machine time.

elementarily equivalent. For example, any sentence true in the field of real numbers is also true in the field of real algebraic numbers or any other real closed field. This is not the case for two simple transcendental extensions of real closed fields, as we shall see below.

A simpler proof of the completeness of the axioms for real closed fields has been given by A. Robinson [56; 63] using model-theoretic methods. However his results do not give as much information about definable sets nor such a direct algorithm for testing the truth of sentences of L in a real closed field.

A. Robinson's method of *model completeness*[4] consists in first showing that every formula φ of L' is P-equivalent to an existential formula ψ, i.e., a formula in prenex form without universal quantifiers, and then using the persistence under extension of such existential formulas to show completeness. It is well known and easy to see that a complete axiomatizable theory is also decidable.

Kochen [61] gave another proof of the completeness of the theory of real closed fields using ultraproducts.

A decision method in the easier case of algebraically closed fields was also obtained first by Tarski by the same method of elimination of quantifiers (unpublished, see Seidenberg [54]). Additional proofs of the completeness of the theory of algebraically closed fields of a given characteristic have been given by A. Robinson [56], Kochen [61], and Vaught [54].

A possible strengthening of these theorems may lie in the open decision problems for the ring of all real algebraic integers and the ring of all algebraic integers. In general, if an integral domain is decidable, so also is its quotient field. For it is clear that every statement in the quotient field can be transformed mechanically into an equivalent statement in the integral domain. We do not know whether or not the converse holds. On the other hand, to show the undecidability of these rings, it would be sufficient to define a family of arbitrarily large finite sets, i.e., to find a formula $\varphi(x, y)$ such that for each x, there are finitely many y's satisfying $\varphi(x, y)$ and for each natural number n, there is some x for which there are more than n numbers y satisfying $\varphi(x, y)$, see J. Robinson [62]. However it seems unlikely that this can be done.

A number of years ago, Tarski conjectured that the fields D1–D3

[4] See A. Robinson [63] for an account of this method which is applicable in some cases where the elimination of quantifiers is not and often is easier to carry out.

were the only decidable fields. In each of these cases, the fields are decidable because in some sense "so many" equations are solvable. This suggests that the p-adic fields are promising candidates for counter-examples to Tarski's conjecture. For like the real field, the p-adic fields are completions of the rational field; i.e., Cauchy sequences in a p-adic field under the appropriate definition of absolute value have a limit.

Hensel introduced p-adic numbers as formal power series

$$\sum_{n=k}^{\infty} c_n p^n \qquad \text{for } k \text{ an integer and } c_n \in \{0, 1, \ldots, p-1\}.$$

The p-adic integers are those p-adic numbers whose expansion does not contain negative powers. Hensel showed that for every prime p, these formal power series form a field. He also found that while their number theory is not trivial, it is simpler and more regular than that of the rational field or of the rational integers. For example, a number is a p-adic square for an odd prime p if and only if the first non-zero coefficient in its expansion corresponds to an even power of p and is a quadratic residue (mod p); a number is a 2-adic square if and only if it is a power of 4 times a number of the form $1+c_3 2^3+c_4 2^4+\ldots$.

We can easily define the p-adic integers in the field of p-adic numbers by the formulas:

$$\text{Int } x \leftrightarrow \bigvee_y y^2 = 1 + px^2 \qquad \text{for odd } p,$$

$$\text{Int } x \leftrightarrow \bigvee_y y^2 = 1 + 8x^2 \qquad \text{for } p = 2.$$

Hence the field of p-adic numbers is decidable if and only if the ring of p-adic integers is decidable. Nerode [63] has recently shown that the existential theory of a p-adic field is decidable.

3. Undecidable fields. The result of Gödel [31], Church [36] and Rosser [36] that the arithmetic of natural numbers is undecidable is the *sine qua non* of known proofs of the undecidability of fields. To show that a given field $\mathfrak{F}$ is undecidable, we merely find a formula φ of L with one free variable such that an element a of F satisfies φ in $\mathfrak{F}$ if and only if a is a natural number. Then any sentence σ of the arithmetic of natural numbers is equivalent to the sentence σ' obtained from σ by restricting the range of the variables to elements satisfying φ. Of course it would also be enough to define a model of the arithmetic of

natural numbers in $\mathfrak{F}$ in such a way that any sentence of number theory could be translated mechanically into an equivalent sentence of $\mathfrak{F}$.

The field first shown to be undecidable was the field of rational numbers. Let $\varphi(a, b, k)$ be the formula

$$\bigvee_{x,y,z} x^2 + ay^2 = bz^2 + abk^2 + 2.$$

Then the integers can be defined in the field of rationals by the formula

$$\mathrm{Int}\ n \leftrightarrow \bigwedge_{a,b} \{\varphi(a, b, 0) \wedge \bigwedge_{m} [\varphi(a, b, m) \rightarrow \varphi(a, b, m+1)] \rightarrow \varphi(a, b, n)\}.$$

Since $\varphi(a, b, -n)$ whenever $\varphi(a, b, n)$, it is easy to see that every integer n satisfies the right side of the equivalence. In the other direction, it is necessary to use some known results about the representation of numbers by quadratic forms in the rational field. Here we will just note that for every prime p, there are numbers a and b so that $\varphi(a, b, 0)$, the set of t satisfying $\varphi(a, b, t)$ is inductive, and $\varphi(a, b, t)$ implies that t is a rational p-adic integer, i.e., the denominator of t in lowest terms is not divisible by p. Thus the denominator of n is not divisible by any prime p so n is an integer. We can use Lagrange's theorem to define the natural numbers as those integers which are the sum of four squares. Hence the field of rationals is undecidable [5].

From this definition of the set of natural numbers in the rational field, we can find an algebraic surface in a many-dimensional space over the rationals such that by starting with this surface and successively taking projections and complements, we will obtain the set of natural numbers. In contrast, we have already seen from Tarski's theorem on real closed fields that no matter what algebraic surface we start with in a space over the real numbers, the only sets which will be obtained by taking successive projections and complements are unions of a finite number of intervals with algebraic end points.

Let θ be a formula of L with one free variable such that a rational number a satisfies θ in the field of rationals if and only if a is a natural number. We can then find an essentially undecidable sentence σ of L which is true in the field of rational numbers [6]. Namely, we obtain σ by taking the conjunction of the axioms of R. M. Robinson's theory Q and then relativizing the quantifiers to θ in the resulting sentence. Hence by Tarski's argument [6], any class $\mathscr{C}$ of fields which contains the

[5] See J. Robinson [49].

[6] See Tarski-Mostowski-Robinson [53].

rational field is undecidable since the set of sentences true in all fields of $\mathscr{C}$ is compatible with the essentially undecidable theory of σ.

In particular, the theory of fields is undecidable, i.e., the set of sentences derivable from the axioms for a field is not recursive. This result can be obtained from the definability of the set of natural numbers in any field. We shall give a proof below of the definability of natural numbers in a pure transcendental extension of a real closed field which does not depend on such deep results of number theory as needed in the rational field. Other classes of fields which we see are undecidable are the real fields, the fields of characteristic 0, the infinite fields, etc.

The definition of the set of natural numbers given for the rational field can be modified so as to hold in any given field of finite degree over the rational field. In this case, we go through an intermediate step of first defining the ring of all algebraic integers of the field and then showing that the subring of rational integers can be defined in this ring [7].

The only infinite algebraic extensions of the rational field known to be undecidable are fields whose descriptions depend on non-recursive sets. For example, if we adjoin to the rational field the square-roots of a non-recursive set of prime numbers, then the set of true sentences in the resulting field of the special form

$$\bigvee_{x} x^2 = n \qquad \text{for } n \text{ a natural number,}$$

is non-recursive. Certainly, then, the field is undecidable.

Hence there are many interesting open cases of the decision problem for algebraic fields of infinite degree over the rational field. In particular, Tarski has mentioned the case of the field of all numbers expressible in terms of radicals as well as that of the field of constructible numbers, i.e., real algebraic numbers which can be expressed in terms of square-roots. Perhaps the most promising case to work on is the field of all totally real algebraic numbers [8]. It is known that the ring of all totally real algebraic integers is undecidable [9].

Now it is not true that the set of natural numbers can be defined in every undecidable field or even that a model of the arithmetic of natural numbers can be defined. In fact, we shall construct an undecidable field $\mathfrak{F}$ such that each sentence true in $\mathfrak{F}$ is also true in some finite field.

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[7] See J. Robinson [59].

[8] An algebraic number is *totally* real if it and all of its conjugates are real.

[9] See J. Robinson [62].

Hence no essentially undecidable sentence σ can hold in $\mathfrak{F}$, whereas a translation of Q can be made whenever a model of the arithmetic of natural numbers can be defined.

Let S be any set of natural numbers and Σ be the set of sentences true in all finite fields. Consider the set Γ_S of the following sentences:

$$(2) \qquad \begin{cases} \bigvee_x x^2 = p_k & \text{for } k \in S, \\ \neg \bigvee_x x^2 = p_k & \text{for } k \notin S, \end{cases}$$

where p_k is an abbreviation for the kth prime number written as a sum of 1's. We can assign the quadratic character of a finite number of primes arbitrarily and find a prime q for which the Legendre symbol (p/q) will have the desired value for each of the given primes p. Thus the field with q elements will satisfy the corresponding sentences of (2). Hence $\Sigma \cup \Gamma_S$ is consistent, since every finite subset is consistent. Now Σ contains the negation of any sentence which has only infinite models. Therefore any sentence true in a model of $\Sigma \cup \Gamma_S$ must have a finite model. Furthermore if S is non-recursive, then every model of $\Sigma \cup \Gamma_S$ is undecidable. In fact, we have shown that for any set S of natural numbers, there is a field $\mathfrak{F}_S$ such that S is recursive in the set of true sentences of $\mathfrak{F}_S$ and every true sentence of $\mathfrak{F}_S$ is true in some finite field. The decision problem for the theory of finite fields, i.e., the question of whether or not Σ is recursive, is an interesting open problem [10].

Before turning to the case of transcendental extensions of real fields, we will discuss an earlier result. R. M. Robinson [51a, section 2] showed that a polynomial ring over a field is undecidable. We shall be concerned here only with the case in which the characteristic of the field is 0. If we define divisibility by

$$x|y \leftrightarrow \bigvee_z y = xz,$$

then the constants (i.e., the elements of the base field) are defined by

$$\mathrm{Con}\, x \leftrightarrow x = 0 \vee x|1.$$

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[10] This problem may be closely connected with unpublished results of Cobham and with extensions of Cobham's work by Vaught [62]. In particular, Cobham strengthens results of Tarski-Mostowski-Robinson [53] concerning the theory R and uses this to show the undecidability of the theory of finite groups. See also a related paper by Malcev [61].

It is not difficult to see that the set of natural numbers is defined by

$$\mathrm{Nat}\ x \leftrightarrow \bigvee_{u,w} \{\neg\, \mathrm{Con}\ u \wedge w \neq 0 \wedge u|w$$

$$\wedge \bigwedge_{y} (\mathrm{Con}\ y \wedge u+y|w \rightarrow u+y+1|w \vee y=x)\}.$$

This definition depends essentially on the concept of divisibility in the ring of polynomials. But in a field of rational functions the relation of divisibility is trivial and some other concept will have to be used.

The first special case of the undecidability of a transcendental extension of a real field is also due to R. M. Robinson [11]. He showed in 1950 that any pure transcendental extension of the field of rational numbers is undecidable.

Malcev [60, 61a] first showed the undecidability of a simple transcendental extension of a real closed Archimedean [12] field by defining the set of natural numbers.

He defined the subring of functions bounded by a constant. In this ring, the relation of divisibility is not trivial and can play roughly the same role as divisibility in rings of polynomials.

In this same paper, Malcev also showed that certain fields of formal power series over undecidable fields are undecidable. In particular, the field of formal power series over the rational field is undecidable.

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[11] An account of this work is to be included in R. M. Robinson [64].

[12] Let $N(x)$ be the formula constructed by Malcev which defines the set of natural numbers in a simple transcendental extension of an Archimedean real closed field. Tarski has shown that $N(x)$ always defines a wider class in case the ground field is a non-Archimedean real closed field. He constructed the following formula which does define the set of natural numbers in a simple transcendental extension of an arbitrary real closed field:

$$N(x) \wedge \bigvee_{u}\bigvee_{v}\bigwedge_{w} (\mathrm{Pos}\ w \rightarrow (u+w||v \leftrightarrow N(w) \wedge w \leqq x)),$$

where Pos u and $u \leqq v$ are defined by Malcev and $u||v$ is given by

$$u||v \leftrightarrow \bigvee_{w}\bigvee_{z} (uw=v \wedge w \leqq z \wedge \mathrm{Pos}\ z).$$

Tarski has also shown that the following formula containing an indeterminate α defines the set of natural numbers in a pure transcendental extension of any real closed field:

$$N(x) \wedge \bigvee_{v}\bigwedge_{w} (\mathrm{Pos}\ w \rightarrow (\alpha+w||v \leftrightarrow N(w) \wedge w \leqq x)).$$

The indeterminate α can be eliminated from this formula (cf. formula (5) in this section).

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be real closed fields and let $\mathfrak{R}_1$ and $\mathfrak{R}_1'$ be simple transcendental extensions of $\mathfrak{R}$ and $\mathfrak{R}'$ respectively. Tarski [63] pointed out that there is a sentence σ of L which is true in $\mathfrak{R}_1$ if and only if $\mathfrak{R}$ is Archimedean, since both the set of natural numbers and the set of elements of $\mathfrak{R}$ are definable in $\mathfrak{R}_1$. Hence if $\mathfrak{R}$ is Archimedean and $\mathfrak{R}'$ is non-Archimedean, then σ will be true in $\mathfrak{R}_1$ but false in $\mathfrak{R}_1'$. Thus $\mathfrak{R}_1$ and $\mathfrak{R}_1'$ are not elementarily equivalent, even though $\mathfrak{R}$ and $\mathfrak{R}'$ are elementarily equivalent, as we have seen.

Since we can define the natural numbers in $\mathfrak{R}_1$, we can also define the set of rational numbers which are less than the real number e and the set of rational numbers which are greater than e. (Both of these sets are recursive and hence arithmetically definable.) So there is a sentence θ of L such that θ is true in $\mathfrak{R}_1$ if and only if $\mathfrak{R}$ contains an element in the Dedekind cut of the rationals corresponding to e. Let $\mathfrak{R}$ be the field of real numbers and $\mathfrak{R}'$ be the field of all real algebraic numbers. Then θ is true in $\mathfrak{R}_1$ but false in $\mathfrak{R}_1'$, so $\mathfrak{R}_1$ and $\mathfrak{R}_1'$ are not elementarily equivalent.

Tarski and Vaught [57] have shown a related, contrasting result that any two transcendental extensions of infinite degree over the same field are elementarily equivalent.

We will now turn to a new proof of the results of Malcev and Tarski. Let $\mathfrak{F}$ be any real closed field and α a transcendental over $\mathfrak{F}$. ($\mathfrak{F}$ may be Archimedean or non-Archimedean.) We will show that the set of natural numbers can be defined in $\mathfrak{F}(\alpha)$. At first we will allow the symbol α in the definition but later show that it can be eliminated.

In the first place, we can easily define the constants by the formula,

$$\text{(3)} \qquad \mathrm{Con}\, x \leftrightarrow \bigvee_y 1 + x^4 = y^4.$$

Clearly, if x is in $\mathfrak{F}$, then there is a y satisfying the right side, since $\mathfrak{F}$ is real closed. Conversely, it is known that the curve $1+x^4=y^4$ does not have a rational parametrization so that any x satisfying the right side must be in the ground field.

Let $x \leqq_n y$ be defined by the formula,

$$x \leqq_n y \leftrightarrow \bigvee_{u_1, \ldots, u_n} x + u_1^2 + \ldots + u_n^2 = y.$$

Now it is well known and easily proved that any positive definite rational function in $\mathfrak{F}(\alpha)$ is the sum of two squares. Thus an inequality $u \leqq v$,

assumed to hold between the two rational functions u and v for all values of α for which they are defined, is equivalent to $u \leqq_2 v$ and hence is definable in $\mathfrak{F}(\alpha)$. These facts are used by Malcev in his proof.

If a is a constant, the inequality $u^2 \leqq (\alpha - a)^2$ guarantees that the rational function u has a zero at a. On the other hand, we can satisfy the inequalities

$$u^2 \leqq (\alpha - a_1)^2,\ u^2 \leqq (\alpha - a_2)^2,\ \ldots,\ u^2 \leqq (\alpha - a_k)^2,$$

simultaneously for any constants $a_1, a_2, \ldots, a_k$ by a rational function u which has zeros only at these points. In fact, it is sufficient to take

$$u = \frac{c(\alpha - a_1)\,(\alpha - a_2) \ldots (\alpha - a_k)}{1 + (\alpha - a_1)^2\,(\alpha - a_2)^2 \ldots (\alpha - a_k)^2},$$

where c is a sufficiently small constant.

We can now define the natural numbers by the formula

$$(4)\quad \mathrm{Nat}\ n \leftrightarrow \bigvee_u [u \neq 0 \wedge u^2 \leqq_2 \alpha^2 \wedge \bigwedge_k (\mathrm{Con}\ k \wedge k \neq n \wedge u^2 \leqq_2 (\alpha - k)^2 \rightarrow u^2 \leqq_2 (\alpha - k - 1)^2)].$$

Indeed, if n is a natural number, then we can find u so that $u^2 \leqq_2 (\alpha - k)^2$ for $k = 0, 1, \ldots, n$ and u has no other zeros. Hence the condition on the right is satisfied. On the other hand, if n is not a natural number, then u would have to have zeros at all the natural numbers in order to satisfy the right side. This is impossible.

We still must eliminate α from the definition of natural numbers. Let $\varphi(\alpha, n)$ be the formula on the right side of the equivalence sign in (4). Then

$$(5)\quad \mathrm{Nat}\ n \leftrightarrow \bigwedge_t \{\varphi(t, 0) \wedge \bigwedge_k [\varphi(t, k) \rightarrow \varphi(t, k+1)] \rightarrow \varphi(t, n)\}$$

is a suitable definition of the natural numbers. The condition on the right is clearly necessary and its sufficiency is seen by taking $t = \alpha$.

This completes the proof that the natural numbers can be defined in a simple transcendental extension of a real closed field $\mathfrak{F}$ and hence this field is undecidable. As mentioned before, this example is also sufficient to prove the undecidability of the elementary theory of fields.

The proof can be extended at once to the case of a transcendental extension $\mathfrak{F}(\alpha)$ of a real field $\mathfrak{F}$ which may not be real closed but which

has a real closed subfield $\mathfrak{F}_0$. It will still be true that the equation $1+x^4=y^4$ can be satisfied for all x in $\mathfrak{F}_0$. The formula $u^2 \leqq_2 (\alpha-a)^2$ for a in $\mathfrak{F}$ will still guarantee that u has a zero at a. A rational function can have only a finite number of zeros, and we can still satisfy any finite number of conditions of the form

$$u^2 \leqq_2 (\alpha-a_j)^2,$$

at least for a_j in $\mathfrak{F}_0$, which is sufficient. Hence the same definition of Nat is valid. As a special case of this result, we see that if any number of indeterminates are adjoined to a real closed field $\mathfrak{F}_0$, we can still define the natural numbers in the resulting field.

Only a slight modification of the above definition of natural numbers is needed in a transcendental extension $\mathfrak{F}(\alpha)$ of a completely arbitrary real field $\mathfrak{F}$ if the set of elements of $\mathfrak{F}$ can be defined (or is considered given). In fact, it is sufficient to define some subset of the constants which includes the natural numbers. Suppose such a set of constants is defined by a formula $\gamma(x)$ so that

$$(\mathrm{Nat}\ x \to \gamma(x)) \wedge (\gamma(x) \to \mathrm{Con}\ x).$$

Then we can define the set of natural numbers by

$$\mathrm{Nat}\ n \leftrightarrow \bigvee_u [u \neq 0 \wedge u^2 \leqq_8 \alpha^2 \wedge \bigwedge_k (\gamma(k) \wedge k \neq n \wedge u^2 \leqq_8 (\alpha-k)^2 \to u^2 \leqq_8 (\alpha-k-1)^2)].$$

The argument is the same as before, except that now we use the theorem of Landau [06] that any positive definite rational function of one variable with rational coefficients can be expressed as the sum of eight squares of rational functions with rational coefficients. Hence we can make up a function u with rational coefficients and prescribed rational zeros $a_1, \ldots, a_k$ so that $u^2 \leqq_8 (\alpha-a_j)^2$ for $j=1, \ldots, k$. Thus the whole problem for a transcendental extension $\mathfrak{F}(\alpha)$ would be solved if we could define a suitable set of constants.

Recently, R. M. Robinson [64] has carried on these methods to show that *every* pure transcendental extension of a real field is undecidable.

The difficult case for transcendental extensions of real fields arises because we do not have a suitable definition of a sufficiently wide class of constants. In the case of a transcendental extension $\mathfrak{C}(\alpha)$ of the field

of all complex numbers $\mathfrak{C}$, we can immediately define the complex numbers by the formula

$$\mathrm{Con}\ x \leftrightarrow \bigvee_y 1+x^4=y^4.$$

However at present we know of no way of defining the natural numbers in $\mathfrak{C}(\alpha)$.

The case of a transcendental extension of a finite field is an interesting open problem. Here, of course, there is no question of defining the natural numbers but there may be a definable model of arithmetic nevertheless. In the case of the ring of polynomials over a finite field, R. M. Robinson [51a] defined a model consisting of powers of a transcendental element. Quite possibly something similar can be done for the field of rational functions over a finite field. As yet there are no known undecidable fields of characteristic different from 0 except those defined using non-recursive sets.

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ON MODELS OF ELEMENTARY ELLIPTIC GEOMETRY

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Introduction. In Tarski [59] and Schwabhäuser [56] the completeness and decidability of certain theories of elementary Euclidean geometry was proved; in Schwabhäuser [59] and Szmielew [59a] (see also [59], [61], [62]) this was done for elementary hyperbolic geometry, in all cases by reduction to the corresponding result for the arithmetic of real numbers, which was established by Tarski [51].

The first aim of this paper was to obtain an analogous result for a theory $\mathscr{E}^*$ of elementary elliptic geometry. By a method applied for Euclidean geometry in Tarski [59] it is sufficient for this purpose to characterize all models of $\mathscr{E}^*$ as isomorphic with elliptic Klein spaces over real closed fields. This suggests consideration of the models of more general theories of elementary elliptic geometry.

In this paper, we consider first a theory $\mathscr{E}$ of elementary elliptic geometry based on axioms which are the analogue to Hilbert's groups I–IV (see Hilbert [56]) or Tarski's axioms A1–A12 (see Tarski [59, p. 19]) for Euclidean geometry (these are all axioms with the exception of the axioms of continuity). Then we consider a theory $\mathscr{E}'$, which arises from $\mathscr{E}$ by adding the circle axiom, and finally a theory $\mathscr{E}^*$, obtained from $\mathscr{E}$ by adding the scheme of elementary continuity axioms. It will be proved that a structure $\mathfrak{M}$ is a model of $\mathscr{E}$ if and only if it is isomorphic to an elliptic Klein space over some Pythagorean ordered field; similarly, for $\mathscr{E}'$, with Euclidean ordered fields; and, for $\mathscr{E}^*$, with real closed fields (which implies the completeness and decidability of $\mathscr{E}^*$ as mentioned above). These results for $\mathscr{E}$ and $\mathscr{E}'$ were noticed by Royden [59, p. 89] but without proof and without specifying axioms. For the axioms adopted here, we can take these results from Podehl-Reidemeister [34], where, essentially, the models of more general axiom systems than for $\mathscr{E}$ are described. However, these results are obtained in this paper by using another method, which may be of interest in itself, namely, by developing elliptic trigonometry (of right triangles) in a very short way, the values of the trigonometric functions being

points in a point calculus which is a special case of the well-known projective number scales.

We restrict ourselves, in this paper, to elementary *plane* elliptic geometry. However, it is not difficult to extend the results to elementary elliptic geometry of higher dimensions.

For better readability, logical symbolism is used sparingly and most of the statements for $\mathfrak{E}$ are formulated in colloquial language; however, they can be expressed by formulas of the theory $\mathfrak{E}$.

1. Axioms. To have a fixed base which is given in the literature for our theory $\mathfrak{E}$ of elementary plane elliptic geometry, we shall use points $(p, q, \ldots, z, o, e)$ and (straight) lines $(a, b, c, d, g, \ldots)$ as basic elements and, as primitive notions, incidence between points and lines $(I(p, a))$, orthogonality between lines $(a \perp b)$, a quaternary congruence relation $(pq \equiv rs)$, and a quaternary order relation $(pq|rs)$ between points. We adopt as axioms the axioms given in terms of incidence, orthogonality, and congruence by Podehl-Reidemeister [34], including the axiom of segment construction (which states that, if $p \neq q$ and r lies on a, then there exists at least one point s on a such that $pq \equiv rs$), and the axioms for order usually adopted in projective geometry (see, e.g., Heffter [50, pp. 16 ff.]).

The intuitive meaning of $pq|rs$ is: p, q, r, s are collinear and the pair $[p, q]$ separates the pair $[r, s]$. The meaning of $pq \equiv rs$ is: the distance from p to q (on a shortest segment joining p and q) equals the distance from r to s[1]. We shall then also say, as usual, that the congruence relation holds for the *segments* pq and rs, which we understand to be unordered pairs of points, or pq and rs are congruent. If $p = q$, we call pq a *zero segment*.

By (pq) we shall denote, if $p \neq q$, the (uniquely determined) line joining p and q.

The following results can be obtained from our axioms (cf. Podehl-Reidemeister [34]).

If there are two lines passing through p and perpendicular to a, then each line passing through p is perpendicular to a. Then p is called a

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[1] Podehl and Reidemeister use a congruence relation between segments having different endpoints. However, one can easily give an equivalent formulation of their axioms in terms of our congruence relation ("equidistance relation"), which is sometimes more convenient for our purposes.

pole of a and a a *polar line* of p. For each point there exists exactly one polar line, and for each line there exists exactly one pole. A point p lies on the polar line of q if and only if q lies on the polar line of p; we shall then call p and q points *at polar distance* (from each other). Then, p' and q' are also at polar distance if and only if $p'q' \equiv pq$. For each segment pq with $p \neq q$ there exists at least one midpoint (i.e., a point r collinear with p, q such that $pr \equiv qr$) (one can even show that there exist exactly two such midpoints and that these points are at polar distance).

2. **Point calculus and coordinates.** From Podehl–Reidemeister [34] we can further take the result that in elliptic geometry $\mathscr{E}^2$ the points and lines form a projective plane and Desargues' and Pappus' theorems hold in this plane. In such a plane a construction is well-known from projective geometry which – if we choose three different points o, e, u on a line g – leads to a commutative field with zero element o and unit element e the elements of which are the points different from u on g. Another choice of the points o, e, u leads to a field isomorphic to the first one.

We shall use this construction also for elliptic geometry but choose the points o, e, u in a special way, namely, so that o and u are points at polar distance and o and e are points at half the polar distance. The point u is then uniquely determined by o and e. Thus, we get to the following definitions.

$$L(p_1, \ldots, p_n) =_{df} \exists a[I(p_1, a) \wedge \ldots \wedge I(p_n, a)]$$

($p_1, \ldots, p_n$ are collinear points).

$$P(p, q) =_{df} \exists a \exists b \exists a'[I(p, a) \wedge I(q, a) \wedge I(q, b) \wedge a \perp b \wedge \wedge I(p, a') \wedge a' \perp b \wedge a \neq a']$$

(p and q are at polar distance).

$$A^*(o, e, u; x_1, \ldots, x_n) =_{df} L(o, e, u, x_1, \ldots, x_n) \wedge P(o, u) \wedge oe \equiv ue \wedge \wedge x_1 \neq u \wedge \ldots \wedge x_n \neq u \qquad (n \geqslant 0)$$

(o and e determine a number scale with the point at infinity u, and $x_1, \ldots, x_n$ are "arithmetical points" [with respect to o and e], i.e., points of this scale [elements of the field determined by o and e]).

$$A(o, e; x_1, \ldots, x_n) =_{df} \exists u A^*(o, e, u; x_1, \ldots, x_n).$$

Then, $A(o, e)$ means that o and e determine a number scale, i.e., they

are at half the polar distance. Of course, we can prove in $\mathfrak{E}$ that such points exist.

$$Su(o, e; x, y, z) =_{df}$$
$$\exists u \exists o' \exists o^* \exists x^* \exists y' [A^*(o, e, u; x, y, z)$$
$$\wedge \neg L(o, e, o') \wedge L(o, o', o^*) \wedge o^* \neq o \wedge o^* \neq o'$$
$$\wedge L(x, o', x^*) \wedge L(u, o^*, x^*) \wedge L(y, o^*, y') \wedge L(u, o', y')$$
$$\wedge L(x^*, y', z)]$$

(z is sum of x and y with respect to o and e, see Figure 8 [2, 3]).

$$Pr(o, e; x, y, z) =_{df}$$
$$\exists u \exists e' \exists e^* \exists x^* \exists y' [A^*(o, e, u; x, y, z)$$
$$\wedge \neg L(o, e, e') \wedge L(e, e', e^*) \wedge e^* \neq e \wedge e^* \neq e'$$
$$\wedge L(x, e', x^*) \wedge L(u, e^*, x^*) \wedge L(y, e^*, y') \wedge L(o, e', y')$$
$$\wedge L(x^*, y', z)$$

(z is product of x and y with respect to o and e, see Figures 3 or 5).

$$Pt(o, e; x) =_{df} \exists u [A^*(o, e, u; x) \wedge x \neq o \wedge \neg ou|ex]$$

(x is positive with respect to o and e).

From the above-mentioned result for projective geometry it follows that, if $A(o, e; x, y)$, then there is exactly one z_1 such that $Su(o, e; x, y, z_1)$ and exactly one z_2 such that $Pr(o, e; x, y, z_2)$ — we shall denote these points by $x \underset{o,e}{+} y$, $x \underset{o,e}{\cdot} y$ respectively — and that the arithmetical points form a commutative field $K_{o,e}$ with respect to the operations $\underset{o,e}{+}, \underset{o,e}{\cdot}$. Then we denote by $x \underset{o,e}{-} y$ the difference of elements x, y in this field and define an order relation by $x \underset{o,e}{<} y =_{df} Pt(o, e; y \underset{o,e}{-} x)$. It is not difficult to obtain from the axioms of order that $\mathfrak{K}_{o,e} = [K_{o,e}, \underset{o,e}{+}, \underset{o,e}{\cdot}, \underset{o,e}{<}]$ is a commutative ordered field. We shall drop the indices o, e and write $\mathfrak{K} = [K, +, \cdot, <]$ if there is no danger of confusion. The notations $\leqq$, $>$, $\geqq$, $-x$, x^2, x/y will be used, if convenient, in their customary sense.

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[2] Diametrically opposed points on the exterior circle in the figures represent one and the same point of elliptic geometry.

[3] The definitions of sum and product are analogous to the definitions usual in affine geometry if we single out the line $g^* = (ux^*)$ as the ideal line.

The coordinates which we shall use in $\mathscr{E}$ are, again, a special case of the homogeneous coordinates well-known from projective geometry as follows: (i) each two of the basic points u_0, u_1, u_2 are at polar distance; (ii) the projections of the unit point e^* from one basic point on the axis joining the two other ones are at half the polar distance from these two basic points. Hence, the points u_0, u_1, u_2, e^* (determinating the system of coordinates) are uniquely determined by one basic point u_0 and the projections e_1, e_2 of e^* on the axes joining u_0 with u_1, u_2 respectively. Thus, we get to the following definitions.

$R(p, q, r) =_{df} \exists a \exists b [a \perp b \wedge I(p, a) \wedge I(q, a) \wedge I(q, b) \wedge I(r, b) \wedge p \neq q \wedge q \neq r]$
(p, q, r form a right angle with vertex at q).

$$Cs(u_0, e_1, e_2) =_{df} R(e_1, u_0, e_2) \wedge A(u_0, e_1) \wedge A(u_0, e_2)$$

(u_0, e_1, e_2 determine a system of coordinates; of course, we can prove in $\mathscr{E}$ that such points exist).

$$\begin{aligned} Cd^*(o, e, u_0, e_1, e_2; p, x_1, x_2) =_{df} \\ \exists u_1 \exists u_2 \exists p_1 \exists p_2 [R(e_1, u_0, e_2) \wedge A^*(u_0, e_1, u_1) \wedge A^*(u_0, e_2, u_2) \\ \wedge L(u_0, u_1, p_1) \wedge L(u_2, p, p_1) \wedge L(u_0, u_2, p_2) \wedge L(u_1, p, p_2) \\ \wedge A(o, e; x_1, x_2) \\ \wedge \bigwedge_{\nu=1}^{2} \{ox_\nu \equiv u_0 p_\nu \wedge [Pt(o, e; x_\nu) \leftrightarrow p_\nu \neq u_0 \wedge \neg\, u_0 u_\nu | e_\nu p_\nu]\}] \end{aligned}$$

(x_1, x_2 are affine coordinates of p with respect to o, e [forming a number scale] and u_0, e_1, e_2 [forming a system of coordinates]).
(A point has such coordinates if and only if it does not lie on the ideal line singled out in this system, i.e., the polar line of the origin u_0, which joins u_1 and u_2.)

$$\begin{aligned} Cd(o, e, u_0, e_1, e_2; p, x_0, x_1, x_2) =_{df} \\ \exists x_1^* \exists x_2^* [Cd^*(o, e, u_0, e_1, e_2; p, x_1^*, x_2^*) \\ \wedge x_0 \neq o \wedge x_1 = x_0 x_1^* \wedge x_2 = x_0 x_2^*] \\ \vee \{\neg\, \exists x_1^* \exists x_2^* Cd^*(o, e, u_0, e_1, e_2; p, x_1^*, x_2^*) \\ \wedge \exists p' \exists x_1^* \exists x_2^* \exists z [L(u_0, p', p) \wedge p' \neq u_0 \wedge Cd^*(o, e, u_0, e_1, e_2; p', x_1^*, x_2^*) \\ x_0 = o \wedge z \neq o \wedge x_1 = z x_1^* \wedge x_2 = z x_2^*]\} \end{aligned}$$

(x_0, x_1, x_2 are homogeneous coordinates of p with respect to o, e and u_0, e_1, e_2).

It is well known from projective geometry that, if $A(o, e)$ and $Cs(u_0, e_1, e_2)$, then each point p has a triple $[x_0, x_1, x_2] \neq [o, o, o]$ of coordinates in

$K_{o,e}$, where y_0, y_1, y_2 are also coordinates of p if and only if the triple $[y_0, y_1, y_2]$ can be obtained from the first one by multiplying it by a constant factor $z \neq o$, and to each triple of this kind there is exactly one point p having these coordinates (everywhere with respect to o, e, u_0, e_1, e_2).

3. **Elliptic trigonometry.** By an *angle* $\sphericalangle aqb$ with vertex q and sides a, b we understand an unordered triple of a point q and (not necessarily distinct) lines a, b passing through q. If p, r are points different from q and lying on a, b, respectively, we shall denote this angle also by $\sphericalangle pqr$. By a *zero angle* we mean an angle having identical sides.

We shall introduce the trigonometric functions for segments and angles in the same way by assigning to each angle $\sphericalangle pqr$ a *corresponding segment* st, which, intuitively speaking, has the same measure as $\sphericalangle pqr$ and is constructed as follows: s, t are points on the sides $a=(qp)$, $b=(qr)$, respectively, and are at polar distance from the vertex q. We say that the angles $\sphericalangle pqr$ and $\sphericalangle p'q'r'$ are *congruent* ($pqr \equiv p'q'r'$) if and only if the corresponding segments are congruent [4].

We say that segments pq, rs are *complementary* (to each other) if and only if they form together a segment whose length is the polar distance (i.e., $\exists t[L(p, q, t) \wedge P(p, t) \wedge rs \equiv qt]$). Likewise, we say that angles $\sphericalangle pqr$, $\sphericalangle p'q'r'$ are complementary if and only if they form together a right angle (i.e., $\exists s[R(p, q, s) \wedge p'q'r' \equiv rqs]$). This holds if and only if the corresponding segments are complementary. We obtain from our axioms that $pq \equiv rs$ if and only if some segments which are complementary to pq and rs, respectively are, congruent. Similarly for angles.

For the following, we single out two points o and e which determine a number scale ($A(o, e)$) and may be fixed for the remainder of this paper. The definitions of our trigonometric functions and all statements involving arithmetical points are to be understood with respect to o and e.

First, we define the *tangent* $T(p, q)$ of a segment pq as the uniquely determined arithmetical point $x \geqslant o$ such that $ox \equiv pq$. Since x shall

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[4] Intuitively speaking, this means that the angles considered here are of measure m such that $0 \leqq m \leqq (\pi/2)$. This may be somewhat unusual but it is sufficient for trigonometry of *right* triangles which is to be developed here. It may be noticed that the measure of angles and segments itself—expressed in our point calculus—is not definable in $\mathfrak{E}$ nor in the theory $\mathfrak{E}'$ or $\mathfrak{E}^*$ (considered in Section 4). This can be proved in the same way that it was done for hyperbolic geometry in Schwabhäuser [62] or, in more detail, in [62a].

be arithmetical, $T(p, q)$ is defined if and only if p and q are not at polar distance. Analogously, we define the tangent $T(p, q, r)$ of an angle $\sphericalangle pqr$ (if and only if this is not a right angle) as $T(s, t)$ where st is the segment corresponding to this angle.

Then we define the *cosine* $C(p, q, r)$ of an arbitrary angle $\sphericalangle pqr$ as follows: Let s be a point on the side (qp) of this angle such that $qs \equiv oe$ and t the perpendicular projection of s on the side (qr). Then we put $C(p, q, r) = T(q, t)$ (Figure 1). We also define the cosine $C(p, q)$ of an arbitrary segment pq as $C(r, s, t)$ where $\sphericalangle rst$ is an angle such that pq is the segment corresponding to this angle.

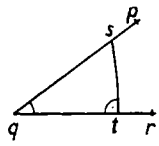


Fig. 1.

Further, we define the *sine* $S(p, q, r)$ or $S(p, q)$ of an arbitrary angle $\sphericalangle pqr$ or segment pq as the cosine of a complementary argument.

Finally we introduce the *cotangent* $Ct(p, q)$ or $Ct(p, q, r)$ of a segment pq or angle $\sphericalangle pqr$ to be the tangent of a complementary argument. This is defined if and only if the argument is not a zero segment or a zero angle.

It is easily seen that an arbitrary one of the "trigonometric functions" S, C, T, Ct has the same values for two segments or angles (or is not defined for both arguments if we take T or Ct) if and only if these segments or angles are congruent. Furthermore, $C(p, q, r) = o$ if and only if $\sphericalangle pqr$ is a right angle. Hence, $S(p, q, r) = o$ if and only if $\sphericalangle pqr$ is a zero angle. Similarly for segments.

We shall now establish some propositions of elliptic trigonometry.

Proposition 1. *Let p, q, r form a right triangle with a right angle at r (i.e., $R(p, r, q)$). Then, the following conditions are equivalent* (cf. Figure 2):

(i) *p, q are not at polar distance;*

(ii) *p, r as well as q, r are not at polar distance;*

(iii) *$\sphericalangle pqr$ as well as $\sphericalangle qpr$ are not right angles.*

This is easily seen from our propositions on poles and polar lines. We shall then say that p, q, r (in this order) form a *proper right triangle.*

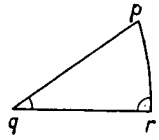


Fig. 2.

Proposition 2. *Let p, q, r form a proper right triangle. Then*

$$T(q, r)=C(p, q, r)\cdot T(q, p).$$

Let (Figure 3) g^* be the polar line of o, u the common point of g^* and (oe), u' a point on g^* such that $u'ou \equiv pqr$, and e' a point on (ou') such that $oe' \equiv oe$ and the perpendicular projection x of e' on (oe) is a positive arithmetical point. Let $y=T(q, p)$, e^* be the common point of g^* and (ee'), and z, x^*, y' points according to the definition of $Pr(o,e;x,y,z)$ (together with u, e', e^* as above). By definition of C, $x=C(u', o, u)=C(p, q, r)$. Since $oe' \equiv oe$, the line (ee') is perpendicular to a certain bisectrix l of the angle $\sphericalangle eoe'$ and e^* is the pole of l. Hence, (yy') is also perpendicular to l and, consequently, $oy' \equiv oy \equiv qp$. Since $(e'x)$ is perpendicular to (oe), x^* is the pole of (oe) and, hence, $(y'z)$ is also perpen-

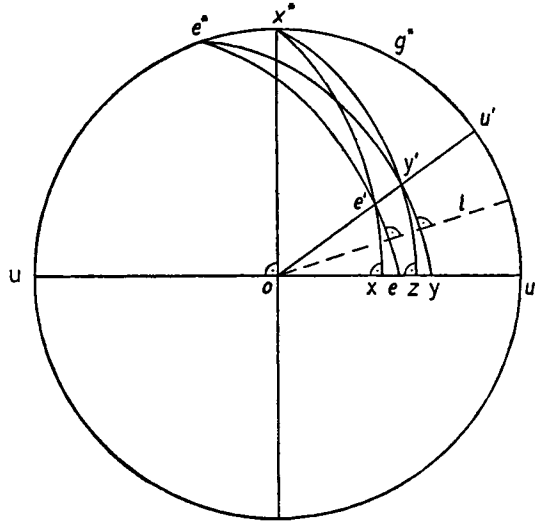


Fig. 3.

dicular to (oe). Thus, the triangles $[y', o, z]$ and $[p, q, r]$ are congruent (homologous elements are congruent), and we have $oz \equiv qr$. Moreover $\neg\, ey|ou$, hence $\neg\, e'y'|ou'$ and $\neg\, xz|ou$. Thus, z is a positive arithmetical point, hence $z=T(q, r)$. Noting that $x \cdot y = z$, we obtain the conclusion.

Lemma 3. *Let p, q, r form a proper right triangle and s be the perpendicular projection of r on (pq)* (Figure 4), *and $r'q'$ a segment complementary to rq. Then*

$$T(r, p) \cdot T(r', q') = T(p, r, s).$$

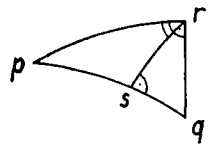


Fig. 4.

Let (Figure 5) $x=T(r, p)$, $y=T(r', q')$, and u, e', e^*, x^*, y', z points according to the definition of $Pr(o, e; x, y, z)$ such that $oe' \equiv oe$, $R(e', o, e)$ and e^*, x^* lie on the polar line g^* of o. Let u' be the pole of (oe) and t the perpendicular projection from u' on (x^*z). Then, $\sphericalangle\, x^*u'e'$ and $\sphericalangle\, xoe'$ are right angles and $u'e' \equiv oe'$, hence $u'x^* \equiv ox \equiv rp$. Since $oe' \equiv oe$, the line (ee') is perpendicular to a certain bisectrix l of the angle $\sphericalangle\, eoe'$

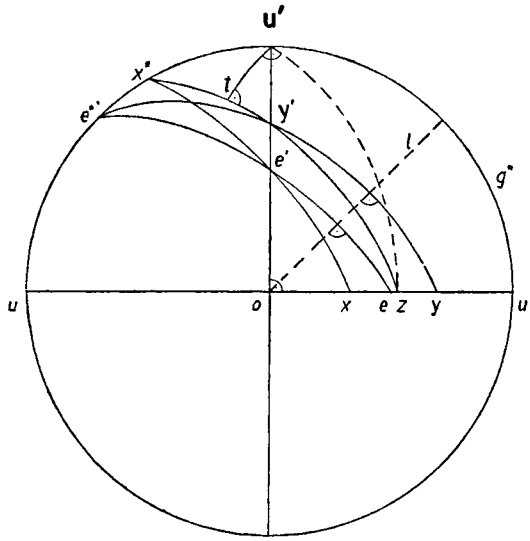


Fig. 5.

and e^* is the pole of l. Hence, (yy') is also perpendicular to l and, consequently, $oy' \equiv oy \equiv r'q'$. The segment $u'y'$ is complementary to oy', hence, $u'y' \equiv rq$. Thus, the triangles $[x^*, y', u']$ and $[p, q, r]$ are congruent, the point t in the first triangle is homologous to s in the second one, and, hence, $x^*u't \equiv prs$. The point z lies on the polar line of u', hence, u' lies on the polar line of z, which is perpendicular to (zx^*) and, consequently, coincides with $(u't)$. Thus, $\sphericalangle zu't$ is a right angle, and, hence, the angles $\sphericalangle x^*u't$ and $\sphericalangle ou'z$, both being complementary to $\sphericalangle tu'y'$, are congruent. Since oz is the segment corresponding to the angle $\sphericalangle ou'z$, we have $z = T(o, z) = T(o, u', z) = T(x^*, u', t) = T(p, r, s)$, which completes the proof.

Let us consider, in Lemma 3, the case that $pr \equiv qr$. Then (rs) is a bisectrix of the right angle $\sphericalangle prq$, and we easily obtain $T(p, r, s) = T(o, e) = e$. Thus, we obtain the following proposition.

Proposition 4. *Let r, q be distinct points not at polar distance and let $r'q'$ be a segment complementary to rq. Then $T(r, q) \cdot T(r', q') = e$ and, hence, $Ct(r, q) = \dfrac{e}{T(r, q)}$. Similarly for angles.*

Applying this to Lemma 3, we obtain:

Proposition 5. *Under the hypothesis of Lemma 3, we have*

$$\frac{T(r, p)}{T(r, q)} = T(p, r, s).$$

Proposition 6. *Let p, q, r form a triangle (be non-collinear points), with p, r as well as q, r not at polar distance, and let s be the perpendicular projection from r on (pq). Then*

$$\frac{T(r, p)}{T(r, q)} = \frac{C(q, r, s)}{C(p, r, s)}.$$

This can easily be obtained from Proposition 2.

Remark. If the hypothesis of Proposition 6 holds and p', q' are further points on (rp), (rq), respectively, different from r and not at polar distance from r, such that $(p'q') \perp (rs)$, we obtain

$$\frac{T(r, p)}{T(r, q)} = \frac{T(r, p')}{T(r, q')}.$$

This means that the relation of proportion, introduced by Wanda

Szmielew [59, p. 215] for Euclidean and hyperbolic geometry, can also be used to characterize our multiplication of points in $\mathscr{E}$.

If the hypothesis of Proposition 5 holds, then so does the hypothesis of Proposition 6 and the angles $\sphericalangle qrs$ and $\sphericalangle prs$ are complementary; hence, $C(q, r, s)=S(p, r, s)$, and, by Propositions 5 and 6, $\dfrac{S(p,r,s)}{C(p,r,s)} = T(p, r, s)$. Obviously, this holds also if $\sphericalangle prs$ is a zero angle. Thus, we obtain:

Proposition 7. *Let p, q, r form an angle which is not a right one. Then*

$$\frac{S(p,q,r)}{C(p,q,r)} = T(p, q, r).$$

Similarly for segments.

Proposition 8. *For each proper right triangle $[p, q, r]$ the following holds:*

(i) $C(p,q,r)=\dfrac{T(q,r)}{T(q,p)}$, (ii) $C(q,p,r)=\dfrac{T(p,r)}{T(p,q)}$,

(iii) $T(p,q,r)=\dfrac{T(p,r)}{S(q,r)}$, (iv) $T(q,p,r)=\dfrac{T(q,r)}{S(p,r)}$,

(v) $C(p,q)=Ct(p,q,r)\cdot Ct(q,p,r)$,

(vi) $C(p,q)=C(p,r)\cdot C(q,r)$,

(vii) $S(p,q,r)=\dfrac{S(p,r)}{S(p,q)}$, (viii) $S(q,p,r)=\dfrac{S(q,r)}{S(q,p)}$,

(ix) $C(p,r)=\dfrac{C(p,q,r)}{S(q,p,r)}$, (x) $C(q,r)=\dfrac{C(q,p,r)}{S(p,q,r)}$.

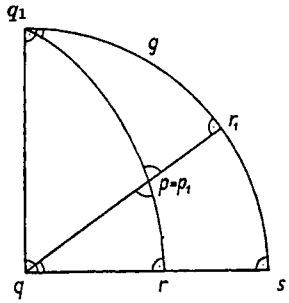


Fig. 6.

For each proper right triangle $[p, q, r]$ we can construct a second such triangle $[p_1, q_1, r_1]$ by the well-known method of the Napier chain (cf. Liebmann [23, pp. 104 ff.]) illustrated in Figure 6 such that $qpr \equiv q_1p_1r_1$, qp is complementary to p_1r_1, q_1p_1 is complementary to pr, qr is complementary to the segment rs corresponding to $\sphericalangle p_1q_1r_1$, and q_1r_1 is complementary to the segment r_1s corresponding to $\sphericalangle pqr$. Now, (i) and (ii) are other versions of Proposition 2. Each of the other formulas is equivalent with one of the preceding formulas formulated for the triangle $[p_1, q_1, r_1]$ except (vi), which follows directly from (iii), (iv), (v).

Proposition 9. *Let p, q, r, s (in this order) "form a proper Lambert quadrangle", i.e., $R(s, p, q)$, $R(p, q, r)$, $R(q, r, s)$, and p, q as well as q, r are not at polar distance* (Figure 7). *Then*

$$C(p, q) = \frac{T(p, s)}{T(q, r)}.$$

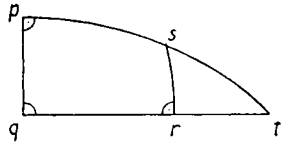


Fig. 7.

Let t be the pole of (pq). Then, we obtain the conclusion by applying Proposition 8 (i) (or 2) to the proper right triangle $[s, t, r]$ noting that ps and qr are complementary to ts and tr, respectively.

Proposition 10. *Let p, q, r be distinct collinear points such that $pr|qq'$, where q' denotes the point on (pr) which is at polar distance from q. Then*

$$S(p, r) = S(p, q) \cdot C(q, r) + C(p, q) \cdot S(q, r).$$

Let (Figure 8) $x = -T(p, q)$, $z = T(q, r)$, $y = z - x$, and u, o', o^*, x^*, y' points according to the definition of $Su(o, e; x, y, z)$ such that $oo' \equiv ox$, $R(x, o, o')$, and o^*, x^* lie on the polar line g^* of o. Let t be the perpendicular projection of o on (x^*z) and v the common point of (x^*z) and (oo^*). Since $oo' \equiv ox$, the line (xo') is perpendicular to a certain bisectrix l of the angle $\sphericalangle xoo'$ and x^* is the pole of l. Hence, (zv) is also perpendicular to l and, consequently, $ov \equiv oz \equiv qr$. It can easily be shown that $ox \equiv pq$. (This amounts to showing that, if $x = -x_1$ and $x \neq o$, then $ox \equiv ox_1$.) Hence $oo' \equiv pq$. Moreover, $x < o$ and $z > o$, so $xz|ou$, and hence (by projection from x^*) $o'v|oo^*$. Thus, $o'v \equiv pr$ (this can be

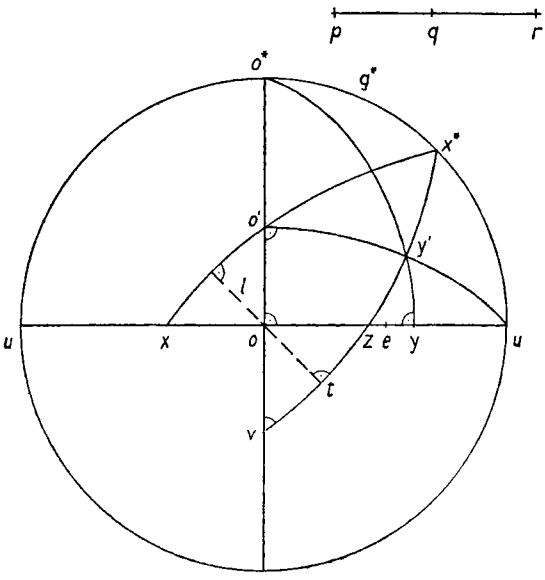


Fig. 8.

obtained from our axioms). Applying Proposition 8 (v) to the right triangle $[o, v, t]$, we obtain $C(q, r)=C(o, v)=Ct(v, o, t)\cdot Ct(o, v, t)$. Since l is a bisectrix of the right angle $\sphericalangle\, voz$, we have $Ct(v, o, t)=e$, hence $C(q, r)=Ct(o, v, t)$ (*). Since u is the pole of (oo') and o^* is the pole of (oy), the angles $\sphericalangle\, oo'y'$ and $\sphericalangle\, oyy'$ are right angles. Applying Proposition 8 (iv) to the right triangle $[v, y', o']$, we obtain $T(o', y')=S(o', v)\cdot T(o, v, t)$ (which also holds if o', v are at polar distance), hence, by (*), $T(o', y')\cdot C(q, r)=S(o', v)=S(p, r)$ (**). Applying Proposition 9 to the Lambert quadrangle $[o', o, y, y']$, we obtain $T(o', y')=C(o, o')\cdot T(o, y)$, hence $T(o', y')=C(p, q)\cdot y$, and, by (**), $S(p, r)=C(p, q)\cdot y\cdot C(q, r)$. Now, by construction, we have $y=z-x=T(p, q)+T(q, r)$, hence $S(p, r)=$ $=C(p, q)\cdot\{T(p, q)+T(q, r)\}\cdot C(q, r)$, which together with Proposition 7 gives the conclusion.

Lemma 11. *For an arbitrary segment pq there is*

$$S^2(p, q)+C^2(p, q)=e.$$

(We use $S^2(p, q)$ as an abbreviation for $S(p, q)\cdot S(p, q)$, similarly for C and T.) Similarly for angles [5].

[5] The proof of Lemma 11 and Proposition 12 on the basis of the preceding propositions is taken, essentially, from Bogomolov [34, pp. 136 f.].

Let r be a point such that p, q, r are collinear and p, r are at polar distance. Then, the segments pq and qr are complementary and $S(p, r)=e$. If $p \neq q$ and $q \neq r$, then the hypothesis of Proposition 10 holds, as can be easily shown. If $p=q$ or $q=r$, then, obviously, the conclusion of this proposition holds as well. From this we obtain our lemma, since $C(q, r)=S(p, q)$ and $S(q, r)=C(p, q)$.

Proposition 12. *Let p, q, r be distinct collinear points such that $pr|qp'$, where p' denotes the point on (pr) which is at polar distance from p. Then*

$$C(p, r)=C(p, q)\cdot C(q, r)-S(p, q)\cdot S(q, r).$$

We can show from our hypothesis that, a fortiori, the hypothesis of Proposition 10 holds. From this proposition and Lemma 11 we obtain (putting $2=e+e$)

$$\begin{aligned} e-C^2(p, r) &= S^2(p, r) \\ &= [e-C^2(p, q)]\cdot C^2(q, r)+[e-S^2(p, q)]\cdot S^2(q, r) \\ &\quad +2\cdot C(p, q)\cdot C(q, r)\cdot S(p, q)\cdot S(q, r), \end{aligned}$$

hence

$$\begin{aligned} C^2(p, r) &= C^2(p, q)\cdot C^2(q, r)+S^2(p, q)\cdot S^2(q, r) \\ &\quad -2\cdot C(p, q)\cdot C(q, r)\cdot S(p, q)\cdot S(q, r) \\ &= [C(p, q)\cdot C(q, r)-S(p, q)\cdot S(q, r)]^2. \end{aligned}$$

Moreover, we can show from our hypothesis that $C(p, q)\cdot C(q, r)-S(p, q)\cdot S(q, r)>o$. (This amounts to showing, intuitively speaking, that C is a decreasing function. From this we obtain $S(p, q)=C(q, p')<C(q, r)$ and, similarly, $S(q, r)<C(p, q)$.) Since also $C(p, r)>o$, the conclusion holds.

Proposition 13. *Under the same hypothesis as in Proposition 12 there is*

$$T(p, r)=\frac{T(p, q)+T(q, r)}{e-T(p, q)\cdot T(q, r)}.$$

In fact, we have $C(p, q) \neq o$, $C(q, r) \neq o$, and the conclusion follows from Propositions 7, 10, and 12.

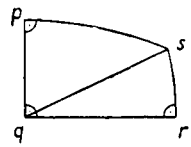


Fig. 9.

Proposition 14. *Let p, q, r, s form a proper Lambert quadrangle* (cf. Proposition 9) (Figure 9). *Then*

$$T^2(q, s) = T^2(q, p) + T^2(q, r).$$

This can easily be obtained from Proposition 2 and Lemma 11.

4. Representation theorems. Using the propositions of elliptic trigonometry, especially Propositions 2, 9, 13, 14, we can prove [6] the following

Proposition 15. *Let x_0, x_1, x_2 and y_0, y_1, y_2 be homogeneous coordinates of p and q, respectively, (in a certain system of coordinates as described in Section 2). Then*

$$C^2(p, q) = \frac{(x_0y_0 + x_1y_1 + x_2y_2)^2}{(x_0^2 + x_1^2 + x_2^2)\,(y_0^2 + y_1^2 + y_2^2)} = \frac{(\mathfrak{x}\mathfrak{y})^2}{\mathfrak{x}^2\mathfrak{y}^2},$$

the latter in the usual notation of vector algebra

$$(\mathfrak{x} = [x_0, x_1, x_2],\ \mathfrak{y} = [y_0, y_1, y_2]).$$

From this, we obtain the following characterization of the congruence relation.

Proposition 16. *Let $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{u}$, $\mathfrak{v}$ be coordinate vectors of p, q, r, s, respectively, (as above). Then $pq \equiv rs$ if and only if* $\dfrac{(\mathfrak{x}\mathfrak{y})^2}{\mathfrak{x}^2\mathfrak{y}^2} = \dfrac{(\mathfrak{u}\mathfrak{v})^2}{\mathfrak{u}^2\mathfrak{v}^2}$.

It is well known from projective geometry how lines, the incidence relation, and the order relation can be described by coordinates ($pq|rs$ if and only if the double ratio of these four collinear points is negative). Finally, we can easily obtain the following description of orthogonality: if $\mathfrak{a}$, $\mathfrak{b}$ are vectors of homogeneous line coordinates (to be introduced in our coordinate system as well known from projective geometry) of a, b, respectively, then $a \perp b$ if and only if $\mathfrak{a}\mathfrak{b} = o$.

From this description and the axiom of segment construction we obtain that the ordered field $\mathfrak{K}_{o,e}$ is Pythagorean (i.e., each sum of two squares is a square itself).

Given any ordered field $\mathfrak{K} = [K, +, \cdot, <]$, we understand by the (two-dimensional) *elliptic Klein space* $\mathfrak{E}(\mathfrak{K})$ over $\mathfrak{K}$ a structure constructed from the elements of K in the same way as we described the basic elements

[6] For hyperbolic geometry, this was done analogously in Schwabhäuser [59 pp. 199 ff.].

and the primitive notions of $\mathscr{E}$ in coordinates (cf. Royden [59, p. 89]). One can easily prove that each such space over a Pythagorean ordered field (and, of course, each isomorphic structure) is a model of $\mathscr{E}$. From this and the above description we obtain the

Representation Theorem for $\mathscr{E}$. *A structure $\mathfrak{M}$ is a model of $\mathscr{E}$ if and only if it is isomorphic with an elliptic Klein space $\mathfrak{E}(\mathfrak{K})$ over some Pythagorean ordered field $\mathfrak{K}$.*

Now, we consider the theory $\mathscr{E}'$ arising from $\mathscr{E}$ by adding the so-called "circle axiom" (cf. Podehl-Reidemeister [34, p. 255]) and the theory $\mathscr{E}^*$ arising from $\mathscr{E}$ by adding the infinite scheme of elementary continuity axioms which consists of *all sentences of the form*

$$\forall s_1 \dots \forall s_n\{\exists z\exists u\forall x\forall y[\varphi \wedge \psi \rightarrow ux|zy] \rightarrow \exists z\exists v\forall x\forall y[\varphi \wedge \psi \wedge x \neq v \wedge y \neq v \rightarrow zv|xy]\},$$

where φ stands for any formula in which the variables $s_1, \dots, s_n, x$ occur free but z, u, v, y do not occur free, and similarly for ψ, with x and y interchanged.

One can easily prove in $\mathscr{E}$ that the circle axiom holds if and only is the ordered field $\mathfrak{K}_{o,e}$ is Euclidean (i.e., each positive element is a square), and that the elementary continuity axioms hold if and only if the elementary continuity axioms for arithmetic hold in $\mathfrak{K}_{o,e}$; this means, by a result of Tarski [51], that $\mathfrak{K}_{o,e}$ is a real closed field. Thus, we obtain representation theorems for $\mathscr{E}'$ and $\mathscr{E}^*$ which result from that for $\mathscr{E}$ by replacing "Pythagorean ordered field" by "Euclidean ordered field" and "real closed field", respectively.

From the last theorem, we can conclude in the same way as it was done for Euclidean geometry by Tarski [59, pp. 21 ff.] that the theory $\mathscr{E}^*$ is complete, decidable, and not finitely axiomatizable.

Of course, the theories $\mathscr{E}$ and $\mathscr{E}'$ are incomplete and, by their definition, finitely axiomatizable. The decision problem for these theories remains open.

Note. When the paper was submitted, the author realized that a similar development of elliptic trigonometry, though in another calculus and without metamathematical applications, was given by G. Hessenberg

in Sitzungsber. d. Berliner Math. Ges. 4 (1905), pp. 69–77. This can be carried over in the calculus used here. Then, there results a simpler proof of Propositions 7 and 4, while the resulting proof of Proposition 10 seems to be more involved than the proof given here. Moreover, by use of stronger propositions of projective geometry, there results a direct proof of Proposition 13 from Hessenberg's paper.

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LOGIC WITH DENUMERABLY LONG FORMULAS AND FINITE STRINGS OF QUANTIFIERS *

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Consider the most obvious and immediate extension of ordinary predicate logic to a logic involving infinitely long expressions: namely, extend the propositional part of the logic by allowing denumerable conjunctions and disjunctions. In making an extension to an infinitary logic, Tarski originally suggested [58b] that infinite strings of quantifiers also be allowed. In some ways, in the denumerable case, this plan is not reasonable. Following the terminology of Karp [a], let us call Tarski's logic $\mathbf{L}_{\omega_1, \omega_1}$, where the two subscripts indicate that both the conjunctions *and* the strings of quantifiers can be of any (denumerable) length $< \omega_1$. The logic we are discussing in this paper is called $\mathbf{L}_{\omega_1, \omega_0}$, where the second subscript indicates that the strings of quantifiers prefixing any subformula of a formula must be *finite* in length. $\mathbf{L}_{\omega_0, \omega_0}$ is the ordinary finitary predicate logic.

As was pointed out by Tarski [58b], in the logic $\mathbf{L}_{\omega_1, \omega_1}$ it is possible to characterize the class of well-ordering relations by a *single* sentence of $\mathbf{L}_{\omega_1, \omega_1}$. For this reason, and also in view of the behavior of $\mathbf{L}_{\omega_1, \omega_1}$ as regards questions of completeness and axiomatization, it seems to the author that $\mathbf{L}_{\omega_1, \omega_1}$ behaves more like *second-order* logic. This is why he feels that Tarski's extension to the countably infinitary language cannot be regarded as a suitable generalization of first-order logic. The purpose of this paper is to summarize the main results presently known about $\mathbf{L}_{\omega_1, \omega_0}$ in order to show that this logic *does* deserve to be called a reasonable infinitary *first-order* logic.

As a first simple example, let us consider formulas of $\mathbf{L}_{\omega_1, \omega_0}$ involving identity and one additional binary predicate $<$. We use $\rightarrow$, $\neg$, $\wedge$, $\vee$, $\leftrightarrow$, $\forall$, $\exists$ for the usual logical connectives and quantifiers, while an in-

* The author is very much indebted to Carol Karp, E. G. K. Lopez-Escobar, and Czesław Ryll-Nardzewski for much stimulating correspondence and many interesting conversations on the topic of this paper.

finite conjunction and an infinite disjunction of formulas A_ξ, $\xi<\alpha$, are denoted respectively by

$$\bigwedge_{\xi<\alpha} A_\xi \text{ and } \bigvee_{\xi<\alpha} A_\xi.$$

If we wanted to be more formal, we could consider infinite conjunctions as infinite well-ordered concatenations of symbols of the form

$$[\bigwedge A_0 A_1 \ldots A_\xi \ldots],$$

where we string out the formulas A_ξ in order as ξ ranges through the ordinals less than α. Of course, in $\mathbf{L}_{\omega_1,\omega_0}$, the length α of the conjunction must be an ordinal less than ω_1. The more abbreviated notation is more convenient, however.

Now, using x and y for individual variables, let us define by transfinite recursion on ordinals $\alpha<\omega_1$ the following sequence of formulas:

$$E_\alpha = \forall y[y<x \leftrightarrow \exists x[x \doteq y \wedge \bigvee_{\xi<\alpha} E_\xi]].$$

Note how as α increases the quantifiers in E_α will become more and more nested even though only two distinct variables occur in the formulas. Each E_α has x as the only free variable. The use of equality in the right half of the formula E_α was only a device to avoid substitution. Allowing the usual inexact notation indicating the free variables and substitutions, what is meant is that these formulas should be logically valid:

$$\forall x[E_\alpha(x) \leftrightarrow \forall y[y<x \leftrightarrow \bigvee_{\xi<\alpha} E_\xi(y)]].$$

Next let $\mathfrak{A}=\langle A, <\rangle$ be a relational system where $<$ simply orders A. Which elements of A will satisfy E_α in $\mathfrak{A}$? The answer, which is easily verified by transfinite induction, is that an element satisfies E_α in $\mathfrak{A}$ if and only if the initial segment of $\mathfrak{A}$ determined by the element is well-ordered by $<$ in type α. (One must realize that the definition of satisfaction of infinitary formulas is as obvious as for finitary formulas —except a transfinite recursion is involved because the formulas are obtained by a transfinite process, as exemplified by the formulas E_α.)

If the set A is countable, then there is a (unique) maximal well-ordered initial segment of $\mathfrak{A}$; say it is of type α. Then the following sentence will be true in $\mathfrak{A}$:

$$[\neg \exists x E_\alpha \wedge \bigwedge_{\xi<\alpha} \exists x E_\alpha].$$

If $\mathfrak{A}$ is actually well-ordered of type α, then the sentence

$$\forall x \underset{\xi<\alpha}{\mathbb{W}} E_\xi$$

is true in $\mathfrak{A}$. Indeed, any well-ordered system will satisfy all of the sentences

$$(*) \qquad [\neg \exists x E_\alpha \rightarrow \forall x \underset{\xi<\alpha}{\mathbb{W}} E_\xi]$$

for arbitrary $\alpha<\omega_1$. Conversely, if $\mathfrak{A}$ is a *countable* ordered system satisfying all of the sentences (*), then $\mathfrak{A}$ is a well-ordered system. There are, however, uncountable ordered systems that are not well-ordered but still satisfy all of the sentences (*): namely, it is shown in Karp [*], and was independently established by Lopez-Escobar, that a well-ordered system of type ω_1 is *equivalent* in $\mathbf{L}_{\omega_1,\omega_0}$ to an ordered system of type $\omega_1+\omega_1\cdot\omega_0{}^*$, which obviously is not well-ordered. (*Equivalence* of course means that the two systems satisfy the same sentences of $\mathbf{L}_{\omega_1,\omega_0}$.) It follows at once that well-orderings cannot be characterized by any set of sentences of $\mathbf{L}_{\omega_1,\omega_0}$. We also see the analogy between the logics $\mathbf{L}_{\omega_i,\omega_0}$, $i=0,1$: for systems of cardinality less than $\aleph_i$, $i=0, 1$, characterizations seem to be possible, but for higher cardinalities this is not so. This analogy will be discussed later, when it is shown it holds in general and not only for well-ordered systems.

Before leaving orderings, let us think for a moment about systems $\mathfrak{A}=\langle A, <\rangle$ satisfying these sentences:

$$(**) \qquad \begin{array}{c} \forall x \forall y [x<y \vee x \doteq y \vee y<x] \\ [[\forall x[\forall y[y<x \rightarrow \exists x[x \doteq y \wedge A]] \rightarrow A] \rightarrow \forall x A] \end{array}$$

where A is an arbitrary formula with x as the only free variable. A more readable version of the second type of formula would be

$$[\forall x[\forall y[y<x \rightarrow A(y)] \rightarrow A(x)] \rightarrow \forall x A(x)],$$

which is nothing more than the schema of transfinite induction. It is easy to argue, using a transfinite induction in the metalanguage, that if $\mathfrak{A}$ satisfies all the sentences (**), then $\mathfrak{A}$ satisfies all the sentences (*). Thus, among *countable* systems at least, the axioms (**) do characterize the well-orderings. Non-standard models of (**) must be uncountable.

We have just seen from the *semantical* view point that the sentences

(*) follow from the set of sentences (**). The deduction of (*) from (**) can also be done *syntactically*, using either logical axioms or Gentzen-style rules. Both versions will be described below.

To make the statement of the systems simpler, we employ a restricted number of logical primitives. In the first instance we shall use only $\rightarrow$, $\neg$, $\bigwedge$, $\forall$, and $\doteq$. Any number of finitary predicate or operation symbols can be used as non-logical constants. The logical axioms are given next.

LOGICAL AXIOMS FOR $\mathbf{L}_{\omega_1, \omega_0}$

$[A \rightarrow [B \rightarrow A]]$,

$[[A \rightarrow [B \rightarrow C]] \rightarrow [[A \rightarrow B] \rightarrow [A \rightarrow C]]]$,

$[[\neg B \rightarrow \neg A] \rightarrow [A \rightarrow B]]$,

$[\bigwedge_{\xi<\alpha} A_\xi \rightarrow A_\eta]$, where $\eta<\alpha<\omega_1$,

$[\forall v A(v) \rightarrow A(t)]$,

$t \doteq t$,

$[t=u \rightarrow [A(u) \rightarrow A(t)]]$,

where A, B, C are arbitrary formulas, v is a variable, t and u are terms, and where A(t) and A(u) denote proper substitutions of terms t and u for free occurrences of the variable v in A.

In addition to these axioms, the following rules are required:

INFERENCE RULES FOR $\mathbf{L}_{\omega_1, \omega_0}$

$$\frac{A,\ [A \rightarrow B]}{B},$$

$$\frac{[A \rightarrow B_0],\ [A \rightarrow B_1],\ \ldots,\ [A \rightarrow B_\xi],\ \ldots,\ \xi<\alpha<\omega_1}{[A \rightarrow \bigwedge_{\xi<\alpha} B_\xi]},$$

$$\frac{[A \rightarrow B(v')]}{[A \rightarrow \forall v B(v)]},$$

where in the last rule the variable v′ is not free in A and every occurrence of v′ in B is free for v in B.

We say that a sentence A *follows syntactically* or is *provable* from a set Φ of sentences if there is a proof of A from Φ; that is, a transfinite sequence of sentences $B_0, B_1, \ldots, B_\xi, \ldots, B_\beta$ where $\beta<\omega_1$ such that

$B_\beta = A$ and for each $\xi \leqslant \beta$, the formula B_ξ is either a logical axiom, a member of Φ, or follows from previous formulas in the sequence by one of three rules of inference. There is no need to consider proofs that are non-denumerably long, since the only infinitary rule of inference makes its conclusion from only denumerably many premises. One other restriction we shall make on the notion of a proof is that every formula B_ξ occurring in a proof has only *finitely* many free variables. This is reasonable because given a sentence A, *every subformula of* A *has only finitely many free variables.* This statement is proved directly by transfinite induction on the length of the formula A and is closely related to the fact that any quantifier prefix of a subformula can involve only finitely many variables. This fact will be important when we discuss the extension of the Löwenheim-Skolem theorem.

Further, we say that a sentence A *follows semantically* from a set Φ of sentences if in every system in which all sentences in Φ are true, the sentence A is also true; that is, every model of Φ is a model of A. We can now state the first main result.

THE COMPLETENESS THEOREM FOR L_{ω_1, ω_0}. *A sentence* A *follows syntactically from a countable set* Φ *of sentences if and only if* A *follows semantically from* Φ.

Taking Φ to be the empty set, we see that a sentence A is logically provable using the above axioms and rules if and only if it is universally valid. This is all in strict analogy with the case of L_{ω_0, ω_0}, except for the restriction on the countability of Φ. That this restriction cannot be removed was shown by Ryll-Nardzewski, who constructed a remarkable *complete* and *consistent* set Ψ of sentences which has no model (that is, for every sentence A, either A or $\neg$ A but not both follows syntactically from Ψ, but there is no system in which all sentences in Ψ are true.) Unfortunately, Ryll-Nardzewski's ingenious construction is too complicated to explain here. Instead, we will have to be satisfied by the less interesting example presented in the next paragraph.

We employ non-logical constants 0, $+$, $\prec$, $<$. The set Θ of sentences we want consists in the first place of the usual axioms for *ordered abelian groups* written in terms of 0, $+$, and $\prec$. To these we adjoin the *Archimedean axiom*, which is the sentence

$$\forall x[0 \prec x \rightarrow \forall y \bigvee_{n<\omega} [y \prec n \cdot x]],$$

where $n \cdot x = x + \ldots + x$ (n times). Next we bring in the relation $<$ with the adjunction of the following non-denumerable number of sentences:

$$\forall x[\mathrm{E}_\alpha(x) \wedge y < x \to y \prec x], \text{ and}$$

$$\exists x \mathrm{E}_\alpha(x), \text{ for each } \alpha < \omega_1,$$

and the usual axioms on $<$ for a simple ordering. Obviously any model of these sentences would have to be an Archimedean-ordered group with a subset ordered in type ω_1 by the group ordering. As is well known, this is impossible. On the other hand, every countable subset of Θ clearly has a model where the group is just the ordered group of rationals. Further, we see that Θ is not as satisfactory as Ryll-Nardzewski's set Ψ, because Θ has *no* complete and consistent extensions. This is established by noticing that from Θ we can prove for each $\alpha < \omega_1$ the sentence

$$\exists x[\mathrm{E}_\alpha(x) \wedge \forall y[\mathrm{E}_\alpha(y) \to x \doteq y]].$$

In a complete extension of Θ, these *unique* elements determined by the formulas E_α would have to generate an Archimedean-ordered group whose group table is completely determined syntactically by the sentences provable from the given extension. In other words, every complete extension would automatically produce in terms of the syntactically definable elements an Archimedean-ordered group of the kind we already know cannot exist.

The proof of the completeness theorem for $\mathbf{L}_{\omega_1, \omega_0}$ was first given by Karp [a]. Lopez-Escobar and the author have verified that there is a very straightforward adaptation of the standard Henkin-style completeness proof to $\mathbf{L}_{\omega_1, \omega_0}$. The success of the method depends heavily on the fact that the sentences of a given countable set have only countably many different subformulas each of which have only finitely many free variables. This same circumstance makes it at once evident that the next result is correct.

THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM FOR $\mathbf{L}_{\omega_1, \omega_0}$. *Every countable set of sentences which has a model of infinite cardinality has models of all smaller infinite cardinalities.*

The proof is immediate when one realizes that the subformulas require Skolem functions for their initial quantifiers, which are each functions of only a finite number of arguments. The corresponding upward theorem is not so obvious.

THE UPWARD LÖWENHEIM-SKOLEM THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *Every countable set of sentences which has a model of cardinality* $\beth_{\omega_1}$ *has models of all higher cardinalities.*

Here $\beth_{\omega_1}$ is the cardinality of the set V_{ω_1} of all sets of rank less than ω_1. The sequence of sets V_α is defined by transfinite recursion; indeed for any ordinal α,

$$V_\alpha = \bigcup_{\xi<\alpha} \mathrm{P}V_\xi,$$

where $\mathrm{P}A = \{X : X \subseteq \mathrm{A}\}$. In the upward theorem the cardinal number $\beth_{\omega_1}$ cannot be improved. This is easily demonstrated using the relational systems $\mathfrak{V}_\alpha = \langle V_\alpha, \in_\alpha \rangle$, where $\in_\alpha$ is the usual membership relation restricted to V_α. To this end consider formulas with the non-logical constant $\in$. Define by recursion the sequence V_α of formulas where

$$\mathrm{V}_\alpha = \bigvee_{\xi<\alpha} \forall y[y \in x \to \exists x[x \doteq y \wedge \mathrm{V}_\xi]].$$

Clearly in the system $\mathfrak{V}_\alpha$ the formula $\mathrm{V}_\beta(x)$, $\beta \leqslant \alpha$, defines the subset V_β of V_α. Now take the formula

$$[\forall x \forall y[\forall z[z \in x \leftrightarrow z \in y] \to x \doteq y] \wedge \forall x \bigvee_{\beta<\alpha} \mathrm{V}_\beta(x)].$$

One can verify by induction that every model of this sentence is isomorphically embeddable in the system $\mathfrak{V}_\alpha$, if $\alpha < \omega_1$. That is, no model of this sentence has cardinality greater than $\beth_\alpha = \overline{\overline{V}}_\alpha$. Since $\beth_{\omega_1} = \lim_{\alpha<\omega_1} \beth_\alpha$, the desired conclusion follows.

For the proof of the upward theorem one uses the result of Morley [*] which shows that the same result with the same cardinal $\beth_{\omega_1}$ holds for ω-logic. By ω-logic we understand the two-sorted predicate logic where variables of one of the sorts in all interpretations run over the set of integers. One can have any type of predicate symbol with mixed arguments and one must have in addition individual constants for each integer. Then given a countable set Φ of sentences of $\mathbf{L}_{\omega_1,\omega_0}$, one constructs a corresponding set $\Phi^{(\omega)}$ of sentences of ω-logic, which uses the same predicate symbols on the non-integer sort of variables that Φ uses, but which in addition has additional mixed predicates. The sentences in $\Phi^{(\omega)}$ can be so arranged that an $\mathbf{L}_{\omega_1,\omega_0}$-model of Φ gives an ω-model of $\Phi^{(\omega)}$ using the same elements and relations but has suitable mixed relations adjoined; and, conversely, every ω-model of $\Phi^{(\omega)}$ gives an

$\mathbf{L}_{\omega_1,\omega_0}$-model of Φ by removing the mixed relations. The details of the construction need not trouble us here.

Returning to syntactical questions, we can also give a Gentzen-style formulation of logical derivability for $\mathbf{L}_{\omega_1,\omega_0}$. In the following I shall use sets rather than sequents and shall write $\Phi \vdash \Psi$ to mean, roughly, that the conjunction of the formulas in Φ implies the disjunction of the formulas in Ψ. In fact let us write $\Phi \models \Psi$ for the semantical notion meaning every model of Φ is a model of at least one sentence of Ψ. The relation $\models$ will be used only between countable sets of sentences, while $\vdash$ can stand between countable sets of *formulas*. The relation $\vdash$ is precisely described as the least relation satisfying the particular rules given below in which we write $\Phi, A_0, A_1, \ldots, A_\xi, \ldots, \xi<\alpha \vdash \Psi$ short for $\Phi \cup \{A_\xi : \xi<\alpha\} \vdash \Psi$. To simplify things we take $\neg$, $\bigwedge$, $\forall$, and $=$ as the only logical constants.

CUT-FREE RULES FOR $\mathbf{L}_{\omega_1,\omega_0}$.

$$\Phi, A \vdash \Psi, A.$$

$$\frac{\Phi, A \vdash \Psi}{\Phi \vdash \Psi, \neg A}. \qquad \frac{\Phi \vdash \Psi, A}{\Phi, \neg A \vdash \Psi}.$$

$$\frac{\Phi, A_0, A_1, \ldots, A_\xi, \ldots, \xi<\alpha \vdash \Psi}{\Phi, \bigwedge\limits_{\xi<\alpha} A_\xi \vdash \Psi}.$$

$$\frac{\Phi \vdash \Psi, A_0 \quad \Phi \vdash \Psi, A_1 \quad \ldots \quad \Phi \vdash \Psi, A_\xi \quad \ldots \quad \xi<\alpha}{\Phi \vdash \Psi, \bigwedge\limits_{\xi<\alpha} A_\xi}.$$

$$\frac{\Phi, A(t) \vdash \Psi}{\Phi, \forall v A(v) \vdash \Psi}. \qquad \frac{\Phi \vdash \Psi, A(v')}{\Phi \vdash \Psi, \forall v A(v)}.$$

$$\Phi \vdash \Psi, t = t.$$

$$\frac{\Phi, A(t) \vdash \Psi, B(t)}{\Phi, t = u, A(u) \vdash \Psi, B(u)} \qquad \frac{\Phi, A(t) \vdash \Psi, B(t)}{\Phi, u = t, A(u) \vdash \Psi, B(u)}$$

where in the rule for the introduction of $\forall$ on the right we must obey the usual restriction on free variables. In addition, we can assume in any step in a deduction of a $\vdash$-relationship that the total number of free variables occurring in the formulas in the sets on both sides of the $\vdash$ is finite.

THE CUT ELIMINATION THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *If Φ and Ψ are countable sets of sentences of $\mathbf{L}_{\omega_1,\omega_0}$, then $\Phi \vdash \Psi$ can be established by the cut-free rules if and only if $\Phi \models \Psi$ holds.*

As a consequence of the elimination theorem we have:

THE INTERPOLATION THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *If Φ and Ψ are countable sets of sentences of $\mathbf{L}_{\omega_1,\omega_0}$ and if $\Phi \vdash \Psi$, then there is a sentence* A *involving as non-logical constants only those occurring in at least one sentence of Φ and at least one sentence of Ψ such that $\Phi \vdash \mathrm{A}$ and $\mathrm{A} \vdash \Psi$.*

These last two results were established by Lopez-Escobar and will appear in his dissertation. Earlier in [61] and [a] Maehara and Takeuti gave cut-elimination theorems for logics $\mathbf{L}_{\theta,\theta}$, where θ is strongly inaccessible, and these proofs encouraged Lopez-Escobar to see if the standard type of argument could be carried over to $\mathbf{L}_{\omega_1,\omega_0}$. Also it must be mentioned that Engeler in [63] gave a Schütte-style formulation of $\mathbf{L}_{\omega_1,\omega_0}$, but in that system formulas with *infinitely many* free variables seem to enter in an essential way. In particular, the author does not see whether Engeler's result can be applied to the proof of the interpolation theorem. Lopez-Escobar feels that the use of only finitely many free variables at each stage of a deduction is quite important in the proof of the interpolation theorem using the method of the cut-elimination theorem. This question, of course, never comes up in $\mathbf{L}_{\omega_0,\omega_0}$.

Once the interpolation theorem is established, then Beth's definability theorem is deduced as usual. Lopez-Escobar has also verified that the more refined results for $\mathbf{L}_{\omega_0,\omega_0}$ about positive and negative sentences also carry over to $\mathbf{L}_{\omega_1,\omega_0}$.

The next step in seeing how results for $\mathbf{L}_{\omega_0,\omega_0}$ generalize to $\mathbf{L}_{\omega_1,\omega_0}$ would be to turn to the initial results of model theory. Some difficulties are at once encountered, however. In the first place, what should one mean by a *universal class* for $\mathbf{L}_{\omega_1,\omega_0}$? The most obvious answer seems to be this: Consider the class of formulae built up from quantifier-free formula by applying the operators $\forall$, $\bigwedge$, $\bigvee$ in any order. As an interesting example (suggested by Lopez-Escobar) take as non-logical constants the individual constant 0 and the singulary function symbol P. The models we have in mind are trees with finitary branching and with paths of length $\leqslant \omega$. These systems can be characterized by the following three axioms, where 0 is interpreted as the *root* of the tree and P(x) as the

immediately preceding node to x. (Draw a picture, and all will be clear.) We take

$$\forall x[x \eqsim 0 \leftrightarrow P(x) \eqsim x],$$

$$\forall x \bigvee_{n<\omega} P^n(x) \eqsim P^{n+1}(x),$$

where $P^n(x) = P(P(\ldots P(x)\ldots))$ (n-times), and finally

$$\bigwedge_{n<\omega} \bigvee_{m<\omega} \forall x_0 \ldots \forall x_m [\bigwedge_{k\leqslant m} P^n(x_k) \eqsim P^{n+1}(x_k) \rightarrow \bigvee_{i<j\leqslant m} x_i \eqsim x_j].$$

Note that this last sentence is our first example of a sentence with an unbounded number of variables. It cannot be simplified directly since we have *no prenex normal form* in $\mathbf{L}_{\omega_1,\omega_0}$. Indeed in view of the results in Tarski [58a] there is no set of universal sentences with a bounded number of variables which are equivalent to the above three sentences. This is so because it is trivial to give a structure $\mathfrak{A} = \langle A, 0, P\rangle$ which is not a model of our three sentences, but every finitely generated subsystem is.

Now the difficulty encountered with universal sentences is this: Suppose a class of systems $\mathscr{K}$ can be characterized by a single sentence of $\mathbf{L}_{\omega_1,\omega_0}$, and suppose $\mathscr{K}$ is closed under the formation of subsystems; then can $\mathscr{K}$ be characterized by a single universal sentence (or even a set of such sentences)? This question has not yet been resolved. The initial stumbling block goes back to the failure of the compactness theorem for $\mathbf{L}_{\omega_1,\omega_0}$, since as we have seen there are uncountable consistent sets of sentences without models. At the present time other topics of model theory seem to share the same fate.

Let us now turn to a general discussion of countable systems in relation to $\mathbf{L}_{\omega_1,\omega_0}$. The author has established the next two results which generalize the corresponding well-known facts about finite systems and $\mathbf{L}_{\omega_0,\omega_0}$. The relational systems considered are to have only a countable number of finitary relations or operations.

THE COUNTABLE ISOMORPHISM THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *Two countable systems satisfying the same sentences of* $\mathbf{L}_{\omega_1,\omega_0}$ *are isomorphic; indeed, for a given countable system* $\mathfrak{A}$ *one can obtain a single sentence* A *true of* $\mathfrak{A}$ *such that all countable systems satisfying* A *are isomorphic to* $\mathfrak{A}$.

THE COUNTABLE DEFINABILITY THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *For a finitary relation* S *on the domain of* $\mathfrak{A}$ *to be* $\mathbf{L}_{\omega_1,\omega_0}$ *definable in the countable*

system $\mathfrak{A}$ it is necessary and sufficient that every automorphism of $\mathfrak{A}$ be an automorphism of S.

The proofs of these results will be given elsewhere. The isomorphism theorem has an interesting application to the theory of Borel sets. Let N be a denumerable set and consider the product space $2^{N\times N}$. With the ordinary product topology this space is homeomorphic to Cantor's discontinuum. From our point of view, the space can be considered as the space of all binary relations on the set N. Now every permutation of N induces a permutation of $2^{N\times N}$. The subsets of $2^{N\times N}$ invariant under these permutations are called *invariant sets*. The invariant sets form a Boolean algebra whose *atoms*, the *minimal invariant sets*, are just the *isomorphism types* of binary relations on N. The isomorphism theorem at once implies that *the minimal invariant sets are Borel sets.* This is clear because the set of all relations in $2^{N\times N}$ satisfying a given sentence A of $\mathbf{L}_{\omega_1,\omega_0}$ is obviously a Borel subset of $2^{N\times N}$.

Furthermore, in view of the interpolation theorem, as was pointed out by Ryll-Nardzewski to the author, the invariant Borel subsets of $2^{N\times N}$ are *exactly* the subsets of $2^{N\times N}$ that can be characterized by single sentences of $\mathbf{L}_{\omega_1,\omega_0}$. Indeed, by virtue of the downward Löwenheim-Skolem theorem, two sentences of $\mathbf{L}_{\omega_1,\omega_0}$ determine the same subset of $2^{N\times N}$ if and only if they are logically equivalent. This remark has a pleasant application: The family of invariant Borel subsets of $2^{N\times N}$ is a sub-σ-field of the σ-field of all Borel sets. Assertion: there is a countably additive $\{0,1\}$-valued measure on this sub-σ-field that *cannot* be extended to a Borel measure. Proof: consider those sentences of $\mathbf{L}_{\omega_1,\omega_0}$ that are true of the well-ordering of type ω_1. This determines a $\{0,1\}$-valued measure on the invariant Borel sets in the obvious way. By making use of our ω_1-sequence of formulas $\mathbf{E}_\alpha$ it is easily checked that this measure cannot be extended to a countably additive, real-valued measure on all the Borel sets.

As another application of the isomorphism theorem the author obtained the solution to a problem of Kuratowski [48, p. 377]. The problem amounts to this: Let Q be the set of rationals and consider the product space 2^Q. Again this is homeomorphic to Cantor's discontinuum, but now we interpret its points as being sets of rational numbers. Two sets of rationals are called *isomorphic* if there is a one-one order-preserving mapping of one set onto the other (this mapping is *not* defined on the complements of the sets!). Kuratowski's question: Is the subset of 2^Q

of all sets of rationals isomorphic to a given set always a Borel subset of $2^{\aleph_0}$? For well-ordered sets of rationals the affirmative answer is given by Kuratowski [48, § 26, XII, 1]. But from our general isomorphism theorem the complete affirmative answer is readily derived. Some of the details of these deductions can be found in Scott [64].

There is also another aspect of the isomorphism theorem that may prove to be of interest. This involves the characterization of elementary equivalence of Fraïssé [54c] and Ehrenfeucht [61]. I will give a somewhat different description which was suggested to me by some recent work of Hanf. If $\mathfrak{A}$ is a relational system with universe A (and, say, a finite number of finitary relations), and if $a \in A^m$ where $m < \omega$, then we let $\mathfrak{A}[a]$ denote the relational system induced from $\mathfrak{A}$ on the set $m = \{0, 1, \ldots, m-1\}$ by the function a; that is, the relations of $\mathfrak{A}[a]$ are the inverse images of the relations of $\mathfrak{A}$ under the function a. With this notation we now define $\tau_\alpha(\mathfrak{A}, a)$ by transfinite recursion on α for all systems $\mathfrak{A}$, all $m < \omega$, all $a \in A^m$:

$$\tau_\alpha(\mathfrak{A}, a) = \begin{cases} \mathfrak{A}[a], & \text{if } \alpha = 0; \\ \{\tau_\beta(\mathfrak{A}, a^\frown\langle x\rangle) : x \in A\}, & \text{if } \alpha = \beta + 1; \\ \{\tau_\xi(\mathfrak{A}, a) : \xi < \alpha\}, & \text{if } \alpha \text{ is a limit } > 0. \end{cases}$$

In addition we let $\tau_\alpha(\mathfrak{A}) = \tau_\alpha(\mathfrak{A}, 0)$ where $0 \in A^0$ is the empty sequence. By the way, if $a \in A^m$ and $x \in A$, then $a^\frown\langle x\rangle$ is that sequence $b \in A^{m+1}$, where $b_m = x$ and $b_i = a_i$ for $i < m$.

From the results of Fraïssé and Ehrenfeucht it can be shown that $\tau_{\omega_0}(\mathfrak{A}) = \tau_{\omega_0}(\mathfrak{B})$ if and only if the two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent in $\mathbf{L}_{\omega_0, \omega_0}$. From the results of Karp [*] we see that $\tau_{\omega_1}(\mathfrak{A}) = \tau_{\omega_1}(\mathfrak{B})$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences with *arbitrary* conjunctions and disjunctions, finitary strings of quantifiers, but with quantifier depth strictly less than ω_1. (See Karp [*] for a fuller explanation.) From the isomorphism theorem we see for countable systems that $\tau_{\omega_1}(\mathfrak{A}) = \tau_{\omega_1}(\mathfrak{B})$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. That is, for countable A the system of sets $\tau_\xi(\mathfrak{A})$, $\xi < \omega_1$, is a complete system of isomorphism invariants. Whether this remark has any useful applications remains to be seen. At least it is interesting that the result can be formulated without reference to the formal languages in a purely set-theoretical way.

In conclusion, then, it seems fair to say that $\mathbf{L}_{\omega_1, \omega_0}$ is a useful infinitary logic, and probably many other applications are waiting to

be found. Further, the author feels he has fully justified the contention that $\mathbf{L}_{\omega_1,\omega_0}$ is the proper generalization of $\mathbf{L}_{\omega_0,\omega_0}$ to denumerable formulas. In fact we have seen several reasons for claiming that $\mathbf{L}_{\omega_1,\omega_0}$ plays the same role for $\mathbf{L}_{\omega_0,\omega_0}$ that the theory of Borel sets and σ-fields plays for the ordinary fields of sets.

Note added in proof:

Recently, Jerome Malitz has obtained the following results which show that the analogy between the languages $\mathbf{L}_{\alpha,\beta}$ for $\alpha > \omega_0$, $\beta \geqslant \omega_0$, and $\mathbf{L}_{\omega_0,\omega_0}$ is not as strong as might be hoped. For example, Beth's theorem (and hence Craig's interpolation theorem) fails for all languages $\mathbf{L}_{\alpha,\beta}$ whenever $\alpha \geqslant \beta \geqslant \omega_1$. For $\mathbf{L}_{\varkappa,\varkappa}$ where $\varkappa$ is strongly inaccessible this contradicts the interpolation theorem in Maehara–Takeuti [61]. In fact, there is a sentence of $\mathbf{L}_{\omega_1,\omega_1}$ which implicitly defines a relation that is not explicitly definable in any language $\mathbf{L}_{\alpha,\beta}$. In addition, one can show that Craig's theorem fails for $\mathbf{L}_{\alpha,\omega_0}$ whenever $\alpha > \omega_1$; a valid sentence $\sigma \to \varrho$ of $\mathbf{L}_{\omega_2,\omega_0}$ exists such that no sentence γ of any language $\mathbf{L}_{\alpha,\omega_0}$ has the property that both $\sigma \to \gamma$ and $\gamma \to \varrho$ are valid. This answers a question posed in Lopez-Escobar's dissertation. Another problem which Lopez–Escobar raised is that of the preservation of $\mathbf{L}_{\omega_1,\omega_0}$ equivalence under cardinal sums; i.e., if $\mathfrak{A}$ is equivalent to $\mathfrak{A}'$ in $\mathbf{L}_{\omega_1,\omega_0}$ does it follow that the cardinal sum of $\mathfrak{A}$ and $\mathfrak{L}$ is $\mathbf{L}_{\omega_1,\omega_0}$ equivalent to the cardinal sum of $\mathfrak{A}'$ and $\mathfrak{L}$ for all $\mathfrak{L}$. Again the answer is no, and in fact $\mathbf{L}_{\alpha,\beta}$ equivalence is preserved under cardinal sums if and only if α is strongly inaccessible.

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NON-STANDARD MODELS FOR FRAGMENTS OF NUMBER THEORY

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Summary. The fragments considered are first-order theories whose (non-logical) constants are among 0, ′ (successor), P (predecessor), +, ·, ∸, $[x/n]$ $(n=2, 3, \ldots)$; whose axioms are the primitive recursive defining equations for the chosen functions and whose sole (non-logical) rule of inference is the rule of induction for open formulae. The main aims are: (1) to find equivalent systems without a rule of induction, i.e., to see what is the algebraic effect of the open rule of induction, (2) to survey the non-standard models, and (3) to use these models to give independence proofs, in particular to see how much weaker the given rule of induction is than the full rule of induction.

1. Introduction. Skolem, over forty years ago, directed attention to free-variable systems consisting of the defining equations for certain primitive recursive functions together with the rule of induction, e.g.,

A1. $x' \neq 0$.

A2. $Px=0$.

A3. $Px'=x$.

A4. $x+0=x$.

A5. $x+y'=(x+y)'$.

A6. $x\cdot 0=0$.

A7. $x\cdot y'=x\cdot y+x$.

A8. $x \dot{-} 0=x$.

A9. $x \dot{-} y'=P(x \dot{-} y)$.

$A10_n$. $[0/n]=0 \quad (n=2, 3, \ldots)$.

$A11_n$. $n([x/n]+1) = x+1 \rightarrow [(x+1)/n]=[x/n]+1 \cdot \wedge \cdot$
$n([x/n]+1) \neq x+1 \rightarrow [(x+1)/n]=[x/n] \quad (n=2, 3, \ldots)$.

Rule: $A(0),\ A(x) \rightarrow A(x') \vdash A(x)$.

He has repeatedly suggested that they represent an important form of constructive, finitist number theory. Herbrand and Hilbert, and more recently Kreisel, have also studied them extensively. In particular Kreisel has suggested they offer an explication of the notion of direct proof, viz., one which can be carried out in a free-variable system without introducing additional functions. He points out that this is one way (and, as far as I know, the only plausible way) of distinguishing the Selberg-Erdös elementary proof of the prime number theorem from the classical proof. This explication of direct proof has some fairly evident weaknesses but perhaps these are inherent in the notion itself, which may be more useful as a heuristic guiding principle than as a precisely defined concept. It led Kreisel to ask in [54] whether the usual proof involving Legendre symbols of

$$x^3+y^3 \neq z^3 \vee xyz=0$$

could be replaced by a direct proof, i.e., a proof in the free-variable system with axioms A1–7. Skolem in [55] asked a similar question—can the irrationality of $\sqrt{2}$, i.e.,

$$x^2 \neq 2y^2 \vee x=0,$$

be proved in the free-variable system based on $0, +, \cdot, <$? Scott [61a] asked the same question about $x^2+x+1 \neq 2y$. The general program of considering free-variable systems based on successively more and more of the usual arithmetic functions was also initiated by Kreisel. Shoenfield [58a] continued it, showing that in the free-variable system based on $0, ', \mathsf{P}, +$, the rule of induction was equivalent to four simple axioms. Here we make a systematic survey of free-variable systems starting with one with $0, '$ only and ending with $0, ', \mathsf{P}, +, \cdot$. We also describe a non-standard model for the system based on all the functions mentioned above, which enables us to answer many independence questions, including the three just mentioned.

Although our main interest is in free-variable systems we will actually work throughout in the full first-order predicate calculus with identity. This will allow us to consider the relative strengths of the free-variable rule of induction and various quantified axioms. The completeness theorem for the free-variable calculus tells us that the theorems of the free-variable system based on certain (quantifier-free) axioms and the rule of induction above coincide with the open (i.e., quantifier-free)

theorems of the first-order system with the same axioms and the restricted rule of induction:

RIO. $A(0),\ A(x) \to A(x') \vdash A(x)$, for all *open* A.

2. Rules and axioms of induction. We first consider to what extent this rule is replaceable by axioms. This is desirable because from the model-theoretic point of view a rule is rather difficult to handle. Thus to say that a first-order theory T (based on certain axioms, and having no (non-logical) rules) is closed under RIO, i.e., that

$$\vdash_T A(0),\ \vdash_T A(x) \to A(x') \text{ imply } \vdash_T A(x)$$

means that if $A(0)$, $A(x) \to A(x')$ are valid in *all* models of T then $A(x)$ is valid in all models of T. Thus we cannot consider each model of T separately. To illustrate the sort of argument used let us solve the problem of finding an axiom system equivalent to the rule RIO in the simplest case of all:

2.0. Definition of first-order theory T_0. Constants: 0, '. Axioms: none. Rules: RIO.

We shall show that T_0 is equivalent to, i.e., has the same class of theorems as, the following theory $\overline{T_0}$.

2.1. Definition of $\overline{T_0}$. Constants: 0, '. Axioms: B_{rs} (see below) for all natural numbers r, s with $r \leqslant s$. Rules: none.

B_{rs}. $$0^{(r)} = 0^{(s+1)} \to x = 0 \vee \ldots \vee x = 0^{(s)}.$$

(Here and in future the $^{(r)}$ stands for $' \ldots '$ (r times).) It is easy to prove that all the B_{rs}, and hence all theorems of $\overline{T_0}$, are theorems of T_0. For the converse we must show that if A is any open formula and $A(0)$, $A(x) \to A(x')$ are valid in all models of T_0 then so is $A(x)$. To say that $A(x)$ is valid in all models of $\overline{T_0}$ means that if M is any model of $\overline{T_0}$ and a any element of M and $A_1(x)$ any formula obtained from $A(x)$ by replacing the free variables other than x (if any) by names for elements of M, then $A_1(a)$ is true in M. If a is a 'natural' element of M, i.e., of the form $0^{(n)}$ this is easily seen to be true, for $A(0)$, $A(x) \to A(x')$ are supposed to be valid in M and the truth of $A_1(0^{(n)})$ in M follows from this. So we may suppose a to be unnatural and since M satisfies the B_{rs} this implies that all of $0, 0', 0'', \ldots$ are distinct in M. Now the non-

trivial atomic formulae of A_1 are of the forms $x^{(p)} = x^{(q)}$ (where $p \neq q$ and $x^{(0)}$ stands for x), $x^{(p)} = b$ where b is an element of M. When x is assigned $0^{(n)}$ these are all F (false) for sufficiently large n. Now form a new model M_1 by adding $m-1$ new elements $c, c', c'', \ldots, c^{(m-1)}$ and putting $c^{(m)} = a$. If m is chosen sufficiently large all atomic formulae of A_1 will be F also if x is assigned c. Thus for sufficiently large n, $A_1(c)$ has in M_1 the same truth value as $A_1(0^{(n)})$, i.e., T. But $A(x) \to A(x')$ is supposed valid in all models so $A_1(x) \to A_1(x')$ is valid in M_1. Hence in M_1 $A_1(c)$ and $A_1(c) \to A_1(c') \to \ldots \to A_1(c^{(m)})$ are T, i.e., $A_1(a)$ is T in M_1 and hence also in M since M_1 is an extension of M. This completes the proof of equivalence of $T_0, \overline{T_0}$.

Let us return now to the question of whether we can replace RIO by axioms and so eliminate the need to consider more than one model at once. The obvious axiom of induction to consider is

AI. $$A(0) \wedge (x)(A(x) \to A(x')) \to A(x).$$

AI certainly implies the rule of induction for the same formula A (i.e., in any theory satisfying this axiom the corresponding rule is a valid derived rule). Conversely it is well known that AI can be proved by the rule of induction. However in order to do this you have to apply the rule of induction not to the formula A but to the formula AI, which contains quantifiers, and it is well known (Rabin [61]) that the introduction of more and more quantifier alternations in AI leads to stronger and stronger systems. It is not true that, for fixed A, AI follows from the rule of induction, nor even that AIO (i.e., all instances of AI for all open A) follow from RIO. E.g., AI applied to the case where A(x) is $x \neq x'$ is

$$0 \neq 0' \wedge (x)\,(x \neq x' \to x' \neq x'') \to (x)\,(x \neq x'),$$

which is not provable from RIO; for if it were it would be a theorem of T_0, and hence of $\overline{T_0}$, whereas the model of $\overline{T_0}$ with $0, 0', 0'', \ldots$ (all distinct), $\omega' = \omega$ does not satisfy it. However

Theorem 2.2. *The set of open sentences provable from open axioms by* AIO (*the axiom of induction for open formulae*) *coincides with that provable by* RIO (*the rule of induction for open formulae*).

We shall obtain this from

Lemma 2.3. *The open consequences of any set of axioms of the form* A_1: $(x)B(x) \to C$ *coincide with those of the corresponding rules*

R_1. $\qquad D \to B(x) \vdash D \to C \qquad (x$ *not free in* $D)$.

Here B, C, D are supposed to be open formulae, possibly containing other free variables. The notation $(x)B(x)$ is also intended to cover the case $(x_1, \ldots, x_n)\, B(x_1, \ldots x_n)$ of several variables.

This in turn will be obtained from

Lemma 2.4. *The open consequences of any set of axioms of the form* $(Ex)A(x)$ *coincide with those of the corresponding rules*

R_2. $\qquad A(x) \to D \vdash D \qquad (x$ *not free in* $D)$.

Here too A, D are supposed to be open and the x stands for any sequence of variables, so the theorem applies to any set of AE axioms.

Proof of 2.2 from 2.3. Take $B(x)$ to be $A(x) \to A(x')$, C to be $A(0) \to A(x)$. Then A_1 is equivalent to AI and R_1 is

$$D \to (A(x) \to A(x')) \vdash D \to (A(0) \to A(x)),$$

which is derivable from RI applied to $D \to (A(0) \to A(x))$; on the other hand RI is derivable from this R_1 by putting $D = A(0)$. Thus, for a fixed open A, AI is equivalent, as far as open consequences are concerned, to RI applied to all open formulae of the form $E \to A(x)$, x not free in E.

Proof of 2.3 from 2.4. Changing the variables of C if necessary so that x is not free in C we see that A_1 is equivalent to $(Ex)\,(B(x) \to C)$. By 2.4 this is equivalent for open consequences to the rule

R_2. $\qquad (B(x) \to C) \to D \vdash D \qquad (x$ *not free in* $D)$.

R_1 for D can be obtained by applying R_2 to $D \to C$ (followed if necessary by a change back to the original variables of C) and R_2 for D by applying R_1 to $\neg D$.

Proof of 2.4. Clearly *all* (not necessarily open) consequences of the rule are consequences of the axiom. For the converse we consider first the case of a single axiom $(Ex)A(x, y)$ where y is supposed to stand for the sequence $y_1, \ldots, y_n$ of all the other free variables of A. Let B be an open consequence of this; we may clearly suppose neither x nor y free

in B. Then, denoting by T the theory based on the remaining (open) axioms, we have

$$\vdash_T (y)(Ex)A(x, y) \to B, \text{ i.e., } \vdash_T (Ey)(x)(A(x, y) \to B).$$

So by Herbrand's theorem,

$$\vdash_T [A(x, t_0) \to B] \vee [A(x_1, t_1(x)) \to B] \vee \ldots \vee [A(x_k, t_k(x, \ldots, x_{k-1})) \to B]$$

for some terms $t_0, t_1, \ldots, t_k$ which in addition to the indicated variables may contain variables free in B, but are such that t_i does not contain any x_p with $p \geqslant i$, $x_1, \ldots, x_k$ being distinct new variables not free in B. Hence

$$\vdash_T A(x_k, t_k) \to [A(x_{k-1}, t_{k-1}) \to [\ldots \to [A(x, t_0) \to B]\ldots]].$$

From this B can be derived by $k+1$ changes of variable and applications of the rule:

R_3. $\qquad A(x, t) \to D \vdash D \qquad (x$ *not free in* t *or* D),

which can be derived from R_2 by taking the D of R_2 to be $y = t \to D$ (after substituting some variable different from x, y for y if it occurs free in D).

This completes the proof for the case of a single axiom. In the general case (which is what we use when we consider the replacement by rules of the instances of the axiom of induction for *all* open A) we note that if B is derivable from a set of axioms of the given form it is derivable from a finite subset, say

$$(y_1)(Ex_1)A_1(x_1, y_1), \ldots, (y_n)(Ex_n)A_n(x_n, y_n)$$

(where $x_1, \ldots, x_n, y_1 \ldots y_n$ are distinct) whose conjunction is equivalent to

$$(y_1, \ldots, y_n)(Ex_1, \ldots, x_n)(A_1(x_1, y_1) \wedge \ldots \wedge A_n(x_n, y_n)).$$

Applying the above result to this axiom we see that if B is open it is derivable by the rule

$$A_1(x_1, y_1) \wedge \ldots \wedge A_n(x_n, y_n) \to D \vdash D \qquad (x_1, \ldots, x_n \textit{ not free in } D).$$

Writing this in the form

$$A_1(x_1, y_1) \to [A_2(x_2, y_2) \to \ldots [A_n(x_n, y_n) \to D]\ldots] \vdash D$$

we see that it is derivable from the cases of R_2 corresponding to the separate axioms $(y_i)(Ex_i)A_i(x_i, y_i)$.

Notes. (1) Although by Herbrand's theorem one can always give a set of free-variable rules which have the same open consequences as a given set of axioms containing quantifiers, these rules are in general very complicated, and the rules corresponding to the conjunction of two axioms are not obtained simply by conjoining the two rules. The AE case is the last case where the rules are simple and have this property. For a *single* formula of EA form, $(Ex)(y)A(x, y)$, the open consequences can be obtained from the rule

$$A(x, t_1) \wedge \ldots \wedge A(x, t_k) \rightarrow D \vdash D \qquad (x \text{ not free in } D),$$

but the open consequences of two or more formulae of this form cannot be obtained from the separate rules.

(2) Theorem 2.2 appears more surprising in the case where A contains other free variables. Thus one might expect the rule

$$A(0, z),\ A(x, z) \rightarrow A(x', z) \vdash A(x, z)$$

to be equivalent in some sense (which it is – see remarks after Theorem 3.4 below) to the axiom

A'I. $$(z)A(0, z) \wedge (z)(x)(A(x, z) \rightarrow A(x', z)) \rightarrow A(x, z),$$

but we assert equivalence (for open consequences), to the stronger

AI. $$A(0, z) \wedge (x)(A(x, z) \rightarrow A(x', z)) \rightarrow A(x, z).$$

From the point of view of models, if $A(0, c)$ is T and $A(a, c)$ is F then AI says there exists b such that $A(b, c)$ is T and $A(b', c)$ is F. A'I merely says there exist b, c_1 such that $A(0, c_1)$ is F or $A(b, c_1)$ is T and $A(b', c_1)$ is F. Thus in testing AI for validity not only can we confine ourselves to a single model at a time but we can replace all free variables of A by names for elements of the model and test the resulting formula $A_1(x)$ with just one free variable for the property that if $A_1(0)$ is T and $A_1(a)$ is F there exists a b such that $A_1(b)$ is T and $A_1(b')$ is F.

In terms of models, Theorem 2.2 says that *every model for the set of open consequences of* RIO *can be embedded in a model for* AIO. And although to be a model for AIO is the stronger property it is easier to test for and understand mathematically, so the easiest way to prove independence of an open formula from RIO is to construct a non-standard model for AIO in which it is false; Theorem 2.2 shows this is always possible.

(3) As we saw above AIO itself is not usually provable from RIO, i.e., it is not usually true that all models for the open consequences of RIO actually satisfy AIO. A straightforward application of Herbrand's theorem shows that AI for an open $A(x)$ is provable from open axioms iff there is a finite set of terms $t_1, \ldots, t_k$ such that

$$A(0) \wedge \neg A(x) \rightarrow \bigvee_{i=1}^{k} A(t_i) \wedge \neg A(t_i')$$

is provable. If there are no functions t_i, obtainable by substitution from the given basic functions, for which this is true in the natural model, then AI for this A cannot be proved from RIO plus any set of true open axioms not involving new constants. If there are for each open A then all AIO are provable from RIO plus a suitably chosen set of true open axioms (e.g., the true Herbrand disjunctions themselves). One would expect that starting with a given set F of functions one would have to enlarge this to F' to include such 'Herbrand' terms for the AIO applied to formulae built up from the F, then to F'' to get Herbrand terms for formulae built up from F' and so on. One would not expect small natural classes F to be closed under this procedure (although sufficiently large classes like the class of primitive or general recursive functions clearly are) but it turns out that this does happen in some of the cases we consider and AIO is actually provable from RIO using only the defining equations of the basic functions.

3. Induction-free equivalents for systems up to and including multiplication.

Theorem 3.1. *The theory* T_1 *with constants* $0, '$, *axiom* A1: $x' \neq 0$, *and rule* RIO *is equivalent to (i.e., has the same theorems as) the theory* $\overline{T_1}$ *obtained from it by dropping* RIO *and adding axioms* B_{rs} *above.*

This is proved by exactly the same argument as we used for T_0. Axioms B_{0s} are superfluous in $\overline{T_1}$ but neither $\overline{T_0}, \overline{T_1}$ are finitely axiomatizable for the structure with $0, 0', \ldots, 0^{(n+1)} = 0^{(n)}$ and an element $\omega = \omega'$ satisfies A1 and all B_{rs} with $r < n$ but not B_{nn}. The example given above (AI applied to $x \neq x'$) shows that AIO is not provable in T_0 or T_1.

Theorem 3.2. *The theory* T_2 *obtained by adding* A3a: $x' = y' \rightarrow x = y$ *to* T_1 *is equivalent to the theory* T_2 *obtained from it by dropping* RIO *and adding axioms* B_n: $x \neq x^{(n)}$ $(n = 1, 2, \ldots)$.

Theorem 3.3. *The theory T_3 with constants* $0, ', \mathsf{P}$, *axioms* A1–3 *and rule* RIO *is equivalent to the theory* $\overline{T_3}$ *obtained from it by dropping* RIO *and adding axioms* B_n $(n=1, 2, \ldots)$ *and* B1: $x \neq 0 \rightarrow (\mathsf{P}x)' = x$.

The proof of Theorem 3.3 is essentially contained in Herbrand's thesis [30] (he omitted P, had $x \neq 0 \rightarrow (\mathsf{E}y)(y' = x)$ instead of B1, and had A3a instead of A2, A3). He showed by elimination of quantifiers that *all true sentences* (i.e., *all sentences true in the natural model*) *are provable* in T_3, $\overline{T_3}$, so not only AIO but all AI are provable. We need to supplement his proof by showing how to eliminate P; this is done by a succession of replacements of $t_1 = t_2$ by $t_1' = t_2'$ (using A3) followed by replacement of $(\mathsf{P}x)' = t$ by $(x = 0 \wedge t = 0') \vee (x \neq 0 \wedge t = x)$ (using A2, B1). $\overline{T_3}$ is clearly not finitely axiomatizable but is decidable; its models consist of a natural part $N = \{0, 0', 0'', \ldots\}$ together with any number of copies of I (integers under $x' = x + 1$). The models of $\overline{T_2}$ differ in that they may also contain any number of copies of N. By extending these to copies of I we can embed any model for $\overline{T_2}$ in one for $\overline{T_3}$. Since all true sentences are provable in $\overline{T_3}$ it follows that all true open formulae are provable in $\overline{T_2}$; this implies that $\overline{T_2}$ is closed under RIO, which, together with the easy proof that all axioms of $\overline{T_2}$ are theorems of T_2, proves Theorem 3.2. In fact all true EA sentences are provable in $\overline{T_2}$, for if $(\mathsf{E}x)(y)\mathrm{A}(x, y)$ is true then $(y)\mathrm{A}(0^{(n)}, y)$ is true for some n (and hence, as above, provable) and it implies $(\mathsf{E}x)(y)\mathrm{A}(x, y)$. This tells us that all A'IO (see Note (2) of Section 2 above) are provable in $\overline{T_2}$. So also are all cases of AIO applied to formulae $\mathrm{A}(x)$ containing no free variables other than x, for AIO coincides with A'IO for such an $\mathrm{A}(x)$. However, if we take the formula $x \neq z$ for $\mathrm{A}(x)$ we get an instance of AIO equivalent to $z \neq 0 \rightarrow (\mathsf{E}y)(y' = z)$, which is not provable in $\overline{T_2}$, and (since all true open formulae in $0, '$ are already provable) the only way to make it provable from open formulae is to include a new function constant from which a Herbrand term for this formula can be built up. The obvious choice is P, which leads to $\overline{T_3}$. Note that although $\overline{T_2}$ is very weak, it is constructively categorical in the sense of Grzegorczyk [62], i.e., every model contains a submodel containing 0 and closed under $'$ which is isomorphic to the natural one. (All the theories which follows are also constructively categorical.)

Theorem 3.4. *The theory T_4 with constants* $0, ', \mathsf{P}, +$, *axioms* A1–5, *and rule* RIO *is equivalent to the theory T_4 obtained from it by dropping*

RIO *and adding axioms:*

B1. $x \neq 0 \rightarrow x = (Px)'$.

B2. $x + y = y + x$.

B3. $(x + y) + z = x + (y + z)$.

B4. $x + y = x + z \rightarrow y = z$.

Shoenfield proved this in [58a]. In fact he showed that AIO is provable in $\overline{T_4}$ which implies that B1–4 exhaust all applications of RIO (with the given constants), i.e., that $T_4, \overline{T_4}$ are equivalent in having not only the same theorems but also the same derivability relation, i.e., the addition of extra axioms to $T_4, \overline{T_4}$ still give equivalent theories. This is not always true; e.g., in passing from T_1 to T_2 by adding A3a we have to add more than A3a to $\overline{T_1}$ to get $\overline{T_2}$, since the addition of new axioms generates new applications of RIO. In fact, for any formula $A(x, z)$ the rule RI: $A(0, z)$, $A(x, z) \rightarrow A(x', z) \vdash A(x, z)$ is equivalent to the axiom A'I: $(z)A(0, z) \wedge (x)(z)(A(x, z) \rightarrow A(x', z)) \rightarrow A(x, z)$ in the sense that if T is a theory with RI and $\overline{T}$ a theory with no rules whose axioms are theorems of T then a necessary and sufficient condition that, *for all classes M of additional axioms*, the set of theorems of $T \cup M$, $\overline{T} \cup M$ should coincide, is clearly that A'I be provable in $\overline{T}$. To put it another way, when we say the rule RI is valid in a model we normally mean that if both premises are valid so is the conclusion, i.e., that A'I is valid. There are three successively stronger ways in which an induction rule RI may be valid in a model: (1) all consequences of the rule may be valid, (2) the rule may be valid (i.e., A'I may be valid), (3) AI may be valid. What we have seen already in Section 2 above is that in the case of RIO every model of type (1) can be embedded in one of type (3). The theory $\overline{T_2}$ above is an example of one where, although AIO is not provable, A'IO is, so that the equivalence to T_2 is preserved under the addition of axioms; e.g., if B_1' is $x \neq 0 \rightarrow (Ey)(y' = x)$ then the theorems of $T_2 \cup B_1'$, $\overline{T_2} \cup B_1'$ coincide (with the set of all true formulae, in fact).

Shoenfield showed that not all true open formulae, e.g., none of

C_n. $\quad nx = ny \rightarrow x = y \qquad (n > 1)$,

$D_{m,n}$. $\quad mx + n = my \qquad (0 < n < m)$,

(here $nx = x + \ldots + x$ (n times), and now that we have $+$ we write n or $n \cdot 1$ for $0^{(n)}$) are provable in T_4 although they are provable by the

full rule of induction, or by a rule of double induction applied to open formulae. A model of T_4 is a subset M of an additive abelian group which is closed under $+$ and contains an element 1 such that, for $x \in M$, $x-1 \in M$ iff $x \neq 0$.

Theorem 3.5. *The theory T_5 with constants* $0, ', \mathsf{P}, +, \cdot$, *axioms* A1–7, *and rule* RIO *is equivalent to the theory $\overline{T_5}$ obtained from it by dropping* RIO *and adding axioms* B1–4 *above and*

B5. $\mathsf{xy} = \mathsf{yx}$.

B6. $\mathsf{x(yz)} = \mathsf{(xy)z}$.

B7. $\mathsf{x(y+z)} = \mathsf{xy} + \mathsf{xz}$.

C′$_d$. $\mathsf{dy} = \mathsf{dy}_1 \rightarrow \bigvee_{i=0}^{d-1} \mathsf{(x+i)y} = \mathsf{(x+i)y}_1 \quad (d = 2, 3, \ldots)$.

A proof of this (again AIO is provable in $\overline{T_5}$) will appear in a forthcoming paper by the author. (M. D. Gladstone (unpublished) had previously given a theory containing $\dot{-}$ which included all theorems of T_5 but was not itself closed under RIO. This gave the first proof of the independence of Kreisel's formula in T_5.) The models for A1–7, B1–7 are obtained from a commutative ring R with unit element by taking a subset M of R which is closed under $+$ and $\cdot$, closed under $x-1$ for $x \neq 0$, and contains 0,1 but not -1. (x' of course is defined to be $x+1$ and Px to be $x-1$ if $x \neq 0$, 0 if $x=0$.) The further axioms C′$_d$ are equivalent to saying that if I_d is the ideal formed by all elements of R which annihilate all elements which d annihilates, then each element of R is congruent modulo I_d to one of $0, 1, 2, \ldots, d-1$. If we take for R the ring of polynomials in two indeterminates u, v with integer coefficients modulo the ideal $(p(u-v))$ (i.e., identify pu, pv but not u, v, where p is a prime), then take for M the images of those polynomials in u, v all of whose terms of highest degree have positive coefficients, we get a model for A1–7, $\mathrm{C}'_2, \ldots, \mathrm{C}'_{p-1}$ (in fact for all C′$_d$ such that p does not divide d) which does not satisfy C′$_p$ (viz., with u, v, u for $\mathsf{y}, \mathsf{y}_1, \mathsf{x}$, respectively), so $\overline{T_5}$ is not finitely axiomatizable. If we use the ideal $(n(u-v), u(u-v), v(u-v))$ instead we get a model for $\overline{T_5}$ which does not satisfy C_n: $\mathsf{nx} = \mathsf{ny} \rightarrow \mathsf{x} = \mathsf{y}$ (clearly C_n implies C′$_n$). If n is prime this model satisfies all of $\mathrm{C}_2, \ldots, \mathrm{C}_{n-1}$ so we see that C_n is independent of these in $\overline{T_5}$ and also in T_5 (since AIO is provable in $\overline{T_5}$). The axiom C*: $\mathsf{zy} = \mathsf{zy}_1 \wedge \mathsf{z} \neq 0 \rightarrow \mathsf{y} = \mathsf{y}_1$ is thus independent in T_5 of all of $\mathrm{C}_1, \mathrm{C}_2, \ldots$, and the axiom C$^+$: $\mathsf{z}_1\mathsf{y}_1 + \mathsf{z}_2\mathsf{y}_2 = \mathsf{z}_1\mathsf{y}_2 + \mathsf{z}_2\mathsf{y}_1 \rightarrow \mathsf{z}_1 = \mathsf{z}_2 \vee \mathsf{y}_1 = \mathsf{y}_2$ (which says

that the ring R in which the model is embedded is an integral domain) is independent of C* (proof: use polynomials in y_1, y_2, z_1, z_2 modulo the ideal $((y_1-y_2)(z_1-z_2))$). We would not expect any recursively enumerable set of axioms to give as theorems all true open formulae involving $+$ and $\cdot$ for this would give a positive solution to Hilbert's tenth problem, but even with C+ the theory T_5 does not yield all true formulae in $+$ only, for none of Shoenfield's $D_{m,n}$: $mx+n \neq my$ $(0<n<m)$ are provable. To see this take the integral domain $R[t]$ of polynomials in t with rational coefficients and the elements of M to be those with leading coefficient a non-negative integer. This gives a model for A1–7, B1–7 and C+ (and hence for all C'_d) which satisfies no $D_{m,n}$, since $mt+n=m(t+n/m)$. Note that in this model not only are some numbers (e.g., $2t+1$) both even and odd but some (e.g., t) are neither. So theorems such as $x^3+y^3=z^3 \to xyz=0$ are clearly way outside the scope of T_5; an obvious modification of the last model will show the unprovability of this; we later give a model showing its unprovability in a far stronger system.

4. A non-standard model for the system including $\dot{-}$.

Definition 4.1. *T_6 is the theory with constants* $0, ', \mathsf{P}, +, \cdot, \dot{-}$, *axioms* A1–9, *and rule* RIO.

We have not succeeded in finding an equivalent induction-free system for T_6 although we can offer a useful non-standard model for it and a fairly simple description of the stronger theory T'_6 obtained from it by replacing RIO by AIO.

Since T_6 is an extension of T_5 all of B1–7 are provable in it; so also are

B8. $\quad x \dot{-} y \neq 0 \to y+(x \dot{-} y)=x.$

B9. $\quad x \neq y \to x \dot{-} y \neq 0 \vee y \dot{-} x \neq 0.$

(I am indebted to Kreisel for pointing out that B9, which can be proved easily enough using double induction or induction on a quantified formula, can be proved using only RIO; you first prove $y' \dot{-} x'=y \dot{-} x$, $y' \dot{-} y=0', y' \dot{-} x=0 \to y \dot{-} x=0$). These imply all the usual properties of $\dot{-}$, e.g., $(z \dot{-} x) \dot{-} y = z \dot{-} (x+y)$, for they imply that any model for them can be embedded in a ring in the same way as the natural model, i.e., simply by adjoining formal inverses (unlike T_5 where the embedding is in a ring which contains further elements and is analogous

to that of the multiplicative semi-group of positive integers in the positive rationals). If we define $\leqslant$ by:

$$x \leqslant y \leftrightarrow x \dot{-} y = 0,$$

they imply that $\leqslant$ is an ordering satisfying the usual recursion equations:

$$x \leqslant 0 \rightarrow x = 0,$$

$$x \leqslant y' \leftrightarrow x \leqslant y \vee x = y',$$

and also that

$$x \leqslant y \rightarrow x + z \leqslant y + z \wedge xz \leqslant yz.$$

A model for A1–9, B1–9 consists of the non-negative elements of any discretely ordered commutative ring with unit. Since an ordered commutative ring is an integral domain we see that A1–9, B1–9 imply C^+ and hence all of C'_d, and hence all instances of AIO for formulae involving $0, ', \mathsf{P}, +, \cdot$ only. They also imply all of Shoenfield's C_n, $D_{m,n}$, which he showed to imply all true open formulae in $0, ', \mathsf{P}, +$ only. The quickest way to see this is to observe that Shoenfield showed that all of these were provable by a rule of double induction:

$$A(x, 0), A(0, y), A(x, y) \rightarrow A(x', y') \vdash A(x, y)$$

applied to open sentences. And in T_6 any multiple induction rule like this is a derived rule, for we can prove $A(x, y+x)$, $A(y+x, x)$ by simple induction (RIO) on x, and these give $A(x, y)$, for either $y = (y \dot{-} x) + x$ or $x = (x \dot{-} y) + y$.

In trying to show the unprovability in T_6 of the irrationality of $\sqrt{2}$ it is easiest to follow the line suggested by Theorem 2.2 and look for a model with non-standard elements t, $t\sqrt{2}$ for the stronger system T'_6 formed by replacing RIO by AIO. Such a model M_0 is given in a forthcoming paper by the author. Its elements are the non-negative elements of the ring of fractional polynomials

$$a_p t^{p/q} + a_{p-1} t^{p-1/q} + \ldots\ldots a_1 t^{1/q} + b$$

in the indeterminate t, where $p \geqslant 0$, $q > 0$ are integers, $a_1, \ldots, a_p$ are real algebraic numbers, b is an integer and the ordering is the one naturally determined by making t infinitely large. Since it also has an element $t\sqrt[3]{2}$ such that $(t\sqrt[3]{2})^3 = t^3 + t^3$, it follows that the formula suggested by Kreisel is unprovable in T_6. It was also shown there that T'_6 had a slightly simpler description: namely, all AIO are derivable from those obtained

from formulae A(x) of the form $f(x) \geqslant g(x)$ where f, g are polynomials, i.e., from the axiom schema

$$f(0) \geqslant g(0) \wedge f(x) < g(x) \rightarrow (Ey)(f(y) \geqslant g(y) \wedge f(y') < g(y')).$$

An alternative schema is one asserting the existence of the integral part of all positive real algebraic functions meaning by this for each n a function of $a_1, \ldots, a_n, b_1, \ldots, b_n$ defined as the least positive real root of the equation $a_1x^n + \ldots a_n = b_1x^n + \ldots b_n$ if one exists and 0 otherwise. If we introduce such functions, e.g., $[a/b]$, $[\sqrt{a}]$, we can express these axioms without quantifiers. But they cannot be so expressed in terms of the original constants; in fact, they are not provable from any set of true open formulae in $0, ', P, +, \cdot, \dot{-}$. Take, for example, the AIO obtained from $2x < y$, i.e., $0 < y \wedge (x)\,(2x < y \rightarrow 2x' < y) \rightarrow 2x < y$. Replacing the free x by y this gives $0 < y \rightarrow (Ex)\,(2x < y \leqslant 2(x+1))$; as we saw in Section 2, for this to be provable from true open formulae some disjunction $0 < y \rightarrow \bigvee_{i=1}^{k} 2t_i(y) < y \leqslant 2(t_i(y)+1)$ would have to be true for some terms $t_i(y)$ built up from the above constants. By letting $y \rightarrow \infty$ this is seen to be impossible, for (apart from the difference between $\dot{-}$ and $-$) the possible terms are polynomials with *integer* coefficients. So the AIO are not provable in T_6; i.e., T'_6 really is stronger than T_6. In fact not even the A'IO are provable in T_6. Perhaps the easiest way to show this is to consider a structure M_1 which is an extension to include $\dot{-}$ of one proposed by Skolem [55]. M_1 consists of polynomials in t with coefficients in $I(\sqrt{2})$, leading coefficient $\geqslant 0$, and coefficient of t^0 an integer. This does not satisfy the A'IO obtained from the formula $A(x): y^2 = 2(z+1)^2 \rightarrow x^2 < 3y^2$, for A(0), $A(x) \rightarrow A(x')$ are valid in M_1 (the latter because $\sqrt{3}$ is not in the ring $I(\sqrt{2})$ so that for all unnatural y if $x^2 < 3y^2$ then $x'^2 < 3y^2$) but A(x) is false, e.g., under the assignment $t\sqrt{2}$ to y, $t-1$ to z, and $3t$ to x. But M_1 is a model for all theorems of T_6, for we have shown that there are models, e.g., M_0, for the system obtained from T_6 by adding new constants t, t_1 and the axiom $t_1^2 = 2t^2$. The minimal substructure of any such model which contains $0, t, t_1$ and is closed under $', P, +, \cdot, \dot{-}$ must satisfy all open formulae satisfied by the model, and hence all theorems of T_6; and this minimal substructure is clearly isomorphic to M_1. Unlike the AIO, however, the A'IO are provable from a suitably chosen set of true open sentences in the given constants, for they are of EA form and are true, so they are always derivable from the set of all true open sentences. However for the given constants this set is probably not recursively enumerable; I do not know

whether there is any recursively enumerable set of true open sentences in the given constants which imply all the A'IO, i.e., which exhaust all applications of RIO.

Another way of describing the theory T'_6 is to say that it is the theory of the non-negative elements of a discretely ordered commutative ring which contains an element within 1 of each element of its real algebraic closure. It is easy to see that all open formulae true of the non-negative real numbers are theorems of T'_6 (and hence of T_6). Since every open formula A of T_6 can be simply transformed into one of the form $f(x_1, \ldots x_n) \neq g(x_1, \ldots, x_n)$ which is a theorem iff A is, one can say that the open formulae which are theorems of T_6, T'_6 correspond to a class of polynomial equations lying between those with no solutions in non-negative real numbers and those with no non-negative integer solutions. Examples of equations which have real (indeed rational) solutions which in T_6, T'_6 can be proved to have no solutions are $2x = 2y + 1$, $3x + 2 = y^2$, and $x^2 + x + 1 = 2y$. (The last answers Scott's question; to prove it in T_6 apply RIO to $z < 2x \rightarrow z^2 + z + 1 \neq 2y$.)

5. Inclusion of the functions $[x/n]$. M_0 remains a model for the theories T_7, T'_7 obtained by adding new function constants $[x/n]$ $(n = 2, 3, \ldots)$ together with their defining equations $A10_n$, $A11_n$ to T_6, T'_6 (and allowing induction on formulae containing these new constants). For using RIO these yield

$B10_n$. $$\bigvee_{r=0}^{n-1} x = n[x/n] + r,$$

which implies

$B11_n$. $$[(nx + y)/n] = x + [y/n].$$

If $[x/n]$ is defined in the obvious way in M_0 then axiom $B10_n$ (which implies $A10_n$, $A11_n$) is satisfied. The proof will be completed by showing that $B10_n$ implies that all cases of AIO containing the new constants can be derived from the previous instances of AIO. Let x, y be all the free variables (the general case is obtained by replacing y by $y_1, \ldots, y_n$ in what follows) of a formula $A(x, y)$ and let $B(y)$ denote the hypothesis of AIO for A, i.e., $A(0, y) \wedge (x)(A(x, y) \rightarrow A(x', y))$. If there are altogether k occurrences of the new functional constants in A with 'denominators' $m_1, \ldots, m_k$ (not necessarily distinct) and if $d = m_1 m_2 \ldots m_k$ there is a set

of d^2 formulae A_{rs} not involving the new constants such that the $B11_n$ (for $n = m_1, \ldots, m_k$) imply

E. $$A(dx+r, dy+s) \leftrightarrow A_{rs}(x, y)$$

for all r, s, $0 \leqslant r, s < d$. From this we can derive

$$B(dy+s) \rightarrow A_{rs}(0, y) \wedge (x)\,(A_{rs}(x, y) \rightarrow A_{rs}(x', y))$$

and using AI for A_{rs} this gives $B(dy+s) \rightarrow A_{rs}(x, y)$, which with E gives $B(dy+s) \rightarrow A(dx+r, dy+s)$, hence

$$\bigwedge_{r,s=0}^{d-1} B(d[y/d]+s) \rightarrow A(d[x/d]+r, d[y/d]+s)$$

which, with $B10_d$ gives $B(y) \rightarrow A(x, y)$, which is the required AIO. Note that a similar argument works for RIO and A'IO. Similarly the Legendre symbols for all fixed p may be introduced. However, although M_0 admits the introduction of the general function $[x/y]$ satisfying

$$y[x/y] \leqslant x < y([x/y]+1)$$

(the existence of such an element $[x/y]$ is, as we noted above, a consequence of the AIO), it does not satisfy AIO applied to formulae containing it, e.g., to $(x+1)[t/(x+1)] = t \wedge (x+2)[t/(x+2)] = t$. But I do not know any *open* formula provable from such AIO which M_0 does not satisfy; nor do I know any open theorems of T'_6 or T'_7 which are not provable from A1–9, $B1–10_n$.

6. Other forms of free-variable induction. We have already mentioned double induction, which is more general than RIO but which becomes equivalent to it in any system containing $\dot{-}$ and suitable axioms. Multiple induction of this kind is a special case of *parameter substitution* induction:

$$A(0, y), A(x, t(x, y)) \rightarrow A(x', y) \vdash A(x, y),$$

which can be further generalized by replacing y by $y_1, \ldots, y_n$ and the premise by a conjunction of similar terms (and in other ways). This is stronger than RIO in T_6, for the irrationality of $\sqrt{2}$ can be proved by it. It has some of the force of an ordinary induction on the quantified formula $(y)A(x, y)$. An open question of Kreisel is whether there is a small number of basic functions (corresponding to $\dot{-}$ above) which allow the reduction of this type of induction to RIO.

The irrationality of $\sqrt{2}$ is also provable in free variable systems including the function $gcd(x, y)$ or 2^x. Kreisel [58] and Mostowski [55] have shown that the theory based on RIO with constants $0, ', <$ and two further special primitive recursive predicates has no recursive non-standard models. In view of the primitive recursiveness of M_0 it would be interesting to know at what stage this phenomenon occurs when one successively adds further basic arithmetical functions.

In conclusion I should like to thank my colleague M. D. Gladstone for bringing these problems to my notice and Kreisel for the stimulus of his enthusiastic and unflagging correspondence.

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APPLICATIONS OF MODEL THEORY TO DEGREES OF UNSOLVABILITY

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During the past decade, the theory of degrees of unsolvability has developed rapidly. Although the concept of degree is implicit in earlier work of Post and Turing, the first study of degrees as abstract objects is the fundamental paper of Kleene and Post [54]. In this and several subsequent papers, the set of degrees was studied as an algebraic structure, namely, a partially ordered set with a unary operation (the jump operation).

On the whole, the results have been rather disappointing. Kleene and Post observed that the set of degrees forms an upper semi-lattice having a smallest element. Almost all further results have been negative results [1]; that is, results which showed that the algebraic structure fails to have some "nice" property. The general conclusion seems to be that there is no hope of describing this structure by algebraic methods.

The thesis of this paper is that the best hope of future progress in this field lies in the application of model-theoretic methods. The main content of the paper consists of some problems; such results as are stated are quite elementary. The problems are probably quite difficult, but not unapproachable; and their solution might cast much more light on the structure than the rather piecemeal results obtained (often at great effort) so far.

Let us recall briefly how model theory is applied to algebra. Corresponding to a type of algebraic structure [2], we introduce a first-order language whose function and predicate symbols (other than the equality symbol) designate the operations and relations of the structure. The structures of this type may then be regarded as models of a suitable

[1] An exception is the result of Friedberg [57]: every degree $\geqslant \mathbf{0}'$ is an image under the jump operation.

[2] By *algebraic structure*, we mean what is usually called a relational structure or relational system. We use this terminology here to emphasize the algebraic nature of the investigation.

theory formulated in this language. It then becomes meaningful to say that a sentence of the language is valid in a certain structure of this type. In this way, the tools of model theory become available. The effect of this is that we may now study certain further relations, defined by the formulas of the language. With these new relations, we can often prove properties of the structures which could not even be stated without the new relations [3].

To simplify matters, we shall ignore the jump operation, and consider the structure $\mathfrak{D}$ of degrees of unsolvability merely as a partially ordered set. The corresponding first-order language has a single [4] predicate symbol $\leqslant$. The theory in this language whose valid sentences are just those sentences which are (in an obvious sense) valid in $\mathfrak{D}$ is called the *elementary theory* of $\mathfrak{D}$.

An obvious problem is: Is the elementary theory of $\mathfrak{D}$ decidable? If the decision problem could be solved, the solution would undoubtedly provide a great deal of information on the structure of $\mathfrak{D}$. It seems more likely, however, that this theory is undecidable.

We recall the notion of *elementary extension* (or *arithmetical extension*) as applied to partially ordered sets (for details, see Tarski–Vaught [57]). Let $\mathfrak{B}$ be a subset of the partially ordered set $\mathfrak{A}$. Then $\mathfrak{A}$ is an elementary extension of $\mathfrak{B}$ if for any formula φ of the above language, and any assignment of elements of $\mathfrak{B}$ to the free variables of φ, φ is true in $\mathfrak{B}$ if and only if it is true in $\mathfrak{A}$.

Let $\mathfrak{D}_{\mathbf{b}}$ be the partially ordered set of degrees $\mathbf{a}$ such that $\mathbf{a} \leqslant \mathbf{b}'$. A question of some interest is: Is $\mathfrak{D}_{\mathbf{b}}$ an elementary extension of $\mathfrak{D}_0$? There is quite a bit of evidence for a positive answer. For in every case in which a degree with some interesting property has been produced, it has been possible to produce such a degree which is in $\mathfrak{D}_0$. Unfortunately, there are no very effective ways of proving that one structure is an elementary extension of another, unless both are models of a theory whose decision problem has been solved.

Many other such problems about the structure $\mathfrak{D}$ are easily formulated. While their solution would throw quite a bit of light on the structure of $\mathfrak{D}$, it is unlikely that we can obtain in this way any sort of complete description of this structure. However, we shall state a conjecture which,

[3] This aspect of model-theoretic methods in algebra is brought out by cylindric and polyadic algebras.

[4] In this case, we do not even need the equality symbol, since it can be defined in terms of $\leqslant$.

if true, would completely describe the structure of an important substructure of $\mathfrak{D}$.

We recall that a degree is *recursively enumerable* if it is the degree of some recursively enumerable set. The partially ordered set $\mathfrak{D}_R$ of recursively enumerable degrees, unlike $\mathfrak{D}$ itself, is countable and has a largest element. In many respects, however, $\mathfrak{D}_R$ is similar to $\mathfrak{D}$. In fact, many of the anomalous properties of $\mathfrak{D}$ have been shown to hold also for $\mathfrak{D}_R$ (often at the cost of much extra work). There is one notable exception. Whereas Spector has shown that there are minimal degrees (greater than **0**), Muchnik [56] and Friedberg [60] have shown that there are no minimal recursively enumerable degrees. By a large extrapolation from this, we arrive at a conjecture about $\mathfrak{D}_R$. To state it, we need some model-theoretic notions.

Let T be a first-order theory (with equality). We suppose that the variable of T are assigned a definite order. Let $a_1, a_2, \ldots, a_n$ be elements in a model of T. We let $\Delta(a_1, a_2, \ldots, a_n)$ be the set of open formulas of T which contain only the first n variables, and which are true in this model when the ith variable is taken to designate a_i. We call this set of formulas the *open type* of $a_1, a_2, \ldots, a_n$. By an *open n-type*, we mean a set of formulas which is the open type of n elements in some model [5] of T. A model $\mathfrak{M}$ of T is *dense* if for any elements $a_1, a_2, \ldots, a_n$ of $\mathfrak{M}$ and any open $(n+1)$-type Δ' including $\Delta(a_1, a_2, \ldots, a_n)$, there is an element a_{n+1} of $\mathfrak{M}$ such that $\Delta(a_1, a_2, \ldots, a_n, a_{n+1}) = \Delta'$.

The terminology is suggested by the following fact: if T is the elementary theory of linearly ordered sets, the dense models are just the densely ordered sets without a first or last element. The notion of a dense model is closely related to the notion of a *saturated* model used by Vaught [61]. In fact, if we omit the restriction to open formulas in the above definition, we obtain the definition of a saturated model. By proofs almost exactly like those of 4.4 and 4.6 of Vaught [61], we can prove the following.

(A) *Any two denumerable dense models of T are isomorphic.*

(B) *A denumerable model $\mathfrak{M}$ of T is dense if and only if it satisfies the following two conditions*: (i) *Every denumerable model of T is isomorphic to a submodel of $\mathfrak{M}$*; (ii) *If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are elements of $\mathfrak{M}$*

[5] Using the completeness theorem, one can easily give a purely syntactical definition of open n-type.

such that $\Delta(a_1, a_2, \ldots, a_n) = \Delta(b_1, b_2, \ldots, b_n)$, *then there is an automorphism of* $\mathfrak{M}$ *which takes* a_i *into* b_i *for* $i = 1, 2, \ldots, n$.

Now consider the theory T formulated as follows. The constants of T are "$\leqslant$", "$\cup$", "0", and "1". The axioms of T guarantee that $\leqslant$ is a partial order, that $x \cup y$ is the least upper bound of x and y, that 0 is the first element, and that 1 is the last element. We may call T the elementary theory of upper semi-lattices with first and last elements. Clearly $\mathfrak{D}_R$ is a denumerable model of T. Moreover, T has a denumerable dense model. For it follows from the results of Jónsson [60] that T has a countable model satisfying the conditions of (B). (The proof that upper semi-lattices with first and last element satisfy the amalgamation condition of Jónsson is like the proof for lattices, but is somewhat simpler.)

Conjecture. *The set* $\mathfrak{D}_R$ *of recursively enumerable degrees is a denumerable dense model for the elementary theory of upper semi-lattices with first and last element.*

In view of (A), this would completely describe the structure of $\mathfrak{D}_R$. Moreover, it would describe it in a useful way. For example, we could use it to solve the decision problem for the elementary theory of $\mathfrak{D}_R$. The method is an obvious analogue of the method used by Langford [26] to solve the decision problem for densely ordered sets.

It would not be difficult to reduce the conjecture to several conjectures about degrees, none of which used model-theoretic concepts. Since these would be rather complicated, it seems more useful at present to give some typical consequences of the conjecture, which illustrate the sort of problems which arise. Three such consequences are the following, in which boldface letters stand for recursively enumerable degrees.

(I) *If* $\mathbf{a} < \mathbf{b}$, *then there is a degree* $\mathbf{c}$ *such that* $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

(II) *If* $\mathbf{0} < \mathbf{a} < \mathbf{b}$, *then there is a degree* $\mathbf{c}$ *such that* $\mathbf{0} < \mathbf{c} < \mathbf{b}$ *and* $\mathbf{b} = \mathbf{a} \cup \mathbf{c}$.

(III) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are incomparable, then* $\mathbf{a}$ *and* $\mathbf{b}$ *have no greatest lower bound (in the set of recursively enumerable degrees).*

Consequence (I) is a generalization of the Muchnik–Friedberg result quoted above. It was conjectured by Sacks in 1962; and it was this conjecture which led the author to the above, more general conjecture.

Just before this symposium, Sacks obtained a proof of (I); it will be published soon[6].

Consequences (II) and (III) are unsolved. They seem similar to results which have been proved, but involve some new difficulties. For example, (III) requires us to be able to construct degrees which are recursive in both **a** and **b**. The following easily proved lemma will probably not provide sufficiently many such degrees to prove (III); but it at least indicates that many such degrees do exist.

Lemma. *Let F be a recursive function which enumerates one-one a set of degree* **a**, *and let G be a recursive function which enumerates one-one a set of degree* **b**. *Let* **c** *be the degree of the set of pairs $(F(n), G(n))$. Then* $\mathbf{c} \leqslant \mathbf{a}$ *and* $\mathbf{c} \leqslant \mathbf{b}$.

An enumeration of some consequences of the conjecture has convinced the author that all the major problems arise already in (I)–(III). In other words, if one could prove (II) and (III), it would probably be merely a matter of patience to prove the conjecture. If this is correct, we may probably expect the conjecture to be settled in the near future.

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[6] *Added in proof*: This result has appeared in *Ann. of Math.*, ser. 2, vol. 80 (1964), pp. 300–312.

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LOGICS APPROPRIATE TO EMPIRICAL THEORIES

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1. Introduction. To those like myself who are mainly concerned with the methodology of the empirical sciences, the present symposium is both sobering and encouraging. It is sobering as one thinks of the scientific contrast between the majority of papers read here and the standard sources in the methodology and philosophy of science. Yet it is encouraging, because the hope is engendered that many of the methods, and perhaps above all, the intellectual standards of these papers, will extend themselves in a natural way to logical investigations of the empirical sciences. The logical and philosophical foundations of physics, for example, seem to be at about the stage where the foundations of mathematics were during most of the nineteenth century. Nearly any physicist and a large number of philosophers are prepared to deliver at a moment's notice a lecture on the foundations of quantum mechanics. The situation is far different with respect to the foundations of mathematics. With an ever-increasing volume of deep and rigorous results, mathematicians unacquainted with the literature are not prone to deliver casually-put-together *obiter dicta* on logic and related topics.

The present paper is meant to be a quite modest contribution to the foundations of physics conceived of as the same sort of discipline as the foundations of mathematics. My objective is to give two sorts of examples of empirical theories for which it is natural and convenient to introduce a logic that is deviant in some respect. The first class of examples is derived from theories of measurement and the more general consideration of physical principles of invariance. The second class is drawn from quantum mechanics.

2. Principles of invariance and three-valued logic. In theories of measurement or in more general physical theories principles of invariance arise in a natural way. For example, the statement that Venus has greater mass than Earth is true or false, independent of what unit of

mass is used. Put another way, the truth value of the statement is invariant under an arbitrary (positive) similarity transformation of the units of mass used in attempting to verify or falsify the statement. On the other hand, the statement that the mass of Venus is greater than 10^{10} has no such invariance, for its truth or falsity will vary with the units of mass selected.

Let us generalize on this example by looking at a language in which we can express the results of n distinct measurements on physical objects. To avoid many repetitious details, I shall use as a basis the formal language of Tarski's monograph [51] enriched by individual variables "a", "b", "c", ..., which take as values physical objects and n function symbols "m_1", ..., "m_n". Terms are constructed from these function symbols and individual variables in the expected fashion; in particular, it is intended that $m_i(a)$ be a real number for $1 \leqslant i \leqslant n$. Let us call this language for a fixed n, $\mathscr{L}_n$. This formalization of $\mathscr{L}_n$ is necessary only to give definiteness to some of the logical results and problems it seems of interest to formulate. With inessential modifications for what is to be stated in this paper, $\mathscr{L}_n$ could easily be adapted to express in canonical form the results of a wide class of physical experiments.

The general semantical notion of a model of a language I shall assume is familiar. By a *model* of $\mathscr{L}_n$, I mean an ordered $(n+2)$-tuple $\mathfrak{M} = \langle \mathfrak{S}, A, m_1, \ldots, m_n \rangle$, where $\mathfrak{S}$ is the usual system of real numbers under the operations of addition and multiplication and the relation less than (technical details about $\mathfrak{S}$ are not critical here), A is a finite, non-empty set, and $m_1, \ldots, m_n$ are real-valued functions on A. The intended interpretation should be obvious. The set A is meant to be a set of physical objects whose various properties are measured by the numerical functions $m_1, \ldots, m_n$.

Associated with each function m_i expressing the results of a particular kind of measurement is a group G_i of (numerical) transformations. The group G_i characterizes the degree of uniqueness of the measurement. If m_i is a mass measurement, then G_i is the group of similarity transformations. If m_i is a classical position measurement on the x-axis, then G_i is the group of linear transformations, and so forth.

We first define when two models of $\mathscr{L}_n$ are appropriately related by these groups of transformations.

Definition 1. *Let* $\mathfrak{M}_1 = \langle \mathfrak{S}, A_1, m_1, \ldots, m_n \rangle$ *and* $\mathfrak{M}_2 = \langle \mathfrak{S}, A_2, m_1', \ldots$

..., $m_n'\rangle$ be two models of $\mathscr{L}_n$, and let $\mathfrak{G}=\langle G_1, \ldots, G_n\rangle$ be an n-tuple of groups of transformations. Then $\mathfrak{M}_1$ is $\mathfrak{G}$-related to $\mathfrak{M}_2$ *if and only if*

(i) $A_1=A_2$;

(ii) *for $1\leqslant i\leqslant n$, there is a φ in $\mathfrak{G}_i$ such that*

$$\varphi \circ m_i = m_i',$$

where $\circ$ is functional composition.

The intuitive idea of truth-value invariance under the appropriate groups of transformations is formalized in the next definition.

Definition 2. *An atomic formula F of $\mathscr{L}_n$ is* $\mathfrak{G}$-empirically meaningful *if and only if F is satisfied in a model $\mathfrak{M}$ of $\mathscr{L}_n$ when and only when it is satisfied in every model $\mathfrak{G}$-related to $\mathfrak{M}$.*

In other words, an atomic formula is said to be empirically meaningful only when it has the appropriate invariance properties—that the condition expressed in Definition 2 is always sufficient and not merely necessary is not a point of argument here. For a more extensive discussion of these matters see Suppes [59].

Definition 3. *A formula of $\mathscr{L}_n$ is* $\mathfrak{G}$-empirically meaningful *if and only if each of its atomic formulas is such.*

The reason for requiring that each atomic formula of a formula be meaningful is to obtain truth-functionality for the three-valued logic characterized in the next theorem. The substance of this logic is to assign the ordinary values truth and falsity to meaningful formulas, and the single value *meaninglessness* (abbreviated μ) to the remainder. The proof of the theorem is straightforward and is omitted.

Theorem 1. *Assign the truth value μ to $\mathfrak{G}$-empirically meaningless formulas of $\mathscr{L}_n$. Then the formulas of $\mathscr{L}_n$ are truth-functional in the three-valued logic T, F and μ defined by the following tables for negation and conjunction.*

S	$\neg S$
T	F
F	T
μ	μ

$\wedge$	T	F	μ
T	T	F	μ
F	F	F	μ
μ	μ	μ	μ

The ordinary definitions of disjunction, implication and so forth in terms of negation and conjunction are assumed in working out additional tables. The resulting truth tables are precisely those which Kleene [52a, p. 334] has called the *weak tables*, and they do seem to be about the weakest possible extension of classical two-valued logic.

A simple example will show why meaningfulness needs to be initially defined in terms of the atomic formulas rather than arbitrary formulas in order to have the truth-functional result of Theorem 1. Suppose we had proceeded the other way, and let m_1 be a mass measurement function with the associated group of similarity transformations. Then the sentence

$$(\exists a)(m_1(a) = 1) \vee \neg\ (\exists a)(m_1(a) = 1)$$

would have the value T although its single component atomic formula has the value μ. On the other hand, the sentence

$$(\exists a)(m_1(a) = 1) \vee (\exists a)(m_1(a) = 2)$$

would have the value μ as does its component atomic formulas, and thus the logic would not be truth-functional.

When we turn our attention to what happens to the set of meaningful formulas under inferences involving quantifiers the results are not as satisfactory as the sentential ones of Theorem 1. In formulating the theorem it is useful to introduce the notion of a *non-trivial* n-tuple $\mathfrak{G}$ of groups for a model $\mathfrak{M}$. By this I mean at least one G_i is such that every model $\mathfrak{G}$-related to $\mathfrak{M}$ may not be obtained by the identity transformation on the positive integers (a weaker notion of non-triviality will suffice to prove the theorem, but is not important for the purposes at hand).

Theorem 2. *Let $\mathfrak{M}$ be a model of $\mathscr{L}_n$ and let $\mathfrak{G}$ be non-trivial for $\mathfrak{M}$. Then the set of $\mathfrak{G}$-meaningful formulas of $\mathscr{L}_n$ whose closures are true in $\mathfrak{M}$ is not a deductive system (i.e., the set is not closed under logical consequence).*

Proof. (I sketch the proof without too great concern for use-mention distinctions.) Let G_i be non-trivial for $\mathfrak{M} = \langle \mathfrak{S}, A, m_1, \ldots, m_n \rangle$. Then there is an atomic formula of $\mathscr{L}_n$, a positive integer k, a model $\mathfrak{M}'$ $\mathfrak{G}$-related to $\mathfrak{M}$ and an a in A such that

$$m_i(a) = k$$
$$m_i'(a) \neq k.$$

Now the integer k may be described in $\mathscr{L}_n$ by an expression of the form "$1+\ \ldots\ +1$", and thus there is a $\mathfrak{G}$-meaningful sentence of $\mathscr{L}_n$ which is true in $\mathfrak{M}$ and which asserts that for every number x if $x=k$ then $x<k+1$. For the variable in the sentence of $\mathscr{L}_n$ corresponding to "x" we now specify a term of the form "$m_i(a)$", and we obtain a formula which is not $\mathfrak{G}$-meaningful and thus not in the original set.

The difficulties posed by Theorem 2 are real. It certainly does not seem satisfactory to have a notion of meaningfulness that is not deductively closed. In Suppes [59] I proposed taking the bull by the horns and redefining the concept of logical consequence. We may say that a formula is a $\mathfrak{G}$-*meaningful logical consequence* of a set A of formulas of $\mathscr{L}_n$ if and only if S is a logical consequence of A and S is $\mathfrak{G}$-meaningful whenever every formula in A is also.

Two problems beset this new notion of meaningful logical consequence. First, it will not in general be possible to give finitary rules of inference, because for many n-tuples of groups the notion of meaningfulness will not be decidable. Secondly, there is a problem of completeness. Putting aside the finitary problem we may ask this question. If a standard set of rules of inference is used, augmented by the requirement that each step of an inference be $\mathfrak{G}$-meaningful, does being a meaningful logical consequence imply derivability by the rules of inference? This problem has not been solved for any significant n-tuples of groups.

The discussion of empirical meaningfulness and the related three-value logic has been restricted to the simple languages $\mathscr{L}_n$, but if we are willing to move from talk about formulas and sentences to talk about propositions or relations, it is easy to generalize the results to richer physical theories. The three-valued logic of T, F and μ applies in a wholly natural way to the system-of-classical-particle-mechanics functions introduced in McKinsey and Suppes [55], for example. In this case the groups of transformations are no longer restricted to one-to-one functions from the real numbers to the real numbers, but include the geometrical and Galilean transformations familiar in classical mechanics. Other examples from classical and relativistic physics are easily constructed.

In the past thirty or forty years a large number of different multi-valued logics have been proposed, and the technical literature on the formal structure of these logics has reached sizable proportions. The philosophical defense of the significance of multi-valued logics has been much less substantial. The interest of the discussion in this paper is

meant to be in the intuitively direct philosophical defense possible for the three-valued logic introduced. The extension of classical two-valued logic developed here is too weak to raise any interesting technical problems at the level of sentential logic, and perhaps too complicated at the level of predicate logic, for which the problem of completeness in terms of meaningful logical consequence is central. But the empirical or methodological defense of the logic is clear. The data of experiments, or the empirical predictions of physical theories, are best expressed not by a particular model of the theory but by an appropriate equivalence class of models. Sentences or propositions of the theory which do not have an invariant truth value (in the two-valued sense) over these equivalence classes simply do not have any clear empirical meaning. A three-valued logic, with the third value being meaninglessness, provides a simple device for segregating these non-invariant propositions that do not have a definite empirical content, and for calling attention to the otiose features of any one model of a theory.

3. Families of Boolean algebras in quantum mechanics[1]. Turning now to the logic of quantum mechanics, I try to develop an approach that hews somewhat closer to classical lines than many discussions. I also frankly admit that many of the proposals made in the literature are hard for me to fathom from an empirical or experimental standpoint. It is a simple enough matter formally to define orthocomplemented modular lattices and relate them to the structure of subspaces that arise in quantum mechanics, but exactly how this logic corresponds to the set of experimental propositions or events is ordinarily not elaborated in any detail. For example, the much-cited article of Birkhoff and von Neumann [36] restricts the consideration of the decisive supporting experimental consequences of quantum-mechanical theory to a few lines (p. 831).

My objective here is to give a simpler and more classical logic for the experimental propositions or events. I mean by an experimental event an event that may at least in principle be observed by some configuration of experimental apparatus. In this sense the Heisenberg uncertainty principle is not an experimental proposition expressing the observation of a single event, for it is in general not possible to measure simultaneously position and momentum. In probabilistic terms,

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[1] I am indebted to Dana Scott for several helpful comments on this section.

as I have emphasized elsewhere (Suppes [61a]), there is no joint probability distribution of position and momentum. It is not a matter of measuring position and probability with zero variance. It is rather that there is no possibility of measuring them jointly at all, because their joint distribution does not exist.

To simplify developments and thereby to avoid a number of technical issues, I shall make the restrictive assumption that all observables may be expressed as functions of position and momentum. Thus for a system of n particles the probability space is the $6n$-dimensional Euclidean space $\mathscr{E}_{6n}$, and I shall represent any point by $6n$ real numbers $(q_1, \ldots, q_{3n}, p_{3n+1}, \ldots, p_{6n})$, where it is understood that the first $3n$ coordinates are the position measurements and the last $3n$ the momentum measurements, with q_1, q_2, q_3 the position coordinates of the first particle, p_{3n+1}, p_{3n+2}, p_{3n+3} its momentum coordinates, etc. A one-dimensional event is just a measurable subset of points of $\mathscr{E}_{6n}$ restricted in some one dimension, and in general, an event is a measurable cylinder set $C_S \subseteq \mathscr{E}_{6n}$, where S is the set of dimensions of $\mathscr{E}_{6n}$ on which C_S is restricted, i.e., $S \subseteq \{1, \ldots, 6n\}$. The following theorem about cylinder sets is easily proved.

Theorem 3. *For any non-empty set $R \subseteq \{1, \ldots, 6n\}$, the collection of measurable cylinder sets C_S such that $S \subseteq R$ forms a Boolean algebra, which we designate $\mathfrak{B}_R$.*

The following theorem can be proved on the basis of the standard formalism of classical quantum mechanics.

Theorem 4. *Let R be a non-empty subset of $\{1, \ldots, 6n\}$. Then there exists a quantum-mechanical probability distribution on $\mathfrak{B}_R$ if and only if there is no number m such that m and $3n+m$ are both in R.*

In other words, a distribution exists on $\mathfrak{B}_R$ whenever no two coordinates representing conjugate observables are in R. It is obvious, of course, that all kinds of proper distributions exist on any $\mathfrak{B}_R$ apart from quantum mechanics. The restriction of the theorem applies only to those distributions derived from quantum-mechanical considerations. I say that a subset R of $\{1, \ldots, 6n\}$ is *admissible* whenever it satisfies the restriction stated in Theorem 4. Analogous to the definitions of Section 2, I then define a measurable cylinder set C_R as (*experimentally*) *meaningful* if and only if R is admissible. A collection $\{C_S\}$ of measurable cylinder sets is *simultaneously meaningful* if and only if there is a subset R of

$\{1, \ldots, 6n\}$ such that for each cylinder set C_S in the collection $S \subseteq R$ and R is admissible.

The departures from classical logic that follow from these definitions are evident. For example, the intersection or union of two meaningful events is not necessarily meaningful. The truth tables given in Theorem 1 certainly do not hold for this logic of meaningfulness.

My own view is that the most natural way to look at this situation is in terms of a family of logics defined in the following manner. An admissible set R of coordinates is *admissibly complete* iff it is a proper subset of no admissible set. A Boolean algebra $\mathfrak{B}_R$ of measurable cylinder sets of $\mathscr{E}_{6n}$ is a *maximal, meaningful* Boolean algebra iff R is admissibly complete. From Theorem 4 it is evident there is no unique maximal Boolean algebra. Moreover, the meaningful Boolean algebras do not form a lattice, because two maximal such algebras, of which there are 2^{3n} in the present setup, do not have a meaningful greatest lower bound or meet. On the other hand, it is evident that any two maximal Boolean algebras as defined here are isomorphic, although it would be most surprising if for any Hamiltonian the quantum-mechanical probability distributions on the two algebras were also isomorphic.

I thus come to the following thesis.

The logic of the experimental propositions of the quantum mechanics of n particles for which all observables are defined in terms of position and momentum, is the family $\mathfrak{F}(\mathscr{E}_{6n})$ *of* 2^{3n} *maximal meaningful Boolean algebras* $\mathfrak{B}_R$ *and their subalgebras* $\mathfrak{B}_S$, *with* $S \subseteq R$.

At least one of these maximal algebras is always appropriate for the logic of a given experiment, and the intersection of all those that are is the unique subalgebra of the experiment.

It may be useful to compare the logic defined here with other proposals that have been made. I first mention the three-valued logic of Reichenbach [44]. His table for conjunction is the following

$\wedge$	T	F	I
T	T	F	I
F	F	F	F
I	I	F	I

which is Kleene's [52a, p. 334] *strong* table for conjunction. But Reichenbach's three-valued logic is highly misleading, for the structure of experimental propositions or events proposed by him is not constructed

so as to be truth-functional in this logic, and the same is true of the family $\mathfrak{F}(\mathscr{E}_{6n})$ of Boolean algebras with $\wedge$ interpreted as intersection. The actual structure Reichenbach seems to be after in terms of what can and cannot be measured is rather close to what has been suggested in this paper.

By adopting a more complicated set of values, a truth-functional scheme for $\mathfrak{F}(\mathscr{E}_{6n})$ rather similar to the three-valued logic proposed in the first part of this paper may be set up. The idea is that the truth values are now not simply T, F and μ, but are ordered pairs (T, R), (F, R) and (μ, R), where R is the set of coordinates of the cylinder set corresponding to the proposition in question. To get truth-functionality, we must, as in the previous case, pay particular attention to the atomic formulas. What I mean is easily illustrated by an example. Let

$$A = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \in (0, 1) \ \& \ x_{3n+1} \in (0, 1)\},$$

$$B = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \notin (0, 1) \ \& \ x_{3n+1} \notin (0, 1)\},$$

where x_i is the ith coordinate of point x in $\mathscr{E}_{6n}$. Then neither A nor B is meaningful, but $A \cup B$ is, being just $\mathscr{E}_{6n}$ itself. The disjunction corresponding to $A \cup B$ is declared meaningless, for the same reasons that a similar decision was made about '$x = 1 \vee x \neq 1$' in the earlier discussion of invariance. Holding these remarks in mind, the intuitive basis for the following truth tables for negation and conjunction is evident. (I shall not formalize an elementary language to make their application precise, for it is clear from the earlier discussion how this may be done.) Because meaningfulness is preserved under complementation of cylinder sets, negation is simple.

P	$\neg P$
(T, R)	(F, R)
(F, R)	(T, R)
(μ, R)	(μ, R)

In the case of conjunction, two natural cases arise according to whether or not the set of coordinates $R \cup S$ is admissible.

Case 1. $R \cup S$ admissible.

$\wedge$	(T, S)	(F, S)
(T, R)	$(T, R \cup S)$	$(F, R \cup S)$
(F, R)	$(F, R \cup S)$	$(F, R \cup S)$

Case 2. $R \cup S$ not admissible.

$\wedge$	(T, S)	(F, S)	(μ, S)
(T, R)			
(F, R)	$(\mu, R \cup S)$ for all entries		
(μ, R)			

A theorem like Theorem 1 may be proved for these tables. We obtain truth-functionality by widening the usual net of considerations that count in determining the truth value of a molecular sentence to include, not the full factual content of the proposition, but the set of coordinates relevant to the factual content of its atomic constituents. This inclusion marks a wider departure from classical logic than is true of the tables of Theorem 1.

The central point of comparison with the logic of quantum mechanics proposed by Birkhoff and von Neumann [36] has already been mentioned. They require that the structure of experimental propositions be a lattice, and thus that the *conjunction* of two meaningful propositions in the lattice also be in the lattice, but for reasons already stated this requirement seems too strict when one proposition expresses a possible result of measuring position and the other of measuring momentum. The fact that the closed linear subspaces of a complex separable Hilbert space form such a lattice is not sufficient, in my view, to maintain that it expresses the more restricted logic of experimental propositions, in spite of the central importance of this lattice of subspaces in the formulation of the theory.

A generalization of the logic of von Neumann and Birkhoff has been given by Varadarajan [62] and also by Mackey [63]. They define a *logic* as an orthocomplemented partially ordered set, which concept is easily made explicit. Let A be a non-empty set, let $\leqslant$ be a binary relation on A, let $'$ be a unary operation on A, and let 0 and 1 be elements of A. Let us also define another binary relation and a binary operation on A. Elements a and b are *disjoint* (in symbols: $a \perp b$) if and only if $a \leqslant b'$. And, if $a \perp b$ we define the sum $a + b$ in the expected fashion: $a + b = c$ if and only if $a \leqslant c$, $b \leqslant c$ and for every d in A, if $a \leqslant d$ and $b \leqslant d$ then $c \leqslant d$. In terms of these notions we then have:

Definition 4. *A structure* $\mathfrak{A} = \langle A, \leqslant, ', 0, 1\rangle$ *is an orthocomplemented partial ordering if and only if the following axioms are satisfied for* a, b *and* c *in* A:

1. $a \leqslant a$.
2. *If* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$.
3. *If* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$.
4. $0 \leqslant a$.
5. $a \leqslant 1$.

6. *If* $a \leqslant b$ *then* $b' \leqslant a'$.
7. $(a')' = a$.
8. $a + a' = 1$.
9. *If* $a \leqslant b$ *then there is a* c *in* A *such that* $c \perp a$ *and* $a + c = b$.
10. *For any sequence of pairwise disjoint elements* $a_1, \ldots, a_n, \ldots$ *of* A *there is a* c *in* A *such that for all* n, $a_n \leqslant c$ *and for every* d *in* A, *if for every* n, $a_n \leqslant d$, *then* $c \leqslant d$.

The purpose of Axiom 10 from the standpoint of probability distributions on logics is obvious; it guarantees the necessary underlying structure for a countably additive measure.

Although an orthocomplemented partial ordering is not a lattice, the requirements are still too strict for the collection of experimentally meaningful events in the family $\mathfrak{F}(\mathscr{E}_{6n})$ of Boolean algebras. The intended interpretation is that A be $\bigcup \mathfrak{F}(\mathscr{E}_{6n})$, $\leqslant$ be set inclusion $\subseteq$, $'$ be set complementation, 0 be the empty set and 1 be $\mathscr{E}_{6n}$. But this structure does not satisfy Axiom 10, because the union of two disjoint meaningful events is not itself meaningful. For example, let

$$C = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \in (0, 1) \ \& \ x_2 \in (0, 1)\},$$
$$D = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \in (1, 2) \ \& \ x_{3n+2} \in (0, 1)\},$$

where x_i is the ith coordinate of point x in $\mathscr{E}_{6n}$. Clearly both C and D are meaningful, disjoint events, but $C \cup D$ is not meaningful, and is thus not in $\bigcup \mathfrak{F}(\mathscr{E}_{6n})$.

The results stated for $\mathfrak{F}(\mathscr{E}_{6n})$ thus far have been rather negative, but there is a positive result related to the work of Kochen–Specker [*], namely, $\mathfrak{F}(\mathscr{E}_{6n})$ with the appropriate structure is a partial Boolean algebra. Roughly speaking, every partial Boolean algebra may be represented by a collection of Boolean algebras satisfying certain consistency conditions. For details see the article of Kochen–Specker [*].

In this sense the family of Boolean algebras constructed here originally from probabilistic considerations provides yet another intuitive argument for Kochen and Specker's concept of a partial Boolean algebra. However, the issue is not entirely simple. From a probability standpoint it is also convenient to have well defined the probability of the union of any two disjoint events, as in the case of $C \cup D$ above. No conflict with Theorem 4 exists, for the theorem is concerned with the existence of a measure on the entire algebra $\mathfrak{B}_R$, not with the much narrower question of

whether or not the event $C \cup D$ has a probability. In any given experiment, C, C', D or D' may be observed, but, of course, never the complex event $C \cup D$. For computational purposes we may then in a natural way extend the probability measure to these events.

It is not possible here to study in any detail the relations between the orthocomplemented partial orderings and partial Boolean algebras, particularly the analysis of the similar but differing notions of simultaneously measurability, one of which is primitive in the theory of partial Boolean algebras, and the other of which is definable in a simple way in the theory of orthocomplemented partial orderings.

Either one of these theories certainly provides a more natural logic than that of classical Boolean algebra by itself as a structural foundation on which to erect the probabilistic superstructure of quantum mechanics because of the difficulties about improper or pseudo-probability distributions that arise in all too direct a fashion in the classical formulation.

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ON THE DENUMERABLE MODELS OF THEORIES WITH EXTRA PREDICATES

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1. General terminology. When P is a set of predicate symbols and A a set of individual symbols, by $\Lambda(P, A)$ we will designate the language in predicate logic with identity determined by these sets of extra-logical constants. A formula in $\Lambda(P, A)$ is then a formula in the usual sense of predicate logic with identity using the standard logical symbols, some preassigned infinite list of individual variables (distinct from the symbols in A), the predicate symbols of P, and the individual constants of A. By a sentence in $\Lambda(P, A)$ we mean a formula with no free individual symbols outside of A.

When we speak of a language we will always understand a language in predicate logic with identity. Moreover, we will always assume the languages to be denumerable.

A set Σ of sentences in a language Λ is said to be *existentially closed in* Λ iff whenever an existential sentence $(\mathsf{E}x)\alpha(x)$ in Λ is a consequence of Σ, then also some instance $\alpha(a)$ in Λ is a consequence of Σ. (Remark: An equivalent definition is: Σ is existentially closed in Λ, iff for every sentence $(\mathsf{E}x)\alpha(x)$ in Λ there is some constant a of Λ such that $(\mathsf{E}x)\alpha(x) \rightarrow \alpha(a)$ is a consequence of Σ.)

When Λ is a language and Σ a set of sentences in Λ, we say that Σ is a *model-description* in Λ, when Σ is consistent, complete in Λ, and existentially closed in Λ. There is a close connection between this concept of model-description and the usual concept of a structure. A model-description may be thought of as the set of all sentences true in a certain structure.

When $\Lambda = \Lambda(P, A)$ is a language, Σ a set of sentences in Λ, (Λ, Σ) is called an *elementary theory about* (P, A). A set of interpretations of

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(P, A) which consists of the models of some elementary theory about (P, A), is called an *elementary class.*

2. **Theories with extra predicates. PC_Δ-classes.** Let P, P′ be two sets of predicates, let A, A′ be two sets of individual symbols, with $P \subseteq P'$, $A \subseteq A'$, and $\Lambda' = \Lambda(P', A')$, and let Σ be a set of sentences in Λ'; then by a (P, A)-*model* of (Λ', Σ) we mean an interpretation (U, f) (where U is a non-empty set and f is an assignment of classes of n-tuples of elements U to the n-place predicate symbols of P, and of elements of U to the individual constants of A) of the constants of (P, A), such that f is a subfunction of some function f' for which (U, f') is a model of (Λ', Σ). (Common terminology: (U, f) is a (P, A)-*reduct* of (U, f').) A class of interpretations of some (P, A), which forms in this sense the (P, A)-models of some theory (Λ', Σ) in a possibly extended language Λ', is called a PC_Δ-*class.* All elementary classes are obviously PC_Δ-classes, but it is well known that not all PC_Δ-classes are elementary.

When $P \subseteq P'$, and $A \subseteq A'$ and $\Lambda' = \Lambda(P', A')$, we will refer to a theory $(\Lambda' \Sigma)$ as a theory with extra predicates with respect to $\Lambda(P, A)$.

The problem naturally arises of characterizing the PC_Δ-classes. We may formulate the question so: What properties of interpretations (U, f) may be described using extra predicates, which can not be described elementarily? The main theorem of this paper (Theorem 2) gives a partial answer to this question; more precisely, it characterizes the classes of denumerable models in a PC_Δ-class.

We give now some examples of simple properties of interpretations which can be expressed using extra predicates.

(a) Given some sets P (of predicate symbols) and A (of individual constants), and an infinite set $\gamma_1(x), \gamma_2(x), \ldots$ of conditions in $\Lambda(P, A)$ with one free variable, we consider the set $\mathscr{M}$ of interpretations of (P, A) such that there is some element in the interpretation which satisfies all the conditions $\gamma_1(x), \gamma_2(x), \ldots$. Such a set is not always elementary (i.e. there is not always a set Σ of sentences in Λ such that the models of (Λ, Σ) are the interpretations in question.) Obviously this set of models can be described with an infinite expression of the simple type $(Ex)\Phi$, where Φ is the infinite conjunction of the elementary formulas $\gamma_1(x), \gamma_2(x), \ldots$. It is easy to see that it can also be described using extra predicates. Namely, if we let $P_1(x)$ be a one-place predicate symbol not in P, Λ' the language obtained by adjoining P_1 to $\Lambda(P, A)$, and Σ the system of sentences

$(\mathsf{E}x)\,\mathsf{P}_1(x)$,

$(\forall x)\,(\mathsf{P}_1 x \rightarrow \gamma_1(x))$,

$(\forall x)\,(\mathsf{P}_1 x \rightarrow \gamma_2(x))$,

$\ldots$,

it is easily verified that the (P, A)-models of (Λ', Σ) are exactly those in the set described.

Slightly more complicated types of infinite expressions which may be of some interest in themselves, and for which there are translations in terms of systems with extra predicates, are:

(b) Denumerable conjunctions of elementary formulas, containing denumerably many free variables, preceded by a denumerable string of existential quantifiers. Schematically: $(\mathsf{E}x_1)\,(\mathsf{E}x_2)\ldots\prod_{i\in\omega}\gamma_i$.

(c) Denumerable conjunctions of elementary formulas, preceded by an infinite string of quantifiers, where all the universal quantifiers precede all the existential ones. Schematically: $(\forall x_1)\,(\forall x_2)\ldots(\forall x_n)\,(\mathsf{E}y_1)\,(\mathsf{E}y_2)\ldots\prod_{i\in\omega}\gamma_i$.

(d) A more general type of infinite expression, for which there are equivalent theories with extra predicates, is the one considered in the next section.

3. Theories with extra predicates of special form. A type of infinite expression which is of central interest for this paper is the one consisting of denumerable conjunctions of elementary formulas preceded by an arbitrary string of quantifiers. We will refer to them as denumerable conjunctive statements, or briefly, conjunctive statements. We will show that the interpretation-classes so described can also be described by means of theories with extra predicates of a certain simple form.

Definition. *Let Λ(P, A) be a language. We will say that a theory (Λ', Σ) is a* theory with extra predicates of special form with respect to Λ(P, A) *iff the symbols in Λ' which are not in Λ(P, A) are the predicates* $\mathsf{P}_1(x_1, \ldots, x_{k_1}, y_1), \ldots, \mathsf{P}_i(x_1, \ldots, x_{k_i}, y_1, \ldots, y_i) \ldots$, *for some non-decreasing sequence $k_1, \ldots, k_i, \ldots$, and the sentences of Σ consist of the two groups:*

Group (1). *The sentences*

$$(\forall x_1) \ldots (\forall x_{k_1})\,(\mathsf{E}y_1)\,\mathsf{P}_1(x_1, \ldots, x_{k_1}, y_1)$$

and $(\forall x_1) \ldots (\forall x_{k_{i+1}}) (\forall y_1) \ldots (\forall y_i) (P_i(x_1, \ldots, x_{k_i}, y_1, \ldots, y_i)$
$\rightarrow (Ey_{i+1})P_{i+1}(x_1, \ldots, x_{k_{i+1}}, y_1, \ldots, y_{i+1}))$
for $i = 1, 2, \ldots$.

Group (2). *An arbitrary class of sentences of the form*

$$(\forall x_1) \ldots (\forall x_{k_i}) (\forall y_1) \ldots (\forall y_i) (P_i(x_1, \ldots, x_{k_i}, y_1, \ldots, y_i) \rightarrow \gamma)$$

where γ *is any formula in* Λ.

Theorem 1. *Let* $\Lambda = \Lambda(\mathrm{P}, \mathrm{A})$ *be a language. Let* Γ *be an infinite expression consisting of a denumerable conjunction* Φ *of elementary formulas in* Λ *preceded by an infinite string of quantifiers. Then there is a theory* T *with extra predicates of special form with respect to* Λ, *which has the same* (P, A)*-models as* Γ.

Proof. We may assume that the quantifier-string in Γ is $(\forall x_1) (\exists y_1) \ldots (\forall x_i) (\exists y_i) \ldots$. If the given Γ does not have a quantifier-string of this form, we can easily, by inserting vacuous quantifiers, form an equivalent expression Γ' with such a quantifier-string.

We form Λ' by adjoining to Λ predicate-symbols $P_1(x_1, y_1), \ldots, P_i(x_1, \ldots, x_i, y_1, \ldots, y_i), \ldots$, which are not symbols in Λ, and we let Σ consist of the sentences

$(\forall x_1) (Ey_1) P_1(x_1, y_1)$

and $(\forall x_1) \ldots (\forall x_{i+1}) (\forall y_1) \ldots (\forall y_i) (P_i(x_1, \ldots, x_i, y_1, \ldots, y_i)$
$\rightarrow (Ey_{i+1})P_{i+1}(x_1, \ldots, x_{i+1}, y_1, \ldots, y_{i+1}))$
for $i = 1, 2, \ldots$,

together with the formulas

$$(\forall x_1) \ldots (\forall x_i) (\forall y_1) \ldots (\forall y_i) (P_i(x_1, \ldots, x_i, y_1, \ldots, y_i) \rightarrow \gamma)$$

where γ is any conjunction of a finite number of conjuncts in Φ with no variables free other than $x_1, \ldots, x_i, y_1, \ldots, y_i$. An easy, intuitive argument shows that the (P, A)-models of the theory (Λ', Σ) are the same as the models of Γ.

Remark. It is easy to see that the converse is also true; i.e., if (Λ', Σ) is a theory of special form with respect to $\Lambda(\mathrm{P}, \mathrm{A})$, then there is a denumerable conjunctive statement in $\Lambda(\mathrm{P}, \mathrm{A})$ which has the same (P, A)-models. What can be expressed by these infinite statements is

then the same as what can be expressed by theories with extra predicates of special form.

The following theorem is the main result of the paper.

Theorem 2. *Let $\Lambda = \Lambda(\mathrm{P}, \mathrm{A})$ be a language. Then for any theory T with extra predicates with respect to $\Lambda(\mathrm{P}, \mathrm{A})$ there is some theory $T^* = (\Lambda', \Sigma)$ of special form with respect to $\Lambda(\mathrm{P}, \mathrm{A})$ which has the same denumerable and finite* (P, A)*-models as the given theory T.*

4. Lemmas for Theorem 2. For the proof of Theorem 2 it is convenient to introduce some new concepts and state some simple propositions about these.

Definition. *A function f is called a* substitution *iff its domain and range consist of individual symbols.*

When γ is a formula and f is a substitution, then by $\mathrm{Subst}_f(\gamma)$ we denote the result of replacing in γ, for every a for which $f(a)$ is defined, every free occurrence of a by $f(a)$, changing the bound variables if necessary to avoid collision of variables. When Σ is a set of formulas, then by $\mathrm{Subst}_f(\Sigma)$ we denote the set of all formulas $\mathrm{Subst}_f(\gamma)$, for γ in Σ. When Λ is a language, with A as its set of individual constants, by $\mathrm{Subst}_f(\Lambda)$ we mean the language with the set ((A − domain of f) $\cup$ f-image of A) as its set of individual constants, and the same predicate symbols as Λ.

Lemma 1. *If Σ, Σ' are two sets of sentences in the language Λ, with Σ included in Σ', then if Σ is existentially closed in Λ, so is Σ'.*

Proof. Assume $\Sigma \subseteq \Sigma'$, Σ existentially closed in Λ. Assume $\Sigma' \vdash (\mathsf{E}x)\alpha(x)$ where $(\mathsf{E}x)\alpha(x)$ is in Λ. $\Sigma \vdash (\mathsf{E}y)[(\mathsf{E}x)\alpha(x) \rightarrow \alpha(y)]$ holds since $\vdash (\mathsf{E}y)[(\mathsf{E}x)\alpha(x) \rightarrow \alpha(y)]$. Since Σ is existentially closed, it follows that $\Sigma \vdash (\mathsf{E}x)\alpha(x) \rightarrow \alpha(a)$ for some a in Λ. Since $\Sigma \subseteq \Sigma'$ and $\Sigma' \vdash (\mathsf{E}x)\alpha(x)$, we get $\Sigma' \vdash \alpha(a)$.

Lemma 2. *Let Λ be a language, Σ a set of sentences in Λ. If Σ is consistent and existentially closed in Λ, then there is a set Σ' of sentences in Λ, $\Sigma \subseteq \Sigma'$, such that Σ' is a model-description* (i.e., such an Σ can be extended to a model-description without addition of any individual constants not already provided by Λ).

Proof. There is an extension Σ' of Σ which is consistent and complete in Λ. Σ' is existentially closed, according to Lemma 1. Therefore Σ' is a model-description.

Lemma 3. *Let Σ be a set of sentences in a language Λ. Let $\mathsf{y}_1, \mathsf{y}_2, \ldots$ be a sequence of distinct individual constants not in Λ, and let Λ' be the language obtained by adjoining the symbols $\mathsf{y}_1, \mathsf{y}_2, \ldots$ to Λ. Then there is a sequence $\alpha_1(\mathsf{y}_1), \alpha_2(\mathsf{y}_1, \mathsf{y}_2), \ldots$ of sentences in Λ' with the following properties:*

(1) $\Sigma \vdash (\mathsf{E}\mathsf{v}_1)\alpha_1(\mathsf{v}_1)$

$\Sigma, \alpha_n(\mathsf{y}_1, \ldots, \mathsf{y}_n) \vdash (\mathsf{E}\mathsf{v}_{n+1})\alpha_{n+1}(\mathsf{y}_1, \ldots, \mathsf{y}_n, \mathsf{v}_{n+1})$

for $n = 1, 2, \ldots$.

(2) *The set $A = \{\alpha_1(\mathsf{y}_1), \alpha_2(\mathsf{y}_1, \mathsf{y}_2), \ldots\}$ is existentially closed.*

(3) $A \vdash \Sigma$ (*Should be read: Every formula in Σ is deducible from the set A.*)

Proof. The process of constructing such a set is familiar from Henkin's completeness proof.

Lemma 4. *Let β be an elementary formula, Σ a set of elementary formulas. Let f be a substitution whose domain contains all individual symbols free in Σ or β. Let I_f be the set of all formulas of the form $\mathsf{a} = \mathsf{b}$, such that $f(\mathsf{a})$ and $f(\mathsf{b})$ are both defined and $f(\mathsf{a}) = f(\mathsf{b})$. Then if $\mathrm{Subst}_f(\Sigma) \vdash \mathrm{Subst}_f(\beta)$ then $\Sigma, I_f \vdash \beta$.*

The proof of this lemma is rather immediate by semantical considerations.

Lemma 5. *Let Λ be a language, A the set of individual constants in Λ, let Σ be a set of sentences in Λ, f a substitution. Then if Σ is existentially closed in Λ, $\mathrm{Subst}_f(\Sigma)$ is existentially closed in $\mathrm{Subst}_f(\Lambda)$.*

Proof. We introduce a new substitution g, whose domain is A, and such that $g(\mathsf{x}) = f(\mathsf{x})$ when $f(\mathsf{x})$ is defined, and $g(\mathsf{x}) = \mathsf{x}$ when $f(\mathsf{x})$ is undefined. We observe that $\mathrm{Subst}_g(\gamma)$ is the same as $\mathrm{Subst}_f(\gamma)$ when γ is a sentence in Λ.

We assume now that $(\mathsf{E}\mathsf{x})\alpha(\mathsf{x})$ is a sentence in $\mathrm{Subst}_f(\Lambda)$ such that

$\mathrm{Subst}_f(\Sigma) \vdash (\mathsf{E}x)\alpha(x)$. Then there is a sentence $(\mathsf{E}x)\beta(x)$ in Λ such that $(\mathsf{E}x)\alpha(x) = \mathrm{Subst}_f((\mathsf{E}x)\beta(x))$. From the relation

$\mathrm{Subst}_f(\Sigma) \vdash \mathrm{Subst}_f((\mathsf{E}x)\beta(x))$ we have

$\mathrm{Subst}_g(\Sigma) \vdash \mathrm{Subst}_g((\mathsf{E}x)\beta(x))$.

Applying Lemma 4, we get

$\Sigma, I_g \vdash (\mathsf{E}x)\beta x$,

and therefore

$\Sigma \vdash (\mathsf{E}x)(\iota^* \to \beta(x))$,

where ι^* is a conjunction of some members of I_g.
Since by assumption Σ is existentially closed in Λ we will also have, for some a

$\Sigma \vdash \iota^* \to \beta(a)$.

Then, since $\mathrm{Subst}_g(\iota^*)$ is a logical truth, we have

$\mathrm{Subst}_g(\Sigma) \vdash \mathrm{Subst}_g(\beta(a))$.

But $\mathrm{Subst}_g(\beta(a)) = \alpha(g(a))$. And since $\mathrm{Subst}_g(\Sigma) = \mathrm{Subst}_f(\Sigma)$ we have derived that

$\mathrm{Subst}_f(\Sigma) \vdash \alpha(g(a))$.

We may conclude that $\mathrm{Subst}_f(\Sigma)$ is existentially closed in $\mathrm{Subst}_f(\Lambda)$.

5. Proof of Theorem 2. We divide the proof into three parts. In Part 1 we show how to construct, for a given theory T with extra predicates, a certain corresponding theory T^* of special form. In Part 2 we show that a finite or denumerable (P, A)-model of the constructed theory T^* is also a (P, A)-model of T. In Part 3 we show that any (P, A)-model of T is a (P, A)-model of T^*.

Part (1) (Construction of the theory T^*).

We let Λ_T be the language of T, Σ_T the set of axioms of T, so that $T = (\Lambda_T, \Sigma_T)$. We let $\Lambda_T(\mathrm{X})$ be the result of adjoining to Λ_T the infinite set $\mathrm{X} = \{x_1, x_2, \ldots\}$ of new individual constants, and we let $\Lambda_T(\mathrm{X}, \mathrm{Y})$ be the result of adjoining to $\Lambda_T(\mathrm{X})$ the infinite set $\mathrm{Y} = \{y_1, y_2, \ldots\}$ of new individual constants.

Applying Lemma 3 to the theory $(\Lambda_T(\mathrm{X}), \Sigma_T)$ and the sequence

$y_1, y_2, \ldots$, we know that there exists some sequence B of formulas

$$\beta_1(x_1, \ldots, x_{k_1}, y_1), \ldots, \beta_i(x_1, \ldots, x_{k_1}, y_1, \ldots, y_i), \ldots$$

with no individual symbols not in A except as shown, and with the following properties:

(1) $\Sigma_T \vdash (Ev_1)\beta_1(x_1, \ldots, x_{k_1}, v_1)$

$\Sigma_T, \beta_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_n) \vdash$

$(Ev_{n+1})\beta_{n+1}(x_1, \ldots, x_{k_{n+1}}, y_1, \ldots, y_n, v_{n+1})$

for $n = 1, 2, \ldots$.

(2) The set B is existentially closed in $A_T(X, Y)$.

(3) $B \vdash \Sigma_T$.

(4) $\beta_{n+1}(x_1, \ldots, x_{k_{n+1}}, y_1, \ldots, y_{n+1}) \vdash \beta_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_n)$

for $n = 1, 2, \ldots$.

By the application of Lemma 3 we get directly a sequence with the first three properties. Taking conjunctions of these formulas we get a sequence which also has the fourth property.

Using the sequence B of formulas so constructed, we construct the new theory T^* as follows.

The language A^* is obtained by adjoining, for $i = 1, 2, \ldots$, the predicates $P_i(x_1, \ldots, x_{k_i}, y_1, \ldots, y_i)$ to A. (The predicate P_i is $(k_i + i)$-place.)

The axioms Σ^* are the following:

Group (1). $(\forall u_1) \ldots (\forall u_{k_1})\,(Ev_1)P_1(u_1, \ldots, u_{k_1}, v_1)$

and $(\forall u_1) \ldots (\forall u_{k_{n+1}})\,(\forall v_1) \ldots (\forall v_n)\,P_n(u_1, \ldots, u_{k_n}, v_1, \ldots, v_n)$

$\rightarrow (Ev_{n+1})\,P_{n+1}(u_1, \ldots, u_{k_{n+1}}, v_1, \ldots, v_{n+1})$

for $n = 1, 2, \ldots$.

Group (2). All sentences

$$(\forall u_1) \ldots (\forall u_{k_n})\,(\forall v_1) \ldots (\forall v_n)\,(P_n(u_1, \ldots, u_{k_n}, v_1, \ldots, v_n) \rightarrow \gamma)$$

such that γ is a formula in A, and such that $T, \beta_n(u_1, \ldots, v_n) \vdash \gamma$.

It is easy to see that the theory $T^* = (A^*, \Sigma^*)$ so constructed is a theory of special form with respect to A.

Part (2). (Proof that if $\mathfrak{M}$ is a finite or denumerable (P, A)-model of T^*, then $\mathfrak{M}$ is a (P, A)-model of T.)

$T=(\Lambda_T, \Sigma_T)$ is the given theory with extra predicates with respect to $\Lambda(\mathrm{P}, \mathrm{A})$. $T^*=(\Lambda^*, \Sigma^*)$ is the corresponding theory of special form constructed as described in Part 1. We assume that $\mathfrak{M}$ is a finite or denumerable (P, A)-model of T^*.

Then $\mathfrak{M}$ is the (P, A)-reduct of some model $\mathfrak{M}^*$ of T^*. We let Δ be the set of sentences describing $\mathfrak{M}^*$ (in some language $\Lambda^{*\prime}$ obtained from Λ^* by adjoining names of all the objects in the model). We let $\mathsf{a}_1, \mathsf{a}_2, \ldots$ be a complete enumeration of the individual constants in Δ (not necessarily without repetitions.) In virtue of the first group of axioms for T_{P} there must be constants $\mathsf{b}_1, \mathsf{b}_2, \ldots$ (among the a's), such that the formulas

$$\mathsf{P}_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i) \quad \text{for } i=1, 2, \ldots$$

belong to Δ.

Choosing a particular such sequence $\mathsf{b}_1, \mathsf{b}_2, \ldots$, we form the set Δ' of formulas as follows: We first remove all formulas containing any of the extra predicates $\mathsf{P}_1, \mathsf{P}_2, \ldots$. (What is left then is a description Δ_n of the given (P, A)-model $\mathfrak{M}$.) Then we add for every i the formula

$$\beta_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i).$$

We claim that the set Δ' so constructed has the properties:

(i) $\Delta' \vdash \Sigma_T$;

(ii) Δ' is consistent;

(iii) Δ' is existentially closed in Λ_T' (the language obtained by adjoining all the individual constants of Δ' to Λ_T).

To see that (i) holds we observe that Δ' includes $\mathrm{Subst}_f(B)$ for some substitution f which only changes the symbols $\mathsf{x}_1, \mathsf{x}_2, \ldots$ and $\mathsf{y}_1, \mathsf{y}_2, \ldots$, which do not occur in the axioms of T. Then since $B \vdash S_T$, also $\mathrm{Subst}_f(B) \vdash \Sigma_T$, and $\Delta' \vdash \Sigma_T$.

We prove now that (ii) holds. If Δ' were inconsistent, there would be a sentence γ belonging to $\Delta_{\mathfrak{M}}$ and an i such that $\beta_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i)$ is inconsistent with γ, i.e. such that $\beta_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i) \vdash \neg\gamma$. (We use here the fact that the formulas β_i are cumulative; see property 4 of the sequence B.) This means that

$$\mathrm{Subst}_f(\beta_i(\mathsf{x}_1, \ldots, \mathsf{x}_{k_i}, \mathsf{y}_1, \ldots, \mathsf{y}_i)) \vdash \mathrm{Subst}_f(\neg\gamma')$$

for some substitution f and some formula γ' such that $\mathrm{Subst}_f(\gamma')=\gamma$. (To make I_f the right set we should choose f so that it is defined for all the free individual symbols of β_i and γ'.) Then according to Lemma 4

we will have $\beta_i, I_f \vdash \neg \gamma'$, and $\beta_i \vdash \iota^* \rightarrow \neg \gamma'$, where ι^* is some conjunction of members of I_f. Therefore according to the construction of T^*, the universal closure of $\mathsf{P}_i(\mathsf{x}_i, \ldots, \mathsf{x}_{k_i}, \mathsf{y}_1, \ldots, \mathsf{y}_i) \rightarrow (\iota^* \rightarrow \gamma')$ is an axiom of T^*. Performing the operation Subst_f on this formula, and observing that $\mathrm{Subst}_f(\iota^*)$ is a logical truth, we deduce that $\mathsf{P}_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i) \rightarrow \neg \gamma$ is deducible from the axioms of T^*. But this is impossible since both formulas $\mathsf{P}_i(\mathsf{a}_1, \ldots, \mathsf{a}_{k_i}, \mathsf{b}_1, \ldots, \mathsf{b}_i)$ and γ belong to Δ, which is a model-description for T. This concludes the proof that Δ' is consistent.

Proof of (iii): We let f be the substitution defined for the symbols $\mathsf{x}_1, \mathsf{x}_2, \ldots$ and $\mathsf{y}_1, \mathsf{y}_2, \ldots$ such that for every i $f(\mathsf{x}_i) = \mathsf{x}_i$ and $f(\mathsf{y}_i) = \mathsf{b}_i$. Then Δ' includes the set $\mathrm{Subst}_f(B)$. Since B is existentially closed in $\Lambda_T(\mathrm{X}, \mathrm{Y})$, by Lemma 5 $\mathrm{Subst}_f(B)$ is existentially closed in $\mathrm{Subst}_f(\Lambda_T(\mathrm{X}, \mathrm{Y}))$. But since $\mathsf{a}_1, \mathsf{a}_2, \ldots$ is a complete enumeration of the individual symbols of Δ, $\mathrm{Subst}_f(\Lambda_T(\mathrm{X}, \mathrm{Y}))$ is evidently Λ_T'. (Notice that here the assumption that $\mathfrak{M}$ is denumerable is used, via the assumption that $\mathsf{a}_1, \mathsf{a}_2, \ldots$ is a complete enumeration of the individual constants of Δ.) Then since Δ' includes $\mathrm{Subst}_f(B)$, and $\mathrm{Subst}_f(B)$ is existentially closed in Λ_T', Δ' itself is existentially closed in Λ_T' (Lemma 1).

From the properties (i), (ii), and (iii), which have now been established, it follows by Lemma 2 that Δ' can be extended to a model-description for T without addition of any new individual constants. This means that the given model $\mathfrak{M}$ is a (P, A)-model of T. This concludes Part 2 of the proof.

Part (3). (Proof that if $\mathfrak{M}$ is a (P, A)-model of T, then $\mathfrak{M}$ is a (P, A)-model of T^*.)

To establish the assertion of this part of the proof it is sufficient to show that any given model of T can be extended, without addition of individuals, to a model of the union-theory $T \cup T^*$. This is seen readily since the axioms of T^* can be made true by letting the interpretations of the predicates P_i coincide with those of the predicates β_i.

6. Uniqueness of T^*. In the construction described for the theory T^* from a given theory T we have left free the choice of sequence B with the required properties 1–4. There are evidently many different ways to choose this sequence. However, we shall show (Theorem 4) that different choices lead to essentially the same result.

Definition. *We let* $T_1=(\Lambda_1, \Sigma_1)$ *and* $T_2=(\Lambda_2, \Sigma_2)$ *be two theories and we let* $\Lambda(\mathrm{P}, \mathrm{A})$ *be some common sublanguage of* Λ_1 *and* Λ_2. *We say that* T_1 *is* (P, A)*-interpretable in* T_2 *iff there is a substitution of expressions in* Λ_2 *for the non-logical constants in* Λ_1 *outside of* (P, A) *such that the axioms of* T_1 *are turned into theorems of* T_2.

We observe that T^* is in this sense (P, A)-interpretable in T. The following theorem shows that T^* is the strongest theory of special form with respect to $\Lambda(\mathrm{P}, \mathrm{A})$ which is (P, A)-interpretable in T.

Theorem 3. *When* T' *is a theory of special form with respect to* $\Lambda(\mathrm{P}, \mathrm{A})$ *which is* (P, A)*-interpretable in* T, *then* T' *is* (P, A)*-interpretable in* T^*.

Proof. We let the extra predicates of T' be $\mathsf{R}_1, \ldots, \mathsf{R}_i, \ldots$ where R_i is (m_i+i)-place. (To facilitate the writing and printing we will write $\mathsf{R}_i(\mathrm{X}, \mathrm{Z})$ as an abbreviation for $\mathsf{R}_i(\mathsf{x}_1, \ldots, \mathsf{x}_{m_i}, \mathsf{z}_1, \ldots, \mathsf{z}_i)$ and $\mathsf{P}_i(\mathrm{X}, \mathrm{Y})$ for $\mathsf{P}_i(\mathsf{x}_1, \ldots, \mathsf{x}_{k_i}, \mathsf{y}_1, \ldots, \mathsf{y}_i)$.) The axioms of T' are assumed to be

$$(\forall \mathsf{x}_1) \ldots (\forall \mathsf{x}_{m_1})\, (\mathsf{E}\mathsf{z}_1)\mathsf{R}_1(\mathrm{X}, \mathrm{Z})$$

and $(\forall \mathsf{x}_1) \ldots (\forall \mathsf{x}_{m_{i+1}})\, (\forall \mathsf{z}_1) \ldots (\forall \mathsf{z}_i)\, (\mathsf{R}_i(\mathrm{X}, \mathrm{Z}) \rightarrow (\mathsf{E}\mathsf{z}_{i+1})\ \mathsf{R}_{i+1}(\mathrm{X}, \mathrm{Z}))$ for $i=1, 2, \ldots,$
together with a set of axioms of the form

$(\forall \mathsf{x}_1) \ldots (\forall \mathsf{x}_{m_i})\, (\forall \mathsf{z}_1) \ldots (\forall \mathsf{z}_i)\, (\mathsf{R}_i(\mathrm{X}, \mathrm{Z}) \rightarrow \gamma)$, where γ is a formula in $\Lambda(\mathrm{P}, \mathrm{A})$.

As before we let $\beta_1, \beta_2, \ldots$ be the formulas used in forming the theory T^* (Section 5, Part 1).

We let $\alpha_1, \alpha_2, \ldots$ be some formulas representing $\mathsf{R}_1(\mathrm{X}, \mathrm{Z}), \mathsf{R}_2(\mathrm{X}, \mathrm{Z}), \ldots$ in some (P, A)-interpretation of T' in T. This means that the system of axioms for T' will be turned into a set of theorems of T, if we everywhere read α_i instead of $\mathsf{R}_i(\mathrm{X}, \mathrm{Z})$.

From the conditions that

$$T \vdash (\mathsf{E}\mathsf{z}_1)\alpha_1$$

and

$$T, \alpha_i \vdash (\mathsf{E}\mathsf{z}_{i+1})\alpha_{i+1}$$

together with the fact that B is existentially closed in $\Lambda_T(\mathrm{X}, \mathrm{Y})$ and stronger than Σ_T ($B \vdash \Sigma_T$), we can deduce that there is a sequence $i_1, i_2, \ldots$ of numbers, and a sequence $\mathsf{u}_1, \mathsf{u}_2, \ldots$ of individual symbols in the language $\Lambda_T(\mathrm{X}, \mathrm{Y})$ such that for every n

$$\beta_{i_n}, \mathsf{z}_1=\mathsf{u}_1, \ldots, \mathsf{z}_n=\mathsf{u}_n \vdash \alpha_n. \tag{1}$$

Now if γ is a formula in $\Lambda(\mathrm{P}, \mathrm{A})$ such that

$$T', \mathsf{R}_n(\mathrm{X}, \mathrm{Z}) \vdash \gamma \tag{2}$$

we know that

$$T, \alpha_n \vdash \gamma.$$

But then because of (1)

$$T, \beta_{i_n} \wedge \mathsf{z}_1 = \mathsf{u}_1 \wedge \ldots \wedge \mathsf{z}_n = \mathsf{u}_n \vdash \gamma.$$

And because of the way T^* is constructed we will have

$$T^*, \mathsf{P}_{i_n}(\mathrm{X}, \mathrm{Y}) \wedge \mathsf{z}_1 = \mathsf{u}_1 \wedge \ldots \wedge \mathsf{z}_n = \mathsf{u}_n \vdash \gamma. \tag{3}$$

We have now proved that (3) holds whenever γ is a formula in $\Lambda(\mathrm{P}, \mathrm{A})$ and $T', \mathsf{R}_n(\mathrm{X}, \mathrm{Z}) \vdash \gamma$.

We denote by Q_n the string of existential quantifiers $(\mathsf{E}\mathsf{v})$ for those variables v which are free in $\mathsf{P}_{i_n}(\mathrm{X}, \mathrm{Y}) \wedge \mathsf{z}_1 = \mathsf{u}_1 \wedge \ldots \wedge \mathsf{z}_n = \mathsf{u}_n$ but not free in $\mathsf{R}_n(\mathrm{X}, \mathrm{Z})$. For every n, we let δ_n be the expression

$$Q_n(\mathsf{P}_{i_n}(\mathrm{X}, \mathrm{Y}) \wedge \mathsf{z}_1 = \mathsf{u}_1 \wedge \ldots \wedge \mathsf{z}_n = \mathsf{u}_n).$$

It will still hold that

$$T^*, \delta_n \vdash \gamma \tag{4}$$

whenever γ is an Λ-formula such that $\mathsf{R}_n(\mathrm{X}, \mathrm{Z}) \rightarrow \gamma$ is deducible in T'. And from the axioms of T^* one can deduce that

$$T^* \vdash (\forall \mathsf{x}_1) \ldots (\forall \mathsf{x}_{m_1})\, (\mathsf{E}\mathsf{y}_1)\delta_1 \tag{5}$$

and $T^* \vdash (\forall \mathsf{x}_1) \ldots (\forall \mathsf{x}_{m_{n+1}})\, (\forall \mathsf{z}_1) \ldots (\forall \mathsf{z}_n)\, (\delta_n \rightarrow (\mathsf{E}\mathsf{z}_{n+1})\delta_{n+1})$

for $n = 1, 2, \ldots$.

We see from this that by interpreting $\mathsf{R}_n(\mathrm{X}, \mathrm{Z})$ as δ_n we get a (P, A)-interpretation of T' in T^* which converts the axioms of T' into theorems of T^*. We conclude that under the conditions of the theorem the theory T^* is (P, A)-interpretable in T^*. This concludes the proof.

Applying this theorem to the case that T' is also a theory T^*, but with a different choice of the sequence B, we get:

Theorem 4. *When $(T^*)_1$ and $(T^*)_2$ are two theories constructed as in Section 5 from a theory T, although with different choices for the sequence B, each of these two theories is* (P, A)*-interpretable in the other.*

7. Application to Craig's lemma.

Theorem 5. *If T^* is a theory of special form with respect to a language $\Lambda = \Lambda(\mathrm{P}, \mathrm{A})$, then for any finite subset Γ of the axioms of T^* there is some Λ-sentence which has the same* (P, A)*-models as Γ.*

Sketch of proof. Instead of writing down a proof for the general case I will prove the statement for an example of a sufficiently general character.

We let the extra predicates P_i be $(i+i)$-place, and we let Γ consist of the universal closures of the following six formulas:

$$(\mathsf{E}y_1)\ \mathsf{P}_1(x_1, y_1),$$

$$\mathsf{P}_1(x_1, y_2) \rightarrow (\mathsf{E}y_2)\ \mathsf{P}_2(x_1, x_2, y_1, y_2),$$

$$\mathsf{P}_2(x_1, x_2, y_1, y_2) \rightarrow (\mathsf{E}y_3)\ \mathsf{P}_3(x_1, x_2, x_3, y_1, y_2, y_3),$$

$$\mathsf{P}_1(x_1, y_1) \rightarrow \alpha_1(x_1, y_1),$$

$$\mathsf{P}_2(x_1, x_2, y_1, y_2) \rightarrow \alpha_2(x_1, x_2, y_1, y_2),$$

$$\mathsf{P}_3(x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow \alpha_3(x_1, x_2, x_3, y_1, y_2, y_3),$$

where α_1, α_2, and α_3 are Λ-formulas.

This system can be seen to have the same (P, A)-models as the following system of three sentences, using extra function symbols f_1, f_2, f_3:

$$(\forall x_1)\alpha_1(x_1, \mathsf{f}_1(x_1)),$$

$$(\forall x_1)\ (\forall x_2)\alpha_2(x_1, x_2, \mathsf{f}_1(x_1), \mathsf{f}_2(x_1, x_2)),$$

$$(\forall x_1)\ (\forall x_2)\ (\forall x_3)\alpha_3(x_1, x_2, x_3, \mathsf{f}_1(x_1), \mathsf{f}_2(x_1, x_2), \mathsf{f}_3(x_1, x_2, x_3)).$$

The conjunction of these sentences has the form

$$(\forall x_1)\ (\forall x_2)\ (\forall x_3)\ \beta(x_1, x_2, x_3, \mathsf{f}_1(x_1), \mathsf{f}_2(x_1, x_2), \mathsf{f}_3(x_1, x_2, x_3)).$$

But this formula is known to have the same (P, A)-models as the Λ-sentence $(\forall x_1)\ (\mathsf{E}y_1)\ (\forall x_2)\ (\mathsf{E}y_3)\ (\forall x_3)\ (\mathsf{E}y_3)\ \beta\ (x_1, x_2, x_3, y_1, y_2, y_3))$.

Combining this theorem with Theorem 2 it is easy to derive Craig's lemma.

Craig's lemma. *Let $\alpha(\mathsf{R}, \mathsf{S})$ and $\beta(\mathsf{R}, \mathsf{T})$ be two elementary formulas, such that $\alpha(\mathsf{R}, \mathsf{S})$ contains no predicate symbols except* R *and* S*, and $\beta(\mathsf{R}, \mathsf{T})$ contains no predicate symbols except* R *and* T*, $\mathsf{S} \neq \mathsf{T}$, and such that $\alpha(\mathsf{R}, \mathsf{S}) \vdash \beta(\mathsf{R}, \mathsf{T})$. Then there is some formula $\gamma(\mathsf{R})$ which contains only the predicate symbol* R*, such that $\alpha(\mathsf{R}, \mathsf{S}) \vdash \gamma(\mathsf{R})$ and $\gamma(\mathsf{R}) \vdash \beta(\mathsf{R}, \mathsf{T})$.*

Proof. Let A be the set of free individual symbols in α and β. By Theorem 2 there is some theory T^* of special form with respect to $\Lambda(\mathsf{R}, \mathrm{A})$ with the same denumerable (R, A)-models as $\alpha(\mathsf{R}, \mathsf{S})$. Then (since $\mathsf{T} \neq \mathsf{S}$) $\beta(\mathsf{R}, \mathsf{T})$ must be true for all denumerable interpretations which make the axioms of T^* true. Then by the Löwenheim-Skolem theorem $\beta(\mathsf{R}, \mathsf{T})$ is a consequence of T^*. Then by compactness it is a consequence of a finite subset of T^*. But according to Theorem 5 such a finite subset has the same (R, A)-models as some sentence $\gamma(\mathsf{R})$ containing only the predicate symbol R. From what we have said about $\gamma(\mathsf{R})$ it is clear that $\alpha(\mathsf{R}, \mathsf{S}) \vdash \gamma(\mathsf{R})$ and $\gamma(\mathsf{R}) \vdash \beta(\mathsf{R}, \mathsf{T})$.

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A LÖWENHEIM-SKOLEM THEOREM FOR CARDINALS FAR APART*

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1. Introduction. The well-known theorem of Löwenheim, Skolem, and Tarski states that:

1.1. *If a set Σ of first-order sentences has a model of infinite power then it has a model of each infinite power.*

(We assume always that there are only countably many symbols.)

If the sentences of Σ involve a one-place relation symbol U, then a model of Σ determines not only the cardinal number $\varkappa$ of its universe but also the cardinal number λ of its U. Thus, if $\varkappa, \lambda, \varkappa', \lambda'$ are infinite cardinals, such that $\varkappa \geqslant \lambda$ and $\varkappa' \geqslant \lambda'$, one may raise the 'two cardinals problem':

1.2. *Does every set of sentences having a model of type $\langle \varkappa; \lambda \rangle$ also have one of type $\langle \varkappa'; \lambda' \rangle$?*

Similar questions can be raised concerning three or more cardinals.

The sentence $\forall x \mathsf{U}x$ has a model of type $\langle \varkappa; \lambda \rangle$ if and only if $\varkappa = \lambda$. R. Robinson pointed out to the author in 1958 that, more generally:

1.3. *For each n, there is a sentence σ_n such that σ_n has a model of arbitrary type $\langle \varkappa; \lambda \rangle$ if and only if $\lambda < \varkappa \leqslant 2^{\lambda, n}$.*

(We write $2^{\lambda, 0} = \lambda$, $2^{\lambda, n+1} = 2^{2^{\lambda, n}}$, and $2^{\lambda, \omega} = \sup\{2^{\lambda, n} / n \in \omega\}$.) Indeed, for σ_1 one may take the 'axiom of extensionality'

$$(\forall x, y)\,\{\forall u[\mathsf{U}u \to (\mathsf{E}ux \leftrightarrow \mathsf{E}uy)] \to x \doteq y\}.$$

σ_2, σ_3, etc., can then be formed by an obvious kind of iteration.

Assuming G. C. H. (the generalized continuum hypothesis), 1.3 still represents the only negative information known about the problems 1.2.

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It may be conjectured that: *all assertions of the form* 1.2 *not immediately blocked by* 1.3 *are true, assuming* G. C. H.

The main theorem of this paper, Theorem 4.1, has the following corollary, establishing that part of the above conjecture which concerns two cardinals 'far apart':

Corollary 4.2. *If Σ has a model of type $\langle \varkappa; \lambda \rangle$, where $\varkappa \geqslant 2^{\lambda, \omega}$ (or even if, for each n, Σ has a model of type $\langle \varkappa; \lambda \rangle$, where $\varkappa \geqslant 2^{\lambda, n}$), then Σ has a model of every type $\langle \varkappa'; \lambda' \rangle$ $(\varkappa' \geqslant \lambda')$.*

Theorem 4.1 itself draws a stronger conclusion from the same hypothesis. We shall see, in Section 4, that, consequently, a number of further corollaries can be obtained by using 4.1 in combination with results concerning the problems 1.2 obtained earlier by the author, Chang-Keisler, and Chang, and with results of Fuhrken concerning the languages Q_α. (Q_α is the first-order language enriched by the new quantifier 'there exist at least $\aleph_\alpha$'. See Fuhrken [*] for a discussion of these languages and references.)

In particular, we shall obtain in this way some results of the form 1.2 concerning three or more cardinals. Moreover, we shall show that *the Hanf number of Q_α is $2^{\aleph_\alpha, \omega}$, for $\alpha = 1, 2, \ldots, \omega + 2, \omega + 3, \ldots$.* This means that any (countable) set of Q_α-sentences having a model of power $2^{\aleph_\alpha, \omega}$ has a model of every higher power, and that $2^{\aleph_\alpha, \omega}$ is the least such power. (In Hanf [62c] it was observed that for a very wide class of languages such a cardinal must exist.)

Since virtually all previously known results concerning the problems 1.2 can thus be used in combination with 4.1, we shall devote Section 3 to a summary of these results.

2. Preliminaries. Letters $\varkappa, \lambda, \mu$ always denote infinite cardinals (initial ordinals); $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta$ ordinals; m, n natural numbers, i.e., members of ω. $c(X)$ is the cardinal number of the set X. $\varkappa^+$ is the smallest $\lambda > \varkappa$.

$\mathfrak{A}$ is a structure of *signature* ϱ or, simply, a ϱ-structure if: ϱ is a function on an ordinal α (less than ω_1) to ω, and $\mathfrak{A} = \langle A, R \rangle = \langle A, R_\xi \rangle_{\xi < \alpha}$, where A is a non-empty set and each R_ξ is a subset of $A^{\varrho(\xi)}$, the set of all $\varrho(\xi)$-tuples of elements of A. The universe $|\mathfrak{A}|$ of $\mathfrak{A}$ is A, and the length of $\mathfrak{A}$ is $\alpha = l(\mathfrak{A})$. $c(\mathfrak{A}) = c(|\mathfrak{A}|)$. If $\alpha = \beta + \gamma$, and $\mathfrak{B} = \langle A, R_\xi \rangle_{\xi < \beta}$ then we say that $\mathfrak{A}$ is an *expansion* of $\mathfrak{B}$ and $\mathfrak{B}$ a *contraction* of $\mathfrak{A}$. We write $\mathfrak{B} = \mathfrak{A} \restriction \beta$ and $\mathfrak{A} = (\mathfrak{B}, R_{\beta+\xi})_{\xi < \gamma}$. Similarly, if $\mathfrak{C} = \langle C, S_\xi \rangle_{\xi < \delta}$ and $|\mathfrak{C}| \subseteq |\mathfrak{A}|$,

then we write $(\mathfrak{A}, \mathfrak{C})$ for $(\mathfrak{A}, C, S_\xi)_{\xi<\delta}$. $\mathfrak{A}'$ is a substructure of $\mathfrak{A}$ ($\mathfrak{A}' \subseteq \mathfrak{A}$) if $\mathfrak{A}$ and $\mathfrak{A}' = \langle A', R'_\xi \rangle_{\xi<\alpha}$ have the same signature ϱ, $A' \subseteq A$, and $R'_\xi = R_\xi \cap A'^{\varrho(\xi)}$ for each $\xi<\alpha$. If $0 \neq X \subseteq A$, $\mathfrak{A} \restriction X$ is the substructure of $\mathfrak{A}$ with universe X.

L^ϱ is the first-order language with equality corresponding to the signature ϱ. $\mathfrak{A} \equiv \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ have the same signature ϱ and have the same true ϱ-sentences (sentences of L^ϱ). If $a_\xi \in |\mathfrak{A}|$ for $\xi<\eta$, we may form the structure $\overline{\mathfrak{A}} = (\mathfrak{A}, \{a_\xi\})_{\xi<\eta}$ of type, say, $\bar{\varrho}$, which will be denoted by $(\mathfrak{A}, a_\xi)_{\xi<\eta}$. (These auxiliary structures may have uncountable length.) In $L^{\bar{\varrho}}$ one can say just what could be said in L^ϱ by adding individual constants. $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ ($\mathfrak{A} \prec \mathfrak{B}$), or $\mathfrak{B}$ an elementary extension of $\mathfrak{A}$, if $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and for any finite (or, equivalently, arbitrary) ordinal sequence $a_0, \ldots, a_\xi, \ldots (\xi<\eta)$ of elements of $|\mathfrak{A}|$, $(\mathfrak{A}, a_\xi)_{\xi<\eta} \equiv (\mathfrak{B}, a_\xi)_{\xi<\eta}$. If Q is the ξ-th predicate of L^ϱ, then $Q^{\mathfrak{A}} = R_\xi$. If U is a unary predicate, we write $\mathfrak{A} \prec_U \mathfrak{B}$ to mean that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ with the same U, i.e., with $U^{\mathfrak{A}} = U^{\mathfrak{B}}$. $\mathfrak{A} \prec_* \mathfrak{B}$ means $\mathfrak{A} \prec \mathfrak{B} \restriction l(\mathfrak{A})$ and $\mathfrak{A} \prec_* \mathfrak{B}$ means $\mathfrak{A} \restriction l(\mathfrak{B}) \prec \mathfrak{B}$.

$\prec$ is transitive and, moreover, if $\mathfrak{A}, \mathfrak{B} \prec \mathfrak{C}$ and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ then $\mathfrak{A} \prec \mathfrak{B}$. Tarski's union theorem asserts that, if $\mathfrak{A}_\xi \prec \mathfrak{A}_\eta$ whenever $\xi<\eta<\zeta$, then there is a unique system $\mathfrak{B} = \bigcup \{\mathfrak{A}_\xi / \xi<\zeta\}$ such that $|\mathfrak{B}| = \bigcup \{|\mathfrak{A}_\xi| / \xi<\zeta\}$ and, for each $\xi<\zeta$, $\mathfrak{A}_\xi \prec \mathfrak{B}$.

We shall frequently make use of the downward Löwenheim-Skolem theorem, in the following detailed form due to Tarski:

2.1. *If $X \subseteq |\mathfrak{A}|$ and $c(X) \leqslant \mu \leqslant c(\mathfrak{A})$, then $\mathfrak{A}$ has an elementary substructure $\mathfrak{B}$ such that $X \subseteq |\mathfrak{B}|$ and $c(\mathfrak{B}) = \mu$.*

3. Summary of known results. Suppose $U_0, \ldots, U_n$ are distinct unary predicates, and $\mathfrak{A}$ is any structure for which $U_0^{\mathfrak{A}}, \ldots, U_n^{\mathfrak{A}}$ are defined. We say $\mathfrak{A}$ is of *reduced type* $\langle \lambda_0, \ldots, \lambda_n \rangle$ if, for $i \leqslant n$, $c(U_i^{\mathfrak{A}}) = \lambda_i$. We say that $\mathfrak{A}$ is of *type* $\langle \varkappa; \lambda_0, \ldots, \lambda_n \rangle$ if, in addition, $c(\mathfrak{A}) = \varkappa$ and, moreover, $U_0^{\mathfrak{A}} \supseteq U_1^{\mathfrak{A}} \supseteq \ldots \supseteq U_n^{\mathfrak{A}}$. (Thus the cardinals of a type are always decreasing). To facilitate discussion of problem 1.2, we agree to say that a type (or reduced type) τ *yields* another type (resp., reduced type) τ' of the same length if every Σ having a model of (reduced) type τ has also one of (reduced) type τ'. It is obvious that no difference in the meaning of 'yields' would result from replacing the $U_i^{\mathfrak{A}}$ by subsets $\theta_i^{\mathfrak{A}}$ of $|\mathfrak{A}|$ defined by fixed formulas $\theta_0, \ldots, \theta_n$ with one free variable. From this fact and 2.1, one infers the useful observation:

3.1. *Suppose* $\langle \varkappa; \lambda_0, \ldots, \lambda_n \rangle$ *and* $\langle \varkappa'; \lambda_0', \ldots, \lambda_n' \rangle$ *are types. Then* $\langle \varkappa; \lambda_0, \ldots, \lambda_n \rangle$ *yields* $\langle \varkappa'; \lambda_0', \ldots, \lambda_n' \rangle$ *if and only if* $\langle \varkappa, \lambda_0, \ldots, \lambda_n \rangle$ *yields* $\langle \varkappa', \lambda_0', \ldots, \lambda_n' \rangle$.

Sketch of proof: From right to left is obvious. Consider the converse for, say, $n=1$. Suppose Σ has a model $\mathfrak{A}$ of reduced type $\langle \varkappa, \lambda_0, \lambda_1 \rangle$. By 2.1, we can assume $c(\mathfrak{A})=\varkappa$. Expand $\mathfrak{A}$ to $\mathfrak{B}$ by adjoining a one-to-one function on $U_0^{\mathfrak{A}}$ onto $|\mathfrak{A}|$, another on $U_1^{\mathfrak{A}}$ onto $U_1^{\mathfrak{A}} \cup U_2^{\mathfrak{A}}$, and also the sets $V_0^{\mathfrak{A}} = U_1^{\mathfrak{A}} \cup U_2^{\mathfrak{A}}$ and $V_1^{\mathfrak{A}} = U_2^{\mathfrak{A}}$. Then $\mathfrak{B}$ has type $\langle \varkappa; \lambda_0, \lambda_1 \rangle$ with respect to V_0, V_1, and the hypothesis can be applied.

The well-known theorems 1.1 and 2.1 imply at once two cases of the two (or even n) cardinals problem:

3.2. *Any type yields* $\langle \mu; \mu, \ldots, \mu \rangle$.

3.3. *A type* $\langle \varkappa; \lambda_0, \ldots, \lambda_n \rangle$ *yields* $\langle \varkappa'; \lambda_0, \ldots, \lambda_n \rangle$, *if* $\varkappa \geqslant \varkappa' \geqslant \lambda_0$.

The first result, beyond these observations, on the two cardinals problem was the proposition:

3.4. *If* $\varkappa > \lambda$, $\langle \varkappa; \lambda \rangle$ *yields* $\langle \aleph_1; \aleph_0 \rangle$.

The proof, which is given in Morley-Vaught [62], involves the homogeneous structures discussed there.

Secondly, Chang-Keisler [62], using ultraproducts, obtained the following theorem:

3.5. (G.C.H.) *If* $\varkappa > \lambda' > \lambda$, $\langle \varkappa; \lambda \rangle$ *yields* $\langle \varkappa; \lambda' \rangle$.

Their method also yielded some other, scattered results, including some special cases of 4.2, e.g.: $\langle 2^{\lambda \cdot \omega}, \lambda \rangle$ yields $\langle (2^{\lambda \cdot \omega})^+, \lambda \rangle$, if $\lambda = \lambda^{\aleph_0}$. (It obviously also gives certain n cardinals theorems, $n \geqslant 3$).

Next, Gebhard Fuhrken obtained a number of results concerning the languages Q_α. His work included the discovery that the results or methods of proof of 3.4 and 3.5 could be used in studying the Q_α or even (in the case of 3.5) the languages having many such added quantifiers. A summary of this work (plus references) can be found in Fuhrken [*]. Here we want only to give an example, 3.6, for the sake of some analogies in Section 4, illustrating the fact that some of Fuhrken's results contain new information with regard to the *old* languages L. Call a class $\mathscr{C}$ of ϱ-structures *compact* if, whenever every finite subset of a set Σ of ϱ-sentences has a model in $\mathscr{C}$, so has Σ. Call $\mathscr{C}$ *complete* if the set of ϱ-

sentences true in all members of $\mathscr{C}$ is recursively enumerable [1]. Let $\mathscr{C}^{\varrho}\tau$ be the class of all ϱ-structures of type or reduced type τ. From results in Fuhrken [*] it follows that:

3.6. *$\mathscr{C}^{\varrho}\langle\aleph_1, \aleph_0\rangle$ is compact and complete.*

Recently, C. C. Chang has established the theorem:

3.7. (G.C.H.) *If $\varkappa > \lambda$ and μ is regular, $\langle\varkappa; \lambda\rangle$ yields $\langle\mu^+; \mu\rangle$.*

Chang's beautiful proof of 3.7 will be published in the near future.

4. The main theorem and its consequences.

Theorem 4.1. *For each signature ϱ there is a signature $\varrho' \supseteq \varrho$ and a recursive set T of ϱ'-sentences such that:*

(.1) *Every model of T has a proper elementary extension with the same U, and*

(.2) *For any set Σ of ϱ-sentences, if Σ has, for each n, a model $\mathfrak{D}_n$ of some type $\langle\varkappa_n; \lambda_n\rangle$ such that $\varkappa_n \geqslant 2^{\lambda_n \cdot n}$, then $\Sigma \cup T$ is consistent, and, indeed, every finite subset of T has a model $\mathfrak{A}$ such that, for some m, $\mathfrak{A} \prec_{*U} \mathfrak{D}_m$.*

The proof of 4.1 will be given in the final section, Section 5. In the remainder of Section 4, we shall assume 4.1 and examine its consequences.

By (.1) every model $\mathfrak{A}$ of T has a proper elementary extension $\mathfrak{A}'$ with the same U; by 2.1 we can further stipulate that $c(\mathfrak{A}) = c(\mathfrak{A}')$. Iterating, and taking unions at limit stages, we see that (.1) implies:

(.1′) *Every model $\mathfrak{A}$ of T has an elementary extension, with the same U, in every power $\varkappa > c(\mathfrak{A})$.*

Corollary 4.2 (stated in Section 1) is immediate from 4.1, since, by Gödel's completeness theorem and 1.1, the consistent set $\Sigma \cup T$ has a model $\mathfrak{A}$ of type $\langle\lambda'; \lambda'\rangle$ and (.1′) then gives a model of type $\langle\varkappa'; \lambda'\rangle$. Morley [*] has subsequently found another, completely different, proof of 4.2. His method also establishes Corollary 4.6, below, but appears not to yield the other corollaries of 4.1 which we shall state below.

[1] When ϱ is infinite, there is some awkwardness involved in defining 'recursively enumerable' or 'recursive' in contexts like this. We are going to ignore this problem, since all results we discuss involving these notions are positive and their proofs exhibit specific, clearly effective procedures.

(On the other hand, of course, it yields his important main result, not obtainable from 4.1.)

In proving 4.2 no use was made of the final clause of 4.1.2 following 'consistent'. We shall now see how that clause can be used.

Corollary 4.3. *If* $\varkappa \geqslant 2^{\lambda \cdot \omega}$, $\lambda > \mu$, *and* $\varkappa' \geqslant \aleph_1$, *then* $\langle \varkappa; \lambda, \mu \rangle$ *yields* $\langle \varkappa'; \aleph_1, \aleph_0 \rangle$.

Proof. We are given a model $\mathfrak{D}$ of Σ such that $c(\mathfrak{D}) = \varkappa$, $c(U_0^{\mathfrak{D}}) = \lambda$, $c(U_1^{\mathfrak{D}}) = \mu$, and $U_0^{\mathfrak{D}} \supseteq U_1^{\mathfrak{D}}$; we can assume Σ contains $\forall x(U_1 x \to U_0 x)$. In 4.1, take U_0 for 'U' and $\mathfrak{D}$ for each '$\mathfrak{D}_n$'. Then, by 4.1, every finite subset Σ' of $\Sigma \cup T$ has a model $\mathfrak{A} \prec_{*U_0} \mathfrak{D}$. Clearly $c(U_0^{\mathfrak{A}}) = \lambda$ and $c(U_1^{\mathfrak{A}}) = \mu$, so, by 3.4 (and 3.1), each Σ' also has a model of reduced type $\langle \aleph_1, \aleph_0 \rangle$. Consequently, by 3.6, $\Sigma \cup T$ itself has such a model. Applying (.1'), we obtain the desired model of Σ of type $\langle \varkappa'; \aleph_1, \aleph_0 \rangle$.

Obviously 4.3 could have been improved by imitating the parenthetical weakening of the hypothesis in 4.2. By applying Fuhrken's normal form theorem for Q_1 (see Fuhrken [*]) to the thus improved 4.3, we obtain at once (in view of 1.3):

Corollary 4.4. *The Hanf number of* Q_1 *is* $2^{\aleph_0 \cdot \omega}$. *Indeed, if a set* Σ *of* Q_1^{ϱ}*-sentences has models of power at least* $\varkappa$ *for every* $\varkappa < 2^{\aleph_0 \cdot \omega}$, *then* Σ *has a model of each power* $\geqslant \aleph_1$.

The part of the proof of 4.3 needed to prove 4.4 uses the compactness theorem 3.6 but not the two cardinals theorem 3.4. In a similar way, one can show:

Corollary 4.5. (G.C.H.) *Let* Q^* *be obtained from* L^{ϱ} *by adding all of the quantifiers 'there exist at least* $\aleph_{\alpha_i}$' *for* $i \in \omega$. *Assume that each* ω_{α_i} *is not confinal with* ω, *and, if* α_i *is of the form* $\beta_i + 1$, *that* ω_{β_i} *is also not confinal with* ω. *Let* $\varkappa = \sup \{\aleph_{\alpha_i + \omega} / i \in \omega\}$. *Then the Hanf number of* Q^* *is* $\varkappa$. *In particular the Hanf number of* Q_{α_0} *is* $\aleph_{\alpha_0 + \omega}$ *(for each such* α_0*).*

Sketch of proof: By 1.3, the Hanf number is obviously at least as big as claimed. For the rest, to simplify the notation, we deal with the case $Q^* = Q_{\omega_1}^{\varrho}$. By Fuhrken's second normal form theorem (see Fuhrken [*]), we may correlate with every sentence σ of $Q_{\omega_1}^{\varrho}$ a sentence $\bar{\sigma}$ of $L^{\bar{\varrho}}$ (for a certain fixed $\bar{\varrho}$) involving a (new) unary predicate U and a (new) binary predicate P, in such a way that the following holds: For any set Σ of $Q_{\omega_1}^{\varrho}$-sentences, and any ϱ-structure $\mathfrak{A}$ of power at least $\aleph_{\omega_1}$,

(1) $\begin{cases} \mathfrak{A} \text{ is a model of } \Sigma \text{ if and only if } \mathfrak{A} \text{ has an expansion } \mathfrak{B} \text{ such that} \\ \mathfrak{B} \text{ is a model of } \bar{\Sigma} = \{\bar{\sigma}/\sigma \in \Sigma\},\ c(U^{\mathfrak{B}}) = \aleph_{\omega_1}, \text{ and } P^{\mathfrak{B}} \text{ is an ordering} \\ \text{of } U^{\mathfrak{B}} \text{ in which the set of predecessors of any element has power} \\ \text{less than } \aleph_{\omega_1}. \end{cases}$

Now suppose Σ has a model $\mathfrak{A}$ of power at least $\aleph_{\omega_1+\omega}$. According to (1), $\mathfrak{A}$ has a certain expansion $\mathfrak{B}$. By 4.1, every finite subset of $\bar{\Sigma} \cup T$ (the T for $\bar{\sigma}$) has a model $\mathfrak{D}$ such that $\mathfrak{D} \prec_{*U} \mathfrak{B}$ and so $U^{\mathfrak{D}}$ and $P^{\mathfrak{D}}$ are still as in (1). Hence, by the compactness of Q_{ω_1} (Fuhrken [*]), the whole $\bar{\Sigma} \cup T$ has a model $\mathfrak{C}$ whose U and P are as in (1). By 2.1 and 4.1.1′, $\mathfrak{C}$ may be taken to have arbitrary power $\geqslant \aleph_{\omega_1}$. Finally, by (1) again, $\mathfrak{C}$ has a contraction which is a model of Σ.

In the general case, Fuhrken's normal form and compactness theorems still hold, as he pointed out to me. If there are infinitely many U_i and P_i, then just when 4.1 is applied one should introduce a new U supposed to include all the U_i; this is possible since any finite subset of $\bar{\Sigma}$ will involve only finitely many U_i.

It is an open problem whether the Hanf number of Q_α is $2^{\aleph_{\alpha,\omega}}$ also for ordinals $\alpha > 0$ which are confinal with ω or have immediate predecessors confinal with ω. Of course the method above would yield a positive answer if it could be shown that these Q_α are compact.

One can also return once more to the proof of 4.3 and use Chang's two cardinals theorem 3.7 in place of 3.4, since the required compactness result to substitute for 3.6 is available. We shall include the results thus obtained in the very general corollary 4.7, below, whose proof also employs another device which we now discuss.

Theorem 4.1 has an aspect not yet pointed out, namely, that it yields, itself, new kinds of compactness and completeness. The simplest such result is:

Corollary 4.6. *$\mathscr{C}^c\langle 2^{\lambda,\omega}, \lambda\rangle$ is compact and complete.*

Proof. By 4.1, for any Σ,

(2) Σ has a model of type $\langle 2^{\lambda,\omega}; \lambda\rangle$ if and only if $\Sigma \cup T$ is consistent.

4.6 follows from (2) by easy arguments analogous to part of the proof of 3.6 (see Fuhrken [*]).

This new compactness gives us another result which can be used in applying 4.1 (that is, in the role of 3.6 in the proof of 4.3); and furthermore, every application of 4.1, relying on some compactness theorem,

yields in a similar way a new compactness theorem. In this way, proceeding by induction, one easily establishes:

Corollary 4.7. *(G.C.H. needed sometimes). Suppose* $\kappa_i \geqslant 2^{\kappa_{i+1}, \omega}$ *(for* $i<n$*) and* $\kappa_i' \geqslant \kappa'_{i+1}$ $(i<n)$. *Then* $\langle \kappa_0, \ldots, \kappa_n \rangle$ *yields* $\langle \kappa_0', \ldots, \kappa_n' \rangle$, *and* $\mathscr{C}^{\varrho}\langle \kappa_0, \ldots, \kappa_n \rangle$ *is compact and complete. If (moreover)* $\kappa_n > \lambda$ *and either* $\kappa_n' = \lambda'^{+}$ *and* λ' *is regular or* $\kappa_n' = \lambda'$, *then* $\langle \kappa_0, \ldots, \kappa_n, \lambda \rangle$ *yields* $\langle \kappa_0', \ldots, \kappa_n', \lambda' \rangle$. *If* $\kappa_n = \lambda^{+}$, *where* λ *is regular, then* $\mathscr{C}^{\varrho}\langle \kappa_0, \ldots, \kappa_n, \lambda \rangle$ *is compact and complete.*

C. C. Chang has constructed an example showing that the problems 1.2 for *three* cardinals have negative answers other than those implied by 1.3. By 4.3, we have: (G.C.H.) $\langle \aleph_{\omega+1}, \aleph_1, \aleph_0 \rangle$ yields $\langle \aleph_2, \aleph_1, \aleph_0 \rangle$. But Chang's (straightforward) example shows: (G.C.H.) $\langle \aleph_{\omega+1}, \aleph_\omega, \aleph_0 \rangle$ does not yield $\langle \aleph_2, \aleph_1, \aleph_0 \rangle$.

5. Proof of the main theorem. If f is a biunique function on a set $A' \supseteq A = |\mathfrak{A}|$, we write f^*A for the image set of A under f, and $f^*\mathfrak{A}$ for the image structure onto which f maps $\mathfrak{A}$ isomorphically. It should be noted that a recursive set of sentences is easily constructed whose models are exactly those structures of the form $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ such that $\mathfrak{A}$ has signature ϱ_1, $\mathfrak{B}$ and $\mathfrak{C}$ have signature ϱ_2, and $\mathfrak{C} \prec \mathfrak{B}$.

In order to avoid long formulas we shall not describe the set T of 4.1 explicitly. Rather, we shall describe the class of models of T in such a way that the reader will easily verify that there is a recursive set T of sentences having just those models.

Definition 5.1. *The models of* T *are to be just those structures* $\mathfrak{A}$ *such that*:

(.1) $\mathfrak{A}$ *has the form* $(\mathfrak{D}, f, \mathfrak{B}^0, \ldots, \mathfrak{B}^n, \ldots)_{n<\omega}$ *where* $\mathfrak{D}$ *is a* ϱ*-structure,* f *a binary relation, and for each* $n<\omega$, $\mathfrak{B}^n$ *has the same signature as the structure* $\mathfrak{A}^n = (\mathfrak{D}, f, \mathfrak{B}^0, \ldots, \mathfrak{B}^{n-1})$;

(.2) f *is a biunique function on* $|\mathfrak{D}|$ *into* $|\mathfrak{D}|$;

(.3) $\mathfrak{B}^n \succ_{\cup} f^*\mathfrak{A}^n$, *for each* $n<\omega$;

(.4) $|\mathfrak{B}^0| \neq f^*|\mathfrak{A}^0|$; *and*

(.5) $\mathfrak{B}^n \succ_* \mathfrak{B}^{n-1}$, *whenever* $0<n<\omega$.

It is easy to show that 4.1.1 holds for T. Indeed, if $\mathfrak{A}$ is as above, then $\mathfrak{A}$ is isomorphic to $f^*\mathfrak{A}$, and a proper elementary extension with the same $\cup$ of $f^*\mathfrak{A}$ is 'built into' $\mathfrak{A}$, namely, the structure $\bigcup\{\mathfrak{B}^n/n \in \omega\}$. (Since the lengths of the $\mathfrak{B}^n$ are increasing rather than constant, this

assertion is not strictly speaking a case of Tarski's union theorem as stated in Section 2, but is easily inferred from it.)

Before turning to the proof of 4.1.2, we want now to point out that the idea of forming a theory T as above can be used in another, simpler, way. Let T' be described exactly as was T in 5.1, but with '$\cup$' omitted in 5.1.3. Then clearly:

(1) every model of T' has a proper ('built-in') elementary extension;

and hence (just as in the proof of 4.1.1'):

(1') every model $\mathfrak{A}$ of T' has an elementary extension in every power $\varkappa > c(\mathfrak{A})$.

In the next paragraph we will outline a proof that:

(2) every denumerable ϱ-structure $\mathfrak{D}$ has an expansion which is a model of T'.

(1') and (2) provide a proof of the upward Löwenheim-Skolem theorem (for structures of countable length) which only involves languages with countably many symbols. The familiar proof, based on the method of diagrams, depends on the use of languages with non-denumerably many symbols, which were introduced by Malcev in 1936 and, later, by Henkin and A. Robinson [2]. The author thought at first (as was stated in his lecture) that the proof above must in essence be the same as the first proof by Tarski in 1928 (the exact details of which Tarski does not recall). However, a look at the discussion in Tarski-Vaught [57, p. 94] shows that this apparently is not the case, since it is said there that Tarski's proof showed that *every* structure, of countable length but arbitrary infinite power, has a proper elementary extension. (Of course, both the above proof and Tarski's rely on the use of Tarski's idea of elementary extensions and unions in order to increase the power.)

Sketch of proof of (2): Choose f biunique on $|\mathfrak{D}|$ into $|\mathfrak{D}|$ with $|\mathfrak{D}| - f^*|\mathfrak{D}|$ infinite. By the usual, diagram, method (for a countable language), $(\mathfrak{D}, f)$ has a proper, denumerable, elementary extension $\mathfrak{D}'$. Let g be a biunique function on $|\mathfrak{D}'|$ to $|\mathfrak{D}|$, extending f and such that $|\mathfrak{D}| - g^*|\mathfrak{D}'|$ is infinite. For $\mathfrak{B}^0$ take $g^*\mathfrak{D}'$. At the next step (and all later ones) the situation is a bit more complicated: what we need is to find a denumerable $\mathfrak{D}''$ such that $\mathfrak{D}'' \succ (\mathfrak{D}, f, \mathfrak{B}^0)$ and, at the same time, $\mathfrak{D}'' \succ_* \mathfrak{D}'$.) However, it is easy to check that the diagram method

[2] For references for this whole discussion, see Tarski–Vaught [57].

again yields such a $\mathfrak{D}''$; indeed, this is an instance of a lemma due to A. Robinson [56e]. The remaining steps are all similar.

It appears that this new proof yields nothing more (and something less) than the usual shorter proof! But comprehending its role on the way toward the main theorem, 4.1, should be an aid to understanding the latter.

What remains for us to do is the proof of 4.1.2. Some preliminaries will be needed which concern a notion $\prec_\varkappa$ stronger than $\prec$. We could use for $\prec_\varkappa$ Hanf's notion of being an elementary substructure with respect to the language involving conjunctions and quantifications of length $\varkappa$ (see Hanf [62c]). Instead, we shall avoid having to discuss such languages by using a weaker notion; but the ideas in 5.3 below are nevertheless due essentially to Hanf.

Definition 5.2. $\mathfrak{A} \prec_\varkappa \mathfrak{B}$ *if* $\mathfrak{A} \prec \mathfrak{B}$ *and, for any elements* $a_0, \ldots, a_\xi, \ldots (\xi < \varkappa)$ *of* $|\mathfrak{A}|$ *and* $b_0, \ldots, b_\eta, \ldots (\eta < \varkappa)$ *of* $|\mathfrak{B}|$, *there exist elements* $c_0, \ldots, c_\eta, \ldots (\eta < \varkappa)$ *of* $|\mathfrak{A}|$ *such that* $(\mathfrak{B}, a_0, \ldots, a_\xi, \ldots, b_0, \ldots, b_\eta, \ldots)_{\xi,\eta<\varkappa} \equiv (\mathfrak{B}, a_0, \ldots, a_\xi, \ldots, c_0, \ldots, c_\eta, \ldots)_{\xi,\eta<\varkappa}$.

Lemma 5.3. (.1) *If* $\mathfrak{A} \prec_\varkappa \mathfrak{C}$ *and* $\mathfrak{A} \prec \mathfrak{B} \prec \mathfrak{C}$ *then* $\mathfrak{A} \prec_\varkappa \mathfrak{B}$.

(.2) *If* $X \subseteq |\mathfrak{A}|$ *and* $c(X) \leqslant 2^\varkappa < c(\mathfrak{A})$, *then, for some* $\mathfrak{B} \prec_\varkappa \mathfrak{A}$, $X \subseteq |\mathfrak{B}|$ *and* $c(\mathfrak{B}) = 2^\varkappa$.

(.1) is easily proved. (.2) may be derived from Hanf's downward Löwenheim-Skolem theorem, or proved directly in a similar way. Finally, we shall need the following fact, which one easily proves by using Tarski's condition for $\mathfrak{A} \prec \mathfrak{B}$ referring to satisfaction only in $\mathfrak{B}$ (i.e., Tarski-Vaught [57, Lemma 1.10]):

5.4. *If* $B = \{b_\xi / \xi < \eta\}$, $C = \{c_\xi / \xi < \eta\}$, $(\mathfrak{A}, b_\xi)_{\xi<\eta} \equiv (\mathfrak{A}, c_\xi)_{\xi<\eta}$, *and* $\mathfrak{A} \restriction B \prec \mathfrak{A}$, *then* $\mathfrak{A} \restriction C \prec \mathfrak{A}$.

Now suppose $N < \omega$ and let T_N be the subset of T whose models are just those structures $\mathfrak{A}$ such that 5.1.1–5.1.5 hold with 'ω' replaced by 'N'. In order to prove 4.1.2 it will clearly suffice to show that:

(3) If Σ has a model $\mathfrak{D}$ of type $\langle 2^{\lambda \cdot N+1}; \lambda \rangle$ then T_N has a model $\mathfrak{A} \prec^*_{\cup} \mathfrak{D}$.

Let f be a biunique function on $|\mathfrak{D}|$ into $|\mathfrak{D}|$ such that $c(|\mathfrak{D}| - f^*|\mathfrak{D}|) = c|\mathfrak{D}|$. Then $(\mathfrak{D}, f)$ is a model of T_0.

By 5.3.2 (choosing, for 'X', $\cup^{\mathfrak{D}}$ plus $2^{\lambda \cdot N}$ elements outside of $f^*|\mathfrak{D}|$) we can find $\mathfrak{A}_0$ such that $\mathfrak{A}_0 \prec_{\cup, 2^{\lambda \cdot N}-1} (\mathfrak{D}, f)$ and $c(\mathfrak{A}_0) = c(|\mathfrak{A}_0| -$

$-f^*|\mathfrak{A}_0|)=2^{\lambda,N}$. Applying 2.1, we can find $\mathfrak{C}_0$ such that $|\mathfrak{A}_0|$ is a proper subset of $|\mathfrak{C}_0|$, $\mathfrak{C}_0 \prec (\mathfrak{D}, f)$ and $c(\mathfrak{C}_0)=2^{\lambda,N}$. By 5.3.1, $\mathfrak{A}_0 \prec_{U,2^{\lambda,N-1}} \mathfrak{C}_0$. Obviously there exists a biunique function g_0 on $|\mathfrak{C}_0|$ into $|\mathfrak{A}_0|$ such that g_0 coincides with f on $|\mathfrak{A}_0|$ and $c(|\mathfrak{A}_0|-g_0^*|\mathfrak{C}_0|)=2^{\lambda,N}$. The structure $(\mathfrak{A}_0, g_0^*\mathfrak{C}_0)$ is clearly a model of T_1.

We shall now define by recursion on n, for $1 \leqslant n < N$, functions g_n and structures $\mathfrak{A}_n$ and $\mathfrak{C}_n$ such that (a)–(d), below, hold:

(a) $\mathfrak{A}_n \prec_{U,2^{\lambda,N-n-1}} \mathfrak{C}_n \prec_{*U} (\mathfrak{D}, f)$,

(b) g_n is biunique, on $|\mathfrak{C}_n|$ into $|\mathfrak{A}_n|$, and coincides with f on $|\mathfrak{A}_n|$,

(c) $c(\mathfrak{A}_n)=c(|\mathfrak{A}_n|-g_n^*|\mathfrak{C}_n|)=c(\mathfrak{C}_n)=2^{\lambda,N-n}$,

(d) $(\mathfrak{A}_n, g_n^*\mathfrak{C}_n)$ is a model of T_{n+1}.

Obviously, the case $n=N-1$ will establish (3). Since we have shown above that the same conditions hold for $n=0$, it will suffice to show that, assuming (a)–(d) hold for a particular $n<N-1$ and given g_n, $\mathfrak{A}_n$, and $\mathfrak{C}_n$, we can find g_{n+1}, $\mathfrak{A}_{n+1}$, and $\mathfrak{C}_{n+1}$ as desired.

For convenience, put $\mu=2^{\lambda,N-n-2}$ and $\varkappa=2^{\lambda,N-n-1}=2^{\mu}$. Form the structure $(\mathfrak{C}_n, (\mathfrak{A}_n, g_n^*\mathfrak{C}_n), g_n)$, of power $2^{\varkappa}$. By 5.3.2, it has a μ-elementary substructure $(\mathfrak{C}, \mathfrak{A}_{n+1}, g)$ with the same U, such that $c(\mathfrak{C})=c(|\mathfrak{A}_{n+1}|-g^*|\mathfrak{C}|)=\varkappa$. Obviously, $\mathfrak{A}_{n+1}$ is of the form $(\mathfrak{A}, g^*\mathfrak{C})$ and $c(\mathfrak{A}_{n+1})=\varkappa$; also, as one easily checks, $\mathfrak{A}_{n+1} \prec_{U,\mu} (\mathfrak{A}_n, g_n^*\mathfrak{C}_n)$. At this point we are blocked from the natural next step by the fact that $|\mathfrak{C}| \not\subset |\mathfrak{A}_n|$. What we shall do is to replace the elements of $|\mathfrak{C}|-|\mathfrak{A}_n|$ by elements of $|\mathfrak{A}_n|$ which will be just as good for our purposes. Let the elements of $|\mathfrak{A}_{n+1}|$ be $a_0, \ldots a_\xi, \ldots (\xi<\varkappa)$ and let those of $|\mathfrak{C}|-|\mathfrak{A}_n|=|\mathfrak{C}|-|\mathfrak{A}_{n+1}|$ be $c_0, \ldots, c_\eta, \ldots (\eta<\varkappa)$. Since, by (a), $\mathfrak{A}_n \prec_\varkappa \mathfrak{C}_n$, there exist elements $b_0, \ldots, b_\eta, \ldots (\eta<\varkappa)$ of $\mathfrak{A}_n$ such that

$$(\mathfrak{C}_n, a_0, \ldots a_\xi, \ldots; c_0, \ldots, c_\eta, \ldots)_{\xi,\eta<\varkappa} \equiv (\mathfrak{C}_n, a_0, \ldots, a_\xi, \ldots, b_0, \ldots b_\eta, \ldots)_{\xi,\eta<\varkappa}.$$

Clearly there is a biunique function F on $|\mathfrak{C}|$ such that for any ξ, η, $F(a_\xi)=a_\xi$ and $F(c_\eta)=b_\eta$. Let $C'=F^*|\mathfrak{C}|$ and let $\mathfrak{C}'=\mathfrak{C}_n \restriction C'$. Clearly F is an isomorphism on $\mathfrak{C}$ onto $\mathfrak{C}'$, and is the identity on $|\mathfrak{A}_{n+1}|$. By 5.4, $\mathfrak{C}' \prec \mathfrak{C}_n$; hence, $\mathfrak{C}' \prec \mathfrak{A}_n$. Let g' be the function on $|\mathfrak{C}'|$ such that, for any $c \in |\mathfrak{C}'|$, $g'(c)=g(F^{-1}(c))$; i.e., $g'=g \circ F^{-1}$. Clearly g' coincides with f on $|\mathfrak{A}_{n+1}|$ and $g'^*\mathfrak{C}'=g^*\mathfrak{C}$.

Now $c(\mathfrak{C}')=\varkappa$ and $|\mathfrak{C}'| \subseteq |\mathfrak{A}_n|$ so by 2.1 we can find an elementary substructure $\mathfrak{C}_{n+1}$ of $(\mathfrak{A}_n, g_n^*\mathfrak{C}_n)$ such that $C' \subseteq |\mathfrak{C}_{n+1}|$ and $c(\mathfrak{C}_{n+1})=\varkappa$. By 5.3.1, we conclude that $\mathfrak{A}_{n+1} \prec_{U,\mu} \mathfrak{C}_{n+1} \prec_{*U} (\mathfrak{D}, f)$ (as desired in (a)).

Obviously, $\mathfrak{C}' {}_* \prec \mathfrak{C}_{n+1}$. Since $c(|\mathfrak{A}_{n+1}| - g'^*|\mathfrak{C}'|) = \varkappa$, we can obviously find a biunique function g_{n+1} on $|\mathfrak{C}_{n+1}|$ into $|\mathfrak{A}_{n+1}|$, coinciding with g' on C' (and hence with f on $|\mathfrak{A}_{n+1}|$), and such that $c(|\mathfrak{A}_{n+1}| - g^*_{n+1}|\mathfrak{C}_{n+1}|) = \varkappa$. Thus (b) and (c) hold as desired. From the assumed (d), we see that $\mathfrak{A}_{n+1}$ is a model of T_{n+1}. Hence in order to show that the new (d) (with '$n+1$' for 'n') holds, it only remains to prove that $g^*_{n+1}\mathfrak{C}_{n+1} \succ_\cup f^*\mathfrak{A}_{n+1}$ and $g^*_{n+1}\mathfrak{C}_{n+1} \succ_* g^*\mathfrak{C}$. But $\mathfrak{C}_{n+1} \succ_\cup \mathfrak{A}_{n+1}$ so $g^*_{n+1}\mathfrak{C}_{n+1} \succ_\cup g^*_{n+1}\mathfrak{A}_{n+1} = f^*\mathfrak{A}_{n+1}$; and $\mathfrak{C}_{n+1} \succ_* \mathfrak{C}'$, so $g^*_{n+1}\mathfrak{C}_{n+1} \succ_* g^*_{n+1}\mathfrak{C}' = g^*\mathfrak{C}$. This completes the proof of 4.1.

In conclusion, we want to discuss briefly the situation when one considers structures of non-denumerable length α. Of course, no questions of completeness are to be considered. The proofs of the main theorem, 4.1, and Corollaries 4.2 and 4.6 extend at once, provided the assumptions $\lambda_n, \lambda, \lambda' \geqslant c(\alpha)$ are added. As their authors have noted, the various results of Chang, Keisler, and Fuhrken which rely on ultraproducts also have immediate extensions, if suitable hypotheses are added, and so does Chang's 3.7 where one apparently needs $\lambda \geqslant c(\alpha)$ and $\mu > c(\alpha)$ instead of $\mu \geqslant c(\alpha)$ (so that 3.7 does not include 3.4). Hence the same applies to 4.5 and parts of 4.7. On the other hand, 3.4 clearly fails if $c(\alpha) > \aleph_0$ and so do 3.6 and 4.3.

It is not obvious that 4.4 fails. However, Fuhrken has pointed out to the author that from proposition (6)(b) of Fuhrken [*], with $\alpha = 1$, it easily follows that, if we allow $c(\alpha) \leqslant \omega_1$, then the Hanf number of Q_1 is at least $2^{\aleph_0 \cdot \omega_2}$. Whether this Hanf number is exactly $2^{\aleph_0 \cdot \omega_2}$ is an open problem, though Morley has told me that by the methods of Morley [*] he can show that it is at most $2^{\aleph_0, ((2^{\aleph_1})^+)}$.

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SYNONYMOUS THEORIES *

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The goal of this communication is a model-theoretic characterization of synonymous theories. Synonymity being defined as a purely syntactical-logical relation between theories, we search for a corresponding, purely mathematical, relation between structures, so that two theories are synonymous if and only if their models are related in this mathematical way. Assuming the conditions required and without going into details we shall consider the concepts of elementary equivalence and elementary class (in the wider sense) as purely mathematical, since their mathematical character has been established [1]. The relation of *coalescence*, which is defined from these concepts, seems to be a mathematical counterpart of synonymity and suitable for the characterization intended.

For reasons of simplicity we confine ourselves to a *standard language* $\mathscr{L}_{RS}$ containing only two non-logical constants: the distinct predicates R and S with finite ranks m and n, respectively. The natural numbers $m, n \geqslant 1$ are kept fixed throughout this whole paper. From $\mathscr{L}_{RS}$ we form the sublanguages $\mathscr{L}_R$ and $\mathscr{L}_S$ in the obvious way, the former by dropping the predicate S and the latter by dropping the predicate R. Let Σ_{RS} be the set of all sentences of $\mathscr{L}_{RS}$, let Σ_R be the set of all sentences of $\mathscr{L}_R$, and let Σ_S be the set of all sentences of $\mathscr{L}_S$. We use Γ_{RS}, Γ_R, Γ_S to refer to subsets of Σ_{RS}, Σ_R, Σ_S, respectively. We put $\Gamma_{RS/S} = \Gamma_{RS} \cap \Sigma_R$ and $\Gamma_{RS/R} = \Gamma_{RS} \cap \Sigma_S$. For our purpose we are interested in special sets of sentences, namely *theories*. Obviously, if Γ_{RS} is a theory, then $\Gamma_{RS/S}$ and $\Gamma_{RS/R}$ are theories also, and in particular subtheories of theory Γ_{RS}.

A theory Γ_{RS} is called a *definitional extension of theory* Γ_R *with* S if and only if there is an explicit definition of S from R, say the explicit

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[1] Cf. Tarski [52b], Keisler [63b], [61], Kochen [59a], [61], Frayne–Morel–Scott [58].

definition ϱ, such that $\Gamma_{RS} = \Gamma_R \cup \{\varrho\}$, where $\cup$ denotes logical union. Obviously, if theory Γ_{RS} is a definitional extension of theory Γ_R, then $\Gamma_R = \Gamma_{RS/S}$. We say that the theories Γ_R and Γ_S are *synonymous* if and only if there exists a theory Γ_{RS} which is a definitional extension of Γ_R with S and of Γ_S with R. In other words, the theories Γ_R and Γ_S are synonymous if and only if there exists an explicit definition of S from R, say ϱ, and an explicit definition of R from S, say σ, such that $\Gamma_R \cup \{\sigma\} = \Gamma_S \cup \{\sigma\}$.

In 1951 A. Robinson introduced the concept of *related algebras*, terminology which in 1963 he changed to *related sets of sentences* [2]. Applying his idea to the subject of this paper one can speak of *related theories*, and if one confines oneself to the example of this paper, one can say that theory Γ_R and theory Γ_S are *related* if and only if they are synonymous. In 1957 R. Montague introduced the concept of *bilateral interpretability* [3]. Two theories are bilaterally interpretable if and only if they are synonymous. D. Kaplan has shown by an example [4] that two theories can be mutually interpretable without being bilaterally interpretable. For the sake of brevity we do not want to go into details of the theory of synonymity, which is part of the theory of definition. We mention only that synonymity is an equivalence relation which reveals some interesting aspects of the theory of definition.

On the mathematical side we consider a *similarity type* $\mathscr{S}_{RS}$, which is the class of all *relational structures* $\langle M, R, S\rangle$, where M is any set, $R \subseteq {}^mM$, and $S \subseteq {}^nM$. If $\mathfrak{M}_{RS} = \langle M, R, S\rangle$, then by $\mathfrak{M}_{RS/S}$ we understand the relational structure $\langle M, R\rangle$ and by $\mathfrak{M}_{RS/R}$ the relational structure $\langle M, S\rangle$. We put $\mathscr{S}_R = \mathscr{S}_{RS/S} = \{\mathfrak{M}_{RS/S}\colon \mathfrak{M}_{RS} \in \mathscr{S}_{RS}\}$ and $\mathscr{S}_S = \mathscr{S}_{RS/R} = \{\mathfrak{M}_{RS/R}\colon \mathfrak{M}_{RS} \in \mathscr{S}_{RS}\}$ and it is obvious that $\mathscr{S}_R$ and $\mathscr{S}_S$ are similarity types, subtypes of $\mathscr{S}_{RS}$. By $\mathfrak{M}_{RS}$, $\mathfrak{M}_R$, $\mathfrak{M}_S$ we refer to relational structures, members of $\mathscr{S}_{RS}$, $\mathscr{S}_R$, $\mathscr{S}_S$, respectively. If $\mathfrak{M}_R = \mathfrak{M}_{RS/S}$, then we say that $\mathfrak{M}_{RS}$ is *an enrichment of* $\mathfrak{M}_R$ *with* S, or that $\mathfrak{M}_R$ is *the impoverishment of* $\mathfrak{M}_{RS}$ *to* R. With every relational structure $\mathfrak{M}_{RS}$ we associate a class of structures, $\mathscr{M}_{RS} \subseteq \mathscr{S}_{RS}$, which is the *elementary type of* $\mathfrak{M}_{RS}$, i.e., $\mathfrak{M}'_{RS} \in \mathscr{M}_{RS}$ if and only if $\mathfrak{M}'_{RS}$ is *elementarily equivalent to* $\mathfrak{M}_{RS}$. Simi-

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[2] Cf. A. Robinson [51a], pp. 70, 71 and A. Robinson [63], pp. 31–33.

[3] Cf. Montague [57], Section 6.

[4] Unpublished.

larly, with every $\mathfrak{M}_R$ we associate its elementary type $\mathscr{M}_R$ and with every $\mathfrak{M}_S$ its elementary type $\mathscr{M}_S$. We put $\mathscr{M}_{RS/S} = \{\mathfrak{M}_{RS/S}\colon \mathfrak{M}_{RS} \in \mathscr{M}_{RS}\}$ and $\mathscr{M}_{RS/R} = \{\mathfrak{M}_{RS/R}\colon \mathfrak{M}_{RS} \in \mathscr{M}_{RS}\}$. Obviously, if $\mathfrak{M}_{RS}$ is an enrichment of $\mathfrak{M}_R$ with S, then $\mathfrak{M}'_{RS/S} \in \mathscr{M}_R$ for every $\mathfrak{M}'_{RS} \in \mathscr{M}_{RS}$.

We shall say that an enrichment $\mathfrak{M}_{RS}$ of $\mathfrak{M}_R$ with S is *a definitional enrichment of* $\mathfrak{M}_R$ *with* S if and only if $\mathscr{M}_{RS/S} = \mathscr{M}_R$ and for every $\mathfrak{M}'_R \in \mathscr{M}_R$ there is one and only one $\mathfrak{M}'_{RS} \in \mathscr{M}_{RS}$ such that $\mathfrak{M}'_{RS/S} = \mathfrak{M}'_R$ [5]. Thus we arrive at a concept which is comparable with synonymity. We shall say that the relational structures $\mathfrak{M}_R$ and $\mathfrak{M}_S$ are *coalescent* if and only if they have a common enrichment $\mathfrak{M}_{RS}$, which is both a definitional enrichment of $\mathfrak{M}_R$ with S and a definitional enrichment of $\mathfrak{M}_S$ with R.

The similarity type $\mathscr{S}_{RS}$ has been chosen so as to suit the language $\mathscr{L}_{RS}$. The elements of $\mathscr{S}_{RS}$ are structures for the language $\mathscr{L}_{RS}$, the elements of $\mathscr{S}_R$ structures for the language $\mathscr{L}_R$, and the elements of $\mathscr{S}_S$ structures for the language $\mathscr{L}_S$. We obtain:

Theorem 1. *The consistent and complete theories* Γ_R *and* Γ_S *are synonymous if and only if* Γ_R *has a model* $\mathfrak{M}_R$ *and* Γ_S *a model* $\mathfrak{M}_S$ *such that* $\mathfrak{M}_R$ *and* $\mathfrak{M}_S$ *are coalescent.*

The theorem follows from the notions involved [6]. In order to arrive at a more general result we make a passage from coalescence of structures to coalescence of classes of structures. By $\mathscr{K}_{RS}$, $\mathscr{K}_R$, $\mathscr{K}_S$ we refer to subclasses of $\mathscr{S}_{RS}$, $\mathscr{S}_R$, $\mathscr{S}_S$, respectively. We put $\mathscr{K}_{RS/S} = \{\mathfrak{M}_{RS/S}\colon \mathfrak{M}_{RS} \in \mathscr{K}_{RS}\}$ and $\mathscr{K}_{RS/R} = \{\mathfrak{M}_{RS/R}\colon \mathfrak{M}_{RS} \in \mathscr{K}_{RS}\}$. We are especially interested in *elementary classes in the wider sense.* Speaking of the elementary class $\mathscr{K}_{RS}$ we refer to a subclass of $\mathscr{S}_{RS}$ which is an elementary class in the wider sense; likewise, if we speak of the elementary class $\mathscr{K}_R$ or the elementary class $\mathscr{K}_S$, then we refer to subclasses of $\mathscr{S}_R$ or $\mathscr{S}_S$, respectively, which are elementary in the wider sense. If $\mathscr{K}_{RS}$ is an elementary class in the wider sense, then it does not follow that $\mathscr{K}_{RS/S}$ is such a class also, as has been proved by examples [7]. We say that *the elementary class* $\mathscr{K}_{RS}$ *is a definitional enrichment of the elementary class* $\mathscr{K}_R$ *with* S if and only if $\mathscr{K}_R = \mathscr{K}_{RS/S}$ and for every $\mathfrak{M}_R \in \mathscr{K}_R$ there is one and only one $\mathfrak{M}_{RS} \in \mathscr{K}_{RS}$ such that $\mathfrak{M}_{RS/S} = \mathfrak{M}_R$. We shall say that *the elementary classes* $\mathscr{K}_R$ *and* $\mathscr{K}_S$ *are coalescent* if and only if there

[5] Cf. de Bouvère [63], p. 268, Definition 3.

[6] Cf. de Bouvère [63], Theorems 4 and 5.

[7] Cf. de Bouvère [59], Chapter III, pp. 25–40.

exists an elementary class $\mathscr{K}_{RS}$ which is a definitional enrichment of $\mathscr{K}_R$ with S and of $\mathscr{K}_S$ with R. We obtain:

Theorem 2. *Let $\mathscr{K}_R$ be the class of all models of a theory Γ_R and $\mathscr{K}_S$ the class of all models of a theory Γ_S. For Γ_R and Γ_S to be synonymous it is necessary and sufficient that $\mathscr{K}_R$ and $\mathscr{K}_S$ are coalescent.*

Proof. (i) Assume that Γ_R and Γ_S are synonymous. Then there exists a theory, say Γ_{RS}, which is a definitional extension of Γ_R with S, and of Γ_S with R. Let $\mathscr{K}_{RS}$ be the class of all those elements of $\mathscr{S}_{RS}$ which are models of Γ_{RS}. Then $\mathscr{K}_{RS}$ is an elementary class in the wider sense. According to the theory of definition [8] the elementary class $\mathscr{K}_{RS}$ is a definitional enrichment of $\mathscr{K}_R$ with S and of $\mathscr{K}_S$ with R. It follows that $\mathscr{K}_R$ and $\mathscr{K}_S$ are coalescent.

(ii) Assume that $\mathscr{K}_R$ and $\mathscr{K}_S$ are coalescent. Then there exists an elementary class in the wider sense, say $\mathscr{K}_{RS}$, which is a definitional enrichment of $\mathscr{K}_R$ with S, and of $\mathscr{K}_S$ with R. This elementary class $\mathscr{K}_{RS}$ is the class of all models of a certain subset of Σ_{RS}, say of theory Γ_{RS}. We consider theory $\Gamma_{RS/S}$ and theory $\Gamma_{RS/R}$ and we remark first of all that these two theories are synonymous, with Γ_{RS} as a common definitional extension. We argue: assume that Γ_{RS} is not a definitional extension of $\Gamma_{RS/S}$ with S. Then, according to the theorem of Beth [9], there are two different elements of $\mathscr{K}_{RS}$ which have the same impoverishment to R, contrary to the hypothesis that $\mathscr{K}_{RS}$ is a definitional enrichment of $\mathscr{K}_R$ with S. Hence, Γ_{RS} is a definitional extension of $\Gamma_{RS/S}$ with S. In a similar way we obtain that Γ_{RS} is a definitional extension of $\Gamma_{RS/R}$ with R. It follows that $\Gamma_{RS/S}$ and $\Gamma_{RS/R}$ are synonymous as asserted. Further we remark that $\Gamma_{RS/S} = \Gamma_R$ and $\Gamma_{RS/R} = \Gamma_S$. We argue: assume $\Gamma_{RS/S} \neq \Gamma_R$. Since $\mathscr{K}_{RS}$ is a definitional enrichment of $\mathscr{K}_R$ with S, we have $\mathscr{K}_R = \mathscr{K}_{RS/S}$, and since every element of $\mathscr{K}_{RS/S}$ is a model of $\Gamma_{RS/S}$ the inequality would imply that $\Gamma_{RS/S}$ has a model which is not in $\mathscr{K}_R$, i.e., which in no way can be enriched to an element of $\mathscr{K}_{RS}$, i.e., to a model of Γ_{RS}. However, according to the method developed in de Bouvère [59] this would imply that Γ_{RS} is not a definitional extension of $\Gamma_{RS/S}$ with S, contrary to our former remark. Hence, $\Gamma_R = \Gamma_{RS/S}$ and similarly $\Gamma_S = \Gamma_{RS/R}$. It follows from the two main remarks that Γ_R is synonymous with Γ_S.

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[8] Cf. de Bouvère [59], p. 4.

[9] Cf. Beth [53].

Various details, such as the relationship between coalescence of the elementary classes $\mathscr{K}_R$, $\mathscr{K}_S$ and coalescence of structures belonging to $\mathscr{K}_R$ with structures belonging to $\mathscr{K}_S$, have been left out as beyond the scope of this communication. The same applies to refinements of the model-theoretic characterization of synonymous theories as expounded above. In particular, it can be seen readily that the coalescence of the elementary classes $\mathscr{K}_R$, $\mathscr{K}_S$ implies a one-to-one correspondence between $\mathscr{K}_R$ and $\mathscr{K}_S$ which preserves certain properties. D. Scott has pointed out [10] how, conversely, this one-to-one correspondence with certain properties can be used to characterize synonymous theories by means of the classes of their models.

[10] In the discussion following this communication. The author is indebted to various participants of the Symposium. In particular he wants to thank R. Montague and D. Scott for their interest in the subject of this paper.

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FINITE-QUANTIFIER EQUIVALENCE

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Arguments like that of Langford [26] showing that two densely ordered systems with no first or last element are elementarily equivalent actually show something more. They show that two such systems are equivalent with respect to finite-quantifier sentences with arbitrarily long conjunctions and disjunctions.

Let α, β be infinite cardinals. Let $L_{\alpha\beta}$ be a first-order language that admits conjunctions and disjunctions of fewer than α formulas and quantifications of sequences of variables having length less than β. Call two structures having the same similarity type (α, β)-*elementarily equivalent* if and only if they satisfy the same sentences of the corresponding language $L_{\alpha\beta}$. Suppose we are given two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same similarity type for which there is a non-empty set I of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that:

(i) Given $a \in A$ and $f \in I$ there is $g \in I$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.

(ii) Given $b \in B$ and $f \in I$ there is $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Rng}(g)$.

Then $\mathfrak{A}$ and $\mathfrak{B}$ are (α, ω)-equivalent for all cardinals α, for an inductive argument shows that whenever φ is a finite-quantifier formula with a finite number n of free variables, $n \geqslant 0$, and $a_1, \ldots, a_n \in \mathrm{Dom(f)}$ for $f \in I$, then the truth value of φ in $\mathfrak{A}$ at $\langle a_1, \ldots, a_n \rangle$ is the same as that of φ in $\mathfrak{B}$ at $\langle f(a_1), \ldots, f(a_n) \rangle$. In the case mentioned, where $\mathfrak{A}$ and $\mathfrak{B}$ are densely ordered systems with no first or last element, there is such a set I, namely, the set of all isomorphisms on finite substructures of $\mathfrak{A}$ into $\mathfrak{B}$.

The existence of such a set I is also a necessary condition for two structures to be (α, ω)-equivalent for all α. This is the content of Theorem 2.

These methods can be sharpened to yield Theorem 1, a finite isomorphism condition for a restricted sort of finite-quantifier equivalence. This condition was suggested in connection with the following problem: Is there a finite-quantifier sentence with a single binary relation symbol

R expressing the condition R well-orders A? One can express this condition with an ω-quantifier by taking the conjunction of the axioms for partial ordering with the sentence

$$\neg \exists_{n<\omega} x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} x_{n+1} R x_n].$$

The most straightforward way to see that this condition is not expressible with finite quantifiers would be to produce a system that is not well-ordered but is equivalent with respect to finite-quantifier sentences to a system that is, but this is clearly not possible without some restriction on the sentences. For the sentence

$$[\forall x [\bigvee_{\xi<\delta} Eq_\xi(x)]],$$

where $Eq_0(x) = \neg \exists y [yRx]$,

$$Eq_\xi(x) = \forall y [yRx \leftrightarrow [\bigvee_{\nu<\xi} Eq_\nu(y)]], \quad \xi > 0,$$

characterizes the sequence of length δ up to isomorphism. The appropriate restriction has turned out to be a limitation on quantifier-rank (for definition see Section 1), not on formula length. We were able to show by finite isomorphisms that the transfinite sequence $\langle \delta < \rangle$ is equivalent with respect to sentences having quantifier-rank at most δ to the ordinal product of $\langle \delta < \rangle$ with any ordered structure with zero, provided $\delta = \bigcup_{\theta<\delta} \omega^\theta$ (ordinal exponentiation). That the notion of well-ordering is not a finite-quantifier condition follows immediately.

1. A characterization of equivalence with respect to finite-quantifier sentences of rank δ. The notion of quantifier-rank is defined by induction as follows:

$qr(\varphi) = 0$ if φ is atomic;

$qr(\neg \varphi) = qr(\varphi)$;

$qr([\bigwedge_\xi \varphi_\xi]) = qr([\bigvee_\xi \varphi_\xi]) = \bigcup_\xi qr(\varphi_\xi)$;

$qr([\forall x \varphi]) = qr([\exists x \varphi]) = qr(\varphi) + 1$.

When we say that two structures are equivalent with respect to finite-quantifier sentences of rank δ (in symbols, $\mathfrak{A} \equiv_\delta \mathfrak{B}$), we mean that they have the same similarity type and satisfy the same sentences having

quantifier-rank δ in any of the correlated languages $L_{\alpha\omega}$. Clearly $\mathfrak{A} \equiv_\delta \mathfrak{B}$ implies $\mathfrak{A} \equiv_\theta \mathfrak{B}$ for any $\theta < \delta$. Note that there is no assumption on the number of places of the operations and relations of the structures.

Theorem 1. *For structures $\mathfrak{A}$ and $\mathfrak{B}$ having the same similarity type the following conditions are equivalent:*

(1) $\mathfrak{A} \equiv_\delta \mathfrak{B}$;

(2) *For each ordinal $\theta \leqslant \delta$ there is a non-empty set I_θ of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that $I_\theta \subseteq I_\nu$ for $\nu \leqslant \theta$ and:*

(i) *If $\theta+1 \leqslant \delta$ then given $a \in A$ and $f \in I_{\theta+1}$ there is $g \in I_\theta$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.*

(ii) *If $\theta+1 \leqslant \delta$ then given $b \in B$ and $f \in I_{\theta+1}$ there is $g \in I_\theta$ such that $f \subseteq g$ and $b \in \mathrm{Rng}(g)$.*

Proof. An inductive argument shows that if (2) holds then whenever $\theta \leqslant \delta$ and φ is a finite-quantifier formula with quantifier-rank θ and a finite number n of free variables, $n \geqslant 0$, then for any $f \in I_\theta$ and $a_1, \ldots, a_n \in \mathrm{Dom}(f)$, the value of φ in $\mathfrak{A}$ at $\langle a_1, \ldots, a_n \rangle$ is the same as that of φ in $\mathfrak{B}$ at $\langle f(a_1), \ldots, f(a_n) \rangle$. Hence (1).

Correlate finite-quantifier languages $L_{\alpha\omega}$ to the common similarity type of $\mathfrak{A}$ and $\mathfrak{B}$ and assume (1). To make the proof simpler, consider the case where $\mathfrak{A}$ and $\mathfrak{B}$ have only relations, no operations. The cases with operations can be handled similarly. For $\theta \leqslant \delta$, $n \geqslant 0$, $s \in A^n$, let $I_{\theta,s} = \{f : f$ maps $\mathrm{Rng}(s)$ to B and $\varphi_{\mathfrak{A}}(s) = \varphi_{\mathfrak{B}}(f \circ s)$ for all formulas φ such that $FV(\varphi) \subseteq \{x_1, \ldots, x_n\}$ and $qr(\varphi) \leqslant \theta\}$. (By $\varphi_{\mathfrak{A}}(s)$ we mean the truth value of φ in $\mathfrak{A}$ for the assignment $s(i)$ to x_i, $i \leqslant n$.) Let $I_\theta = \bigcup \{I_{\theta,s} : s \in A^n, n \geqslant 0\}$. Condition (1) guarantees that the I_θ are not empty since they all contain $\emptyset$. The mappings are isomorphisms because they preserve the values of formulas of rank 0. It remains to check conditions (i) and (ii). Suppose $a \in A$, $s \in A^n$, $f \in I_{\theta+1,s}$. List formulas φ with free variables in $\{x_1, \ldots, x_{n+1}\}$ such that $qr(\varphi) \leqslant \theta$ and $\varphi_{\mathfrak{A}}(s^\frown\langle a \rangle) = \top$:

$$\varphi_0, \ldots, \varphi_\nu, \ldots, \quad \nu \in \mathrm{OR}.$$

It is not difficult to see that there is an ordinal λ such that $\models_{\mathfrak{A}} [\bigwedge_{\xi<\lambda} \varphi_\xi] \to \varphi_\nu$ for all ν. For if there were no such λ it would be possible to make a strictly increasing sequence $\lambda_0, \ldots, \lambda_\nu, \ldots$ of any length such that $[\bigwedge_{\xi<\sigma_\nu} \varphi_\xi] \to \varphi_{\lambda_\nu}$ does not hold in $\mathfrak{A}$, where $\sigma_\nu = (\bigcup_{\mu<\nu} \lambda_\mu) + 1$. To each such

formula could be associated an assignment to $x_1, \ldots, x_{n+1}$ making it false in $\mathfrak{A}$. But the number of such assignments is card(A) and no assignment could be used twice. Therefore such a sequence would have to stop eventually. For such an ordinal λ, $[\exists x_{n+1}[\bigwedge_{\xi<\lambda} \varphi_\xi]]_{\mathfrak{A}}(s) = \top$ and the formula has quantifier-rank $\theta+1$. Since $f \in I_{\theta+1,s}$, $[\exists x_{n+1}[\bigwedge_{\xi<\lambda} \varphi_\xi]]_{\mathfrak{B}}(f \circ s) = \top$. Therefore there is $b \in B$ such that $[\bigwedge_{\xi<\lambda} \varphi_\xi]_{\mathfrak{B}}((f \circ s)^\frown\langle b\rangle) = \top$. We can show that $g = f \cup \{\langle a, b\rangle\}$ is in $I_{\theta, s^\frown\langle a\rangle}$. If φ is a formula with quantifier-rank at most θ and free variables in $\{x_1, \ldots, x_{n+1}\}$ and if $\varphi_{\mathfrak{A}}(s^\frown\langle a\rangle) = \top$ then by the choice of λ, $\models_{\mathfrak{A}}[\forall x_{n+1}[[\bigwedge_{\xi<\lambda} \varphi_\xi] \to \varphi]]$. Since this formula has quantifier-rank at most $\theta+1$, it is true in $\mathfrak{B}$ for the assignment $f \circ s$. Since $[\bigwedge_{\xi<\lambda} \varphi_\xi]$ is true for $(f \circ s)^\frown\langle b\rangle$, it follows that $\varphi_{\mathfrak{B}}((f \circ s)^\frown\langle b\rangle) = \varphi_{\mathfrak{B}}(g \circ (s^\frown\langle a\rangle)) = \top$. The case $\varphi_{\mathfrak{A}}(s^\frown\langle a\rangle) = \bot$ can be treated similarly. Hence (i). The proof of (ii) is nearly the same.

As an easy corollary we have

Theorem 2. *For structures $\mathfrak{A}$ and $\mathfrak{B}$ having the same similarity type the following conditions are equivalent:*

(1) *$\mathfrak{A}$ and $\mathfrak{B}$ are (α, ω)-elementarily equivalent for all infinite α.*

(2) *There is a non-empty collection I of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that:*

(i) *Given $a \in A$ and $f \in I$ there is $g \in I$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.*

(ii) *Given $b \in B$ and $f \in I$ there is $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Rng}(g)$.*

The proof that (2) implies (1) in Theorem 1 would have been shorter if we had used the fact that for a given similarity type a cardinal $d(\delta)$ can be computed such that every finite-quantifier formula with quantifier-rank δ is equivalent to one of length less than $d(\delta)$. Such a bound on length was found by Lopez-Escobar using the δ-isomorphism types, themselves generalized Fraissé types (see Fraissé [54b]), discussed in Scott [*]. Two structures have the same type τ_δ if and only if (2) of Theorem 1 holds. A bound on the number of possible δ-isomorphism types can be computed, and each of the types can be described by a δ-rank formula. Since every formula of quantifier-rank δ is equivalent to a disjunction of these describing formulas, we can then compute a bound $d(\delta)$. It follows from Theorem 1 of course, that $\tau_\delta(\mathfrak{A}) = \tau_\delta(\mathfrak{B})$ if and only if $\mathfrak{A} \equiv_\delta \mathfrak{B}$.

2. An application to well-orderings. Let δ be an ordinal number such that $\delta = \bigcup_{\theta<\delta} \omega^\theta$ (ordinal exponentiation). Note that then $\xi, \nu < \delta$ implies $\xi + \nu < \delta$ and $\xi \cdot \nu < \delta$. Let $\mathfrak{A} = \langle A, R\rangle$ be any ordered structure with zero (asymmetric formulation). By the ordinal product $\delta \otimes \mathfrak{A}$ we mean the structure with elements $\langle \xi, a\rangle$, $\xi < \delta$, $a \in A$, ordered by last differences; i.e., $\langle \xi, a\rangle < \langle \nu, b\rangle$ if and only if aRb or ($a = b$ and $\xi < \nu$). An ω^θ-interval of δ is a left-closed interval of the form $[\omega^\theta \cdot \xi,\ \omega^\theta \cdot (\xi+1))$, $\xi < \delta$. An ω^θ-interval of $\delta \otimes \mathfrak{A}$ is one of the form $[\langle \omega^\theta \cdot \xi, a\rangle, \langle \omega^\theta \cdot (\xi+1), a\rangle)$, $\xi < \delta$.

Theorem 3. *If* $\delta = \bigcup_{\theta<\delta} \omega^\theta$ *then* $\delta \equiv_\delta \delta \otimes \mathfrak{A}$.

Proof. For $\theta \leqslant \delta$ let $I_\theta = \{f : f$ is a restriction to a finite subset of δ containing 0, of an order-preserving, one-to-one translation of a union of ω^θ-intervals of δ to a union of ω^θ-intervals of $\delta \otimes \mathfrak{A}$ which carries $[0, \omega^\theta)$ to $[\langle 0, 0\rangle, \langle \omega^\theta, 0\rangle)\}$. Functions in I_θ are clearly isomorphisms and the sets are non-empty since the function taking 0 to $\langle 0, 0\rangle$ is in every I_θ. We have the inclusion $I_\theta \subseteq I_\nu$ for $\nu \leqslant \theta$ since in this case every ω^θ-interval is a union of ω^ν-intervals. It remains to establish conditions (i) and (ii) of (2), Theorem 1. We check (ii); (i) is similar. Suppose $\theta + 1 \leqslant \delta$, $\langle \xi, a\rangle \in \delta \otimes \mathfrak{A}$, $f \in I_{\theta+1}$. Then f is a finite restriction of a one-to-one, order-preserving translation of a union of $\omega^{\theta+1}$-intervals to a union of $\omega^{\theta+1}$-intervals. List those intervals containing elements of Dom(f) in increasing order:

$$
\begin{array}{ll}
 & I_0, \ldots, I_k, \ldots, I_n \quad \text{where } I_0 = [0, \omega^{\theta+1}). \\
(*) & \quad | \qquad | \qquad | \\
 & I_0', \ldots, I_k', \ldots, I_n' \quad \text{where } I_0' = [\langle 0, 0\rangle, \langle \omega^{\theta+1}, 0\rangle).
\end{array}
$$

If $\langle \xi, a\rangle$ is already in one of the intervals I_k' then the same translation can be used to extend f to a function $g \in I_{\theta+1}$ with $\langle \xi, a\rangle$ in its range. From now on assume that $\langle \xi, a\rangle$ is not in any of the I_k'. Let I_m' be the rightmost of the intervals to the left of $\langle \xi, a\rangle$. Two cases arise.

Case 1. $m = n$ or I_{m+1} is not adjacent to I_m. Then f can be extended to a function $g \in I_{\theta+1}$ with $\langle \xi, a\rangle$ in its range by extending the translation (*) to include the translation of the $\omega^{\theta+1}$-interval to the right of I_m to the $\omega^{\theta+1}$-interval containing $\langle \xi, a\rangle$.

Case 2. I_{m+1} is adjacent to I_m. In this case f cannot be extended to a function in $I_{\theta+1}$ with $\langle \xi, a\rangle$ in its range, but it can be extended to

a function in I_θ and still satisfy (ii). The $\omega^{\theta+1}$-interval I_m is a union of ω ω^θ-intervals, among which only finitely many are occupied by elements of Dom(f). Call these $I_{m,1}, \ldots, I_{m,p}$. Let $I_{m,p+1}$ be the first ω^θ-interval to the right of $I_{m,p}$ and let J be the ω^θ-interval of $\delta \otimes \mathfrak{A}$ containing $\langle \xi, a \rangle$. Then J is to the right of I_m' and to the left of I_{m+1}'. Let the translation (**) agree with (*) on I_k, $k \neq m$, and on $I_{m,1}, \ldots, I_{m,p}$, and carry $I_{m,p+1}$ to J. Then (**) is a one-to-one, order-preserving translation of a union of ω^θ-intervals which carries $[0, \omega^\theta)$ to $[\langle 0, 0\rangle, \langle \omega^\theta, 0\rangle)$ and has $\langle \xi, a \rangle$ in its range. Hence f can be extended to a function $g \in I_\theta$ with $\langle \xi, a \rangle$ in its range.

Theorem 4. *There is no finite-quantifier sentence in R expressing the condition R well-orders A.*

Proof. If there were such a formula φ it would have some ordinal ε as its quantifier-rank. Choose $\delta \geqslant \varepsilon$ such that $\delta = \bigcup_{\theta<\delta} \omega^\theta$. For example, any cardinal larger than $\varepsilon \cup \omega$ would suffice. Let $\mathfrak{B}$ be an ordered structure that is not well-ordered. Then φ is true in δ, false in $\delta \otimes \mathfrak{B}$, contradicting Theorem 3.

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ALGEBRAIC THEORIES, ALGEBRAIC CATEGORIES, AND ALGEBRAIC FUNCTORS

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The theory of categories and functors (Eilenberg-MacLane [45], Freyd [60], Kan [58], MacLane [50]) extends to the various fields of mathematics a methodological injunction which may be briefly phrased: "Look to the maps." That such a program can lead to interesting results in the field of model theory is partly borne out in the case of the study of "equational classes of algebras" by the results discussed below. There are at least four distinct levels in general algebra where maps are to be seen: (1) Homomorphisms between algebraic structures of a given equational type. This suggests the notion of "algebraic category." (2) "Equational" interpretations between algebraic theories; for example, the obvious interpretation of the theory of Lie algebras into that of associative algebras. (3) "Algebraic" functors between algebraic categories induced by the interpretations just mentioned, as well as adjoints (Kan [58]) to these. The notion of reduct provides a simple, but by no means trivial, class of examples of algebraic functors; previously known examples of adjoints to reducts include the constructions of tensor algebras and semi-group rings as well as the embedding of a distributive lattice in a Boolean algebra. (4) The semantical assignment of algebraic categories to algebraic theories. This assignment is a functor which, we discover, also has an adjoint when properly construed. The adjoint provides us with a useful tool for giving a new intrinsic characterization of algebraic categories and also for partially analyzing categories of models of more general theories. The results to be discussed appeared in an earlier form in the author's Columbia University doctoral dissertation (Lawvere [63]) written under the direction of Professor Eilenberg. For more recent related material see Isbell [a], Lawvere [a], and unpublished work of F. E. J. Linton.

By an *algebraic theory* we mean a small category $\mathbf{A}$ whose objects are the natural numbers 0, 1, 2, ... and in which each object n is the categorical direct product of the object 1 with itself n times. By an *n-ary*

operation of $\mathbf{A}$ is meant any map $n \to 1$ in $\mathbf{A}$. Since n is a product, the projections $\pi_i^{(n)}: n \to 1$, $i = 0, 1, \ldots, n-1$ are always n-ary operations for each n in any algebraic theory, but in general there will be more. The maps $n \to m$ in an algebraic theory $\mathbf{A}$ are in one-to-one correspondence with the m-tuples of n-ary operations of $\mathbf{A}$. Any "presentation" of a concept of algebraic structure (e.g., groups, modules over a given ring, Jordan algebras, lattices, etc.), which involves a set of symbols denoting finitary operations together with a set of equations (= identities) relating composite operations, determines an algebraic theory, and conversely every algebraic theory has such presentations. By a mapping between algebraic theories we will understand a functor that preserves products and takes 1 into 1. Algebraic theories and the mappings between them thus form a category $\mathscr{T}$.

Each algebraic theory $\mathbf{A}$ determines a large category $\mathscr{S}^{(\mathbf{A})}$ whose class of objects is just the equational class (variety) of all algebras of type $\mathbf{A}$, and whose maps are all (into) homomorphisms between these. An algebra of type $\mathbf{A}$ can be viewed as a product preserving functor $\mathbf{A} \to \mathscr{S}$ from $\mathbf{A}$ to the category of sets; a homomorphism of algebras is then just a natural transformation between such functors. If $\mathbf{A}$ is the algebraic theory whose *only* n-ary operations are projections (i.e., $\mathbf{A}$ is equivalent to the dual of the category of finite sets) then the category of algebras $\mathscr{S}^{(\mathbf{A})}$ is just the category $\mathscr{S}$ of sets. Every map $f: \mathbf{A} \to \mathbf{B}$ of algebraic theories determines in an obvious way a functor $\mathscr{S}^{(f)}: \mathscr{S}^{(\mathbf{B})} \to \mathscr{S}^{(\mathbf{A})}$ which preserves underlying sets, i.e., for which $U_{\mathbf{B}} = \mathscr{S}^{(f)} U_{\mathbf{A}}$, where $U_{\mathbf{A}}: \mathscr{S}^{(\mathbf{A})} \to \mathscr{S}$, $U_{\mathbf{B}}: \mathscr{S}^{(\mathbf{B})} \to \mathscr{S}$ are the underlying set functors (notice the order in which we write composition.) We call any functor of the form $\mathscr{S}^{(f)}$ an *algebraic functor*, and we call any category of the form $\mathscr{S}^{(\mathbf{A})}$ an *algebraic category*. Any algebraic theory $\mathbf{A}$ is equivalent to the dual of the full category of finitely generated free algebras in its associated algebraic category.

Theorem. *Every algebraic functor has an adjoint.*

For example, the category of rings (with unit) and that of monoids are algebraic, and the functor which assigns to each ring the monoid consisting of the same elements under multiplication alone is an algebraic functor. The adjoint to this functor is the well-known construction of the monoid ring. Other instances of this theorem include abelianization of groups, enveloping algebras of Lie algebras, and a generalized quaternion construction. The proof of the above theorem may be conveniently

divided into two steps. Given $f: \mathbf{A} \to \mathbf{B}$ in $\mathscr{T}$, call $\mathscr{S}^{\mathbf{A}}$, the category of *all* functors $\mathbf{A} \to \mathscr{S}$ and natural transformations, the category of $\mathbf{A}$-*prealgebras;* similarly $\mathscr{S}^{\mathbf{B}}$ is the category of $\mathbf{B}$-prealgebras. The induced functor $\mathscr{S}^{\mathbf{B}} \to \mathscr{S}^{\mathbf{A}}$ has an adjoint which may be calculated by a direct limit procedure (see Lawvere [63, p. 39].) Secondly, the inclusion $\mathscr{S}^{(\mathbf{B})} \to \mathscr{S}^{\mathbf{B}}$ of algebras into prealgebras has an adjoint which may be calculated by considering, for a given prealgebra $\mathbf{B} \to \mathscr{S}$, the free algebra over the one-dimensional part reduced modulo the relations implied by the structure of the zero- and higher-dimensional parts. Note that for $f: \mathbf{A} \to \mathbf{B}$ in $\mathscr{T}$, the natural transformation from the identity functor on $\mathscr{S}^{(\mathbf{A})}$ to $F\mathscr{S}^{(f)}$, where F is the adjoint of $\mathscr{S}^{(f)}$, need not be a monomorphism (for example, not every Jordan algebra is special—on the other hand every distributive lattice can be embedded in a Boolean ring); an important problem remaining is an intrinsic characterization of those f in $\mathscr{T}$ for which the natural transformation in question is in fact a monomorphism.

If we write $\mathbf{A}\mathfrak{S} = U_{\mathbf{A}}$, $f\mathfrak{S} = \mathscr{S}^{(f)}$, we obtain a functor $\mathfrak{S}$ which we call *algebraic semantics;* the domain of $\mathfrak{S}$ is the dual $\mathscr{T}^*$ of the category of algebraic theories, and we take as its codomain the category $\mathscr{K}$ whose objects are functors $U: \mathscr{X} \to \mathscr{S}$ with arbitrary domain category and with the category of sets as codomain, subject only to the restriction that for each natural number n, the class of all natural transformations $U^n \to U$ is small, where U^n assigns to each X the nth Cartesian power of XU; a map $T: U \to U'$ in $\mathscr{K}$ is to be any functor $T: \mathscr{X} \to \mathscr{X}'$ for which $U = TU'$.

Theorem. *Algebraic semantics has an adjoint* $\hat{\mathfrak{S}}: \mathscr{K} \to \mathscr{T}^*$ *(which we call* algebraic structure*), and furthermore* $\mathfrak{S}\hat{\mathfrak{S}}$ *is naturally equivalent to the identity functor on* $\mathscr{T}^*$. *Explicitly, for any* U *in* $\mathscr{K}$ *the n-ary operations of the algebraic theory* $U\hat{\mathfrak{S}}$ *are the natural transformations* $U^n \to U$.

Thus, any category $\mathscr{X}$ equipped with an "underlying set functor" U determines an algebraic category $\mathscr{S}^{(U\hat{\mathfrak{S}})}$ together with a functor: $\Phi: \mathscr{X} \to \mathscr{S}^{(U\hat{\mathfrak{S}})}$ which preserves underlying sets, and given any other such functor $\Psi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$, there is a unique: $f: \mathbf{A} \to U\hat{\mathfrak{S}}$ in $\mathscr{T}$ such that $\Psi = \Phi\mathscr{S}^{(f)}$. Also the operations which define an algebraic category are in natural one-to-one correspondence with the natural operations on its underlying set functor. This second assertion of the theorem is a

consequence of Yoneda's lemma (Yoneda [54]) and the fact that $U_{\mathbf{A}}{}^{n} = \mathrm{Hom}(A_n, ?)$ where A_n is a free $\mathbf{A}$-algebra on n generators. Thus the functor $\hat{\mathfrak{S}}$ becomes most interesting when applied to "underlying set functors" on non-algebraic categories. For example, if we take for $\mathscr{X}$ the dual of the category of sets and for U the (contravariant) power set functor, the algebraic structure of U is the theory of Boolean algebras, and Φ assigns to each set the Boolean algebra of its subsets. If $\mathscr{X}$ is the category of models for some given first-order theory T (maps in $\mathscr{X}$ are elementary embeddings) and if U is the "universe" functor, then the set of n-ary operations of the algebraic theory $U\hat{\mathfrak{S}}$ is identified as the set of all formulas of T with at most $v_0, \ldots, v_n$ free which provably express v_0 as an everywhere well-defined function of $v_1, \ldots, v_n$, reduced modulo provable equivalence. If the theory T can be logically generated by formulas of this type, then the functor Φ may be thought of as an inclusion; in general Φ measures semantically the departure of T from its nearest "algebraic approximation", roughly speaking.

The above theorem implies that a category $\mathscr{X}$ is equivalent to some algebraic category iff it has some underlying set functor U such that the particular functor $\Phi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ described above is an equivalence where $\mathbf{A} = U\hat{\mathfrak{S}}$. In that case we must actually have $U = \mathrm{Hom}(G, ?)$ where G is an object in $\mathscr{X}$ such that $G\Phi$ is a free $\mathbf{A}$-algebra on one generator. These observations enable us to completely characterize algebraic categories in the theorem below. We first define some terms.

By an *equalizer* and *coequalizer* of a pair of maps

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$$

in a category $\mathscr{X}$, we mean respectively an inverse limit $j: K \to X$ and a direct limit $p: Y \to K^*$ of the above diagram in the sense of Kan [58]. A map p is called a *regular epimap* iff it is a coequalizer of some pair of maps. A category *has finite limits* iff it has an initial and a final object and if all possible binary products and coproducts (=free products), as well as all possible equalizers and coequalizers, exist. A pair f_0, f_1 as above is called a *precongruence* iff the corresponding $f: X \to Y \times Y$ is a monomorphism as well as reflexive, symmetric, and transitive in an obvious sense; the pair is called a *congruence* iff f is the equalizer of the pair $\pi_0 p, \pi_1 p$, where $\pi_i: Y \times Y \to Y$ are the projections, and p is the coequalizer of f_0, f_1. In a category with finite limits, every congruence

is a precongruence, while in any algebraic category the converse is also true.

An object G in a category $\mathscr{X}$ is *abstractly finite* iff the following three conditions hold: (a) for every object X, the class $\mathrm{Hom}(G, X)$ of maps $G \to X$ is small (i.e., is a "set"); (b) for every set I, the I-fold coproduct $I \cdot G$ esists; (c) any map $G \to I \cdot G$ factors through some $I' \cdot G$, where $I' \subseteq I$ and I' is finite. A map $f: Y \to Z$ in $\mathscr{X}$ is *G-surjective* iff every map $G \to Z$ factors across f. An object G is a *regular projective generator* iff the G-surjective maps are precisely the regular epimaps.

Finally, a functor Φ is said to be *full and faithful* iff for every pair X, Y of objects in its domain category, Φ induces a one-to-one correspondence of the set of maps $X \to Y$ onto the set of maps $X\Phi \to Y\Phi$ in the codomain category.

Theorem. *Let $\mathscr{X}$ be a category with the following properties:* (0) *$\mathscr{X}$ has finite limits;* (1) *$\mathscr{X}$ has an abstractly finite regular projective generator G. Then there is an algebraic theory $\mathbf{A}$ and a functor $\Phi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ which is full, faithful, and has an adjoint, and the free objects in $\mathscr{X}$ coincide with those in $\mathscr{S}^{(\mathbf{A})}$ (i.e., $(I \cdot G)\Phi = I \cdot (G\Phi)$). Furthermore Φ is an equivalence iff* (2) *every precongruence in $\mathscr{X}$ is a congruence. Conditions* 0, 1, 2 *are necessary and sufficient that $\mathscr{X}$ be equivalent to some algebraic category.*

For the proof, choose G as guaranteed by condition 1, set $U = \mathrm{Hom}(G, ?)$ and let $\mathbf{A} = U\hat{\mathfrak{S}}$. Then by the previous theorem, there is a functor $\Phi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ with a universal mapping property. By using the properties of G, we can show that Φ is full and faithful and preserves coproducts of G with itself. Also (as was pointed out by Isbell [a]) every sub-$\mathbf{A}$-algebra of any $X\Phi$ is also of the form $X'\Phi$. To construct the adjoint to Φ, consider a presentation of an $\mathbf{A}$-algebra Y as a quotient of a free $\mathbf{A}$-algebra; as the associated congruence relation actually comes from $\mathscr{X}$ we may perform the "same" quotient operation in $\mathscr{X}$ to obtain $\overline{Y}$, the value at Y of the adjoint of Φ. That $\overline{Y}\Phi \cong Y$ need not hold is shown by the following example: $\mathscr{X}$ = category of torsion-free abelian groups, G = rational integers, where $\overline{Y}$ is the result of reducing the arbitrary abelian group Y modulo its torsion subgroup. However, if condition 2 also holds, then the adjoint of Φ is (up to natural equivalence) actually inverse to Φ.

Corollary. *If $\mathscr{X}$ is an algebraic category and if $\mathscr{C}$ is any small category with finitely many objects, then the full category $\mathscr{X}^{\mathscr{C}}$ of functors $\mathscr{C} \to \mathscr{X}$*

and natural transformations thereof is itself equivalent to an algebraic category, provided either that $\mathscr{X} \cong \mathscr{S}^{(\mathbf{A})}$ *where* $\mathbf{A}$ *has at least one nullary operation or that for any two objects* C, C^1 *in* $\mathscr{C}$, *there exists a map* $C \to C^1$ *in* $\mathscr{C}$.

In proving this corollary, we define the underlying set of a functor $\mathscr{C} \xrightarrow{F} \mathscr{X}$ to be the product, indexed by the objects C in $\mathscr{C}$, of the sets CFU, where U is a given underlying set functor for $\mathscr{X}$. When $\mathscr{C}$ has only one object, then $\mathscr{C}$ is a monoid and the corollary refers to the well-known notion of a monoid acting by endomorphisms of an algebra to form a new type of algebra.

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EXTENSIVE ULTRAPRODUCTS AND HAAR MEASURE *

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A Haar measure is one which is positive on nonempty open sets, finite on compact sets, regular, and invariant under translation. Real-valued Haar measures exist for topological groups which are locally compact; but for the non-locally compact groups there are, in a sense, not enough real numbers for us to be able to give positive real measures to all the deserving sets.

In this paper we follow the lead of Charles Loewner, who in 1939 began the construction of an extension of the reals to serve as the range for a measure on Hilbert space (see Loewner [39]). We construct a translation invariant measure which takes values in an ultrapower of the reals. Ultrapower-bounded measures are given to sets which have compact intersections with sufficiently many locally compact subgroups of the given topological group.

A preliminary section is given to the introduction of the *extensive ultraproduct* and to the definition of infinite sums. The measure is constructed in Section 2. In Section 3 we take a brief look at the resulting integration theory.

1. Extensive ultraproducts and summation. Let $\{\mathfrak{S}_\iota\}_{\iota\in I}$ be a family of similar relational structures: $\mathfrak{S}_\iota=\langle S_\iota, P_{\iota,\alpha}\rangle_{\alpha\in A}$, where each $P_{\iota,\alpha}$ is an n_α-place relation on the set S_ι. Let $a_I=\{a_\iota\}_{\iota\in I}$ denote a function on I with each $a_\iota\in S_\iota$. For a given ultrafilter $\mathscr{D}$ on I let $a_I\equiv_{\mathscr{D}} b_I$ iff $\{\iota|a_\iota=b_\iota\}\in\mathscr{D}$ and let $\boldsymbol{a}=[a_I]_{\mathscr{D}}$ be the equivalence class $\{b_I|b_I=_{\mathscr{D}}a_I\}$. On such equivalence classes define relations P_α by $P_\alpha(\boldsymbol{a}^1,\ldots,\boldsymbol{a}^n)$ iff $\{\iota|P_{\iota,\alpha}(\boldsymbol{a}_\iota^1,\ldots,\boldsymbol{a}_\iota^n)\}\in\mathscr{D}$, where $n=n_\alpha$. Then the *ultraproduct* $\prod_I\mathfrak{S}_{\mathscr{D}}$ is the structure $\langle\prod_I S_{\iota\mathscr{D}}, P_\alpha\rangle_{\alpha\in A}$ where $\prod_I S_{\iota\mathscr{D}}$ is the set of all the equivalence classes $[a_I]_{\mathscr{D}}$. If the $\mathfrak{S}_\iota$ are isomorphic copies of a given

* This is a revised form of the paper, entitled "Ultraproduct Haar measure over topological linear spaces," presented at the Symposium. The author is indebted to Peter Freyd for pointing out that the original formulation was more restricted than it needed to be.

structure $\mathfrak{S}$, then $\prod_I \mathfrak{S}_{\iota/\mathscr{D}}$ is called an *ultrapower* of $\mathfrak{S}$, and denoted by $\mathfrak{S}^I/\mathscr{D}$. (Canonical references for the properties and uses of ultraproducts are Frayne-Morel-Scott [62], Kochen [61], and Keisler [61].)

We may also construct relations on $\prod_I S_{\iota/\mathscr{D}}$ which are not tied to the index α, and obtain a richer structure for which many of the basic properties of $\prod_I \mathfrak{S}_{\iota/\mathscr{D}}$ continue to hold. Let $\Phi = \{P_\iota\}_{\iota \in I}$ where each P_ι is chosen from the relations $P_{\iota,\alpha}$, with $\{\iota | P_\iota \text{ is an } n\text{-place relation on } S\} \in \mathscr{D}$. Then we may define an n-place *ultraproduct relation* P_Φ by $P_\Phi(\mathbf{a}^1, \ldots, \mathbf{a}^n)$ iff $\{\iota | P_\iota(a_\iota^1, \ldots, a_\iota^n)\} \in \mathscr{D}$. Let the *extensive ultraproduct* ${}^*\prod_I \mathfrak{S}_{\iota/\mathscr{D}}$ be $\prod_I S_{\iota/\mathscr{D}}$ together with all ultraproduct relations which can be formed from the relations $P_{\iota,\alpha}$, $\iota \in I$, $\alpha \in A$. If the $\mathfrak{S}_\iota$ are isomorphic copies of a given $\mathfrak{S}$ call ${}^*\prod_I \mathfrak{S}_{\iota/\mathscr{D}} = {}^*\mathfrak{S}^I/\mathscr{D}$ an *extensive ultrapower*.

We embed S in the ultrapower set $S^I/\mathscr{D}$ by identifying a in S with $[a_I]/\mathscr{D}$ where $a_\iota = a$ for each ι. Let $\varphi(\ldots)$ be any formula in n variables, constructed from the relations in ${}^*\mathfrak{S}^I/\mathscr{D}$. Let $\varphi_\iota(\ldots)$ be the formula obtained by substituting for each P_Φ in $\varphi(\ldots)$ the relation $P_\iota \in \Phi$. Then we have as corollaries to Theorem 2.2 of Frayne-Morel-Scott [62],

$$ {}^*\mathfrak{S}^I/\mathscr{D} \models \varphi([a_I]^1, \ldots, [a_I]^n) \text{ iff } \{\iota | \mathfrak{S} \models \varphi_\iota(a_\iota^1, \ldots, a_\iota^n)\} \in \mathscr{D} $$

and $$ {}^*\mathfrak{S}^I/\mathscr{D} \models \varphi(a^1, \ldots, a^n) \text{ iff } \{\iota | \mathfrak{S} \models \varphi_\iota(a_\iota^1, \ldots, a_\iota^n)\} \in \mathscr{D}. $$

$\mathfrak{R}$ will denote the set R of real numbers together with all relations on R (i.e., $\mathfrak{R}$ will be the complete structure of the reals). $\mathfrak{R}^*$ will denote a two-element extension of $\mathfrak{R}$ consisting of $R \cup \{-\infty, +\infty\}$ together with suitable extensions of the relations of $\mathfrak{R}$, abiding by the usual conventions for $-\infty$ and $+\infty$. It is not difficult to prove that an ultrapower $\mathfrak{R}^{*I}/\mathscr{D}$ is a similar two-element extension of the ultrapower $\mathfrak{R}^I/\mathscr{D}$. N will be the nonnegative integers and N^* will denote $N \cup \{\infty\}$.

Let I and $\mathscr{D}$ be fixed and let $\Psi = \{f_\iota\}_{\iota \in I}$ be a family of real-valued functions with domain N. Set $\mathbf{N} = N^I/\mathscr{D}$. Then Ψ determines an ultrapower-sequence $\mathbf{f}_\Psi = \{\mathbf{a}_n\}_{n \in \mathbf{N}}$. (Think of the f_ι as two-place relations in $\mathfrak{R}$; $\mathbf{f}_\Psi$ is the relation they determine in the extensive ultrapower ${}^*\mathfrak{R}^I/\mathscr{D}$.) Over ultrapower sequences of this kind we can define a summation:

$$ \sum_{n<\mathbf{k}} \mathbf{a}_n = [\{\textstyle\sum_{n<k_\iota} f_\iota(n)\}_{\iota \in I}]/\mathscr{D} $$

where $\mathbf{k}$ is any element of $N^I/\mathscr{D}$ and $\{k_\iota\}_{\iota \in I}$ is any element of $\mathbf{k}$. (The sum is independent of the choice of $\{k_\iota\}_{\iota \in I}$.) It is not difficult to show that $\lim_{\mathbf{k} \in \mathbf{N}} \sum_{n<\mathbf{k}} \mathbf{a}_n$ and $[\{\sum_{n<\infty} f_\iota(n)\}_{\iota \in I}]/\mathscr{D}$ exist and are equal if and only if the $\sum_{n<\infty} f_\iota(n)$ exist for all ι in some member of $\mathscr{D}$. (The limit

is with respect to the order topology of $\Re^I_{\mathscr{D}}$.) For brevity set $\lim_{k\in N}\sum_{n<k} a_n = \sum_{n<\infty} a_n$.

(Ultraproduct relations may also be used to define the quasi-standard functions of Abraham Robinson [61a], [63], and W. A. J. Luxemburg in his notes "Non-standard analysis", California Institute of Technology, 1962).

2. The measure. Let $\mathfrak{G}$ be a Hausdorff topological abelian group with a collection $\{\mathfrak{H}_\iota\}_{\iota\in I}$ of subgroups such that

(i) each $\mathfrak{H}$ is locally compact in its relative topology,

(ii) if $\mathfrak{H}_\iota$ and $\mathfrak{H}_\eta$ are in the collection, so is $\mathfrak{H}_\iota\cdot\mathfrak{H}_\eta$, and

(iii) as a group, disregarding topology, $\mathfrak{G}=\bigcup_{\iota\in I}\mathfrak{H}_\iota$.

A subset M of $\mathfrak{G}$ will be said to be *intersection-open* (*intersection-compact*) with respect to $\{\mathfrak{H}_\iota\}_{\iota\in I}$ if $M\cap\mathfrak{H}_\iota$ is open (compact) for every ι. A function μ will be said to be an *ultraproduct Haar measure* on $\mathfrak{G}$ with respect to $\{\mathfrak{H}_\iota\}_{\iota\in I}$ and an ultrafilter $\mathscr{D}$ if the following are satisfied:

(0) μ is a nonnegative $\Re^{*I}_{\mathscr{D}}$-valued function on a collection $\mathscr{M}$ (of subsets of $\mathfrak{G}$) containing the σ-algebra generated by the intersection-open sets of $\mathfrak{G}$. The sets in $\mathscr{M}$ will be the *measurable* sets.

(1) $\mu(\emptyset)=0$.

(2) If $\{M_\beta\}_{\beta\in B}$ is a family of disjoint measurable sets such that for every $\iota\in I$ the collection of nonempty members of $\{M_\beta\cap\mathfrak{H}_\iota\}_{\beta\in B}$ is at most denumerable, then $\{M_\beta\}_{\beta\in B}$ can be indexed by elements of $N^I_{\mathscr{D}}$ so that for some $k\in N^{*I}_{\mathscr{D}}$, $\sum_{n<k}\mu(M_n)=\mu(\bigcup_{\beta\in B}M_\beta)$.

(3) $\mu(U)\neq 0$ for every nonempty intersection-open set U.

(4) $\mu(M)<\infty$ for every intersection compact set M.

(5) For some choice of Haar measures μ_ι on the subgroups $\mathfrak{H}_\iota$, for every intersection-open set U, $\mu_\iota(U)=[\{\sup\{\mu_\iota(C\cap H_\iota)|C\subseteq U$ and C is compact in $\mathfrak{G}\}\}_{\iota\in I}]_{\mathscr{D}}$, and for every measurable set M, $\mu(M)=\{\inf\{\mu_\iota(V\cap H_\iota)|V\supseteq M$ and V is open in $\mathfrak{G}\}\}_{\iota\in I}]_{\mathscr{D}}$.

(6) μ is invariant under translation: $\mu(M)=\mu(g\cdot M)$ for every measurable set M and every $g\in\mathfrak{G}$.

Theorem. *For every Hausdorff topological abelian group $\mathfrak{G}$ with a collection $\{\mathfrak{H}_\iota\}_{\iota\in I}$ of subgroups satisfying* (i), (ii), (iii), *there exist ultrafilters $\mathscr{D}$ on I and ultraproduct Haar measures μ on $\mathfrak{G}$, with respect to $\{\mathfrak{H}_\iota\}_{\iota\in I}$ and $\mathscr{D}$.*

Proof. With $\mathfrak{G}$ as given, for each ι let μ_ι be a Haar measure on $\mathfrak{H}_\iota$. Let $\mathscr{F}$ be the filter on I generated by the sets $F_\eta = \{\iota | \mathfrak{H}_\eta \subseteq \mathfrak{H}_\iota\}$, $\mu \in I$. (By virtue of (ii) these sets do generate a filter.) Let $\mathscr{D}$ be an ultrafilter containing $\mathscr{F}$. The measurable sets will be those sets M such that for some $D \in \mathscr{D}$, $M \cap \mathfrak{H}_\iota$ is μ_ι-measurable for every $\iota \in D$. For measurable M, define $\mu(M) = [\{\mu_\iota(M \cap \mathfrak{H}_\iota)\}_{\iota \in I}]_{\mathscr{D}}$.

(A similar construction is in the Luxemburg notes. Another approach, by Abraham Robinson, avoids the use of ultrapowers.)

The properties (0) to (6) are consequences of the corresponding properties for the measures μ_ι. Proofs will be given only for (2), (5), and (6), since the remainder are almost immediate.

(2) Let $\{M_\beta\}_{\beta \in B}$ be as in (2). For each ι index the sets M_β for which $M_\beta \cap \mathfrak{H}_\iota$ is nonempty by an initial section of the positive integers, giving $n_\iota(\beta)$ to M_β. Let $n_\iota(\beta) = 0$ if $M_\beta \cap \mathfrak{H}_\iota = \emptyset$. For every $\beta \in B$, let $n_I(\beta)$ be the function $\{n_\iota(\beta)\}_{\iota \in I}$. Then, by (iii), if $\beta \neq \beta'$ there is an $\eta \in I$ for which $n_\eta(\beta) \neq n_\eta(\beta')$. Further, setting $F_\eta = \{\iota | \mathfrak{H}_\eta \subseteq \mathfrak{H}_\iota\}$, $n_\iota(\beta) \neq n_\iota(\beta')$ for every $\iota \in F_\eta$. Since $F_\eta \in \mathscr{D}$, $[n_I(\beta)]_{\mathscr{D}} \neq [n_I(\beta')]_{\mathscr{D}}$, and we may index the sets M_β by the ultrapower integers $[n_I(\beta)]_{\mathscr{D}}$.

Let $k_\iota = \sup_{\beta \in B} n_\iota(\beta) + 1$. Then $\boldsymbol{k} = [\{k_\iota\}_{\iota \in I}]_{\mathscr{D}}$ is an element of $N^{*I}{}_{\mathscr{D}}$. Define $\{f_\iota\}_{\iota \in I}$ by $f_\iota(n_\iota(\beta)) = \mu_\iota(M_\beta \cap \mathfrak{H}_\iota)$, $\beta \in B$, and let $\{\boldsymbol{a}_n\}_{n < \boldsymbol{k}}$ be the ultrapower sequence determined by $\{f_\iota\}_{\iota \in I}$. Since the measures μ_ι are σ-additive, we have for each ι, $\sum_{n < k_\iota} f_\iota(n) = \mu_\iota(\bigcup_{\beta \in B} M_\beta \cap \mathfrak{H}_\iota)$, and

$$\sum_{n < \boldsymbol{k}} \mu(M_n) = \sum_{n < \boldsymbol{k}} \boldsymbol{a}_n = [\{\sum_{n < k_\iota} f_\iota(n)\}_{\iota \in I}]_{\mathscr{D}} = \mu\left(\bigcup_{\beta \in B} M_\beta\right).$$

(5) Given an intersection-open set U, for each $\iota \in I$, the regularity of μ_ι gives us a family $\{C_{\iota,\alpha}\}_{\alpha \in A}$ of compact sets in $\mathfrak{H}_\iota$ with $C_{\iota,\alpha} \subseteq U$ and $\mu_\iota(U \cap \mathfrak{H}_\iota) = \sup_{\alpha \in A} \mu_\iota(C_{\iota,\alpha})$. Let $\mathscr{C} = \{C_{\iota,\alpha}\}_{\iota \in I,\ \alpha \in A}$. Then $\mu(U) = [\{\sup\{\mu_\iota(C) | C \in \mathscr{C}\}\}_{\iota \in I}]_{\mathscr{D}}$.

Similarly, given a measurable set M, for each $\iota \in I$, there is a family $\{W_{\iota,\beta}\}_{\beta \in B}$ of sets open in $\mathfrak{H}_\iota$ with $W_{\iota,\alpha} \supseteq M \cap \mathfrak{H}_\iota$ and $\mu_\iota(M \cap \mathfrak{H}_\iota) = \inf_{\beta \in B} \mu_\iota(W_{\iota,\beta})$. Let $V_{\iota,\beta}$ be the union of $W_{\iota,\beta}$ with the complement of $\mathfrak{H}_\iota$. Since $\mathfrak{H}_\iota$ is locally compact in $\mathfrak{G}$ and $W_{\iota,\beta}$ is open in $\mathfrak{H}_\iota$, $V_{\iota,\beta}$ is open in $\mathfrak{G}$. Let $\mathscr{V} = \{V_{\iota,\beta}\}_{\iota \in I,\ \beta \in B}$. Then $\mu(M) = [\{\inf\{\mu_\iota(V) | V \in \mathscr{V}\}\}_{\iota \in I}]_{\mathscr{D}}$. Since μ is monotonic, this completes the proof for (5).

(6) Let $g \in \mathfrak{G}$. By (iii) there is an $\eta \in I$ with $g \in \mathfrak{H}_\eta$. Let $F_\eta = \{\iota | \mathfrak{H}_\eta \subseteq \mathfrak{H}_\iota\}$. Then $g \in \mathfrak{H}_\iota$ for every $\iota \in F_\eta$. Hence, by the translation invariance of the μ_ι, for every measurable M we have $\mu_\iota(M \cap \mathfrak{H}_\iota) = \mu_\iota(g \cdot (M \cap \mathfrak{H}_\iota)) = \mu_\iota(g \cdot M \cap \mathfrak{H}_\iota)$ for every $\iota \in F_\eta$. Since $F_\eta \in \mathscr{D}$, this gives us $\mu(M) = \mu(g \cdot M)$.

Remarks. For $\mathfrak{G}$ a nonabelian group we modify the theorem in the obvious way to obtain a *left* and a *right* translation invariant measure.

If $\mathscr{D}$ is a nonprincipal ultrafilter (always the case if $\mathfrak{G}$ is not itself locally compact), then $\mathfrak{R}^I{}_{\mathscr{D}}$ is an η_1-field, in which denumerable sequences cannot converge. It is Cauchy-complete but not Dedekind-complete in its order topology (see, e.g., Gillman and Jerison [60, Chapter 13].) Further, if the index set B in (2) is infinite, the embedding of B in $N^I{}_{\mathscr{D}}$, although one-to-one is not onto. In a sense, our method of summation fills in the appropriate missing terms so that the sequence of partial sums for $\sum_{n<k}\mu(M_n)$ converges. For example, if B is denumerable and each $\mu(M_\beta)$ is real, the sequence of partial sums for $\sum_{n<k}\mu(M_n)$ is a nonstandard extension of a standard sequence, in the sense of Abraham Robinson [61a].

The real-valued Haar measure on a given locally compact group is unique only to within a multiplicative constant. An alteration in the measures μ_ι by real constants induces an alteration in μ by a constant in $\mathfrak{R}^I{}_{\mathscr{D}}$. But our μ is also dependent on the collection $\{\mathfrak{H}_\iota\}_{\iota\in I}$ and on the ultrafilter $\mathscr{D}$. Hence, we can hardly hope for an analogue for the classical uniqueness.

Frequently we may choose the $\mathfrak{H}_\iota$ to be the finitely generated subgroups. In this case μ, in an ultrapower way, *counts* the elements of the sets it measures. It is not difficult, however, to produce examples of groups in which there are finitely generated subgroups which are not locally compact.

If $\mathfrak{G}$ is a topological linear space over a locally compact field, we may always choose the $\mathfrak{H}_\iota$ to be the finite-dimensional subspaces.

3. Integration. The measure leads to an integration theory which contains analogues for many of the classical theorems. For sufficiently well-behaved real-valued functions f defined on $\mathfrak{G}$, we obtain an $\mathfrak{R}^I{}_{\mathscr{D}}$-valued integral $\int_{\mathfrak{G}} f = [\{\int_{\mathfrak{H}_\iota} f | \iota \in I\}]_{\mathscr{D}}$, and for $\mathfrak{R}^I{}_{\mathscr{D}}$-valued $\mathbf{f} = [\{f_\iota | \iota \in I\}]_{\mathscr{D}}$, $\int_{\mathfrak{G}} \mathbf{f} = [\{\int_{\mathfrak{H}_\iota} f_\iota | \iota \in I\}]_{\mathscr{D}}$. Lebesgue spaces of functions in the new setting usually must be interpreted in the larger sense (all $\mathfrak{R}^I{}_{\mathscr{D}}$-valued functions, satisfying analogues of the classical conditions). Where classical theorems speak of "sup" or "sequence", we must substitute "convergent net" or "sequence over $N^I{}_{\mathscr{D}}$". On the other hand, the Hölder and Minkowski inequalities, for example, hold virtually in their classical form.

ABSTRACTS pp. 424–437

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RELATIVELY CATEGORICAL AND NORMAL THEORIES

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Let $\Sigma(Q, x_1, \ldots, x_n)$ be an elementary formula on the predicate letters Q and free variables $x_1, \ldots, x_n$. The function $\mathfrak{c}Q$, which to the relational system Q assigns the relation $\hat{x}_1 \ldots \hat{x}_n\Sigma$ as value, is called an *elementary concept.* As a consequence of our general model-theoretic Theorem 1 we obtain,

Theorem. *Let* $\mathfrak{c}(\|, \perp)$ *be an elementary concept of elementary Euclidean geometry* $\mathfrak{E}(\|, \perp)$. *If, in every model of* $\mathfrak{E}$, $\mathfrak{c}$ *is an affine invariant (i.e., is invariant under automorphisms of* $\|$*), then* $\mathfrak{c}$ *is elementarily definable in affine geometry (i.e., there is an elementary concept* $\mathfrak{c}'(\|)$ *such that* $\mathfrak{c}(\|, \perp) = \mathfrak{c}'(\|)$ *holds in* $\mathfrak{E}$). *Similar statements hold for any pair of geometries in Klein's hierarchy* [1872].

It should be noted that the accent in this theorem lies on "elementary". If this word is dropped and we deal with a categorical situation, as Klein does in the Erlangerprogramm, the theorem becomes rather trivial. In fact, it is our main concern here to indicate how Klein's ideas can be made compatible with the axiomatic method, and may eventually lead to a general theory of invariants of elementary concepts in elementary model theory. Theorems 1 and 2 should be viewed as basic contributions to this new branch of model theory.

Definition 1. *A theory* $\mathfrak{A}(R, S)$ *is* categorical relative *to the primitive concepts* R, *if any two models* $<R, S_1>$ and $<R, S_2>$ of $\mathfrak{A}$ (*on the same domain and with identical* R) *are isomorphic.*

That this notion may be the key to a more precise formulation of Klein's program was suggested by Büchi and Wright [55], [57]. These papers and Wright [53] provide further illustrations of Theorem 1.

Theorem 1. *Let* $\mathfrak{A}(R, S)$ *be an elementary theory which is categorical relative* R. *Let* $\mathfrak{c}(R, S)$ *be an elementary concept. If, in every model of* $\mathfrak{A}$, $\mathfrak{c}$ *is invariant under all automorphisms of* R, *then* $\mathfrak{c}$ *is elementarily definable*

in $\mathfrak{A}$ *from* R *alone (i.e., there is an elementary concept* $\mathfrak{c}'(R)$ *such that* $\mathfrak{c}(R, S) = \mathfrak{c}'(R)$ *holds in* $\mathfrak{A}$*).*

If $\mathfrak{A}(R, S)$ is categorical relative R, we shall call the theory $\mathfrak{B}(R) = (\exists S)\mathfrak{A}(R, S)$ an *abstraction* of $\mathfrak{A}$. In this terminology Theorem 1 says: If $\mathfrak{B}(R)$ is an abstraction of the elementary theory $\mathfrak{A}(R, S)$, then every elementary concept $\mathfrak{c}(R, S)$ which is R-invariant in $\mathfrak{A}$ is represented by an elementary concept $\mathfrak{c}'(R)$ in $\mathfrak{B}$, in the sense that for every elementary proposition $\mathfrak{P}(C)$, $\mathfrak{A}(R, S) \models \mathfrak{P}(\mathfrak{c}(R, S))$ if and only if $\mathfrak{B}(R) \models \mathfrak{P}(\mathfrak{c}'(R))$.

Theorem 1 is a simple consequence of the definability theorem of Beth [53], or Craig's lemma [57]. An important task is to obtain general results on the hierarchy of abstractions of an elementary theory $\mathfrak{A}(Q)$, i.e., to give a survey of elementary concepts $\mathfrak{r}(Q)$ relative to which $\mathfrak{A}$ is categorical.

S_1 and S_2 may be called *conjugate relative* R, in symbols $S_1 \cong S_2(R)$, if $<R, S_1>$ is isomorphic to $<R, S_2>$, i.e., if there is an automorphism of R which takes S_1 onto S_2. That $\mathfrak{A}(R, S)$ is categorical relative R is to say that, in every model of $\mathfrak{A}$, there is relative to R but one conjugate class of S's; in Galois' terminology one would say that $\mathfrak{A}(R, S)$ is irreducible over $(\exists S)\,\mathfrak{A}(R, S)$. There is a second equivalence relation $S_1 \backsim S_2(R)$, which is even more fundamental for Galois theory, and which is defined as "$<R, S_1>$ and $<R, S_2>$ have the same automorphisms".

Definition 2. *The theory* $\mathfrak{A}(R, S)$ *is normal relative to the primitive concepts* R *in case any* R*-conjugate models of* $\mathfrak{A}$ *have the same automorphism group, i.e.,* $\mathfrak{A}(R, S_1) \wedge S_1 \cong S_2(R) \supset S_1 \backsim S_2(R)$.

If $\mathscr{K}(Q)$ denotes the automorphism group (*Klein-group*) of the relational system Q one easily proves;

Lemma. *The theory* $\mathfrak{A}(R, S)$ *is normal relative* R *just in case in every model of* $\mathfrak{A}$, $\mathscr{K}(R, S)$ *is a normal subgroup of* $\mathscr{K}(R)$.

The quotient $\mathscr{K}(R)/\mathscr{K}(R, S) = \mathscr{G}(R)$ (which exists in the normal case, and is independent of S) is properly called the *Galois group* of $\mathfrak{A}$ relative R, in the model $<R, S>$ of $\mathfrak{A}$.

Theorem 2. *Let* $\mathfrak{A}(R, S)$ *be an elementary theory which is normal relative* R*. There is a number* n*, and there are elementary concepts* $\mathfrak{d}_1(R, S), \ldots, \mathfrak{d}_n(R, S)$ *such that in every model* $<R, S>$ *of* $\mathfrak{A}$ *every conjugate* S' *of* S *relative* R *is one of the relations* $\mathfrak{d}_1(R, S), \ldots, \mathfrak{d}_n(R, S)$*, i.e.,* $\mathfrak{A}(R, S) \wedge S' \cong S(R) . \supset . S' = \mathfrak{d}_1(R, S) \vee \ldots \vee S' = \mathfrak{d}_n(R, S)$.

Corollary. *Let $\mathfrak{A}(R, S)$ be an elementary theory which is normal relative R. Then in every model of $\mathfrak{A}$ the Galois group $\mathcal{G}(R)$ is finite, and in fact has $\leqslant n$ members, and hence n depends on $\mathfrak{A}$ only.*

Theorem 2 is a consequence of the definability theorem of Svenonius [59a]. The definability results of Beth and Svenonius thus are basic for a theory of invariance of elementary concepts. Do these tools suffice to translate a substantial part of classical Galois theory? If not, a careful investigation might lead to additional basic results on elementary definability.

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FREE STRUCTURES AND CATEGORIES

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The aim of this report is to give (a) a description in usual mathematical terms of Bourbaki's [57] *espèces de structure*, which will correspond to our *systems of structures*, and (b) necessary and sufficient conditions for the existence of free structures in such systems with respect to certain good mappings. As to (a), this may be done in the terminology of categories (cf. similar approaches by Ehresmann [57], [60], Freyd [*] and Chevalley [1]); for the sake of convenience we prefer here a formulation using a set theory with classes. Along these lines one can reproduce all of Bourbaki's developments, simplified, e.g., by the use of duality principles. Also the work of Krishnan [51], [53] can be reformulated and shortened considerably, making use of – in Bourbaki's terminology – the transitivity of initial structures. As to (b), our solutions grew out of an analysis of Bourbaki's sufficient criteria (cf. also Freyd [*]).

Let K, K_0 be two categories with classes of units K^e, $K_0{}^e$, respectively. (For definitions cf. Grothendieck [57].) If f is in K or K_0 denote by e_f the right unit and by e_f' the left unit of f; it should be remembered that for units a, b the class Hom (a, b) of all f with right unit a and left unit b is assumed to be a set. Let p be a functor from K into K_0; (K, K_0, p) is called a *structural triplet* if and only if:

(1) if $f, g \in K$, $e_f = e_g$, $e_f' = e_g'$, and $p(f) = p(g)$, then $f = g$;
(2) if $a \in K^e$, $x \in K_0$, $e_x = p(a)$, and x is invertible, then there exists a unique $f \in K$ such that $e_f = a$, $p(f) = x$, and f is invertible;
(3) if $x \in K_0{}^e$, then $p^{-1}(x)$ is a set.

For an interpretation take K_0 to be the category of all functions between sets and $K_0{}^e$ to be the universal class $\mathcal{U}$; let $\mathcal{S}$ be a class, let q be a function from $\mathcal{S}$ into $\mathcal{U}$, assigning to each "structure" S in $\mathcal{S}$ its basic

[1] In his talk "Definition of Algebraic Structures" to a joint Berkeley–Stanford Mathematics Colloquium on November 1, 1962.

set $q(S)$, and let Hom be a function, assigning to each pair S, T of structures the set Hom (S, T) of morphisms from S to T as a subset of $q(T)^{q(S)}$; if we demand that the composition of two morphisms is again a morphism, we obtain a category K with $K^e = \mathscr{S}$ and a functor p from K into K_0 fulfilling (1); $(\mathscr{S}, q, \text{Hom})$ is a *system of structures* if, moreover, (2) and (3) hold.

Let (K, K_0, p) be a structural triplet and assume, further, that K_0 is a category with subobjects (cf. Grothendieck [57]) such that for every $x \in K_0{}^e$ the class $P(x)$ of all right units of subobjects of x is a set. Let $B \subseteq K^e$; by definition let B-erz (z, c) if and only if $z \in K_0$, $c \in B$, $e_z{}' = p(c)$ and for all $f, g \in K$: if $e_f = e_g = c$, $e_f{}' = e_g{}'$, $e_f{}' \in B$, and $(p(f))z = (p(g))z$, then $f = g$. Let $e \in K_0{}^e$; assume that for any $b \in K^e$ there is given a class Good $(e, b) \subseteq$ Hom $(e, p(b))$ such that $y \in$ Good (e, b), $f \in K$, and $e_f = b$ imply $(p(f))y \in$ Good $(e, e_f{}')$ – this situation naturally arises if there is a third category K_1 between K and K_0 and p splits into functors from K into K_1 and from K_1 into K_0. A pair (a, x) is called *B-semifree* if and only if $a \in B$, $x \in$ Good (e, a), and for any $b \in B$ and $y \in$ Good (e, b) there exists $f \in$ Hom (a, b) such that $(p(f))x = y$; if, moreover, f is unique, (a, x) is called *B-free*. By definition let bound (d, B) if and only if $d \in K_0{}^e$ and for any $b \in B$ and $y \in$ Good (e, b) there exist h, z, j such that $h \in K$, $e_h{}' = b$, $z \in$ Good (e, e_h), $(p(h))z = y$, B-erz(z, e_h), $j \in$ Hom $(p(e_h), d)$ and j is monomorphic. The main result then is the equivalence of the following three conditions:

(i) there exists a K^e-free pair (a, x);

(ii) there exists a d such that bound (d, K^e) and for any $B \in \mathscr{U}$, if $B \subseteq K^e$ then there exists a B-semifree pair (a, x);

(iii) there exists a d such that bound (d, K^e) and there exists a $(p^{-1}(P(d)) \cap K^e)$-semifree pair (a, x).

Defining in the obvious manner the notion of *L-semifree* for classes $L \subseteq K$, one may replace B and K^e in (ii) by L and K. This modified condition (ii) will hold if K admits products and if, for a family of good functions, the product function is again good: if L is a set of elements y, $y \in$ Good (e, b_y), then the product of $(b_y)_{y \in L}$ together with the product function of $(y_y)_{y \in L}$ is L-semifree. For systems of structures these are just Bourbaki's sufficient criteria.

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FINITE-DIMENSIONAL ANALOGUES TO BOOLEAN ALGEBRAS

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Problems concerning the representability of finite-dimensional cylindric and polyadic algebras are for the most part difficult and unsolved. Some of these questions are here answered for a simpler yet closely related class of algebras. All the undefined model-theoretic notation used here can be found in Tarski [54, 55], and the letters "n" and "k" will always denote positive integers.

For each n, the class $\mathscr{T}_n$ of transformation equality algebras of degree n consists of those algebraic structures $\langle A, +, \cdot, -, S, E\rangle$ which satisfy all the axioms not mentioning quantifiers for a polyadic algebra of degree n with equality E (see Halmos [57, pp. 3, 4]). The class $\mathscr{SRT}_n$ of strongly representable $\mathscr{T}_n$ algebras consists of those $\mathscr{T}_n$ algebras isomorphic to concrete algebras of n-ary relations (see Howard [63]).

Lemma. *For all n, $\mathscr{SRT}_n$ is an (axiomatizable) $UC_\Delta(A)$ class.*

This is proved by first showing $\mathscr{SRT}_n$ to be a PC' class (i.e., a class of relativized reducts of an EC class) by adding the condition of strong representability, and then using Theorem 1.9 of Tarski [54].

Theorem. *For $n>2$, $\mathscr{SRT}_n$ is not a $UC(A)$ class.*

Before sketching the proof it may be remarked that the theorem is false for $n=2$ (see Howard [62]), and in neither case is $\mathscr{SRT}_n$ equational since it is not closed under direct products.

The theorem can be proved using a modified version of a condition of Vaught [54b, p. 589] according to which it is sufficient to prove that for each k, there is a $\mathscr{T}_n$ algebra $\mathfrak{A}_k$ such that:

(i) $\mathfrak{A}_k \notin \mathscr{SRT}_n$,
(ii) $\mathscr{G}_k(\mathfrak{A}_k) \subseteq \mathscr{SRT}_n$,

where $\mathscr{G}_k(\mathfrak{A}_k)$ is the set of all subalgebras of $\mathfrak{A}_k$ generated by k or fewer elements.

The heart of the construction of $\mathfrak{A}_k$ is the use of $\mathfrak{B}_k$, the concrete algebra over the set of integers $m+n+1$, where $m=2^{n!k}$, generated by the following relations: $P=\bigcup\{\{j\}^n | j\leqslant m\}$; $R_i=\{m+1+i\}^n$ for $i<n$; $S_i=\{\langle m+1, i, i, \ldots, i\rangle\}$ and $b_i=\{\langle i, m+2, m+3, \ldots, m+n\rangle\}$ for $i\leqslant m$.

It is clear that the elements $b_0, \ldots, b_m$ are not distinguishable in $\mathfrak{B}_k$; that is, any permutation of them can be extended to an automorphism of $\mathfrak{B}_k$. Roughly, $\mathfrak{A}_k$ is formed by adding a new element b_{m+1}, together with the minimum necessary to form a $\mathscr{T}_n$ algebra, such that the elements $b_0, \ldots, b_{m+1}$ are not distinguishable in $\mathfrak{A}_k$. In $\mathfrak{A}_k$ the b_i's determine ideals I_i such that:

(a) any member of $\mathscr{G}_k(\mathfrak{A}_k)$ is disjoint from some I_i;

(b) for each i, $\mathfrak{A}_k$ (modulo I_i) $\cong \mathfrak{B}_k$.

Condition (ii) above follows easily from (a) and (b). It remains to prove (i) above, and it suffices to give a sentence $\varphi \in Th(\mathscr{SRT}_n)$ which fails in $\mathfrak{A}_k$. One such φ is formed as follows. Let η be a diagram of $\mathfrak{A}_k$ containing the variables $\mathsf{v}_0, \ldots, \mathsf{v}_p$, in which $\mathsf{v}_0, \ldots, \mathsf{v}_{m+1}$ are associated, respectively, with $b_0, \ldots, b_{m+1}$, and let 0 be the zero element of the algebra. Let μ be the disjunction of the formulas $\mathsf{v}_0 \doteq 0, \ldots, \mathsf{v}_{m+1} \doteq 0$. Finally, let φ be

$$\forall \mathsf{v}_0 \ldots \forall \mathsf{v}_p(\eta \rightarrow \mu).$$

It is clear that φ fails in $\mathfrak{A}_k$. The fact that $\varphi \in Th(\mathscr{SRT}_n)$ can be proved using a "natural" correspondence between universal sentences in $Th(\mathscr{SRT}_n)$ and a certain type of theorem of logic. This finishes the sketch of the proof.

It may further be remarked that the subdirectly indecomposable $\mathscr{T}_n$ algebras are strongly representable, and hence all $\mathscr{T}_n$ algebras are representable in the sense used in the theory of cylindric algebras. This was discovered independently by J. D. Monk.

The problem of a good axiomatization and an algebraic characterization of $\mathscr{SRT}_n$ remain. As to the latter, Craig has pointed out that the characterization can not be expressed by a prenex sentence of second-order logic in which all predicate quantifiers are universal (as can the properties of being directly or subdirectly indecomposable). This follows from the theorem, the fact that $\mathscr{SRT}_n$ is a PC' class, and Theorem 1.13 of Keisler [61d].

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BOOLEAN RECURSIVE FUNCTIONS AND CLOSURE ALGEBRA

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Let $x=x(n)$ be a variable two-valued (either 0 or 1) function of n whose domain is $\{0, 1, 2, \ldots\}$ and $u=u(n)$ a two-valued recursive function defined by

$$\begin{cases} u(0)=u_0=0 \text{ or } 1, \\ u(n+1)=G\{u(n), x(n)\}. \end{cases} \tag{1}$$

where G is a Boolean function of u and x. As this "state" function is uniquely determined by x when G is known, we can write u as

$$u=g(x). \tag{2}$$

Further, if we take any two-valued function $y=y(n)$ defined by

$$y(n)=F\{u(n), x(n)\}, \tag{3}$$

where F denotes any Boolean function of u and x, then this "output'' function y can be written as follows:

$$y=F(u, x)=F\{g(x), x\}=f(x). \tag{4}$$

We see, therefore, that y is also uniquely determined by the "input" function $x=x(n)$ when F is given.

Now let us choose G and F as follows:

$$G(u, x)=F(u, x)=u \vee x \tag{5}$$

and denote the above output function $f(x)$ by "$\Diamond x$".

The $\Diamond x$ function thus defined can be shown to satisfy the following four axioms of closure algebra:

$$\Diamond 0=0, \tag{I}$$

$$\Diamond 1=1, \tag{II}$$

$$\Diamond \Diamond x=\Diamond x, \tag{III}$$

$$\Diamond(x \vee y)=\Diamond x \vee \Diamond y, \tag{IV}$$

where "0" and "1" denote the two constant functions whose values are 0 and 1, respectively.

The set of Boolean recursive functions with the closure (modal) operation $\Diamond$ defined above gives a model for closure algebra and also for Lewis' modal logic S4.

From the above considerations we can also derive a decision procedure for any given formula in the closure algebra or the modal logic S4 which can be realized by sequential circuits.

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A UNIFYING PRINCIPLE IN QUANTIFICATION THEORY

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For any subset M of the domain $D(R)$ of a binary relation R, call M *conjunctively closed under* R iff for every a in M all members of $R(a)$ (i.e., all x such that aRx) are also in M. Call M *disjunctively closed under* R iff for every a in M there is at least one member of $R(a)$ in M. Now consider an abstract system $\langle E, R_1, R_2, \Sigma, \Gamma\rangle$, where E is a denumerable set, R_1 and R_2 are binary relations on E with disjoint domains D_1 and D_2, Σ is a collection of subsets of E, and Γ is a property defined on *finite* subsets of E. Let Σ' be the collection of all elements W of Σ such that all finite subsets of W have the property Γ. Now suppose that for every W in Σ' and every a in W the following two conditions hold.

(C_1) If $a \in D_1$, then for every x in $R_1(a)$, $W \cup \{x\} \in \Sigma'$.

(C_2) If $a \in D_2$, then for at least one x in $R_2(a)$, $W \cup \{x\} \in \Sigma'$.

Lemma 1. *Under conditions* (C_1) *and* (C_2), *every element* W *of* Σ' *can be extended to a subset* W^* *of* E *such that:* (1) *every finite subset of* W^* *has the property* Γ; (2) W^* *is conjunctively closed under* R_1 *and disjunctively closed under* R_2.

Now let E be the set of all sentences (closed formulas) of quantification theory, as formulated with individual variables x_i and individual constants a_i for $i \in \omega$. For present brevity, assume $\sim, \wedge, \forall$ are the primitive logical symbols. Let R_1 be the set of all ordered pairs in $E \times E$ of one of the forms $\langle F \wedge G, F\rangle$, $\langle F \wedge G, G\rangle$, $\langle \sim\sim F, F\rangle$, $\langle (\forall \mathsf{x})F, F^{\mathsf{x}}_{\mathsf{a}_i}\rangle$; let R_2 be the set of all those of one of the forms $\langle \sim(F \wedge G), \sim F\rangle$, $\langle \sim(F \wedge G), \sim G\rangle$, $\langle \sim(\forall \mathsf{x})F, \sim F^{\mathsf{x}}_{\mathsf{a}_i}\rangle$. Call a property Γ (of *finite* subsets of E) an *abstract* (*cut-free*) *consistency property* iff conditions (C_1), (C_2) hold, as well as:

(C_3) No set containing an atomic sentence and its negation has the property Γ.

Take for Σ the collection of all subsets W of E such that only finitely many constants occur in (the sentences of) W. A set W^* obeying the

conclusion of Lemma 1 is a *model set*, in the sense of Hintikka, and hence is denumerably satisfiable. Thus Lemma 1 yields:

Theorem 1. *Let Γ be any abstract consistency property, and let W be any set of sentences which involves only finitely many parameters. Then if all finite subsets of W have property Γ, then W is simultaneously satisfiable in a denumerable domain.*

By considering various appropriate abstract consistency properties, we obtain several well-known theorems (and some new ones) in one fell swoop. For example, satisfiability itself (restricted to finite sets) is easily seen to be an abstract consistency property, hence we at once have both the Skolem-Löwenheim theorem and the compactness theorem for first-order logic. For any of the usual Hilbert-type axiomatic systems for quantification theory, formal consistency within the system is an abstract consistency property; hence Gödel's completeness theorem results. Consistency in Gentzen's classical system G1 *without* the cut rule is an abstract consistency property, hence Theorem 1 yields a (non-finitary) proof of Gentzen's Hauptsatz. For Craig's interpolation lemma, define $\Gamma(M)$ to mean that there exists a partition $\langle M_1, M_2\rangle$ of M for which there is no interpolation formula—i.e., no sentence F for which: (i) every constant and predicate of F occurs in at least one element of M_1 and in at least one element of M_2; (ii) neither $M_1 \cup \{F\}$ nor $M_2 \cup \{\sim F\}$ is satisfiable. We also obtain some new variants of Herbrand's theorem and Gentzen's extended Hauptsatz, and some results concerning a new form of linear reasoning.

Our present version of Lemma 1 is slightly stronger than that of Smullyan [63], and our present definition of "abstract consistency property" is a bit weaker; hence our present Theorem 1 is a bit stronger.

Errata to Smullyan [63]. (1) Application H contains a basic error, and should be entirely deleted. (2) In clause (ii) of the definition of "associate" in (F), replace "subformula" by "subformula or the negation of a subformula".

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$2^{\aleph_0}$ CAN BE ANYTHING IT OUGHT TO BE

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We consider countable models for Zermelo–Fraenkel set theory (including the axiom of choice), which we call, for brevity, ZF*-models. Let $\mathfrak{M}$ $(=(M, \in_M))$ and $\mathfrak{N}$ be ZF*-models. Then $\mathfrak{N}$ is an *excellent extension* of $\mathfrak{M}$ if and only if: (a) $\mathfrak{M}$ is a complete submodel of $\mathfrak{N}$; (b) the ordinals (cardinals) of $\mathfrak{N}$ are exactly the ordinals (cardinals) of $\mathfrak{M}$; and (c) the confinality, $\mathrm{cf}(\aleph)$, of a cardinal $\aleph$ has the same value in $\mathfrak{M}$ as in $\mathfrak{N}$.

The proofs of the following three theorems use the new techniques introduced in Cohen [63, 64]. (The first theorem is due, independently, to Cohen.) Let $\mathfrak{M}$ be a ZF*-model in which $V=L$ is valid, fixed once for all.

Theorem 1. *Let $\aleph$ be an infinite cardinal of $\mathfrak{M}$ with $\aleph_0<\mathrm{cf}(\aleph)$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $2^{\aleph_0}=\aleph$.*

Theorem 2. *Let $\aleph$ and $\aleph'$ be infinite cardinals of $\mathfrak{M}$ with $\aleph=\mathrm{cf}(\aleph)<$ $<\mathrm{cf}(\aleph')$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which:*

(i) $2^{\aleph}=\aleph'$;

(ii) if $\aleph_\alpha<\aleph$, then $2^{\aleph_\alpha}=\aleph_{\alpha+1}$.

Theorem 3. *Identify the ordinary integers with an initial segment of the integers of $\mathfrak{M}$. Let $k, n_0, \ldots, n_k$ be ordinary integers and suppose that $i<n_i$ (for $0\leqslant i\leqslant k$) and $n_0\leqslant n_1\leqslant \ldots \leqslant n_k$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $2^{\aleph_i}=\aleph_{n_i}$ (for $0\leqslant i\leqslant k$).*

Remarks on the proof Theorem 2. The essential point is to make sure that no "new" sets of cardinality less than $\aleph$ land in $\mathfrak{N}$. This is insured as follows: (1) we add a generic subset of $\aleph'$, say A; (2) a set of conditions on A will be a set (of $\mathfrak{M}$) of conditions of the form "$\alpha\in A$" (or "$\neg\,\alpha\in A$") whose *cardinality in $\mathfrak{M}$ is less than $\aleph$*. (Of course, outside of $\mathfrak{M}$, the sets of conditions are denumerable since $\mathfrak{M}$ is, so Cohen's diagonal construction applies.) The crucial observation is the following:

Lemma. *Let Σ be a set (in $\mathfrak{M}$) of limited statements whose cardinality (in $\mathfrak{M}$) is less than $\aleph$. Then any set of conditions P has an extension forcing each member of Σ.*

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CONSTRUCTION OF A MODEL FOR GÖDEL–BERNAYS SET THEORY FOR WHICH THE CLASS OF NATURAL NUMBERS IS A SET OF THE MODEL AND A PROPER CLASS IN THE THEORY

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We consider Gödel–Bernays set theory. The primitive notions of this theory are *Cls* (class), M (set), and $\in$ (element relation). We denote this theory by Σ (axiom groups A, B, C, D) or Σ^* (groups A, B, C, D, E).

Let $\bar{E}$ be a relation. We write

$$x \in \Phi(y) \;.\equiv.\; \langle xy \rangle \in \bar{E}.$$

DEFINITION. *An* inner semimodel *of the theory Σ consists of a relation $\bar{E}$ and a propositional function $\bar{\psi}(X)$. We write:*

$$Cls^*(X) \;.\equiv.\; \bar{\psi}(X);$$
$$M^*(X) \;.\equiv.\; (\exists y)[X = \Phi(y)];$$
$$X \in^* Y \;.\equiv.\; Cls^*(Y) \;\&\; (\exists y)[X = \Phi(y) \;\&\; y \in Y].$$

This semimodel will be denoted by $(\bar{E}, \bar{\psi})$.

DEFINITION. *Let $(\bar{E}, \bar{\psi})$ be a semimodel. We say that $(\bar{E}, \bar{\psi})$ is an* inner model *of the theory Σ if and only if the axioms of Σ are true for the propositions Cls^*, M^*, $\in^*$.*

In Vopěnka [62] I have found necessary and sufficient conditions relative to the relation $\bar{E}$ that there exist a propositional function $\bar{\psi}$ such that $(\bar{E}, \bar{\psi})$ is a model of the theory Σ.

Let $(\bar{E}, \bar{\psi})$ be an inner model of the theory Σ. We denote by On^* the class of all ordinal numbers of the model $(\bar{E}, \bar{\psi})$ and by ω_0^* the class of all natural numbers. Obviously $Cls^*(On^*)$ and $M^*(\omega_0^*)$. The class On^* is ordered by the relation $\bar{E}$.

DEFINITION. *We say that the model $(\bar{E}, \bar{\psi})$ is* standard *if and only if the class On^* is well ordered by the relation $\bar{E}$. In the contrary case we say that the model $(\bar{E}, \bar{\psi})$ is* nonstandard.

DEFINITION. *We say that the model* $(\bar{E}, \bar{\varphi})$ *is* regular *if and only if* $(x)M(\Phi(x))$. *In the contrary case we say that the model* $(\bar{E}, \bar{\varphi})$ *is* nonregular.

EXAMPLES. (1) The model (E, Cls) is standard and regular. (2) The model $(\prod_{x \in s} E_x/j$ (where $E_x = E$), $X \subseteq \mathfrak{W}(\bar{E})$ & $(x)(\exists y)[x \in \mathfrak{W}(\bar{E}) \rightarrow$ $\rightarrow y \in \mathfrak{W}(\bar{E})$ & $\Phi(y) = X \cap \Phi(x)]])$ is nonstandard and regular. (See Vopěnka [62a].)

DEFINITION. *Suppose* $\theta \in On$ *or else* $\theta = On$. *For each* $\alpha \in \theta$ *let* E_α *be a relation. Suppose that for* $\alpha \in \beta \in \theta$ Ω_β^α *is a monomorphism of* E_α *into* E_β. *Suppose* $\Omega_\gamma^\beta(\Omega_\beta^\alpha(x)) = \Omega_\gamma^\alpha(x)$ *for* $\alpha \in \beta \in \gamma \in \theta$ *and* $\Omega_\alpha^\alpha(x) = x$. *Let* $x \in E_x$. *Now set*

$$i(x) = min\ \{\beta | (\exists y)[y \in \mathfrak{W}(E_\beta)\ \&\ x = \Omega_\alpha^\beta(y)]\},$$
$$\langle xy \rangle \in Sp_\theta E_\alpha \Omega_\beta^\alpha .\equiv. x \in E_{i(x)}\ \&\ y \in E_{i(y)}\ \&\ (\exists \alpha)[i(x), i(y) \in \alpha \in \theta$$
$$\&\ \langle \Omega_\alpha^{i(x)}(x) \Omega_\alpha^{i(y)}(y) \rangle \in E_\alpha].$$

This construction is described in detail in Vopěnka [63a].

CONSTRUCTION OF A NONSTANDARD, NONREGULAR MODEL. Let j be a nontrivial ultrafilter on an arbitrary infinite set s. Let $E_0 = E$, $E_\alpha = \prod_{x \in s} E_x/j$, where $E_x = E_\beta$ for $\alpha = \beta + 1$. $\Omega_\alpha^\beta(x)$ is the function f defined on s such that $f(u) = x$ for each $u \in m$, where $m \in j$. For $\gamma \in \beta$ we set $\Omega_\alpha^\gamma(x) = \Omega_\alpha^\beta(\Omega_\beta^\gamma(x))$. $E_\alpha = Sp_\alpha E_\beta \Omega_\beta^\gamma$ for limit α. $\Omega_\alpha^\beta(x) = y$, where $y \in \mathfrak{W}(E_{i(y)})$ & $\Omega_\beta^{i(y)}(y) = x$. We set $\bar{E} = Sp_{On} E_\alpha \Omega_\beta^\alpha$.

The semimodel
$(\bar{E}, X \subseteq \mathfrak{W}(\bar{E})$ & $(\alpha)(x)[x \in \mathfrak{W}(E_\alpha) \rightarrow (\exists y)[y \in \mathfrak{W}(E_\alpha)$ & $\Phi(y) =$
$= \Phi(x) \cap X \cap \mathfrak{W}(E_\alpha)]])$ is a model of the theory Σ. (See Vopěnka [63b].)

In this model we have $Pr(\omega_0{}^*)$, but $M^*(\omega_0{}^*)$. If the ultrafilter j is a measure on the set s, we obtain a nonstandard, nonregular model such that $\omega_0{}^* = \omega_0$.

BIBLIOGRAPHY WITH EXPLANATORY NOTES
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SOME NOTES ON THE THEORY OF MODELS [1]

J. W. ADDISON

What is "the theory of models"? Few words used in science today have meanings which vary so much from one context to another as the word "model". In deciding whether to include or exclude particular items from this "bibliography of the theory of models", it has been necessary to delineate, at least roughly, some boundary lines for the subject.

It is well known that science is multidimensional and continuous, and that the boundary of any domain within it inevitably divides organically related subject areas. To attempt to find a "natural" boundary which avoids such ad hoc divisions runs counter to all of our experience with developing science.

The boundary lines that we have chosen, therefore, are rather to be understood in terms of their relation to presently existing notions, than by reference to their inherent suitability. And indeed it is one of the ironies of our task that we have in the past bespoken the artificiality of some of the very boundary lines here adopted.

Some decisions were clearcut and obvious. As between the uses of the word "model" by logicians on the one hand and most other scientists on the other, we intend the former usage. That is, a model is considered to be a structure related to a given theory, rather than a theory intended to explain a given realm of phenomena. The theory of models is thus to be considered as a branch of logic in the wide sense of the term that includes metalogic and metamathematics.

Now in the present state of the science most individual logical investigations begin with the specification of one or more grammars (or uninterpreted formalized languages) which will be involved in the study. Beyond this common base for investigations the study of logic splits roughly into two directions. In one, the grammars are used as a basis for formal deductive systems (generally specified by axioms and rules

[1] The author is indebted to Leon Henkin, who read an earlier draft of this essay and suggested improvements, and to Alfred Tarski for long discussions on the theory of models.

of inference); in the other, the grammars are converted to interpreted languages (by specification of meanings to the symbols, and to the methods of combining them, which make up the grammar). The adjective "proof-theoretic" is often applied to the first direction, "semantic" to the second.

Sometimes "model-theoretic" is used interchangeably with "semantic", and this suggests that "the theory of models" might simply be identified with "semantics", i.e., with the study of the relationship between grammars and schemes of interpreting them. However, there is also a much narrower sense in which "theory of models" is used. And since there is already the well-established usage of "semantics" for the broader sense it seems best to reserve "the theory of models" for its narrower meaning. Moreover, in the present context this becomes a practical necessity in order to hold the length of the bibliography within reasonable bounds.

It should be emphasized here that in order to qualify as model-theoretic (or even as semantic) a study must make *"essential" use of languages*, not merely mention them superficially. For example, the mathematical notion of a group can be (and often is) defined as "a structure satisfying the following conditions . . .", which is, strictly speaking, an informal but nonetheless *metamathematical* definition, involving as it does the notion of satisfaction. But since this definition is easily and naturally rephrased in purely *mathematical* terms, we do not regard group theory as a branch of model theory. Indeed, to decide otherwise would lead us to regard all of mathematics (and a great deal more) as a branch of the theory of models. For a similar reason a study of "models of Riemannian geometry" will usually not be, despite the name, a part of the theory of models.

On the other hand, a study of models of Zermelo-Fraenkel set theory or of models for first-order number theory normally will fall within the theory of models — here, among other reasons, the axiomatization by a schema tends to make the genuine use of formal languages virtually essential.

It remains to indicate what proper part of semantics we reserve for "the theory of models". One way to mark out such a region is to conceive the whole of semantics as constituting a space determined by various linear components, and to select "intervals" on each of the coordinate axes.

Along one axis we can measure the relative logical power of the language which underlies a study, ranging from propositional through first-, second-, and higher-order predicate languages. Here we find that the

division between propositional and first-order languages tends to be a well-defined boundary dividing the subject fairly well according to the "flavor" or "spirit" of the work. Although the elements of propositional semantics all play a role in predicate semantics, the latter have in addition certain ingredients that give the work a distinctive character. In order to achieve a bibliography of workable size we here restrict "the theory of models" to predicate semantics.

Along another axis we can gauge the degree of abstractness, or generality, with which a study is carried out, ranging from the concrete to the highly algebraic. For example, should some or all of the theory of cylindric and polyadic algebras be included? Here we draw a somewhat vague, admittedly subjective, line as follows: The algebraic theory is included provided it is used to make applications to, or to gain direct insight into, the concrete theories from which it is abstracted. On the other hand where the algebraic questions are investigated purely for their own sake we exclude a study from the theory of models.

Along a third axis we consider how much is fixed by the scheme of interpretation of a language – i.e., what symbols are genuine *constants* whose meanings are completely fixed. In a first category are those languages where varying interpretations are allowed even to the usual "logical" symbols. For example, such languages are natural for use in the investigation of the independence of logical axioms. In a second group are those "pure" languages where the usual logical symbols are given a fixed meaning but the other, "nonlogical," symbols present are allowed varying interpretations. For example, in the study of nonstandard models of number theory it is natural to use languages where the symbols for addition and multiplication are *parameters* (i.e., symbols whose meanings are allowed to vary, although they are not allowed to be quantified in the formulas of the language). In a third category are "applied" languages where the logical symbols as well as certain nonlogical symbols have fixed meanings. For example, in the study of the definability of sets of natural numbers it is convenient to use languages where symbols for addition and multiplication, as well as the usual logical symbols, are constants. In this group of languages it is typical that the *universe is also fixed* by the scheme of interpretation (even though there is normally not a special symbol for the universe in the language). The study of this last group, including as it does most of the work on hierarchies in recursive function theory, is of considerable extent and (again in part simply to preserve a workable limitation on size) we make our division between the

second and third groups so as to exclude this work. Note that this division does not exclude studies of the socalled ω-models of second-order number theory, for although the universe of individuals is fixed in these studies as the set of natural numbers, and there may even be constants present for, say, addition and multiplication, the universes corresponding to the other sorts of variables are not fixed; on the other hand studies of second-order definability of sets of natural numbers would be excluded.

Along a fourth axis we can consider what group of logical symbols are in the language, and in case they are given a fixed meaning what that meaning is – here we range across the spectrum of classical, intuitionistic, modal, many-valued, infinitary, etc., languages. It seems desirable to include this entire axis within the theory of models.

Now the region we have delineated thus far might very roughly be described as concrete, pure, predicate semantics. But this is still a very large domain, and includes a great deal that, at least to many observers, would not fall within the theory of models. What more is required? For one thing, to lie properly in the theory of models a study should really make "*essential*" *use of structures* as well as of languages. Vaguely, it is necessary that the structures (or more general kinds of entities) which underlie the interpretation of the grammars under consideration should be considered as individuals – that we construct them, perform operations on them, classify them, or determine relations that hold among them. For example, we would not consider as part of the theory of models any usual proof (either expressed semantically or given deductively with the help of the completeness theorem) that each first-order formula is semantically equivalent to a formula in prenex normal form – even though to say that two formulas are semantically equivalent means simply that they have the same *models*. Or as another example, a paper showing by the construction of a model that axiom φ of set theory is independent of axiom ψ would normally be included, especially if a metamathematical element, such as induction on formulas, were involved in the proof; but a paper showing that φ is independent of axiom ψ because if it were not then θ (which has earlier been shown by construction of a model to be independent of ψ) would not be independent after all, would not be included.

A BIBLIOGRAPHY OF THE THEORY OF MODELS

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During the symposium Karel de Bouvère made available to participants a 402-item preliminary version of a bibliography of the theory of models. The suggestion that this useful work be made the basis of a unified bibliography for this volume was made by the Organizing Committee and adopted by the Editors. Following the Symposium the Editors assigned to J. W. Addison responsibility for perfecting and completing the bibliography, after setting the basic policies regarding areas of research to be included and excluded, as well as bibliographic notation and conventions.

A questionnaire was sent to all participants in the Symposium asking for their collaboration in the completion of the bibliography; many thoughtful and helpful replies were received. These replies, additional suggestions from de Bouvère, and a very substantial amount of independent searching by William Pitt led to the present version of the work; in addition, Pitt undertook to verify by inspection as many of the items as possible and to put all entries into a uniform style. The final decision on whether or not to include each particular item was made by the editors.

The present 841-entry bibliography serves not only as a reference for the theory of models, but also as a unified bibliography for all of the papers in this volume. Some of the papers cited by authors are, of course, outside the theory of models as outlined in the preceding essay. These papers, and a number of others that were added because they are frequently used in work in model theory, have been marked with a "+" to distinguish them from the central body of the bibliography. Altogether there are 168 "auxiliary" entries thus marked with a "+".

Items in the bibliography are first arranged alphabetically by the last name of the author. The papers of a given author are then arranged chronologically by year of publication, and are cited in the articles by the last name of the author and the last two digits in the year of publication — e.g. "Skolem [20]". In those cases where a given author has more than one paper included for a given year, these items are arranged alphabetically by title and labeled, for example, "Beth [53]", "Beth [53a]", "Beth [53b]", and "Beth [53c]". In general only items which had been published or accepted for publication by the end of 1963 are included. Forthcoming items were labeled, for example, "Keisler [a]", "Keisler [b]", and "Keisler [c]". Papers in this volume itself are listed in the bibliography and signaled by "[*]".

Since abstracts are sometimes more accessible than papers, and since they are occasionally useful even when a paper on the subject has appeared, they have been included. Where it was known, however, that they had been essentially subsumed under a later paper, abstracts are listed in brackets under the same item as the paper. Similar conventions are followed for doctoral dissertations later published in a journal, for reprintings, for translations, etc. Papers appearing in two parts published in the same year have been combined under one heading to save space. Altogether the 841 entries represent in total more than 940 separate references.

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